Check for updates

Statistical Analysis and Data Mining: An ASA Data Science Journal

WILEY

RESEARCH ARTICLE OPEN ACCESS

Statistical Detection of Adversarial Compliance With Benford's Law

Lucio Barabesi[1] | Andrea Cerioli[2] | Andrea Cerasa[3] | Domenico Perrotta[3]

[1]Department of Economics and Statistics, University of Siena, Siena, Italy | [2]Department of Economics and Management, University of Parma, Parma, Italy | [3]European Commission, Joint Research Centre (JRC), Ispra, Italy

Correspondence: Andrea Cerioli (andrea.cerioli@unipr.it)

Received: 25 August 2025 | **Revised:** 3 January 2026 | **Accepted:** 2 March 2026

Keywords: anomaly detection | benford hypothesis | data integrity | digitwise contamination | significand's fractional part

ABSTRACT

We address the task of identifying anomalous observations by analyzing digits under the lens of Benford's law. Motivated by the crucial objective of providing reliable statistical analysis of customs declarations, we answer one major and still open question: How can we detect the behavior of operators who are aware of the prevalence of the Benford's pattern in the digits of regular observations and try to manipulate their data in such a way that the same pattern also holds after data fabrication? This challenge arises from the ability of highly skilled and strategically minded manipulators in key organizational positions or criminal networks to exploit statistical knowledge and evade detection. For this purpose, we write a specific contamination model for digits, obtain new relevant distributional results and derive appropriate goodness-of-fit statistics for the considered adversarial testing problem. Along our path, we also unveil the peculiar relationship between two simple conformance tests based on the distribution of the first digit. We show the empirical properties of the proposed tests through a simulation exercise and application to data from international trade transactions. Although we cannot claim that our results are able to anticipate data fabrication with certainty, they surely point to situations where more substantial controls are needed. Furthermore, our work can reinforce trust in data integrity in many critical domains where mathematically informed misconduct is suspected.

1 | Introduction

Investigators in law-enforcement services, as well as auditors in insurance companies, financial institutions and public bodies, are concerned with intentional manipulations of recorded financial transactions, when there is an economic gain to falsify the actual amount of money exchanged. Our main motivation comes from the important application area of the statistical analysis of customs declarations, an established research track of the Joint Research Centre (JRC) of the European Commission stemming from the commitment to fight against fraud for the defense of the budget of the European Union (EU). This domain raises challenges that cannot be easily solved by statistical learning methods [1, 2]. One reason is that the number of detected violations is typically very low when compared to the total number of transactions, leading to severe class imbalance problems. Second, while information on detected illegal activities can be trusted almost with certainty, great uncertainty should be attached to the remaining class, since investigations only cover a small minority of the available transactions and many infringements can go undetected. Last, but not least, the availability of data on certified illegal activities after the final court judgment is sometimes so much deferred in time that learning rules likely become outdated: therefore, advancements in anti-fraud approaches have to face with concurrent adaptations of the fraudulent practices. Although some partial remedies for some of these issues have

This is an open access article under the terms of the Creative Commons Attribution License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

© 2026 The Author(s). Statistical Analysis and Data Mining published by Wiley Periodicals LLC.

been recently outlined in other contexts [3–6], the most effective methods conceive illegal customs declarations as anomalous transactions and identify them through formal outlier detection rules [7–9].

In this work we address the task of identifying fabricated declarations in customs data through the perspective of analyzing transaction digits instead of transaction values. Anomaly detection in the space of transaction digits can be a powerful companion to more classical outlier detection tools, as proven among others by Barabesi et al. [10–12]. The approach that we adopt relies on the availability of a suitable model for the digits of genuine transactions, that is, of transactions that originate from regular trade flows. Cerioli et al. [13] have shown that Benford's law can provide such a model under fairly general and easily verifiable conditions of trade, related to the ratio between the number of transactions and the number of traded goods of the subject under investigation. Furthermore, an important bonus of the digit approach is its ability to pinpoint serial misconduct, by focusing on the whole amount of transactions made by a single operator instead of looking at the potential anomaly of each transaction within the reference market, as done by the methods that deal with transaction values [8].

The main goal of the present work is to answer one major and still open question: How can we detect the behavior of economic operators who are aware of the prevalence of the Benford's pattern in the digits of regular transactions and try to manipulate their data in such a way that the same pattern also holds after data fabrication? Such a challenging manipulation scheme undermines the available methodologies for fraud identification based on digits, which have been shown to be effective under more standard schemes of data fabrication. Lacasa [14] emphasizes this potential threat in the specific context of the analysis of international trade data, while similar concerns exist in other financial scenarios ([15]). More generally, the possibility that fraudsters adapt their approaches to state-of-the-art analytical methods today is known as "concept drift" in the machine learning literature and is now recognized as a crucial open issue in most disciplines [1]. This challenge crucially affects the area of customs declaration, where greater awareness of Benford's law can invite adversarial efforts to engineer spurious conformity. Our aim is to tackle the consequences of this subtle adversarial attack by unveiling malicious Benford-savvy behavior, thus extending the Benford framework to remain effective under adaptive manipulation. The statistical tools proposed in the present work are thus not intended to replace the existing ones. Rather, they could pave the way for the development of hybrid strategies that combine digit-based invariants with statistical and machine learning methods for more robust fraud detection.

Although our focus is on the detection of infringements in international trade, our work should also be of interest in all the mathematically informed scenarios where lack of conformance to Benford's law is advocated as a signal of potential misconduct. These include accounting and finance [16–18], electoral processes [19–21], economics and national accounts [22, 23], and even the delicate field of evaluation of integrity of scientific research [24, 25]. Moreover, Benford's law has been taken up in broader economic and policy domains as a lens on data reliability. International Monetary Fund work has shown that many key macroeconomic aggregates tend to follow Benford patterns, though deviations can as easily reflect structural shifts as weaknesses in statistical practice [26]. At the same time, applications to survey data reveal systematic discrepancies in reported incomes, underscoring the method's value in highlighting measurement error and reliability concerns [27]. These studies illustrate how a tool first deployed in fraud detection has become a widely scrutinized benchmark for assessing the credibility of quantitative information across diverse contexts.

We achieve our stated goal by means of four main methodological developments. First, we formalize the Benford-savvy malicious behavior that we aim to contrast through a new contamination model for digits, baptized the "manipulated-Benford" scheme, which extends the one presented in Cerioli et al. [13] to the Benford-savvy context. We then study the distributional properties of the fractional part of the significand, which is the most informative random quantity under the manipulated-Benford model. This study opens the door to the construction of new nonparametric tests of the Benford hypothesis which are able to unmask the postulated adversarial Benford-savvy behavior. Our third contribution is a general result which unveils the theoretical relationship between two available statistics for testing conformance to Benford's law, the first-digit chi-square statistic [18] and the Hotelling statistic proposed in Barabesi et al. [11]. The nature of this relationship turns out to be very peculiar and amenable to a surprising simplification. It then leads us to suggest a new test statistic which is again effective to detect data fabrication under the manipulated-Benford model. Since none of our new tests can be expected to dominate the others under all possible configurations of the manipulated-Benford model, we finally combine them into an exact test of conformance to Benford's law which proves to be powerful under various specifications of this contamination scheme.

We investigate the empirical properties of the proposed tests through a simulation exercise and application to real customs data. The former is designed to depict different possible specifications of the manipulated-Benford model and shows that our new tests can greatly outperform the existing ones, including the likelihood-ratio test of Barabesi et al. [28] which is instead optimal under local alternatives to Benford's law. We then represent the application potential of our approach in the case of suspicious economic operators analyzed by investigators in a member state of the European Union (EU). The values declared by these traders do conform to Benford's law when only their first digit is considered, but they are picked as highly suspicious by our tests assuming a manipulated-Benford scheme. Although we cannot claim that our results are able to anticipate data manipulation with certainty, they surely point to situations where more substantial controls are needed in view of a possible serial and mathematically-informed illicit behavior.

A final contribution of our work consists in a more realistic and robust Monte Carlo scheme for estimating the exact p-values of the test statistics. The validity of Benford's law implies that transaction values are generated from a continuous random variable. This assumption is tenable under idealized trade conditions [13], but it might be mildly violated in practice due to the possible effects of truncation and rounding. The main consequence is that some of the transaction values may be recorded

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàÂ Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

with a limited number of digits, even in the absence of substantial irregularities. Although our simulation exercise proves that only extreme approximations strongly affect inferential conclusions, we also propose a refined simulation scheme for computing the exact distribution function of each test statistic, given the observed behavior of the economic operator in terms of truncation (or rounding) of values. The advantage of our proposal, at the expense of a limited loss of power, is a much more accurate control of the false-positive rate under regular trade conditions that do not strictly adhere to Benford's law because of truncation or rounding.

The structure of the paper is as follows. In Section 2 we introduce the problem of testing conformance to Benford's law. Section 3 defines the manipulated-Benford contamination model. In Sections 4 and 5 we derive the properties of the random quantities of interest under the manipulated-Benford model, obtain novel results on first-digit tests and develop a battery of new tests specifically targeted to the Benford-savvy alternative. We then propose to combine our tests in Section 6 and validate them through a simulation exercise in Section 7. Applications to real data from trade transactions are given in Section 8. Proofs are reported in the Supporting Information, together with our simulation algorithm for the truncated-Benford scheme and some complementary results, both theoretical and empirical.

2 | The Benford Set-Up

2.1 | Basic Definitions and Properties

We define the "significand function" $S : \mathbb{R}\backslash\{0\} \to [1, 10)$ as

$$S(x) = 10^{\langle \log_{10} |x| \rangle},$$

where $\langle x \rangle = x - \lfloor x \rfloor$ and $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \le x\}$ are the fractional part and the floor function of $x \in \mathbb{R}$, respectively. By assuming the probability space $(\Omega, \mathcal{F}, P)$, an absolutely-continuous random variable X is defined to be Benford ([29]) if, for $u \in [1, 10)$, the distribution function of $S(X)$ is

$$F_{S(X)}(u) = P(S(X) \le u) = \log_{10} u. \tag{1}$$

The first significant digit of x, denoted as $D(x)$, can be rephrased in terms of the significand function since $D(x) = \lfloor S(x) \rfloor$, and similar relationships also hold for the subsequent significant digits.

Two fundamental properties of Benford random variables stem from Equation (1). The first one is the celebrated first-digit law, stating that

$$p_d = P(D(X) = d) = \log_{10}\left(\frac{d+1}{d}\right) \tag{2}$$

for $d = 1, \ldots, 9$. Under Benford's law the distribution of the first significant digit is thus far from being uniform, but follows the logarithmic-type distribution ruled by Equation (2). Furthermore, for $d = 1, \ldots, 9$, we let

$$Z_{1,d} = \mathbf{1}_{[d,d+1)}(S(X)), \tag{3}$$

where $\mathbf{1}_B$ is the indicator function of the set B, so that the occurrence probability for first-digit d is given by $p_d = \mathrm{E}[Z_{1,d}]$.

The second feature of Benford random variables is called sum-invariance and is even more amazing. By defining

$$Z_{2,d} = S(X)\mathbf{1}_{[d,d+1)}(S(X)) \tag{4}$$

for $d = 1, \ldots, 9$, the (first-digit) sum-invariance property of Benford's law states that

$$\mathrm{E}[Z_{2,d}] = C, \tag{5}$$

where $C = \log_{10} e$. The validity of Benford's law thus implies that the expected value of the significand of X when the first digit is d, that is, the expectation given in Equation (5), does not depend on d and is equal to C. We refer to Berger and Hill [29] for a comprehensive and authoritative account of these and other mathematical characterizations of the law, while Barabesi et al. [11] provide a detailed investigation of the relationship between Equation (2) and Equation (5).

Benford's law has been challenging mathematicians and practitioners for decades [30]. It has also attracted the interest of statisticians mainly thanks to the limit theorem presented by Hill [31], which motivates the adoption of the law as the digit-generating model in many real-world situations. Theoretical results on the relationship between Benford random variables and classical univariate models for X are increasingly studied [28, 29, 32], while an empirical investigation of the accuracy of the Benford property of Equation (2) in the specific domain of international trade is detailed in Cerioli et al. [13]. In that context, alternative ways are suggested to analyze digits when Benford's law does not hold.

2.2 | Tests of the Benford Hypothesis

In view of Equation (1), we define as "Benford hypothesis" the null hypothesis

$$H_0 : S(X) \overset{\mathcal{L}}{=} 10^U, \tag{6}$$

where U is a Uniform random variable on $[0, 1)$. The assessment of hypothesis (6), given a sample $X_1, \ldots, X_n$ of n realizations of X, is the main inferential target of the statistical investigations on transaction digits described in Section 8.

Since the validity of Benford's law implies the first-digit properties displayed in Equation (2) and Equation (5), many popular tests of the Benford hypothesis are based on first-digit statistics. The rationale is that disagreement with either Equation (2) or Equation (5) contradicts the validity of Equation (6), provided that null distributions are obtained under H_0. The simplest and perhaps most adopted strategy for testing Equation (6) in applications (see, e.g., [10, 18], and the references therein) is to compute the first-digit Pearson statistic

$$Q_1 = \sum_{d=1}^{9} \frac{\left(n\hat{p}_d - np_d\right)^2}{np_d},$$

where p_d is given by Equation (2), while

$$\hat{p}_d = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{d\}}\left(D(X_i)\right)$$

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàCà Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

is its sample estimate. It is convenient for the developments that follow to express Q_1 as a quadratic form. If we recall Equation (3) and let $Z_1 = (Z_{1,1} \ldots, Z_{1,8})^{\mathrm{T}}$, $\mu_1 = \mathrm{E}[Z_1]$, $\Sigma_1 = \mathrm{var}[Z_1]$, under H_0 we have

$$\mu_1 = (p_1, \ldots, p_8)^{\mathrm{T}}$$

and

$$\Sigma_1 = (p_d \delta_{d,d'} - p_d p_{d'}),$$

where $\delta_{d,d'}$ is the Kronecker delta function. Therefore,

$$Q_1 = n\left(\overline{Z}_1 - \mu_1\right)^{\mathrm{T}} \Sigma_1^{-1} \left(\overline{Z}_1 - \mu_1\right), \tag{7}$$

where $\overline{Z}_1 = \left(\overline{Z}_{1,1}, \ldots, \overline{Z}_{1,8}\right)^{\mathrm{T}}$ and

$$\overline{Z}_{1,d} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{[d,d+1)}(S(X_i)).$$

Another powerful first-digit tool for testing H_0 is the Hotelling statistic proposed by Barabesi et al. [11] on the basis of the sum-invariance property of Equation (5). If we recall Equation (4) and let $Z_2 = (Z_{2,1}, \ldots, Z_{2,9})^{\mathrm{T}}$, $\mu_2 = \mathrm{E}[Z_2]$, $\Sigma_2 = \mathrm{var}[Z_2]$, it follows that under H_0

$$\mu_2 = (C, \ldots, C)^{\mathrm{T}}$$

and

$$\Sigma_2 = \left(C(d + 1/2)\delta_{d,d'} - C^2\right).$$

The Hotelling statistic is thus given by the quadratic form

$$Q_2 = n\left(\overline{Z}_2 - \mu_2\right)^{\mathrm{T}} \Sigma_2^{-1} \left(\overline{Z}_2 - \mu_2\right), \tag{8}$$

where $\overline{Z}_2 = \left(\overline{Z}_{2,1}, \ldots, \overline{Z}_{2,9}\right)^{\mathrm{T}}$ and

$$\overline{Z}_{2,d} = \frac{1}{n} \sum_{i=1}^{n} S(X_i) \mathbf{1}_{[d,d+1)}(S(X_i)).$$

Even if it relies on the first-digit sum-invariance property given in Equation (5), Q_2 takes into account part of the information provided by all the recorded digits, through the sample averages in $\overline{Z}_2$ and through their covariance structure. This is a major advantage against Benford-savvy manipulations that we further exploit in Section 5.

Strictly speaking, both Q_1 and Q_2 would test weaker hypotheses than the Benford one, since neither Equation (2) nor Equation (5) imply Equation (1). A test statistic specifically tailored to assess Equation (6) is instead the Kolmogorov–Smirnov (two-sided) statistic defined as

$$KS_1 = \max\{A_1, B_1\}, \tag{9}$$

where

$$A_1 = \max_{1 \le i \le n}\left(\frac{i}{n} - \log_{10} S_{(i)}\right), \quad B_1 = \max_{1 \le i \le n}\left(\log_{10} S_{(i)} - \frac{i-1}{n}\right)$$

and $S_{(1)} \le \cdots \le S_{(n)}$ are the order statistics of the significands $S(X_1), \ldots, S(X_n)$. A scale invariant version of KS_1 is the Kuiper statistic

$$KU_1 = A_1 + B_1. \tag{10}$$

Furthermore, Barabesi et al. [28] derive two locally optimal test statistics (i.e., a likelihood-ratio and a score statistic) through a semiparametric approximation to the distribution function of $\log_{10} S(X)$. These statistics are scale invariant and improve over both KS_1 and KU_1 under standard departures from Equation (6), when the law of X is a member of large classes of distributions of wide applicability. We refer to Barabesi et al. [28] for a precise definition of such statistics and a description of their properties, as well as to Cerasa [33] for other discrepancy measures of potential interest in very small samples and to Mumic and Filzmoser [34] for an alternative approach based on the concept of compositional data. We also note that in principle a myriad of alternative statistics could be inspired by the ideas of goodness-of-fit testing for directional data, but they are not considered here.

3 | Manipulations that Mimic Benford's Law

All the test statistics described in Section 2.2 are (to varying degrees) effective under "regular" alternatives to the Benford hypothesis, that is, alternatives where all or a fraction of the random variables $X_1, \ldots, X_n$ are sampled from a distribution different from the Benford one. This setting can be formalized through a contamination model assuming that the distribution function $F_{S(X)}$ of the significand is an element within the following family of distribution functions

$$F_{S(X)} = (1 - \lambda)F_{\mathrm{G}} + \lambda F_{\mathrm{C}}, \tag{11}$$

where $\lambda \in [0, 1]$. In such a model, F_{G} is the distribution function of the "good" part of the data, that is, F_{G} represents the postulated null model, F_{C} is the contaminant distribution, which is left unspecified except at most for the assumption of some regularity conditions, and λ is the contamination rate. With emphasis on the first digit, Cerioli et al. [13] show that in the context of international trade it may be plausible to assume the validity of Benford's law under fairly general and easily verifiable conditions of trade, related to the ratio between the number of transactions n and the number of traded goods, say m. Specifically, both n and m should be large enough, as anticipated by the fundamental limit theorem of Hill [31], and it is suggested to take $m \ge 0.2n$ as a practically sensible rule of thumb. Under the conditions that ensure the validity of Benford's law, it thus follows from Equation (1) that we can take

$$F_{\mathrm{G}}(u) = \log_{10} u \tag{12}$$

in Equation (11), with $u \in [1, 10)$. Barabesi et al. [11] and Barabesi et al. [28] provide extensive empirical evidence of the performance of many tests of the Benford hypothesis under Equation (11) and Equation (12), when $\lambda = 1$ and the contaminant distribution F_{C} corresponds to models for X of popular choice in applications. Furthermore, the power of Q_1 in some cases where $\lambda < 1$ is shown in Cerioli et al. [13].

The Benford-savvy manipulation that we target encompasses digit-wise contamination and replaces the first digit of a fabricated value with the first digit of the realization of a Benford

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàCà Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

random variable. This is arguably the simplest and most attractive way to make up data following the Benford pattern for at least two major reasons. The first one is the existence of legal and administrative constraints (e.g., for balance sheets or customs declarations) that may prevent from producing a fully-fabricated sample of Benford values. The second motivation is the increasing popularity of the first-digit tests introduced in Section 2.2, to which dishonest economic operators may try to adapt. If some of them were willing to fill administrative documents with fake data for the purpose of achieving an economic gain, the feeling that these data might be scrutinized for conformance with Equation (2) could lead to select first digits actually following Benford's law. Indeed, the development of new data manipulation schemes in response to existing fraud detection techniques is recognized as a general and widespread challenge in most applications [1].

It is apparent that the contamination model described by Equation (11) is not suitable to represent digit-wise departure from the Benford hypothesis and must then be replaced by a new one.

We recall Equation (1) and define a random variable X_{A} such that

$$X_{\mathrm{A}} \overset{\mathcal{L}}{=} 10^{U},$$

where U is a Uniform random variable on $[0,1)$. Therefore, X_{A} is a Benford random variable and the probability distribution of $D(X_{\mathrm{A}})$ follows the first-digit law given in Equation (2). We also consider a random variable X_{B} independent of X_{A}. The Benford-savvy manipulated response is then

$$S(X) \overset{\mathcal{L}}{=} D(X_{\mathrm{A}}) + \langle S(X_{\mathrm{B}}) \rangle. \tag{13}$$

We define Equation (13) as the (first-digit) "manipulated-Benford" contamination model. Note that the stated condition of independence between X_A and X_B is not technically necessary for analyzing digits under our manipulated-Benford scheme. Nevertheless, this assumption is retained in Equation (13) because it provides a worst-case scenario for the purpose of contrasting first-digit fabrication.

It is worth remarking that the manipulated-Benford model does not yield Equation (1) when X_{B} is itself a Benford random variable. More details on this, perhaps surprising, result are given in Section 4, but intuition can anticipate that the manipulation devised in Equation (13) destroys the dependence structure among the digits of a Benford random variable. We also note that contamination models similar to the manipulated-Benford one are becoming of prominent interest for anomaly detection in other branches of statistical research. We refer to the framework of cellwise contamination in multivariate estimation problems [35] and we leave the study of its relationship with Equation (13) to further research.

4 | Tests for the Manipulated-Benford Alternative

Our goal consists in developing tests of the Benford hypothesis that can detect departures from it when the subtle manipulated-Benford alternative holds. It is clear from Equation (13) that statistics solely involving the first digit $D(X)$, such as Q_1 in Equation (7), cannot be effective for this purpose. Empirical investigation of the behavior of many existing statistics under model Equation (13) is deferred to Section 7, where details about notation are provided. Nevertheless, we anticipate that even the most powerful scale-invariant test developed by Barabesi et al. [28] is barely effective against such a contamination, as shown in Table 1 for a few selected (and rather extreme) choices of distributions of X_{B} in Equation (13) and for moderately large sample sizes.

TABLE 1 | Estimated power, based on 5000 simulations, of the likelihood-ratio test of Barabesi et al. [28], there denoted as $\Lambda_{\hat{N},n}$, when X follows the manipulated-Benford contamination model for some (rather extreme) choices of distributions of X_{B} in Equation (13) and for different sample sizes. The exact test size is 0.01.

n	Lognormal (0.3,1)	Weibull (3.4,1)	Uniform [0,1)	Generalized Benford(3)
200	0.013	0.014	0.022	0.036
500	0.017	0.020	0.044	0.084
1000	0.023	0.023	0.082	0.205

A direct approach to detect violations of the Benford hypothesis ruled by Equation (13) is to base inferences on the fractional part of $S(X)$, whose main distributional properties are given in the results that follow. Their proof is reported in the Supporting Information.

Proposition 1. *Assume that Equation (6) holds. For $v \in [1,10)$ and $u \in [0,1)$, the distribution function of the bivariate random vector $(D(X), \langle S(X) \rangle)^{\mathrm{T}}$ is*

$$F_{(D(X),\langle S(X)\rangle)}(v,u) = \sum_{j=1}^{\lfloor v \rfloor} \log_{10}\left(\frac{j+u}{j}\right).$$

For $u \in [0,1)$, the distribution function of $\langle S(X) \rangle$ is

$$F_{\langle S(X)\rangle}(u) = \sum_{d=1}^{9} \log_{10}\left(\frac{d+u}{d}\right),$$

with corresponding probability density function

$$f_{\langle S(X)\rangle}(u) = \sum_{d=1}^{9} \frac{C}{d+u}.$$

For $u \in [0,1)$, the conditional distribution function of $\langle S(X) \rangle$ given $\{D(X) = d\}$ is

$$F_{\langle S(X)\rangle|\{D(X)=d\}}(u) = \frac{1}{p_d} \log_{10}\left(\frac{d+u}{d}\right),$$

with corresponding probability density function

$$f_{\langle S(X)\rangle|\{D(X)=d\}}(u) = \frac{C}{p_d(d+u)}.$$

Even if the properties of the distribution of $D(X)$ are well known (indeed, the corresponding probability function is plotted in

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàCà Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

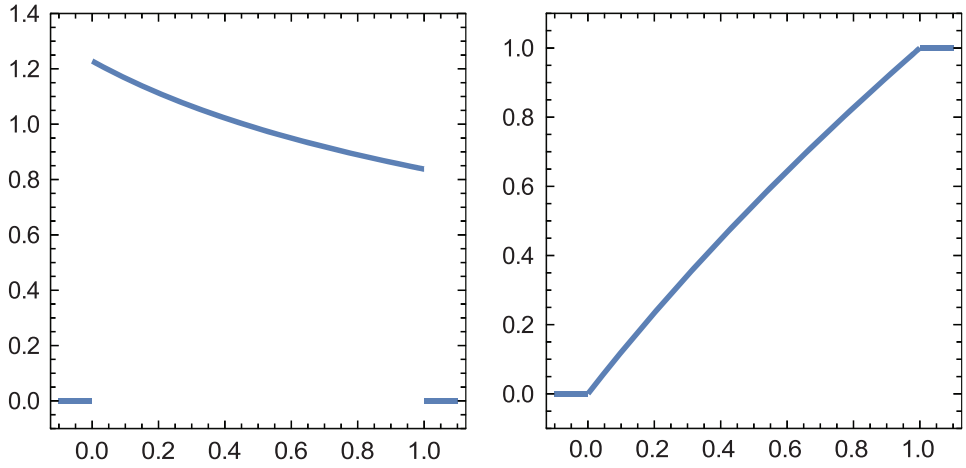


FIGURE 1 | Probability density function (left) and distribution function (right) of $\langle S(X)\rangle$ under Proposition 1.

almost all articles dealing with Benford's law), the features of the distribution of the fractional part $\langle S(X)\rangle$ are not usually noted in the literature. Actually, $\langle S(X)\rangle$ is an absolutely continuous random variable defined on $[0, 1)$, whose digits constitute the sequence of significant digits of X starting from the second significant one. The probability density function and the distribution function of $\langle S(X)\rangle$ are shown in Figure 1. Even if the probability density function of $\langle S(X)\rangle$ is slightly decreasing on $[0, 1)$, the law of $\langle S(X)\rangle$ is not too far from uniformity. This is due to the fact that the probability function of the second significant digit is much less spiked than that of the first significant digit, being nearly "flat" (see [29], Figure 1.1). Intuitively, these properties are likely to provide a small correlation between $D(X)$ and $\langle S(X)\rangle$, as formally shown in Proposition 2 below.

Proposition 1 opens the door to the construction of nonparametric tests of the Benford hypothesis, which can be expected to be especially effective under the manipulated-Benford alternative. As in Equation (9) and Equation (10), we focus both on the Kolmogorov–Smirnov statistic.

$$KS_2 = \max\left\{A_2, B_2\right\} \tag{14}$$

and on the Kuiper statistic

$$KU_2 = A_2 + B_2, \tag{15}$$

where

$$A_2 = \max_{1\le i\le n}\left(\frac{i}{n} - F_{\langle S(X)\rangle}\left(S_{2,(i)}\right)\right),$$
$$B_2 = \max_{1\le i\le n}\left(F_{\langle S(X)\rangle}\left(S_{2,(i)}\right) - \frac{i-1}{n}\right),$$

and $S_{2,(1)} \le \cdots \le S_{2,(n)}$ represent the order statistics of the significand fractional parts $\left\langle S\left(X_1\right)\right\rangle, \ldots, \left\langle S\left(X_n\right)\right\rangle$.

Proposition 2. *Under the same assumptions of Proposition 1, for $r, s \in \mathbb{N}$, we have*

$$\mathrm{E}\left[D(X)^r\langle S(X)\rangle^s\right]$$
$$= (-1)^s\mathrm{E}\left[D(X)^{r+s}\right] + C\sum_{d=1}^{9} d^{r+s}\sum_{j=1}^{s}\binom{s}{j}(-1)^{s-j}\frac{(1+1/d)^j - 1}{j}.$$

Proposition 2 yields the mixed moments of the joint distribution of $D(X)$ and $\langle S(X)\rangle$ when X is a Benford random variable. In particular, after some algebra deferred to the Supporting Information, where its formal expression is given, the numerical value of the correlation coefficient between $D(X)$ and $\langle S(X)\rangle$ turns out to be

$$\mathrm{cor}[D(X), \langle S(X)\rangle] \simeq 0.05636. \tag{16}$$

We also obtain that the conditional expectation

$$\mathrm{E}[\langle S(X)\rangle | \{D(X) = d\}] = \frac{C}{p_d} - d$$

is nearly constant, since it takes values close to 0.5 for all $d = 1, \ldots, 9$. Both measures thus show limited dependence between $D(X)$ and $\langle S(X)\rangle$ under Equation (6).

As an interesting corollary of Proposition 2, Equation (16) also explains why the manipulated-Benford contamination model defined in Equation (13) does not lead to the Benford hypothesis when X_B is itself a Benford random variable. Indeed, this choice of X_B yields $\mathrm{cor}\left[D\left(X_A\right), \left\langle S\left(X_B\right)\right\rangle\right] = 0$, X_A and X_B being independent, and contradicts Equation (16).

5 | Properties of First-Digit Tests and a New Test

5.1 | The Joint Null Distribution of Q_1 and Q_2

We provide new insight on the properties of the first-digit test statistics Q_1 and Q_2 under the Benford hypothesis. Although a good Monte Carlo approximation to their exact joint distribution is available under Equation (6), as described in Section 7, the result presented below gives their asymptotic joint null distribution. A major advantage of this large-sample result is that it suggests a very simple approximation which in Section 5.2 leads to a novel test statistic specifically tailored to the manipulated-Benford alternative.

Let $Z = \mathrm{vec}\left(Z_1, Z_2\right)$, $\mu = \mathrm{E}[Z]$, $\Sigma_{12} = \mathrm{Cov}\left[Z_1, Z_2\right]$ and $\Sigma = \mathrm{var}[Z]$. Under (6),

$$\mu = \mathrm{vec}\left(\mu_1, \mu_2\right)$$

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàÂ Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and

$$\Sigma = \begin{pmatrix} \Sigma_1 & \Sigma_{12} \\ \Sigma_{12}^{\mathrm{T}} & \Sigma_2 \end{pmatrix},$$

where μ_1, μ_2, Σ_1, Σ_2 are given in Section 2.2. In addition, since

$$\mathrm{E}\left[Z_{1,d} Z_{2,d'}\right] = \delta_{d,d'} \mathrm{E}\left[Z_{2,d}\right] = C\delta_{d,d'},$$

it also follows that $\Sigma_{12} = \left(C\delta_{d,d'} - Cp_d\right)$. Correspondingly, if $\overline{Z} = \mathrm{vec}\left(\overline{Z}_1, \overline{Z}_2\right)$, we have $\mathrm{E}[\overline{Z}] = \mu$ and $\mathrm{var}[\overline{Z}] = n^{-1}\Sigma$. The following general result, which is proved in the Supporting Information, leads to the large-sample distribution of the bivariate random vector $Q = \left(Q_1, Q_2\right)^{\mathrm{T}}$.

Theorem 1. *Let $\left(Y_n\right)_{n\geq 1}$ be a sequence of centered random vectors defined on the probability space $(\Omega, \mathcal{F}, P)$, such that $Y_n = \mathrm{vec}\left(Y_{1,n}, Y_{2,n}\right)$, where $Y_{1,n}$ and $Y_{2,n}$ are random vectors of dimensions r_1 and r_2, with $r_1 \leq r_2$. Moreover, assume that*

$$\Psi = \mathrm{var}\left[Y_n\right] = \begin{pmatrix} \Psi_1 & \Psi_{12} \\ \Psi_{12}^{\mathrm{T}} & \Psi_2 \end{pmatrix}$$

is a full-rank matrix with finite entries and that Y_n converges in distribution to the random vector $Y = \mathrm{vec}\left(Y_1, Y_2\right)$ with the Normal law $N_{r_1+r_2}(0, \Psi)$. If $\left(S_n\right)_{n\geq 1}$ is a sequence of bivariate random vectors such that $S_n = \left(S_{1,n}, S_{2,n}\right)^{\mathrm{T}}$ with $S_{1,n} = Y_{1,n}^{\mathrm{T}} \Psi_1^{-1} Y_{1,n}$ and $S_{2,n} = Y_{2,n}^{\mathrm{T}} \Psi_2^{-1} Y_{2,n}$, then S_n converges in distribution to the bivariate random vector

$$V = \left(V_1, V_2\right)^{\mathrm{T}} = \left(Y_1^{\mathrm{T}} \Psi_1^{-1} Y_1, Y_2^{\mathrm{T}} \Psi_2^{-1} Y_2\right)^{\mathrm{T}}$$

with Laplace transform given by

$$L_V\left(t_1, t_2\right) = \left(1 + 2t_2\right)^{-\frac{1}{2}(r_2 - r_1)} \prod_{j=1}^{r_1} \left(1 + 2t_1 + 2t_2 + 4\left(1 - \rho_j^2\right) t_1 t_2\right)^{-\frac{1}{2}},$$

where $1 \geq \rho_1 \geq \cdots \geq \rho_{r_1} \geq 0$ are the canonical correlations between Y_1 and Y_2, that is, the singular values of the matrix $\Psi_1^{-\frac{1}{2}} \Psi_{12} \Psi_2^{-\frac{1}{2}}$.

Theorem 1 implies that $L_V\left(t_1, 0\right) = \left(1 + 2t_1\right)^{-\frac{1}{2} r_1}$ and $L_V\left(0, t_2\right) = \left(1 + 2t_2\right)^{-\frac{1}{2} r_2}$, so that the marginal distributions of V_1 and V_2 are the $\chi^2_{r_1}$ and $\chi^2_{r_2}$ laws, respectively. Furthermore, since $\sqrt{n}(\overline{Z} - \mathrm{E}[\overline{Z}])$ converges in distribution to the Normal law $N_{17}(0, \Sigma)$, by applying Theorem 1 with $r_1 = 8$ and $r_2 = 9$ we obtain that Q converges in distribution to the bivariate random vector V with Laplace transform

$$L_V\left(t_1, t_2\right) = \left(1 + 2t_2\right)^{-\frac{1}{2}} \prod_{j=1}^{8} \left(1 + 2t_1 + 2t_2 + 4\left(1 - \rho_j^2\right) t_1 t_2\right)^{-\frac{1}{2}}.$$

We thus have the stochastic representation $V \overset{\mathcal{L}}{=} \sum_{j=1}^{9} W_j$, where $W_j = \left(W_{1,j}, W_{2,j}\right)^{\mathrm{T}}$ are independent bivariate random vectors such that, for $j = 1, \ldots, 8$, each W_j is distributed with Kibble's law of parameters ρ_j and $\frac{1}{2}$ [36], while $W_{1,9}$ is a Dirac random variable concentrated at zero and $W_{2,9}$ is distributed with the χ^2_1 law. Kibble's law is a bivariate generalization of the Gamma distribution and is in turn a special case of the family discussed in Torricelli et al. [37]. Finally, by using algebraic software, we obtain

$$\mathrm{E}\left[V_1 V_2\right] = \left.\frac{\partial L_V\left(t_1, t_2\right)}{\partial t_1 \partial t_2}\right|_{t_1 = 0, t_2 = 0} = 72 + 2\sum_{d=1}^{8} \rho_d^2.$$

The correlation coefficient between V_1 and V_2 is then

$$\mathrm{cor}\left[V_1, V_2\right] = \frac{\sqrt{2}}{12} \sum_{j=1}^{8} \rho_j^2,$$

by recalling that the marginal distributions of V_1 and V_2 are the χ^2_8 and χ^2_9 laws, respectively.

Since the entries of Σ are known, the exact expression of $\rho_1, \ldots, \rho_8$ can be obtained by means of algebraic software and the corresponding numerical computation yields the values reported in Table 2, in such a way that $\mathrm{cor}\left[V_1, V_2\right] \simeq 0.9381$. Table 3 displays the Monte Carlo estimates, based on 10^5 replicates, of $\mathrm{E}\left[Q_1\right]$, $\mathrm{E}\left[Q_2\right]$, $\mathrm{var}\left[Q_1\right]$, $\mathrm{var}\left[Q_2\right]$ and $\mathrm{cor}\left[Q_1, Q_2\right]$ for selected values of n. It also compares these estimates with the moments of V, reached when $n \to \infty$. The agreement is remarkably close even in the case of small samples.

5.2 | A Simplified Statistic

The Laplace transform L_V cannot be inverted analytically. However, the values of $\rho_1, \ldots, \rho_8$ displayed in Table 2 suggest that a good approximation to the distribution of V can be achieved by inverting L_V under the assumption that $\left(\rho_1, \ldots, \rho_8\right) = (1, \ldots, 1)$. Such an assumption gives rise to the bivariate probability density function of a new random vector, say $T = \left(T_1, T_2\right)^{\mathrm{T}}$, such that

$$f_T\left(x_1, x_2\right) = \frac{1}{96\sqrt{2\pi}} \frac{x_1^3}{\sqrt{x_2 - x_1}} e^{-\frac{1}{2} x_2}, \tag{17}$$

for $x_1 > 0, x_2 > x_1$. The details of the derivation of f_T are given in the Supporting Information. The probability density function provided by Equation (17) is shown in the left-hand panel of Figure 2, while the right-hand panel displays the probability density function of V obtained by numerical inversion. The agreement between the two plots is visually striking. Further peculiar features of Equation (17) are that the random variables T_1 and $\left(T_2 - T_1\right)$ are independent under the Benford hypothesis and that T_1 is distributed with the χ^2_8 law, while $\left(T_2 - T_1\right)$ is distributed with the χ^2_1 law. Therefore, this approximation keeps the same χ^2_8 and χ^2_9 laws for the marginal distributions, while the corresponding correlation becomes $\mathrm{cor}\left[T_1, T_2\right] = \frac{2\sqrt{2}}{3} \simeq 0.9428$, which is very close to the value of $\mathrm{cor}\left[V_1, V_2\right]$.

The approximation to the asymptotic joint null distribution of Q_1 and Q_2 offered by Equation (17) suggests that the difference statistic can be adopted to assess the Benford hypothesis under the manipulated-Benford alternative. The rationale is that the information provided by $\left(T_2 - T_1\right)$, being independent of that of T_1, does not look at the first digit but at the subsequent ones. The same intuition also applies to the original test statistics Q_1 and Q_2, since computation of Q_2 involves all the significant digits through

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàÂ Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

TABLE 2 | Numerical approximation to the canonical correlations between Y_1 and Y_2.

ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8
0.9995	0.9994	0.9992	0.9990	0.9985	0.9977	0.9959	0.9906

TABLE 3 | Monte Carlo estimates of some moments of Q, based on 10^5 replicates, for selected sample sizes and comparison with asymptotic values.

n	30	50	100	200	∞
$\mathrm{E}[Q_1]$	7.984	7.989	8.011	8.013	8
$\mathrm{E}[Q_2]$	8.979	8.999	9.019	9.014	9
$\mathrm{var}[Q_1]$	16.538	16.366	16.466	16.413	16
$\mathrm{var}[Q_2]$	18.355	18.336	18.190	18.167	18
$\mathrm{cor}[Q_1, Q_2]$	0.939	0.937	0.938	0.938	0.938

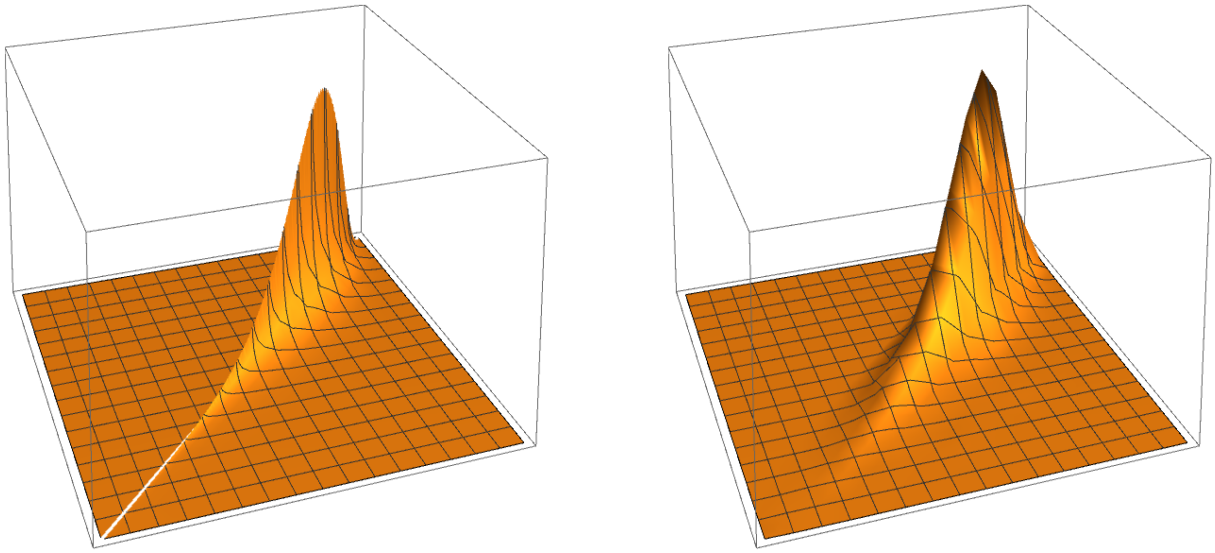


FIGURE 2 | Probability density functions of T (left-hand panel) and V (right-hand panel).

$\overline{Z}_2$ and the dispersion matrix Σ_2, while Q_1 is solely a function of the first digit.

$$Q_\Delta = Q_2 - Q_1 \tag{18}$$

Asymptotically, we approximate the quantiles in the right-hand tail of the null distribution of Q_Δ through those of the χ^2_1 law, as it holds for its approximated counterpart $(T_2 - T_1)$. Although we cannot expect this approximation to be valid across the whole support of Q_Δ, as it happens that $\{Q_1 > Q_2\}$ and hence $\{Q_\Delta < 0\}$ with non-null probability, the fit of the χ^2_1 distribution is amazingly good for testing purposes. For instance, Figure 3 contrasts the quantiles in the 10% right-hand tail of the distribution of Q_Δ under the Benford hypothesis, estimated through the Monte Carlo algorithm described in Section 7 and 10^6 simulations, with those of a χ^2_1 random variable. The match between the two sets of quantiles is very accurate even when $n = 100$.

Independence between $(T_2 - T_1)$ and T_1 is also closely mirrored by Q_Δ and Q_1, when we look at the right-hand tail of their distributions under the Benford hypothesis. Figure 4 compares a zoom of the estimated null probability density of Q_Δ, say f_{Q_Δ}, with the same portion of the estimated null conditional probability density of Q_Δ, given that Q_1 is below the 0.99 quantile of χ^2_1, denoted as $f_{Q_\Delta|\{Q_1<\chi^2_{1,0.99}\}}$, again for $n = 100$ and $n = 500$. The plotted kernel estimates are practically indistinguishable and show that the outcome of the first-digit test based on Q_1 has virtually no impact on the right-hand tail of the null distribution of Q_Δ. This result supports computation of Q_Δ when there is no evidence of data fabrication in the first digit. In that case the observed value of the test statistic can be safely compared with the tail quantiles of its marginal distribution, in contrast to what happens when performing sequential tests on multiple digits and more complicated conditional distributions must be taken into account [10].

6 | Combination of Tests

Since we want to leave the distribution of X_B in Equation (13) unspecified, none of our tests on the fractional part of $S(X)$ can be expected to dominate the others under all possible specifications of the manipulated-Benford model. Our proposal is then to consider a combined test statistic of Equation (6) tailored to detect the existence of Benford-savvy contamination under a variety of choices for X_B in Equation (13).

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàCâ Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

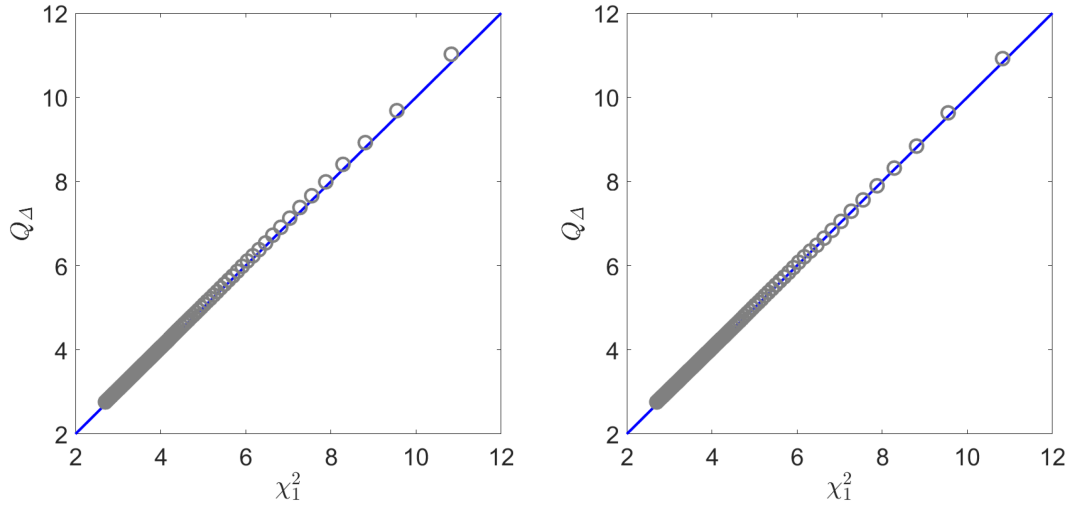


FIGURE 3 | QQ-plot comparing the largest quantiles of Q_Δ under Equation (6) with those of a χ^2_1 random variable for $n = 100$ (left-hand panel) and $n = 500$ (right-hand panel). The plotted quantiles belong to the 10% right-hand tail of each distribution.

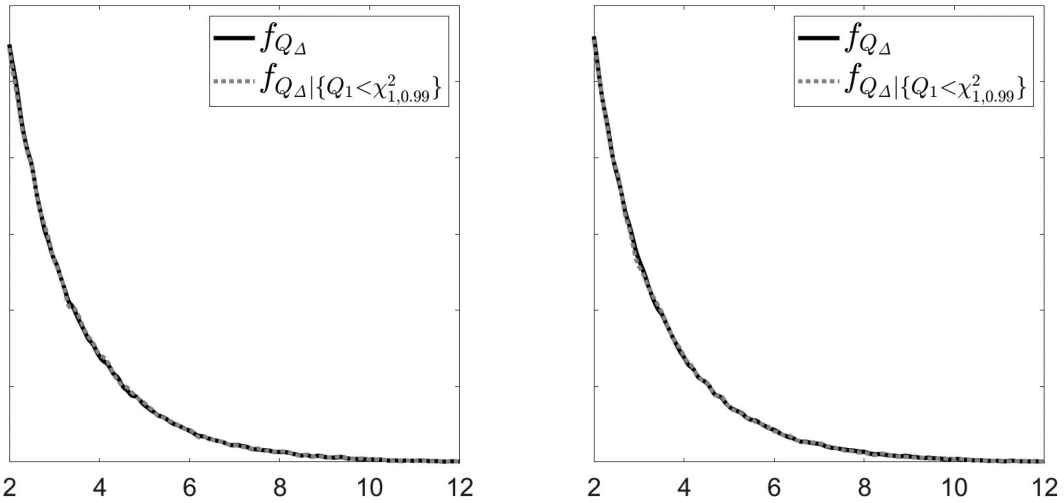


FIGURE 4 | Zoom in the right-hand tail of the estimated null probability density of Q_Δ (black solid line) and of the estimated null conditional probability density of Q_Δ, given that $Q_1 < \chi^2_{1,0.99}$ (gray dotted line). Left-hand panel: $n = 100$; Right-hand panel: $n = 500$. Estimates based on Matlab function `ksdensity` and 10^6 simulated samples under H_0.

For concreteness, we focus on the combination of the Kolmogorov–Smirnov statistic given in Equation (14) with Q_Δ. The choice of a suitable combining function $\phi : [0,1]^2 \to \mathbb{R}$ yields the test statistic

$$G_{KS} = \phi\left(1 - F_{KS_2}(KS_2), 1 - F_{Q_\Delta}(Q_\Delta)\right), \tag{19}$$

where F_{KS_2} and F_{Q_Δ} are the distribution functions of KS_2 and Q_Δ, respectively, under the Benford hypothesis. Specifically, we adopt the minimum p-value principle and let $\phi(u, v) = \min(u, v)$, so that the combined test rejects for small realizations of G_{KS}. The corresponding p-value is then $p_{G_{KS}}(g) = F_{G_{KS}}(g)$, where g is a realization of G_{KS} and $F_{G_{KS}}$ is the distribution function of G_{KS} under Equation (6). Distribution function $F_{G_{KS}}$ can be approximated to an arbitrary degree of accuracy through the simple, but computationally efficient, Monte Carlo approach relying on Equation (20) and through its more robust variant to be described in the sections below. Therefore, a valid combined test is obtained through the estimate of $p_{G_{KS}}(g)$ and no further multiplicity correction is needed. A similar test statistic is obtained by replacing KS_2 with the Kuiper statistic of Equation (15). We denote as G_{KU} the analog of Equation (19) combining KU_2 with Q_Δ.

7 | Simulation Study

7.1 | Aims of the Study

In this section we provide simulation evidence of the empirical advantages of our tests under the manipulated-Benford contamination model. Specifically, we compare the tests based on statistics of Equations (14), (15) and (18), as well as of their combinations, to that of the existing tests introduced in Section 2.2. We include in our comparison also the two-digit version of the Pearson statistic, here written as Q_{12}, and the likelihood-ratio statistic of Barabesi et al. [28], denoted as $\Lambda_{\hat{N},n}$. The former is often

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàÂ Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

advocated to test hypothesis (6) in moderately large samples, as it extends consideration beyond the first digit ([18]), while the latter is a scale-invariant statistic which enjoys the optimality properties of a likelihood approach.

We emphasize that we cannot expect our statistics KS_2, KU_2 and Q_Δ to be particularly effective under the standard contamination scheme defined in Equation (11), since they neglect information on the first digit $D(X)$. We argue that such a limitation does hinder the usefulness of our proposal, the main ability of KS_2, KU_2 and Q_Δ being to detect data fabrication in the fractional part of $S(X)$ when $D(X)$ indeed follows Equation (2). Our new tests are not designed to improve over the existing tests of the Benford hypothesis in any scenario, but to complement them in situations where they loose their effectiveness. Nevertheless, our tests can be also combined with any first-digit test to detect departures from the Benford hypothesis in the absence of a specific suspicion of Benford-savvy manipulations. The advantage of this approach, whose investigation is left for further research, is to obtain simple procedures with good power against both the naive manipulations which agree with Equation (11) and the more subtle data fabrication schemes following Equation (13), without the need to select the type of departure from H_0 in advance.

In our power comparisons, we rely on Monte Carlo estimates of the exact quantiles of each test statistic. For this purpose, B Monte Carlo replicates of the test statistic under consideration, say $T = T\left(X_1, \ldots, X_n\right)$, are generated under the Benford hypothesis as

$$T_b^\dagger = T\left(10^{U_{b,1}}, \ldots, 10^{U_{b,n}}\right), \tag{20}$$

for $b = 1, \ldots, B$, where $U_{b,1}, \ldots, U_{b,n}$ are independent Uniform random variables on $[0, 1)$. For a realization t of T, the Monte Carlo estimates of the exact p-values are then computed as $p_T^\dagger(t) = 1 - F_T^\dagger(t)$, where

$$F_T^\dagger(t) = \frac{1}{B}\sum_{b=1}^{B} \mathbf{1}_{(-\infty,t]}\left(T_b^\dagger\right).$$

The estimates of some relevant quantiles of our test statistics are given in the Supporting Information using $B = 10^6$ replicates.

In the simulations that follow we take $\gamma = 0.01$ to be the exact test size. Results obtained with different exact quantiles from Equation (20) are qualitatively very similar. We only show power results for $n = 500$, while those for different sample sizes are given in the Supporting Information, together with further details about the test statistics not described in Section 2.2 and the adopted simulation algorithm (including the link to the code that can be used to replicate our simulations). Power comparisons are performed using 5000 independent runs of the manipulated-Benford model of Equation (13), for different selections of X_B.

7.2 | Evidence Under the Manipulated-Benford Model

Our choices for X_B in the manipulated-Benford model are displayed in Table 4. The first two scenarios, the Lognormal and the Weibull laws, follow Barabesi et al. [11] and assume that the contaminant random variable X_B comes from distributional models which are popular in many fields, including the analysis of economic aggregates that arise in trade [38]. In both instances, we denote by α the shape parameter of the law and take the scale parameter equal to 1. We focus on the effect of α since it is known that Lognormal and Weibull random variables become practically indistinguishable from a Benford one for suitable choices of their shape parameter [29, 32]. Our subsequent scenario assumes that X_B is a Uniform random variable defined on interval $[0, \alpha)$. This setting can be taken as a representative of "human" data fabrication, as it is often argued that number invention by humans might be biased toward simple distributions such as the Uniform law (see [18] for some examples of human number invention). Here we are not interested in the effect of α itself, but we consider different values of the range to represent our lack of knowledge about the support chosen for the purpose of data invention. Our final power scenario widens the perspective by considering the possibility that data manipulation directly involves a model for the digits of X. We thus assume that X is a Generalized Benford random variable with parameter $\alpha \in \mathbb{R}$. The Generalized Benford model has been introduced to represent digit distributions when X follows a power-law [39, 40]. For instance, the first-digit masses span from $p_1 = 0.556$ to $p_9 = 0.012$ when $\alpha = -1$, while we recover the Uniform first-digit distribution $p_1 = \ldots = p_9 = 1/9$ when $\alpha = 1$. The resulting expressions for $F_{S(X)}$ and $F_{\langle S(X)\rangle}$ are reported in the Supporting Information.

The manipulated-Benford scenario is particularly challenging for all the test statistics derived under Equation (11). Taking into account the full significand $S(X)$, as is done by KS_1, KU_1 and $\Lambda_{\hat{N},n}$, does not help in any instance. Only the mixed approach provided by Q_2 (and to a much lesser extent by Q_{12}) is able to keep power to a reasonable level, a result anticipated in Barabesi et al. [11] for the case of a Generalized Benford random variable. Nevertheless, our new tests clearly outperform Q_2, as well as the others competitors, in all the selected specifications of Equation (13). It is also seen that the ordering among KS_2, KU_2 and Q_Δ can change according to the specific contaminant distribution, but that the power of G_{KS} and G_{KU} is always close to that of the best performer. We thus conclude that our new tests are to be generally recommended for the purpose of detecting Benford-savvy manipulations and that the loss of power induced by combination of tests is minor, especially if compared to the resulting increased robustness against different specifications of the nature of the manipulating digit distribution.

The same qualitative picture is preserved when only a portion τ of the observations come from the manipulated-Benford model, while the remaining fraction $(1 - \tau)$ are generated by a Benford random variable, reflecting only partial fabrication of the data. Of course, the power of our new tests cannot be expected to remain as high as in Table 4 under this less extreme—and thus even more challenging—contamination scheme, but their detection performance is still appreciable. For instance, Table 5 reports the results for the case $\tau = 0.6$, again with $n = 500$ and $\gamma = 0.01$.

Comparing the results displayed in Table 4 with those reported in the Supporting Information shows the effect of the sample size on the power of our tests under the manipulated-Benford

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàÂ Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

TABLE 4 | Estimated power, based on 5000 simulations, under the manipulated-Benford model for different choices of X_B in Equation (13), when $n = 500$. The exact test size is $\gamma = 0.01$. Null quantiles of test statistics are estimated with $B = 10^6$ replicates in Equation (20).

α	Q_1	Q_{12}	Q_2	KS_1	$\Lambda_{\hat{N},n}$	KU_1	KS_2	KU_2	Q_Δ	G_{KS}	G_{KU}
Lognormal$(\alpha, 1)$											
0.3	0.011	0.451	0.630	0.034	0.017	0.038	0.997	0.986	0.927	0.996	0.981
0.4	0.010	0.096	0.122	0.015	0.010	0.014	0.759	0.575	0.381	0.702	0.528
0.5	0.009	0.033	0.021	0.014	0.010	0.013	0.304	0.167	0.076	0.241	0.135
0.6	0.010	0.023	0.011	0.013	0.011	0.015	0.105	0.051	0.018	0.079	0.039
Weibull$(\alpha, 1)$											
2.2	0.010	0.040	0.034	0.016	0.011	0.015	0.426	0.259	0.123	0.365	0.216
2.6	0.010	0.104	0.130	0.016	0.011	0.014	0.819	0.661	0.406	0.771	0.615
3.0	0.010	0.246	0.319	0.027	0.016	0.025	0.970	0.922	0.721	0.959	0.895
3.4	0.010	0.466	0.533	0.036	0.020	0.043	0.997	0.991	0.883	0.996	0.986
Uniform$[0, \alpha)$											
5	0.010	0.047	0.234	0.015	0.041	0.050	0.364	0.211	0.614	0.564	0.535
20	0.009	0.042	0.235	0.019	0.039	0.047	0.360	0.203	0.598	0.559	0.530
40	0.010	0.044	0.235	0.018	0.045	0.047	0.368	0.199	0.600	0.554	0.533
60	0.010	0.047	0.238	0.018	0.040	0.047	0.353	0.200	0.604	0.557	0.533
Generalized Benford(α)											
−1.0	0.011	0.153	0.225	0.018	0.011	0.016	0.884	0.722	0.589	0.852	0.703
1.0	0.010	0.048	0.250	0.018	0.045	0.050	0.356	0.200	0.593	0.546	0.514
2.0	0.010	0.124	0.577	0.025	0.065	0.078	0.768	0.568	0.903	0.887	0.867
3.0	0.012	0.217	0.774	0.049	0.084	0.109	0.930	0.796	0.974	0.969	0.960

TABLE 5 | As Table 4, but now only 60% of the observations come from the manipulated-Benford model, while the remaining 40% are generated by a Benford random variable.

α	Q_1	Q_{12}	Q_2	KS_1	KU_1	KS_2	KU_2	Q_Δ	G_{KS}	G_{KU}
Lognormal$(\alpha, 1)$										
0.3	0.009	0.071	0.142	0.016	0.013	0.728	0.549	0.436	0.688	0.541
0.4	0.008	0.027	0.030	0.013	0.011	0.270	0.151	0.112	0.227	0.144
0.5	0.011	0.017	0.014	0.009	0.009	0.083	0.043	0.030	0.067	0.042
0.6	0.009	0.014	0.009	0.012	0.012	0.038	0.022	0.011	0.027	0.017
Weibull$(\alpha, 1)$										
2.2	0.009	0.016	0.016	0.012	0.014	0.133	0.072	0.043	0.106	0.060
2.6	0.013	0.024	0.037	0.012	0.012	0.298	0.178	0.114	0.244	0.159
3.0	0.010	0.040	0.072	0.015	0.015	0.528	0.373	0.239	0.471	0.345
3.4	0.009	0.082	0.124	0.016	0.018	0.745	0.589	0.392	0.686	0.559
Uniform$[0, \alpha)$										
5	0.009	0.019	0.064	0.011	0.017	0.111	0.056	0.199	0.173	0.156
20	0.009	0.022	0.053	0.012	0.019	0.106	0.054	0.184	0.163	0.147
40	0.008	0.021	0.053	0.011	0.017	0.098	0.049	0.187	0.160	0.148
60	0.012	0.021	0.057	0.014	0.022	0.103	0.051	0.184	0.160	0.143
Generalized Benford(α)										
−1.0	0.012	0.033	0.056	0.016	0.011	0.386	0.218	0.185	0.339	0.217
1.0	0.011	0.019	0.057	0.012	0.017	0.107	0.052	0.199	0.177	0.159
2.0	0.012	0.034	0.132	0.013	0.028	0.272	0.141	0.405	0.370	0.337
3.0	0.011	0.044	0.205	0.016	0.030	0.432	0.243	0.551	0.529	0.487

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàÂ Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

TABLE 6 | Estimated maximum absolute difference between the quantiles of $\langle S(X)\rangle$ under Benford's law and those under a Lognormal random variable, for different values of α and n. Estimates are based on 10^6 replicates.

α	$n = 200$	$n = 500$	$n = 1000$
0.3	0.114	0.115	0.115
0.4	0.070	0.071	0.071
0.5	0.044	0.045	0.045
0.6	0.029	0.028	0.028

model, for any given choice of X_B. To control for the effect of n, we can instead investigate the discrepancy between the null and each alternative model from a descriptive point of view, especially in the cases where the contaminant distribution approaches Benford-like behavior for specific parameter values. For instance, based on 10^6 replicates, the estimated maximum absolute difference between the quantiles of $\langle S(X)\rangle$ under Benford's law and those under a Lognormal random variable is reported in Table 6. As expected, the discrepancy between the two models decreases as α grows irrespectively of the sample size, thus providing complementary empirical evidence to the power values given in Table 4 and in the Supporting Information. In particular, this additional information provides a link between numerical deviations from Benford's law in terms of $\langle S(X)\rangle$ and the detection rates of our tests based on the distribution of $\langle S(X)\rangle$ at a given sample size.

7.3 | Truncation and Rounding

We carry on with our simulation exercise by investigating the effect of contamination in the last significant digits, as it happens with rounding errors or other numerical inaccuracies. Although the Benford hypothesis assumes $S(X)$ to be a continuous random variable, rounding and truncation to a rather limited number of significant digits often occur in practice even in the absence of malicious data manipulation. We require our tests to be fairly robust against such violations of H_0 and need to devise an alternative practical strategy when it is not the case, since rounding and inaccuracies typically have very different implications than data fabrication.

In Table 7 we compute the proportion of rejections of H_0 when the Benford significand $S(X)$, simulated according to Equation (20), is truncated to have k digits. We see that the effect of truncation is negligible, unless k is very small. This result is perhaps surprising, since our tests look at the fractional part of $S(X)$ and the chosen simulation scheme depicts a worst-case scenario for them, by assuming that the same truncation level affects all the simulated significands. Nevertheless, in Section 8 we apply a more robust simulation algorithm which takes truncation (or rounding) into account and allows us to control the exact size of the tests of Equation (6) for any specific truncation (or rounding) scheme. This robust simulation algorithm is detailed in the Supporting Information. To convey the effect of discretization, Figure 5 contrasts the quantiles of $\langle S(X)\rangle$ under the Benford hypothesis with those obtained using our truncated-Benford algorithm, in two instances where $n = 500$ and $B = 10^5$ replicates

TABLE 7 | Proportion of rejections of H_0, based on 5000 simulations, under a truncated-Benford model with k significant digits in $S(X)$, when $n = 500$. The exact test size is $\gamma = 0.01$. Null quantiles are the same as in Table 4.

k	KS_2	KU_2	Q_Δ	G_{KS}	G_{KU}
6	0.009	0.010	0.009	0.008	0.009
5	0.010	0.011	0.010	0.010	0.010
4	0.011	0.009	0.010	0.011	0.011
3	0.020	0.024	0.016	0.016	0.021
2	1.000	1.000	0.756	1.000	1.000

are used to estimate quantiles. If, for $k = 1, \ldots, K$, we define n_k to be the number of sample values with k significant digits in their significand, the left-hand panel displays the case $n_1 = n_2 = 50$ and $n_k = 100$ for $k = 3, \ldots, 6$, while the right-hand panel is obtained with $n_1 = 0$ and $n_k = 100$ for $k = 2, \ldots, 6$. It is seen that only extreme forms of truncation may be expected to have considerable impact on inferential conclusions based on $\langle S(X)\rangle$, while even a moderate fraction of sample values with $k = 2$ (as in the right-hand panel) does not distort the qualitative impression of the distribution of $\langle S(X)\rangle$ gathered under the Benford hypothesis. However, we stress that replacing the Benford simulation scheme based on Equation (20) with the truncation-robust algorithm described in the Supporting Information will provide estimates of exact quantiles and exact p-values for any observed realization of $n_1, \ldots, n_K$, even in situations (like the one depicted in the left-hand panel of Figure 5) where wild truncation occurs without being related to deliberate data manipulation. Additional empirical evidence is displayed in the Supporting Information, where we also cover the case of rounding.

8 | Applications in International Trade

That Benford's law has an almost immeasurable potential in the detection of various types of digit manipulations is becoming part of the collective imagination, with a rapidly increasing scientific literature [12, 41] complemented by recent narrative and documentary production inspired from it [42, 43]. At the present time, it is then almost obvious to consider the possibility that dishonest operators in the international trade circuit may consider having their movements scrutinized under Benford's law. Bockel-Rickermann et al. [1] recognize that the ability of fraudsters to modify their patterns in response to progresses in analytical research is one of the reasons why fraud has not yet disappeared.

The Joint Research Centre (JRC) of the European Commission has established a research track on international trade data and Benford's law both under its own institutional activities and alongside several Customs services through bilateral pilot activities. The declaration of the traded value at customs has direct financial consequences for the economic operator, who has to pay the duties, and for the public bodies collecting them. A dishonest operator has the motivation to manipulate the value to minimize the taxable amount or to get more favorable treatments. In the former case, the typical amounts involved are relatively big and the price is under-declared. The latter case can also occur with

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàÂ Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

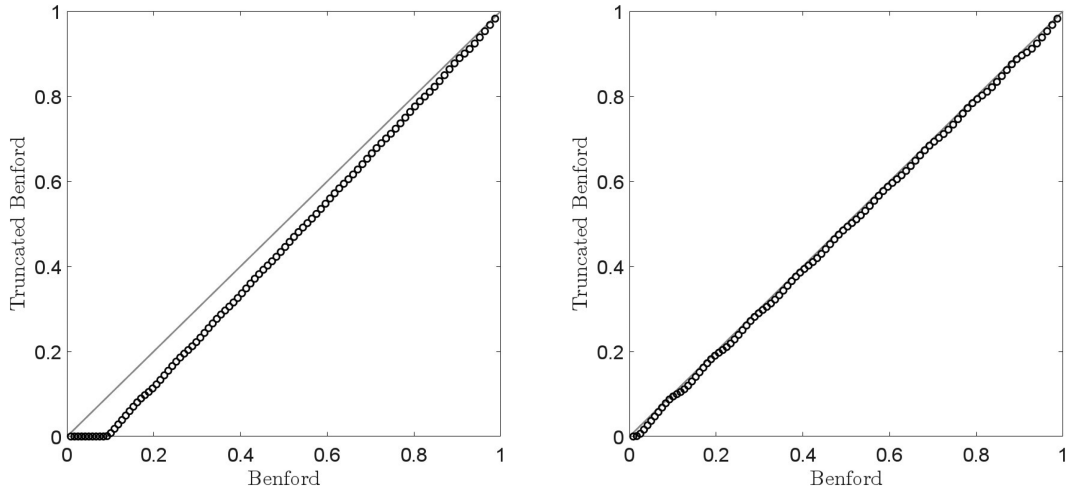


FIGURE 5 | QQ-plots comparing the empirical distribution of $\langle S(X) \rangle$ under the Benford hypothesis with that obtained using the truncated-Benford algorithm described in the Supporting Information, in two cases with $n = 500$. Left: $n_1 = n_2 = 50$ and $n_k = 100$ for $k = 3, \ldots, 6$. Right: $n_1 = 0$ and $n_k = 100$ for $k = 2, \ldots, 6$.

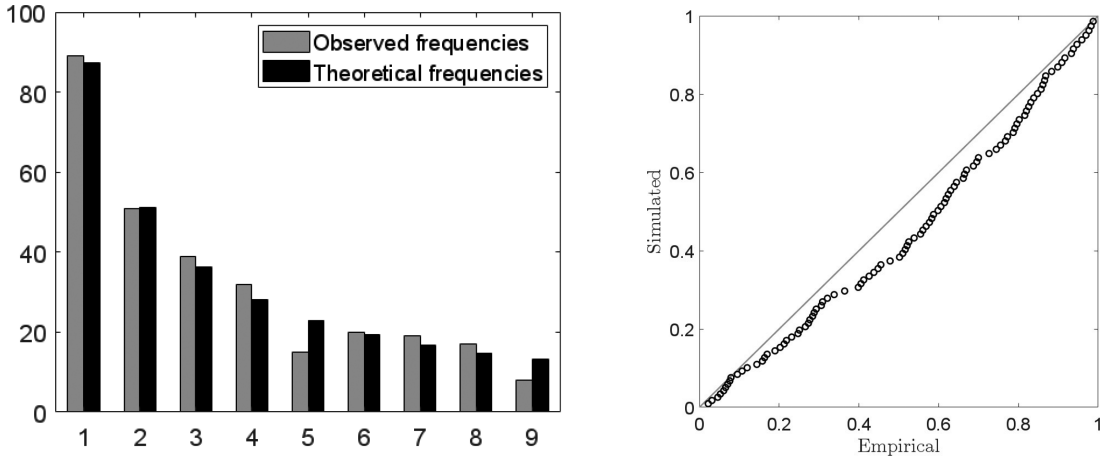


FIGURE 6 | Left: Histogram comparing the first-digit distribution of Operator A with the theoretical frequencies computed under the Benford probabilities Equation (2). Right: QQ-plot comparing the empirical quantiles of $\langle S(X) \rangle$ for the same operator with their Monte Carlo estimates under the truncated-Benford model.

small amounts when involving tax exemption thresholds. In our experimentation, we look at the declared value in euro and we use an anonymized identifier both for the operator and for the commodity code. The discussed application is indeed a component of a pilot exercise conducted in collaboration with a partner Customs service.

In the customs declarations of the first economic operator that we analyze, say Operator A, we observe $n = 290$ distinct values of $S(X)$ and only a limited degree of truncation, since $n_1 = 0$, $n_2 = 2$ and $n_3 = 5$. The conditions described in Cerioli et al. [13] for the validity of Equation (12) are broadly satisfied for this trader and indeed its empirical first-digit distribution, displayed in the left-hand panel of Figure 6, visually confirms the impression of close conformance to the theoretical counts predicted by Equation (2). The corresponding exact p-value of Q_1 is around 0.6, based on $B = 10^5$ replicates of Equation (20). Similarly, the exact p-values of Q_{12}, KS_1 and KU_1 are close to 0.19, 0.25 and 0.10, respectively. None of these tests casts doubts on the validity of hypothesis (6), a conclusion driven by the major effect of the empirical distribution of $D(X)$ on the test statistics. A preliminary signal of potential disagreement with the Benford hypothesis in the digits following the first one is provided by Q_2, whose exact p-value boils down to less than 1%. We then sharpen the conclusion suggested by the Hotelling statistic through a detailed analysis of the fractional part of the distinct significands available for this agent. The right-hand panel of Figure 6 compares the estimated quantiles of $\langle S(X) \rangle$ to those obtained under the truncated-Benford algorithm introduced in Section 7 and detailed in the Supporting Information, again based on $B = 10^5$ simulations. Strong disagreement is now paramount. This qualitative finding is supplemented by the exact p-values of our tests, which are all much smaller than 0.01 under the truncated-Benford model, and in particular by Q_Δ, whose p-value is below 0.0001. Therefore, our approach reinforces the idea that the values of $\langle S(X) \rangle$ do not follow the Benford hypothesis for this operator, even when the observed (mild) truncation is taken into account.

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàÂ Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

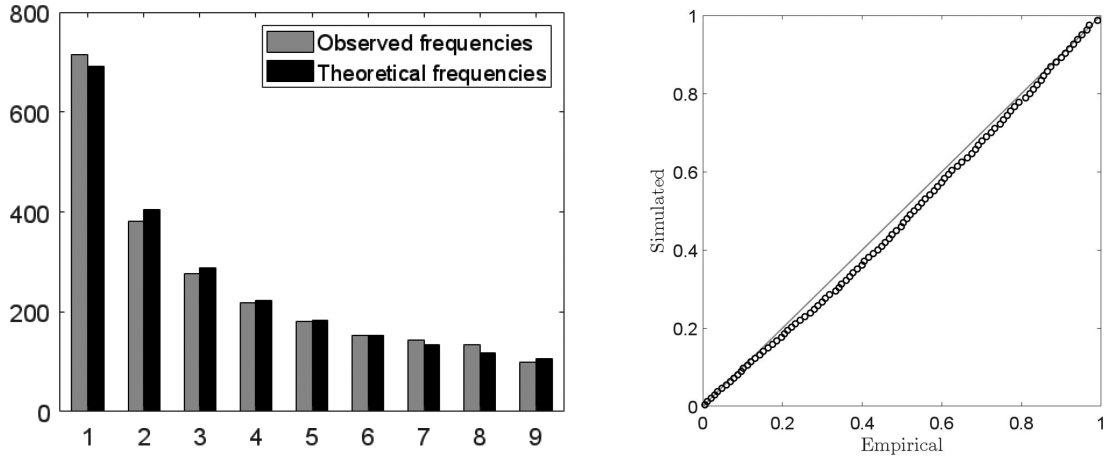


FIGURE 7 | As Figure 6, but now for Operator B.

The situation is even more intriguing in the case of Operator B, an instance for which Benford's law might be considered a questionable model for the digits of trade flows in view of the findings of Cerioli et al. [13], being $m \simeq 0.05n$. It is then puzzling to see that the observed distribution of $D(X)$, pictured in the left-hand panel of Figure 7, closely adheres to the theoretical Benford distribution. Visual inspection is confirmed by statistical analysis, since the asymptotic p-value of the first digit statistic Q_1 is 0.58, based on a quite large sample of $n = 2298$ distinct values of $S(X)$. Similar information is provided by the other statistics described in Section 2.2, whose p-values are all higher than 0.15 after allowing for truncation through our algorithm in the Supporting Information (with $n_1 = 9$, $n_2 = 39$, $n_3 = 100$ and $B = 10^5$). None of the standard statistics, including Q_2, thus provides evidence against the Benford hypothesis for this agent, in spite of the fact that the number of traded goods is relatively low with respect to the number of transactions and the theoretical conditions established by Hill [31] for the validity of Equation (1) are brought into question, especially in the case of a large sample. However, our conclusion becomes very different when looking at the fractional part of $S(X)$, displayed in the right-hand panel of Figure 7. The exact p-values of KS_2 and KU_2 under the truncated-Benford scheme are both close to 0.002. The difference statistic Q_Δ is less extreme (p-value around 0.04), but G_{KS} and G_{KU} inherit much of the evidence provided by $\langle S(X) \rangle$, their exact p-values being 0.004 and 0.003, respectively. The versatility and good power of these combined tests are not surprising in view of the results of the simulation study of Section 7.

To conclude, we have seen that our approach proves to be very informative in the two analyzed examples, but we cannot claim to be able to anticipate Benford-savvy manipulations with certainty, since unforeseen—but licit—economic and trading factors might possibly affect the significands of the financial flows under investigation. Nevertheless, our results surely point to situations where more substantial controls are needed and where comparison with the findings of other statistical techniques designed to signal possible financial infringements, such as those described in Perrotta et al. [8], could be particularly useful. In essence, the Benford detection framework seeks to provide supplementary insights that enhance the robustness of existing fraud detection systems, and the contribution of the present work provides a further step toward the final goal. As an additional reinforcing signal, the integration of Benford flags among other established customs risk criteria can be anticipated to mitigate the overall number of false positives, an important target both for scientists and practitioners [3, 10, 44, 45]. It is now recognized that fraud analysts can only investigate a limited number of transactions and our approach can thus assist law-enforcement agents in prioritizing follow-up actions and in allocating investigative resources more effectively. Although at the time of writing such an assistance can only be informal, we envisage the development of formal cost-sensitive risk criteria based on Benford's law as a challenging and practically useful task for future research.

Author Contributions

All authors contributed to designing this overarching research agenda. Barabesi, who in 2016 motivated the co-authors to embark research on Benford's ideas, developed the mathematical derivations; Cerioli structured the paper, drafted a significant portion of its content and contributed to simulations and empirical analysis; Cerasa and Perrotta, working within a public institution applying the theory and results of the paper, conducted simulations to validate the framework and demonstrated its practical relevance using real data from impactful policy domains.

Acknowledgments

The authors gratefully acknowledge Ivano Azzini's skillful assistance in interfacing Fortran and MATLAB codes, ensuring computational reproducibility. They also thank Emmanuele Sordini for developing WebAriadne, making our new tests accessible and user-friendly for authorized analysts. The Joint Research Centre (JRC) of the European Commission is funded by Horizon Europe and the Euratom Research and Training Programme. JRC has studied the data sets and problems of the paper in the course of bilateral pilot activities with some Customs services of the European Union. The research of Andrea Cerioli has been financially supported by the Italian Project of Relevant National Interest (PRIN) n. 2022LANNKC, titled "Innovative statistical tools for the analysis of large and heterogeneous customs data" and funded by NextGenerationEU (https://next-generation-eu.europa.eu). Open access publishing facilitated by Università degli Studi di Parma, as part of the Wiley - CRUI-CARE agreement.

Funding

This work was supported by NextGenerationEU. Horizon Europe. Euratom Research and Training Programme.

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàÂ Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The Benford-savvy GitHub repository https://github.com/AndreaCerioliUNIPR/Benford-savvy contains the anonymized significands used in our applications.

References

1. C. Bockel-Rickermann, T. Verdonck, and W. Verbeke, "Fraud Analytics: A Decade of Research: Organizing Challenges and Solutions in the Field," *Expert Systems With Applications* 232 (2023): 120605.

2. H. Hassani, X. Huang, E. S. Silva, and M. Ghodsi, "A Review of Data Mining Applications in Crime," *Statistical Analysis and Data Mining* 9 (2016): 139–154.

3. S. De Vos, T. Vanderschueren, T. Verdonck, and W. Verbeke, "Robust Instance-Dependent Cost-Sensitive Classification," *Advances in Data Analysis and Classification* 17 (2023): 1057–1079.

4. Y. Feng, M. Zhou, and X. Tong, "Imbalanced Classification: A Paradigm-Based Review," *Statistical Analysis and Data Mining* 14 (2021): 383–406.

5. F. A. Vandervorst, W. Verbeke, and T. Verdonck, "Claims Fraud Detection With Uncertain Labels," *Advances in Data Analysis and Classification* 18 (2024): 219–243.

6. Y. Zhang, T. Lu, Y. Huang, and Y. Ma, "Smart Data Augmentation: One Equation is All You Need," *Statistical Analysis and Data Mining* 17 (2024): e11672.

7. A. Cerioli, "Multivariate Outlier Detection With High-Breakdown Estimators," *Journal of the American Statistical Association* 105 (2010): 147–156.

8. D. Perrotta, A. Cerasa, F. Torti, and M. Riani, *The Robust Estimation of Monthly Prices of Goods Traded by the European Union* (Publications Office of the European Union, 2020).

9. P. J. Rousseeuw and M. Hubert, "Robust Statistics for Outlier Detection," *WIREs Data Mining and Knowledge Discovery* 1 (2011): 73–79.

10. L. Barabesi, A. Cerasa, A. Cerioli, and D. Perrotta, "Goodness-Of-Fit Testing for the Newcomb-Benford Law With Application to the Detection of Customs Fraud," *Journal of Business and Economic Statistics* 36 (2018): 346–358.

11. L. Barabesi, A. Cerasa, A. Cerioli, and D. Perrotta, "On Characterizations and Tests of Benford's Law," *Journal of the American Statistical Association* 117 (2022): 1887–1903.

12. L. Barabesi, A. Cerioli, and D. Perrotta, "Forum on Benford's Law and Statistical Methods for the Detection of Frauds," *Statistical Methods and Applications* 30 (2021): 767–778.

13. A. Cerioli, L. Barabesi, A. Cerasa, M. Menegatti, and D. Perrotta, "Newcomb-Benford Law and the Detection of Frauds in International Trade," *PNAS* 116 (2019): 106–115.

14. L. Lacasa, "Newcomb-Benford Law Helps Customs Officers to Detect Fraud in International Trade," *PNAS* 116 (2019): 11–13.

15. A. E. Kossovsky, *Benford's Law: Theory, the General Law of Relative Quantities, and Forensic Fraud Detection Applications* (World Scientific, 2015).

16. J. Ensminger and J. Leder-Luis, "Detecting Corruption: Evidence From a World Bank Project in Kenya," *World Development* 188 (2025): 106858.

17. R. Liu, L. Sheng, and J. Wang, "Faking Trade for Capital Control Evasion: Evidence From Dual Exchange Rate Arbitrage in China," *Journal of International Money and Finance* 138 (2023): 1–20.

18. M. J. Nigrini, *Benford's Law* (Wiley, 2012).

19. J. Fernández-Gracia and L. Lacasa, "Bipartisanship Breakdown, Functional Networks, and Forensic Analysis in Spanish 2015 and 2016 National Elections," *Complexity* 2018 (2018): 9684749.

20. W. R. Mebane, Jr., "Fraud in the 2009 Presidential Election in Iran?," *Chance* 23 (2010): 6–15.

21. L. Pericchi and D. Torres, "Quick Anomaly Detection by the Newcomb-Benford Law, With Applications to Electoral Processes Data From the USA, Puerto Rico and Venezuela," *Statistical Science* 26 (2011): 502–516.

22. T. Michalski and G. Stoltz, "Do Countries Falsify Economic Data Strategically? Some Evidence That They Might," *Review of Economics and Statistics* 95 (2013): 591–616.

23. D. Wang, F. Chen, J. Mao, N. Liu, and F. Rong, "Are the Official National Data Credible? Empirical Evidence From Statistics Quality Evaluation of China's Coal and Its Downstream Industries," *Energy Economics* 114 (2022): 106310.

24. G. M. Eckhartt and G. D. Ruxton, "Investigating and Preventing Scientific Misconduct Using Benford's Law," *Research Integrity and Peer Review* 8 (2023): 1.

25. J. Horton, D. K. Kumar, and A. Wood, "Detecting Academic Fraud Using Benford Law: The Case of Professor James Hunton," *Research Policy* 49 (2020): 104084.

26. J. González-García and G. Pastor, "Benford's Law and Macroeconomic Data Quality. IMF Working Paper 09/10, International Monetary Fund," (2009).

27. M. Kaiser, "Benford's Law as an Indicator of Survey Reliability – Can We Trust Our Data?," *Journal of Economic Surveys* 33, no. 5 (2019): 1602–1618.

28. L. Barabesi, A. Cerioli, and M. Di Marzio, "Statistical Models and the Benford Hypothesis: A Unifying Framework," *Test* 32 (2023): 1479–1507.

29. A. Berger and T. P. Hill, *An Introduction to Benford's Law* (Princeton Univ. Press, 2015).

30. A. Berger and T. P. Hill, "Benford's Law Strikes Back: No Simple Explanation in Sight for Mathematical Gem," *Mathematical Intelligencer* 33 (2011): 85–91.

31. T. P. Hill, "A Statistical Derivation of the Significant-Digit Law," *Statistical Science* 10 (1995): 354–363.

32. L. Dümbgen and C. Leuenberger, "Explicit Bounds for the Approximation Error in Benford's Law," *Electronic Communications in Probability* 13 (2008): 99–112.

33. A. Cerasa, "Testing for Benford's Law in Very Small Samples: Simulation Study and a New Test Proposal," *PLoS One* 17 (2022): e0271969.

34. N. Mumic and P. Filzmoser, "A Multivariate Test for Detecting Fraud Based on Benford's Law, With Application to Music Streaming Data," *Statistical Methods and Applications* 30 (2021): 819–840.

35. J. Raymaekers and P. Rousseeuw, "The Cellwise Minimum Covariance Determinant Estimator," *Journal of the American Statistical Association* 119, no. 548 (2024): 2610–2621.

36. W. F. Kibble, "A Two-Variate Gamma Type Distribution," *Sankhya* 5 (1941): 137–150.

37. L. Torricelli, L. Barabesi, and A. Cerioli, "Tempered Positive Linnik Processes and Their Representations," *Electronic Journal of Statistics* 16 (2022): 6313–6347.

38. L. Barabesi, A. Cerasa, D. Perrotta, and A. Cerioli, "Modeling International Trade Data With the Tweedie Distribution for Anti-Fraud and Policy Support," *European Journal of Operational Research* 248 (2016): 1031–1043.

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàCâ Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

39. L. Barabesi and L. Pratelli, "On the Generalized Benford Law," *Statistics & Probability Letters* 160 (2020): 108702.

40. L. Pietronero, E. Tosatti, V. Tosatti, and A. Vespignani, "Explaining the Uneven Distribution of Numbers in Nature: The Laws of Benford and Zipf," *Physica A* 293 (2001): 297–304.

41. A. Berger, T. P. Hill, and E. Rogers, "Benford Online Bibliography," (2009), http://www.benfordonline.net.

42. J. Murtagh, "This Unexpected Pattern of Numbers Is Everywhere," *Scientific American Magazine* 329 (2023): 82.

43. L. Nasser, "Digits. Netflix Series – Connected: The Hidden Science of Everything, Episode 4," (2020), https://www.netflix.com/title/81031737.

44. L. Le and E. Mantelaers, "Benford's Law and Beyond: A Framework for Auditors," *Maandblad Voor Accountancy en Bedrijfseconomie* 98 (2024): 427–438.

45. T. Vanderschueren, B. Baesens, T. Verdonck, and W. Verbeke, "A New Perspective on Classification: Optimally Allocating Limited Resources to Uncertain Tasks," *Decision Support Systems* 179 (2024): 114151.

Supporting Information

Additional supporting information can be found online in the Supporting Information section. **Data S1:** Supplementary Information.

19321872a, 2026, 3, Downloaded from https://onlinelibrary.wiley.com/doi/10.1002/sam.70081 by UniversitàCÂ Degli Studi Di, Wiley Online Library on [14/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License