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Boundary controllability for a degenerate beam equation

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The paper deals with the controllability of a degenerate beam equation. In particular, we assume that the left end of the beam is fixed, while a suitable control f acts on the right end of it. As a first step, we prove the existence of a solution for the homogeneous problem, then we prove some estimates on its energy. Thanks to them, we prove an observability inequality, and using the notion of solution by transposition, we prove that the initial problem is null controllable.

KEYWORDS

boundary observability, degenerate beam equation, fourth-order operator, null controllability

MSC CLASSIFICATION

35L80, 93B05, 93B07

1 | INTRODUCTION

We consider a boundary controllability problem for a system modeling the bending vibrations of a degenerate beam of length $L = 1$. Denote by u the deflection of the beam and assume that the left end of the beam is fixed, while a suitable shear force f is exerted on the right end of the beam; thus, the motion describing beam bending is given by the following problem

$$\begin{cases} u_{tt}(t,x) + Au(t,x) = 0, & (t,x) \in Q_T, \\ u(t,0) = 0,\ u_x(t,0) = 0, & t \in (0,T), \\ u(t,1) = 0,\ u_x(t,1) = f(t), & t \in (0,T), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in (0,1), \end{cases} \tag{1.1}$$

where $Q_T := (0,T) \times (0,1)$, $T > 0$, and $Au := au_{xxxx}$.

An equation similar to the one of (1.1) can be found, for example, in models that describe the vibrations of a bridge. Indeed, a suspension bridge may be seen as a beam of given length L, with hinged ends and whose downward deflection is measured by a function $u(t,x)$ subject to three forces. These forces can be summarized as the stays holding the bridge up as nonlinear springs with spring constant k, the constant weight per unit length of the bridge W pushing it down, and the external forcing term $f(t,x)$. This leads to the equation

$$u_{tt} + \gamma u_{xxxx} = -ku^+ + W + f(t,x),$$

where γ is a physical constant depending on the beam, on Young's modulus and on the second moment of inertia. If γ is a function that depends on the variable x and the external function acts only on the boundary, then we have exactly the equation of (1.1) (see, e.g., Camasta & Fragnelli [1] for other applications of (1.1)).

The novelty of this paper is that $a : [0,1] \to \mathbb{R}$ is such that $a(0) = 0$ and $a(x) > 0$ for all $x \in (0,1]$. If there exists a boundary function f that drives the solution of (1.1) to 0 at a given time $T > 0$, in the sense that

$$u(T,x) = u_t(T,x) = 0$$

for all $x \in (0,1)$, then the problem is said null controllable.

Boundary exact controllability on linear beam problems has been studied for many years by a lot of authors; see, for example, earlier studies [2–9] and the references therein. For quasilinear beams or nonlinear beams, we refer to Yao and Weiss [10] and previous research [11, 12], respectively.

In all the previous papers, the equation is always nondegenerate. The first results on boundary controllability for degenerate problems can be found in earlier work [13, 14] and in [15]. In particular, in [15] considers the equation in divergence form

$$u_{tt} - (x^\alpha u_x)_x = 0$$

for $\alpha \in (0,1)$, and the control acts in the degeneracy point $x = 0$. In the same period, Alabau-Boussouira et al. [13] consider the equation

$$u_{tt} - (a(x)u_x)_x = 0, \tag{1.2}$$

where $a \sim x^K$, $K > 0$. In this case, the authors establish observability inequalities when $K < 2$; if $K \geq 2$, a negative result is given. We remark that in Gueye [15], the observability inequality, and hence null controllability, is obtained via spectral analysis, while in Alabau-Boussouira et al. [13] via suitable energy estimates. In Boutaayamou et al. [14], the same problem of (1.2) in nondivergence case with a drift term is considered. Clearly, the presence of the operator in nondivergence form as well as the presence of a drift term leads the authors to use different spaces with respect to the ones in Alabau-Boussouira et al. [13] or in Gueye [15] and gives rise to some new difficulties. However, thanks to some suitable assumptions on the drift term, the authors prove some estimates on the energy that are crucial to prove an observability inequality and hence null controllability for the initial problem.

As far as we know, this is the first paper where the boundary controllability for a *degenerate* beam equation is considered. For the function a, we consider two cases: the weakly degenerate case and the strongly degenerate one. More precisely, we have the following definitions:

Definition 1.1. A function a is *weakly degenerate at 0*, (*WD*) for short, if $a \in C[0,1] \cap C^1(0,1]$, $a(0) = 0$, $a > 0$ on $(0,1]$ and if

$$K := \sup_{x\in(0,1]} \frac{x|a'(x)|}{a(x)}, \tag{1.3}$$

then $K \in (0,1)$.

Definition 1.2. A function a is *strongly degenerate at 0*, (*SD*) for short, if $a \in C^1[0,1]$, $a(0) = 0$, $a > 0$ on $(0,1]$ and in (1.3) we have $K \in [1,2)$.

We underline that, contrary to degenerate wave equations for which null controllability fails if $K \geq 2$, for degenerate beam equations null controllability is an open problem if $K \geq 2$ is an open problem; indeed, the assumption $K < 2$ is essential in this paper just for technical reasons.

Clearly, the presence of the degenerate operator $Au := au_{xxxx}$ leads us to use different spaces with respect to the ones in the previous papers [13, 14] or [15], and in these new spaces, we prove an estimate similar to the following one

$$E_y(0) \leq C \int_0^T y_{xx}^2(t,1)dt,$$

where y and E_y are the solution and the energy of the homogeneous adjoint problem associated to (1.1), respectively, and C is a strictly positive constant. Then, thanks to the introduction of the solutions by transposition for (1.1), we prove that (1.1) is null controllable for T sufficiently large. Actually, in Komornik [16], null controllability is proved for *nondegenerate* beam equations also for arbitrary short times. We expect that this result still holds for *degenerate* beam equations, but it will be the subject of a forthcoming paper.

We underline also that, as for wave equations or for parabolic models, in this paper we consider the same boundary conditions in the weakly and in the strongly degenerate case since here we consider the operator in *nondivergence form*. Different boundary conditions are considered if the operator is in *divergence form*; see Camasta and Fragnelli [17]. Observe that one can rewrite the operator in divergence form using the operator in nondivergence form in the following way

$$(a(x)y_{xx}(t,x))_{xx} = a''(x)y_{xx}(t,x) + 2a'(x)y_{xxx}(t,x) + a(x)y_{xxxx}(t,x).$$

But in order to apply the results of this paper to the same problem in divergence form, one has to make stronger assumptions on the function a. For this reason, the null controllability for a degenerate beam equation in divergence form is studied in Camasta and Fragnelli [17].

The paper is organized in the following way: in Section 2, we consider the homogeneous problem associated to (1.1) and we prove that this problem is well-posed in the sense of Theorem 2.2; in Section 3, we consider the energy associated to it and we prove two estimates on the energy from below and from above. In Section 4, thanks to these estimates and to the boundary observability (see Corollary 4.1), we prove that the original problem has a unique solution by transposition and this solution is null controllable. Section 5 presents some open problems. The paper ends with the Appendix where we give two proofs to make the article self-contained.

We underline that in this paper, C denotes universal positive constants which are allowed to vary from line to line.

2 | WELL-POSEDNESS FOR THE PROBLEM WITH HOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS

In this section, we study the well-posedness of the following degenerate hyperbolic problem with Dirichlet boundary conditions

$$\begin{cases} y_{tt}(t,x) + a y_{xxxx}(t,x) = 0, & (t,x) \in (0,+\infty)\times(0,1), \\ y(t,0) = y(t,1) = 0, & t \in (0,+\infty), \\ y_x(t,0) = y_x(t,1) = 0, & t \in (0,+\infty), \\ y(0,x) = y_T^0(x), & x \in (0,1), \\ y_t(0,x) = y_T^1(x), & x \in (0,1). \end{cases} \tag{2.5}$$

We underline the fact that the choice of denoting initial data with T-dependence is connected to the approach for null controllability used in the next sections.

As in earlier work [1, 18] or [19], let us consider the following weighted Hilbert spaces:

$$L^2_{\frac{1}{a}}(0,1) := \left\{ u \in L^2(0,1) \, : \, \int_0^1 \frac{u^2}{a}\,dx < +\infty \right\}$$

and

$$H^i_{\frac{1}{a}}(0,1) := L^2_{\frac{1}{a}}(0,1) \cap H^i_0(0,1), \ \ i = 1,2,$$

with the related norms

$$\|u\|^2_{L^2_{\frac{1}{a}}(0,1)} := \int_0^1 \frac{u^2}{a}\,dx \quad \forall u \in L^2_{\frac{1}{a}}(0,1)$$

and

$$\|u\|^2_{H^i_{\frac{1}{a}}(0,1)} := \|u\|^2_{L^2_{\frac{1}{a}}(0,1)} + \sum_{k=1}^{i} \|u^{(k)}\|^2_{L^2(0,1)} \quad \forall u \in H^i_{\frac{1}{a}}(0,1),$$

$i = 1,2$, respectively. We recall that $H^i_0(0,1) := \{u \in H^i(0,1) \, : \, u^{(k)}(j) = 0, \, j = 0,1, \, k = 0, \dots, i-1\}$, with $u^{(0)} = u$ and $i = 1,2$. Observe that for all $u \in H^i_{\frac{1}{a}}(0,1)$, using the fact that $u^{(k)}(j) = 0$ for all $k = 0, \dots, i-1$ and $j = 0,1$, it is easy to prove that $\|u\|^2_{H^i_{\frac{1}{a}}(0,1)}$ is equivalent to the following one

$$\|u\|_i^2 := \|u\|^2_{L^2_{\frac{1}{a}}(0,1)} + \|u^{(i)}\|^2_{L^2(0,1)}, \; i = 1, 2.$$

If $i = 1$, the previous assertion is clearly true.

Moreover, under an additional assumption on a, one can prove that the previous norms are equivalent to the following one

$$\|u\|_{i,\sim} := \|u^{(i)}\|_{L^2(0,1)}, \; i = 1, 2.$$

Indeed, assume the following.

Hypothesis 2.1. *The function* $a : [0, 1] \to \mathbb{R}$ *is continuous in* $[0, 1]$, $a(0) = 0$, $a > 0$ *on* $(0, 1]$, *and there exists* $K \in (0, 2)$ *such that the function*

$$x \mapsto \frac{x^K}{a(x)} \tag{2.6}$$

is nondecreasing near $x = 0$.

Observe that if a is weakly or strongly degenerate, then (1.3) implies that the function

$$x \mapsto \frac{x^\gamma}{a(x)}$$

is nondecreasing in $(0, 1]$ for all $\gamma \geq K$; in particular, (2.6) holds globally. Moreover,

$$\lim_{x\to 0} \frac{x^\gamma}{a(x)} = 0 \tag{2.7}$$

for all $\gamma > K$. The properties above will play a central role in the next sections.

Thanks to Hypothesis 2.1, one can prove the following equivalence.

Proposition 2.1. *Assume Hypothesis 2.1. Then for all* $u \in H^i_{\frac{1}{a}}(0, 1)$ *the norms* $\|u\|_{H^i_{\frac{1}{a}}(0,1)}$, $\|u\|_i$ *and* $\|u\|_{i,\sim}$, $i = 1, 2$, *are equivalent.*

Proof. By Cannarsa et al. [20, Proposition 2.6], one has that there exists $C > 0$ such that

$$\int_0^1 \frac{u^2}{a} dx \leq C \int_0^1 (u')^2 dx,$$

for all $u \in L^2_{\frac{1}{a}}(0, 1) \cap H^1_0(0, 1)$. Thus, the thesis follows immediately if $i = 1$.

Now, assume $i = 2$. Proceeding as for $i = 1$ and applying the classical Hardy inequality (see, e.g., Fragnelli & Mugnai [21]) to $z := u'$ (observe that $z \in H^1_0(0, 1)$), we have

$$\int_0^1 \frac{u^2}{a} dx \leq C \int_0^1 (u')^2 dx \leq C \int_0^1 \frac{z^2}{x^2} dx \leq 4C \int_0^1 (z')^2 dx = 4C \int_0^1 (u'')^2 dx$$

and the thesis follows. □

Hence, assuming Hypothesis 2.1 in the rest of the paper, we can use indifferently $\|\cdot\|_i$ or $\|u\|_{i,\sim}$ in place of $\|\cdot\|_{H^i_{\frac{1}{a}}(0,1)}$, $i = 1, 2$.

Using the previous spaces, it is possible to define the operator $(A, D(A))$ by

$$Au := au'''' \text{ for all } u \in D(A) := \left\{ u \in H^2_{\frac{1}{a}}(0, 1) : au'''' \in L^2_{\frac{1}{a}}(0, 1) \right\}.$$

Moreover,

$$\langle Au, v\rangle_{L^2_{\frac{1}{a}}(0,1)} = \int_0^1 u''v''dx,$$

that is,

$$\int_0^1 u''''v\,dx = \int_0^1 u''v''dx \tag{2.8}$$

for all $(u, v) \in D(A) \times H^2_{\frac{1}{a}}(0, 1)$ (see Camasta & Fragnelli [18, Proposition 2.1]).

Another important Hilbert space, related to the well-posedness of (2.5), is the following one

$$\mathcal{H}_0 := H^2_{\frac{1}{a}}(0, 1) \times L^2_{\frac{1}{a}}(0, 1),$$

endowed with the inner product

$$\langle (u, v), (\tilde{u}, \tilde{v})\rangle_{\mathcal{H}_0} := \int_0^1 u''\tilde{u}''dx + \int_0^1 \frac{v\tilde{v}}{a}dx$$

and with the norm

$$\|(u, v)\|^2_{\mathcal{H}_0} := \int_0^1 (u'')^2dx + \int_0^1 \frac{v^2}{a}dx$$

for every (u, v), $(\tilde{u}, \tilde{v}) \in \mathcal{H}_0$. Then, consider the matrix operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H}_0 \to \mathcal{H}_0$ given by

$$\mathcal{A} := \begin{pmatrix} 0 & Id \\ -A & 0 \end{pmatrix}, \quad D(\mathcal{A}) := D(A) \times H^2_{\frac{1}{a}}(0, 1).$$

Using this operator, we rewrite (2.5) as a Cauchy problem. Indeed, setting

$$\mathcal{U}(t) := \begin{pmatrix} y(t) \\ y_t(t) \end{pmatrix} \quad \text{and} \quad \mathcal{U}_0 := \begin{pmatrix} y_T^0 \\ y_T^1 \end{pmatrix},$$

one has that (2.5) can be formulated as

$$\begin{cases} \dot{\mathcal{U}}(t) = \mathcal{A}\,\mathcal{U}(t), \ t \geq 0, \\ \mathcal{U}(0) = \mathcal{U}_0. \end{cases} \tag{2.9}$$

Theorem 2.1. *Assume Hypothesis 2.1. Then the operator $(\mathcal{A}, D(\mathcal{A}))$ is nonpositive with dense domain and generates a contraction semigroup $(S(t))_{t\geq 0}$.*

Proof. According to Engel and Nagel [22, Corollary 3.20], it is sufficient to prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}_0$ is dissipative and $\mathcal{I} - \mathcal{A}$ is surjective, where

$$\mathcal{I} := \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}.$$

$\mathcal{A}$ is dissipative: Take $(u, v) \in D(\mathcal{A})$. Then $(u, v) \in D(A) \times H^2_{\frac{1}{a}}(0, 1)$ and so (2.8) holds. Hence,

$$\begin{aligned} \langle \mathcal{A}(u, v), (u, v)\rangle_{\mathcal{H}_0} &= \langle (v, -Au), (u, v)\rangle_{\mathcal{H}_0} \\ &= \int_0^1 u''v''dx - \int_0^1 vAu\frac{1}{a}dx = 0. \end{aligned}$$

By Engel and Nagel [22, Chapter 2.3], the operator $\mathcal{A}$ is dissipative.

$\mathcal{I}-\mathcal{A}$ is surjective: Take $(f,g)\in\mathcal{H}_0=H^2_{\frac{1}{a}}(0,1)\times L^2_{\frac{1}{a}}(0,1)$. We have to prove that there exists $(u,v)\in D(\mathcal{A})$ such that

$$(\mathcal{I}-\mathcal{A})\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}f\\ g\end{pmatrix}\iff\begin{cases}v=u-f,\\ Au+u=f+g.\end{cases}\tag{2.10}$$

Thus, define $F:H^2_{\frac{1}{a}}(0,1)\to\mathbb{R}$ as

$$F(z)=\int_0^1(f+g)z\frac{1}{a}dx,$$

for all $z\in H^2_{\frac{1}{a}}(0,1)$. Obviously, $F\in\left(H^2_{\frac{1}{a}}(0,1)\right)^*$, being $\left(H^2_{\frac{1}{a}}(0,1)\right)^*$ the dual space of $H^2_{\frac{1}{a}}(0,1)$ with respect to the pivot space $L^2_{\frac{1}{a}}(0,1)$. Now, introduce the bilinear form $L:H^2_{\frac{1}{a}}(0,1)\times H^2_{\frac{1}{a}}(0,1)\to\mathbb{R}$ given by

$$L(u,z):=\int_0^1 uz\frac{1}{a}dx+\int_0^1 u''z''dx$$

for all $u,z\in H^2_{\frac{1}{a}}(0,1)$. Clearly, thanks to the equivalence of the norms given before, $L(u,z)$ is coercive. Moreover, $L(u,z)$ is continuous. Indeed, for all $u,z\in H^2_{\frac{1}{a}}(0,1)$, we have

$$|L(u,z)|\le\|u\|_{L^2_{\frac{1}{a}}(0,1)}\|z\|_{L^2_{\frac{1}{a}}(0,1)}+\|u''\|_{L^2(0,1)}\|z''\|_{L^2(0,1)},$$

and the conclusion follows again by the equivalence of the norms.

As a consequence, by the Lax–Milgram Theorem, there exists a unique solution $u\in H^2_{\frac{1}{a}}(0,1)$ of

$$L(u,z)=F(z)\ \text{ for all }\ z\in H^2_{\frac{1}{a}}(0,1),$$

namely,

$$\int_0^1 uz\frac{1}{a}dx+\int_0^1 u''z''dx=\int_0^1(f+g)z\frac{1}{a}dx\tag{2.11}$$

for all $z\in H^2_{\frac{1}{a}}(0,1)$.

Now, take $v:=u-f$; then $v\in H^2_{\frac{1}{a}}(0,1)$. We will prove that $(u,v)\in D(\mathcal{A})$ and solves (2.10). To begin with, (4.38) holds for every $z\in C_c^\infty(0,1)$. Thus, we have

$$\int_0^1 u''z''dx=\int_0^1(f+g-u)z\frac{1}{a}dx$$

for every $z\in C_c^\infty(0,1)$. Hence, $(u'')''=(f+g-u)\frac{1}{a}$ a.e. in $(0,1)$, that is, $Au=f+g-u$ a.e. in $(0,1)$. Thus, as in Camasta and Fragnelli [18, Theorem 2.1], $u\in D(A)$; hence, $(u,v)\in D(\mathcal{A})$ and $u+Au=f+g$. Recalling that $v=u-f$, we have that (u,v) solves (2.10). □

Now, if $\mathcal{U}_0\in\mathcal{H}_0$, then $\mathcal{U}(t)=S(t)\mathcal{U}_0$ is the mild solution of (2.9). Also, if $\mathcal{U}_0\in D(\mathcal{A})$, then the solution is classical and the equation in (2.5) holds for all $t\ge0$. Hence, by Bensoussan et al. [23, Propositions 3.1 and 3.3], one has the following theorem.

Theorem 2.2. *Assume Hypothesis 2.1. If $\left(y_T^0,y_T^1\right)\in\mathcal{H}_0$, then there exists a unique mild solution*

$$y\in C^1\left([0,+\infty);L^2_{\frac{1}{a}}(0,1)\right)\cap C\left([0,+\infty);H^2_{\frac{1}{a}}(0,1)\right)$$

of (2.5) which depends continuously on the initial data $\left(y_T^0, y_T^1\right) \in \mathcal{H}_0$. *Moreover, if* $\left(y_T^0, y_T^1\right) \in D(\mathcal{A})$, *then the solution* y *is classical in the sense that*

$$y \in C^2\left([0,+\infty); L^2_{\frac{1}{a}}(0,1)\right) \cap C^1\left([0,+\infty); H^2_{\frac{1}{a}}(0,1)\right) \cap C([0,+\infty); D(A))$$

and the equation of (2.5) holds for all $t \geq 0$.

Remark 1.

1. Due to the reversibility (in time) of the equation, solutions exist with the same regularity also for $t < 0$.
2. Observe that the proofs of Theorems 2.1 and 2.2 are independent of (2.6).

3 | ENERGY ESTIMATES

In this section, we prove some estimates of the energy associated to the solution of (2.5). To this aim, we give the next definition.

Definition 3.1. Let y be a mild solution of (2.5) and consider its energy given by the continuous function defined as

$$E_y(t) := \frac{1}{2}\int_0^1 \left(\frac{y_t^2(t,x)}{a(x)} + y_{xx}^2(t,x)\right) dx \quad \forall\, t \geq 0.$$

The definition above guarantees that the classical conservation of the energy still holds also in this degenerate situation.

Theorem 3.1. *Assume Hypothesis 2.1 and let* y *be a mild solution of (2.5). Then*

$$E_y(t) = E_y(0) \quad \forall\, t \geq 0. \tag{3.12}$$

Proof. First of all, suppose that y is a classical solution. Then multiplying the equation

$$y_{tt} + Ay = 0$$

by $\frac{y_t}{a}$, integrating over $(0,1)$ and using the formula of integration by parts (2.8), one has

$$0 = \frac{1}{2}\int_0^1 \left(\frac{y_t^2}{a}\right)_t dx + \int_0^1 y_{xxxx}\, y_t dx = \frac{1}{2}\frac{d}{dt}\left(\int_0^1 \left(\frac{y_t^2}{a} + y_{xx}^2\right) dx\right) = \frac{d}{dt}E_y(t).$$

Consequently, the energy E_y associated to y is constant.

If y is a mild solution, we approximate the initial data with more regular ones, obtaining associated classical solutions for which (3.12) holds. Thanks to the usual estimates, we can pass to the limit and obtain the thesis. □

In the next results, we establish some inequalities for the energy from above and from below; these inequalities will be used in the next section to establish the controllability result. First of all, we start with the following theorem, whose proof is based on the next lemma.

Lemma 3.1. *Assume Hypothesis 2.1.*

1. *If* $y \in H^1_{\frac{1}{a}}(0,1)$, *then* $\lim_{x\to 0} \frac{x}{a} y^2(x) = 0$.
2. *Assume* a *(SD) at* 0. *If* $y \in D(A)$, *then* $y'' \in W^{1,1}(0,1)$.

The previous results are proved in Boutaayamou et al. [14, Lemma 3.2.5] and Camasta and Fragnelli [1, Proposition 3.2], respectively; anyway, we rewrite their proof in the Appendix to make the paper self-contained.

Theorem 3.2. *Assume a (WD) or (SD) at 0. If y is a classical solution of (2.5), then $y_{xx}(\cdot,1) \in L^2(0,T)$ for any $T > 0$ and*

$$\frac{1}{2}\int_0^T y_{xx}^2(t,1)dt = \int_0^1 \left[y_t \frac{x^2}{a} y_x\right]_{t=0}^{t=T} dx + \frac{1}{2}\int_{Q_T} \frac{x}{a} y_t^2 \left(2 - \frac{xa'}{a}\right) dx\,dt + 3\int_{Q_T} x y_{xx}^2 dx\,dt, \tag{3.13}$$

Proof. Multiply the equation in (2.5) by $\frac{x^2 y_x}{a}$ and integrate over Q_T. Integrating by parts, we obtain

$$\begin{aligned}
0 &= \int_{Q_T} y_{tt}\frac{x^2 y_x}{a} dx\,dt + \int_{Q_T} x^2 y_x y_{xxxx} dx\,dt \\
&= \int_0^1 \left[y_t \frac{x^2 y_x}{a}\right]_{t=0}^{t=T} dx - \int_{Q_T} y_t \frac{x^2}{a} y_{xt} dx\,dt + \int_{Q_T} x^2 y_x y_{xxxx} dx\,dt \\
&= \int_0^1 \left[y_t \frac{x^2 y_x}{a}\right]_{t=0}^{t=T} dx - \int_{Q_T} \frac{1}{2}\frac{x^2}{a}(y_t^2)_x dx\,dt + \int_{Q_T} x^2 y_x y_{xxxx} dx\,dt \\
&= \int_0^1 \left[y_t \frac{x^2 y_x}{a}\right]_{t=0}^{t=T} dx - \frac{1}{2}\int_0^T \left[\frac{x^2}{a} y_t^2\right]_{x=0}^{x=1} dt + \frac{1}{2}\int_{Q_T} \left(\frac{x^2}{a}\right)' y_t^2 dx\,dt \\
&\quad + \int_{Q_T} x^2 y_x y_{xxxx} dx\,dt.
\end{aligned} \tag{3.14}$$

Now, $\left(\frac{x^2}{a}\right)' = \frac{2xa - x^2 a'}{a^2} = \frac{x}{a}\left(2 - \frac{xa'}{a}\right)$. Hence, (3.14) reads

$$\int_0^1 \left[y_t \frac{x^2}{a} y_x\right]_{t=0}^{t=T} dx - \frac{1}{2}\int_0^T \left[\frac{x^2}{a} y_t^2\right]_{x=0}^{x=1} dt + \frac{1}{2}\int_{Q_T} \frac{x}{a} y_t^2 \left(2 - \frac{xa'}{a}\right) dx\,dt + \int_{Q_T} x^2 y_x y_{xxxx} dx\,dt = 0. \tag{3.15}$$

Furthermore, by the regularity of the solution, $y_t \in H^2_{\frac{1}{a}}(0,1) \subset H^1_{\frac{1}{a}}(0,1)$, thus,

$$\lim_{x\to 0} \frac{x^2}{a(x)} y_t^2(t,x) = 0$$

by Lemma 3.1; therefore, by the boundary conditions of y, one has $\frac{1}{a(1)} y_t^2(t,1) = 0$. Now, consider the term $\int_{Q_T} x^2 y_x y_{xxxx} dx\,dt$, which is well-defined; indeed, using the fact that $\frac{x^2}{\sqrt{a}}$ is nondecreasing, we have that there exists a positive constant C such that

$$\left| x^2 \frac{1}{\sqrt{a}} y_x \sqrt{a}\, y_{xxxx} \right| \le C\,|y_x| \left|\sqrt{a} y_{xxxx}\right|.$$

By hypothesis, y_x and $\sqrt{a} y_{xxxx}$ belong to $L^2(0,1)$; thus, by the Hölder inequality, $x^2 y_x y_{xxxx} \in L^1(0,1)$.

Let $\delta > 0$ and write

$$\int_{Q_T} x^2 y_x y_{xxxx} dx\,dt = \int_0^T \int_0^\delta x^2 y_x y_{xxxx} dx\,dt + \int_0^T \int_\delta^1 x^2 y_x y_{xxxx} dx\,dt. \tag{3.16}$$

Obviously, by the absolute continuity of the integral,

$$\lim_{\delta\to 0}\int_0^T\int_0^\delta x^2 y_x y_{xxxx}\,dx\,dt = 0. \tag{3.17}$$

Now, we will estimate the second term in (3.16). By definition of $D(A)$, setting $I := (\delta, 1]$, we have $y_{xxxx} \in L^2(I)$; thus, $y \in H^4(I)$ by Camasta and Fragnelli [18, Lemma 2.1]. Hence, we can integrate by parts

$$\begin{aligned}
\int_0^T\int_\delta^1 x^2 y_x y_{xxxx}\,dx\,dt &= \int_0^T \left[x^2 y_x y_{xxx}\right]_{x=\delta}^{x=1} dt - \int_0^T\int_\delta^1 (x^2 y_x)_x y_{xxx}\,dx\,dt\\
&= \int_0^T \left[x^2 y_x y_{xxx}\right]_{x=\delta}^{x=1} dt - \int_0^T\int_\delta^1 (2xy_x + x^2 y_{xx}) y_{xxx}\,dx\,dt\\
&= \int_0^T \left[x^2 y_x y_{xxx}\right]_{x=\delta}^{x=1} dt - \int_0^T\int_\delta^1 2xy_x y_{xxx}\,dx\,dt - \frac{1}{2}\int_0^T\int_\delta^1 x^2\left(y_{xx}^2\right)_x dx\,dt\\
&= \int_0^T \left[x^2 y_x y_{xxx}\right]_{x=\delta}^{x=1} dt - \int_0^T [2xy_x y_{xx}]_{x=\delta}^{x=1} dt + 2\int_0^T\int_\delta^1 (xy_x)_x y_{xx}\,dx\,dt\\
&\quad - \frac{1}{2}\int_0^T \left[x^2 y_{xx}^2\right]_{x=\delta}^{x=1} dt + \frac{1}{2}\int_0^T\int_\delta^1 2xy_{xx}^2\,dx\,dt\\
&= \int_0^T \left[x^2 y_x y_{xxx}\right]_{x=\delta}^{x=1} dt - \int_0^T [2xy_x y_{xx}]_{x=\delta}^{x=1} dt - \frac{1}{2}\int_0^T \left[x^2 y_{xx}^2\right]_{x=\delta}^{x=1} dt\\
&\quad + 2\int_0^T\int_\delta^1 y_x y_{xx}\,dx\,dt + 2\int_0^T\int_\delta^1 xy_{xx}^2\,dx\,dt + \int_0^T\int_\delta^1 xy_{xx}^2\,dx\,dt\\
&= \int_0^T \left[x^2 y_x y_{xxx}\right]_{x=\delta}^{x=1} dt - \int_0^T [2xy_x y_{xx}]_{x=\delta}^{x=1} dt - \frac{1}{2}\int_0^T \left[x^2 y_{xx}^2\right]_{x=\delta}^{x=1} dt\\
&\quad + \int_0^T\int_\delta^1 \left(y_x^2\right)_x dx\,dt + 3\int_0^T\int_\delta^1 xy_{xx}^2\,dx\,dt\\
&= \int_0^T \left[x^2 y_x y_{xxx}\right]_{x=\delta}^{x=1} dt - \int_0^T [2xy_x y_{xx}]_{x=\delta}^{x=1} dt - \frac{1}{2}\int_0^T \left[x^2 y_{xx}^2\right]_{x=\delta}^{x=1} dt\\
&\quad + \int_0^T \left[y_x^2\right]_{x=\delta}^{x=1} dt + 3\int_0^T\int_\delta^1 xy_{xx}^2\,dx\,dt.
\end{aligned} \tag{3.18}$$

But $xy_{xx}^2 \in L^1(0,1)$ and using the absolute continuity of the integral, we obtain

$$\lim_{\delta\to 0}\int_0^T\int_\delta^1 xy_{xx}^2\,dx\,dt = \int_0^T\int_0^1 xy_{xx}^2\,dx\,dt.$$

Now, we evaluate the boundary terms that appear in (3.18). To this aim, observe that thanks to the boundary conditions of y,

$$\left[x^2 y_x y_{xxx}\right]_{x=\delta}^{x=1} - [2xy_x y_{xx}]_{x=\delta}^{x=1} - \frac{1}{2}\left[x^2 y_{xx}^2\right]_{x=\delta}^{x=1} + \left[y_x^2\right]_{x=\delta}^{x=1} = -\delta^2 y_x(t,\delta) y_{xxx}(t,\delta)$$

$$+ 2\delta y_x(t,\delta) y_{xx}(t,\delta) - \frac{1}{2} y_{xx}^2(t,1) + \frac{1}{2}\delta^2 y_{xx}^2(t,\delta) - y_x^2(t,\delta).$$

Hence, we have to estimate the following quantities:

$$\delta^2 y_x(t,\delta) y_{xxx}(t,\delta),$$

$$\delta y_x(t,\delta) y_{xx}(t,\delta),$$

$$\delta^2 y_{xx}^2(t,\delta),$$

$$y_x^2(t,\delta)$$

as δ goes to 0. Naturally, since $y \in H_0^2(0,1)$, y_x is a continuous function. This implies that

$$\lim_{\delta\to 0} y_x^2(t,\delta) = y_x^2(t,0) = 0. \tag{3.19}$$

Thanks to Lemma 3.1,

$$\lim_{\delta\to 0} \delta^2 y_{xx}^2(t,\delta) = 0 = \lim_{\delta\to 0} \delta y_x(t,\delta) y_{xx}(t,\delta).$$

It remains to prove that

$$\exists \lim_{\delta\to 0} \delta^2 y_x(t,\delta) y_{xxx}(t,\delta) = 0. \tag{3.20}$$

By (3.19), it is sufficient to prove that $\exists \lim_{\delta\to 0} \delta y_{xxx}(t,\delta) \in \mathbb{R}$. To this aim, we rewrite $\delta y_{xxx}(t,\delta)$ as

$$\delta y_{xxx}(t,\delta) = y_{xxx}(t,1) - \int_\delta^1 (x y_{xxx}(t,x))_x dx = y_{xxx}(t,1) - \int_\delta^1 y_{xxx}(t,x)dx - \int_\delta^1 x y_{xxxx}(t,x)dx. \tag{3.21}$$

Note that $x y_{xxxx}(t,x) = \sqrt{a(x)} y_{xxxx}(t,x)\frac{x}{\sqrt{a(x)}} \in L^2(0,1) \subseteq L^1(0,1)$ (indeed, $\sqrt{a} y_{xxxx} \in L^2(0,1)$ and $\frac{x}{\sqrt{a(x)}} \in L^\infty(0,1)$, thanks to (2.6)). Hence, by the absolute continuity of the integral $\lim_{\delta\to 0}\int_\delta^1 x y_{xxxx}(x)dx = \int_0^1 x y_{xxxx}(x)dx$. On the other hand,

$$\int_\delta^1 y_{xxx}(t,x)dx = \int_\delta^1 \left(y_{xxx}(t,1) - \int_x^1 y_{xxxx}(t,s)ds \right) dx$$

$$= (1-\delta) y_{xxx}(t,1) - \int_\delta^1 \int_x^1 y_{xxxx}(t,s)ds\,dx.$$

Now, we estimate the last term in the previous equation

$$\begin{aligned}\int_\delta^1 \int_x^1 y_{xxxx}(t,s)ds\,dx &= \int_\delta^1 \int_\delta^s y_{xxxx}(t,s)dxds = \int_\delta^1 y_{xxxx}(t,s)(s-\delta)ds \\ &= \int_\delta^1 s y_{xxxx}(t,s)ds - \delta \int_\delta^1 y_{xxxx}(t,s)ds.\end{aligned} \tag{3.22}$$

As before, $\lim_{\delta\to 0}\int_\delta^1 s y_{xxxx}(t,s)ds = \int_0^1 s y_{xxxx}(t,s)ds$. Moreover, as far as the second term in the last member of (3.22) is concerned, we have

$$\begin{aligned}
0 < \delta \int_\delta^1 |y_{xxxx}(t,s)|ds &= \delta \int_\delta^1 \sqrt{a(s)}\frac{|y_{xxxx}(t,s)|}{\sqrt{a(s)}}ds \\
&\le \delta\left(\int_\delta^1 \frac{1}{a(s)}ds\right)^{\frac{1}{2}} \left\|\sqrt{a}y_{xxxx}\right\|_{L^2(0,1)} \\
&= \delta^{1-\frac{K}{2}}\left(\int_\delta^1 \frac{\delta^K}{a(s)}ds\right)^{\frac{1}{2}} \left\|\sqrt{a}y_{xxxx}\right\|_{L^2(0,1)} \\
&\le \delta^{1-\frac{K}{2}}\left(\int_\delta^1 \frac{s^K}{a(s)}ds\right)^{\frac{1}{2}} \left\|\sqrt{a}y_{xxxx}\right\|_{L^2(0,1)} \\
&\le C\delta^{1-\frac{K}{2}}(1-\delta)^{\frac{1}{2}}\left\|\sqrt{a}y_{xxxx}\right\|_{L^2(0,1)},
\end{aligned}$$

for a positive constant C. Thus, since $K < 2$, it follows that $\lim_{\delta\to 0}\delta \int_\delta^1 y_{xxxx}(t,s)ds = 0$. Consequently,

$$\lim_{\delta\to 0}\int_\delta^1\int_x^1 y_{xxxx}(t,s)ds\,dx = \lim_{\delta\to 0}\int_\delta^1 y_{xxxx}(t,s)(s-\delta)ds = \int_0^1 s y_{xxxx}(t,s)ds$$

As a consequence, coming back to (3.21),

$$\exists \lim_{\delta\to 0}\delta y_{xxx}(t,\delta) \in \mathbb{R},$$

and, in particular, (3.20) is proved.

Thus, by (3.18), one has

$$\lim_{\delta\to 0}\int_0^T\int_\delta^1 x^2 y_x y_{xxxx}dx\,dt = -\frac{1}{2}\int_0^T y_{xx}^2(t,1)dt + 3\int_0^T\int_0^1 x y_{xx}^2 dxdt.$$

By the previous equality, (3.15)–(3.17), the thesis follows. □

As a consequence of the previous equality on $\frac{1}{2}\int_0^T y_{xx}^2(t,1)dt$, we have the next estimate from below on the energy.

Theorem 3.3. *Assume a (WD) or (SD) at 0. If y is a mild solution of (2.5), then*

$$\int_0^T y_{xx}^2(t,1)dt \le \left(12T + 4\max\left\{\frac{4}{a(1)},1\right\}\right)E_y(0) \tag{3.23}$$

Proof. As a first step, assume that y is a classical solution of (2.5); thus, (3.13) holds. Now, set $z(t,x) := y_x(t,x)$; since $y_x(t,0) = 0$, by the classical Hardy inequality, for any $T > 0$ we obtain

$$\int_0^1 y_x^2dx = \int_0^1 z^2dx = \int_0^1 \frac{z^2}{x^2}x^2dx \le \int_0^1 \frac{z^2}{x^2}dx \le 4\int_0^1 z_x^2dx = 4\int_0^1 y_{xx}^2dx. \tag{3.24}$$

Thus, applying (2.6), one has

$$\left|\int_0^1 \frac{x^2 y_x(\tau,x)y_t(\tau,x)}{a(x)}dx\right| \le \frac{1}{2}\int_0^1 \frac{x^4}{a(x)}y_x^2(\tau,x)dx + \frac{1}{2}\int_0^1 \frac{y_t^2(\tau,x)}{a(x)}dx$$
$$\le \frac{1}{2a(1)}\int_0^1 y_x^2(\tau,x)dx + \frac{1}{2}\int_0^1 \frac{y_t^2(\tau,x)}{a(x)}dx$$
$$\le \frac{2}{a(1)}\int_0^1 y_{xx}^2(\tau,x)dx + \frac{1}{2}\int_0^1 \frac{y_t^2(\tau,x)}{a(x)}dx$$

for all $\tau \in [0,T]$. By Theorem 3.1, we get

$$\begin{aligned}\left|\int_0^1 \left[\frac{x^2 y_x(\tau,x)y_t(\tau,x)}{a(x)}\right]_{\tau=0}^{\tau=T} dx\right| &\le \frac{2}{a(1)}\int_0^1 y_{xx}^2(T,x)dx + \frac{1}{2}\int_0^1 \frac{y_t^2(T,x)}{a(x)}dx \\ &\quad + \frac{2}{a(1)}\int_0^1 y_{xx}^2(0,x)dx + \frac{1}{2}\int_0^1 \frac{y_t^2(0,x)}{a(x)}dx \\ &\le \max\left\{\frac{4}{a(1)},1\right\}(E_y(T)+E_y(0)) \\ &= 2\max\left\{\frac{4}{a(1)},1\right\}E_y(0).\end{aligned} \tag{3.25}$$

Moreover, using the fact that $x|a'| \le Ka$, we find

$$\begin{aligned}\left|\int_{Q_T} \frac{x}{a}y_t^2\left(2-\frac{xa'}{a}\right)dxdt\right| &\le \int_{Q_T}\frac{x}{a}y_t^2(2+K)dxdt \\ &\le (2+K)\int_{Q_T}\frac{y_t^2}{a}dxdt.\end{aligned} \tag{3.26}$$

Clearly,

$$\int_{Q_T} xy_{xx}^2 dxdt \le \int_{Q_T} y_{xx}^2 dxdt \tag{3.27}$$

and from (3.13), (3.25), (3.26), and (3.27), we get (3.23) if y is a classical solution of (2.5). Now, let y be the mild solution associated to the initial data $(y_0, y_1) \in \mathcal{H}_0$. Then, consider a sequence $\{(y_0^n, y_1^n)\}_{n\in\mathbb{N}} \subset D(\mathcal{A})$ that converges to (y_0, y_1) and let y^n be the classical solution of (2.5) associated to (y_0^n, y_1^n). Clearly, y^n satisfies (3.23); then, we can pass to the limit and conclude. □

Now, we will prove an inequality on the energy from above. To this aim, we need on $\int_0^T y_{xx}^2(t,1)dt$ an equality different from (3.13).

Theorem 3.4. *Assume a (WD) or (SD) at 0. If y is a classical solution of (2.5), then $y_{xx}(\cdot,1) \in L^2(0,T)$ for any $T>0$ and*

$$\frac{1}{2}\int_0^T y_{xx}^2(t,1)dt = \int_0^1 \left[\frac{xy_ty_x}{a}\right]_{t=0}^{t=T} dx + \frac{1}{2}\int_{Q_T}\frac{y_t^2}{a}\left(1-\frac{xa'}{a}\right)dxdt + \frac{3}{2}\int_{Q_T} y_{xx}^2 dxdt, \tag{3.28}$$

Proof. Multiplying the equation in (2.5) by $\frac{xy_x}{a}$ and integrating over Q_T, we obtain

$$\begin{aligned} 0 &= \int_0^1 \left[\frac{xy_xy_t}{a}\right]_{t=0}^{t=T} dx - \int_{Q_T} \frac{1}{2}\frac{x}{a}(y_t^2)_x dxdt + \int_{Q_T} xy_xy_{xxxx}dxdt \\ &= \int_0^1 \left[\frac{xy_xy_t}{a}\right]_{t=0}^{t=T} dx - \frac{1}{2}\int_0^T \left[\frac{x}{a}y_t^2\right]_{x=0}^{x=1} dt + \frac{1}{2}\int_{Q_T}\left(\frac{x}{a}\right)' y_t^2 dxdt + \int_{Q_T} xy_xy_{xxxx}dxdt. \end{aligned} \tag{3.29}$$

Now, $\left(\frac{x}{a}\right)' = \frac{a-xa'}{a^2} = \frac{1}{a}\left(1-\frac{xa'}{a}\right)$. Hence, (3.29) reads

$$\int_0^1 \left[\frac{xy_xy_t}{a}\right]_{t=0}^{t=T} dx - \frac{1}{2}\int_0^T \left[\frac{x}{a}y_t^2\right]_{x=0}^{x=1} dt + \frac{1}{2}\int_{Q_T}\frac{y_t^2}{a}\left(1-\frac{xa'}{a}\right) dxdt + \int_{Q_T} xy_xy_{xxxx}dxdt = 0. \tag{3.30}$$

As before,

$$\lim_{x\to 0}\frac{x}{a(x)}y_t^2(t,x) = 0$$

and $\frac{1}{a(1)}y_t^2(t,1) = 0$, so that $\int_0^T \left[\frac{x}{a}y_t^2\right]_{x=0}^{x=1} dt = 0$. In addition, the term $\int_{Q_T} xy_xy_{xxxx}dxdt$ is well-defined since $xy_xy_{xxxx} = \frac{x}{\sqrt{a}}y_x\sqrt{a}y_{xxxx} \in L^1(0,1)$. Thus, we take $\delta > 0$, and as in the proof of Theorem 3.2, we rewrite

$$\int_{Q_T} xy_xy_{xxxx}dxdt = \int_0^T\int_0^\delta xy_xy_{xxxx}dxdt + \int_0^T\int_\delta^1 xy_xy_{xxxx}dxdt.$$

Since $xy_xy_{xxxx} \in L^1(0,1)$, we have $\lim_{\delta\to 0}\int_0^T\int_0^\delta xy_xy_{xxxx}dxdt = 0$. Moreover, integrating by parts the second term of the previous equality and thanks to the boundary conditions on y, we have

$$\begin{aligned} \int_0^T\int_\delta^1 xy_xy_{xxxx}dxdt &= \int_0^T [xy_xy_{xxx}]_{x=\delta}^{x=1} dt - \int_0^T\int_\delta^1 (xy_x)_x y_{xxx}dxdt \\ &= -\int_0^T \delta y_x(t,\delta)y_{xxx}(t,\delta)dt - \int_0^T\int_\delta^1 y_xy_{xxx}dxdt - \frac{1}{2}\int_0^T\int_\delta^1 x\left(y_{xx}^2\right)_x dxdt \\ &= -\int_0^T \delta y_x(t,\delta)y_{xxx}(t,\delta)dt - \int_0^T [y_xy_{xx}]_{x=\delta}^{x=1}dt + \int_0^T\int_\delta^1 y_{xx}^2dxdt - \frac{1}{2}\int_0^T [xy_{xx}^2]_{x=\delta}^{x=1}dt + \frac{1}{2}\int_0^T\int_\delta^1 y_{xx}^2dxdt \\ &= -\int_0^T \delta y_x(t,\delta)y_{xxx}(t,\delta)dt + \int_0^T y_x(t,\delta)y_{xx}(t,\delta)dt - \frac{1}{2}\int_0^T y_{xx}^2(t,1)dt \\ &\quad + \frac{1}{2}\int_0^T \delta y_{xx}^2(t,\delta)dt + \frac{3}{2}\int_0^T\int_\delta^1 y_{xx}^2dxdt. \end{aligned}$$

Proceeding as in the proof of Theorem 3.2, one can prove that $\exists \lim_{\delta\to 0}\delta y_{xxx}(t,\delta) \in \mathbb{R}$; hence,

$$\lim_{\delta\to 0}\delta y_x(t,\delta)y_{xxx}(t,\delta) = 0.$$

Moreover, by Lemma 3.1, we get

$$\lim_{\delta\to0} y_x(t,\delta)y_{xx}(t,\delta)=0,\quad \lim_{\delta\to0}\delta y_{xx}^2(t,\delta)=0.$$

Hence,

$$\lim_{\delta\to0}\int_0^T\int_\delta^1 xy_xy_{xxxx}dx\,dt=-\frac{1}{2}\int_0^T y_{xx}^2(t,1)dt+\frac{3}{2}\int_{Q_T} y_{xx}^2dx\,dt.$$

Coming back to (3.30), it follows that

$$\int_0^1\left[\frac{xy_xy_t}{a}\right]_{t=0}^{t=T}dx+\frac{1}{2}\int_{Q_T}\frac{y_t^2}{a}\left(1-\frac{xa'}{a}\right)dx\,dt-\frac{1}{2}\int_0^T y_{xx}^2(t,1)dt+\frac{3}{2}\int_{Q_T}y_{xx}^2dx\,dt=0$$

and (3.28) holds. □

Thanks to (3.28), we can prove the following estimate on the energy from above.

Theorem 3.5. *Assume a (WD) or (SD) at 0. If y is a mild solution of (2.5), then*

$$\int_0^T y_{xx}^2(t,1)dt\ge\left(T(2-K)-4\max\left\{1,\frac{4}{a(1)},\frac{4K}{a(1)}\right\}\right)E_y(0)$$

for any $T>0$.

Proof. Multiplying the equation in (2.5) by $\frac{-Ky}{2a}$ and integrating over Q_T, we have

$$0=-\frac{K}{2}\int_{Q_T}\frac{y_{tt}y}{a}dx\,dt-\frac{K}{2}\int_{Q_T}yy_{xxxx}dx\,dt=-\frac{K}{2}\int_0^1\left[y\frac{y_t}{a}\right]_{t=0}^{t=T}dx+\frac{K}{2}\int_{Q_T}\frac{y_t^2}{a}dx\,dt-\frac{K}{2}\int_{Q_T}y_{xx}^2dx\,dt,$$

thanks to (2.8). Summing the previous equality to (3.28) multiplied by 2 and using the degeneracy condition (1.3), we have

$$\begin{aligned}
\int_0^T y_{xx}^2(t,1)dt&=2\int_0^1\left[\frac{xy_ty_x}{a}\right]_{t=0}^{t=T}dx+\int_{Q_T}\frac{y_t^2}{a}\left(1-\frac{xa'}{a}\right)dx\,dt+3\int_{Q_T}y_{xx}^2dx\,dt\\
&\quad-\frac{K}{2}\int_0^1\left[y\frac{y_t}{a}\right]_{t=0}^{t=T}dx+\frac{K}{2}\int_{Q_T}\frac{y_t^2}{a}dx\,dt-\frac{K}{2}\int_{Q_T}y_{xx}^2dx\,dt\\
&=2\int_0^1\left[\frac{xy_ty_x}{a}\right]_{t=0}^{t=T}dx-\frac{K}{2}\int_0^1\left[y\frac{y_t}{a}\right]_{t=0}^{t=T}dx\\
&\quad+\int_{Q_T}\frac{y_t^2}{a}\left(1-\frac{xa'}{a}+\frac{K}{2}\right)dx\,dt+\left(3-\frac{K}{2}\right)\int_{Q_T}y_{xx}^2dx\,dt\\
&\ge2\int_0^1\left[\frac{xy_ty_x}{a}\right]_{t=0}^{t=T}dx-\frac{K}{2}\int_0^1\left[y\frac{y_t}{a}\right]_{t=0}^{t=T}dx\\
&\quad+\left(1-\frac{K}{2}\right)\int_{Q_T}\frac{y_t^2}{a}dx\,dt+\left(1-\frac{K}{2}\right)\int_{Q_T}y_{xx}^2dx\,dt\\
&=2\int_0^1\left[\frac{xy_ty_x}{a}\right]_{t=0}^{t=T}dx-\frac{K}{2}\int_0^1\left[y\frac{y_t}{a}\right]_{t=0}^{t=T}dx+(2-K)TE_y(0).
\end{aligned}$$

Now, we analyze the boundary terms that appear in the previous relation. By (3.25),

$$2\left|\int_0^1 \left[\frac{xy_x(\tau,x)y_t(\tau,x)}{a(x)}\right]_{\tau=0}^{\tau=T} dx\right| \le 4\max\left\{\frac{4}{a(1)},1\right\} E_y(0).$$

Furthermore, by (2.6),

$$\left|\frac{y(\tau,x)y_t(\tau,x)}{a(x)}\right| \le \frac{1}{2}\frac{y_t^2(\tau,x)}{a(x)} + \frac{1}{2a(1)}\frac{y^2(\tau,x)}{x^2}$$

for all $\tau \in [0,T]$; in particular, by Theorem 3.1 and (3.24), we have

$$\begin{aligned}
\left|\int_0^1 \frac{y(\tau,x)y_t(\tau,x)}{a(x)}dx\right| &\le \frac{1}{2}\int_0^1 \frac{y_t^2(\tau,x)}{a(x)}dx + \frac{1}{2a(1)}\int_0^1 \frac{y^2(\tau,x)}{x^2}dx\\
&\le \frac{1}{2}\int_0^1 \frac{y_t^2(\tau,x)}{a(x)}dx + \frac{4}{2a(1)}\int_0^1 y_x^2(\tau,x)dx\\
&\le \frac{1}{2}\int_0^1 \frac{y_t^2(\tau,x)}{a(x)}dx + \frac{16}{2a(1)}\int_0^1 y_{xx}^2(\tau,x)dx\\
&\le \max\left\{1,\frac{16}{a(1)}\right\}E_y(0),
\end{aligned}$$

for all $\tau \in [0,T]$. Hence,

$$\frac{K}{2}\left|\int_0^1 \left[y\frac{y_t}{a}\right]_{\tau=0}^{\tau=T} dx\right| \le K\max\left\{1,\frac{16}{a(1)}\right\}E_y(0),$$

and the thesis follows if y is a classical solution. If y is a mild solution, then we can proceed as in Theorem 3.3. □

4 | BOUNDARY OBSERVABILITY AND NULL CONTROLLABILITY

Inspired by Alabau-Boussouira et al. [13], we give the next definition.

Definition 4.1. Problem (2.5) is said to be observable in time $T > 0$ via the second derivative at $x = 1$ if there exists a constant $C > 0$ such that for any $\left(y_T^0, y_T^1\right) \in \mathcal{H}_0$, the classical solution y of (2.5) satisfies

$$CE_y(0) \le \int_0^T y_{xx}^2(t,1)dt. \tag{4.31}$$

Moreover, any constant satisfying (4.31) is called observability constant for (2.5) in time T.

Setting

$$C_T := \sup\{C > 0 : C \text{ satisfies } (4.31)\},$$

we have that problem (2.5) is observable if and only if

$$C_T = \inf_{(y_T^0,y_T^1)\neq(0,0)} \frac{\int_0^T y_{xx}^2(t,1)dt}{E_y(0)} > 0.$$

The inverse of C_T, that is, $c_T := \frac{1}{C_T}$, is called the *cost of observability* (or the *cost of control*) in time T.

Theorem 3.5 admits the following straightforward corollary.

Corollary 4.1. *Assume that a is (WD) or (SD) at 0. If*

$$T > \frac{4}{2-K} \max \left\{ 1, \frac{4}{a(1)}, \frac{4K}{a(1)} \right\},$$

then (2.5) is observable in time T. Moreover,

$$T(2-K) - 4 \max \left\{ 1, \frac{4}{a(1)}, \frac{4K}{a(1)} \right\} \leq C_T.$$

In the following, we will study the problem of null controllability for (1.1). As a first step, we give the definition of a solution for (1.1) by *transposition*, which permits low regularity on the notion of solution itself. Precisely, we have the following:

Definition 4.2. Let $f \in L^2_{loc}[0,+\infty)$ and $(u_0, u_1) \in L^2_{\frac{1}{a}}(0,1) \times \left(H^2_{\frac{1}{a}}(0,1) \right)^*$. We say that u is a solution by transposition of (1.1) if

$$u \in C^1 \left(\left([0,+\infty); \left(H^2_{\frac{1}{a}}(0,1) \right)^* \right) \cap C \left([0,+\infty); L^2_{\frac{1}{a}}(0,1) \right) \right.$$

and for all $T > 0$,

$$\begin{aligned} \left\langle u_t(T), v_T^0 \right\rangle_{\left(H^2_{\frac{1}{a}}(0,1)\right)^*, H^2_{\frac{1}{a}}(0,1)} - \int_0^1 \frac{1}{a} u(T) v_T^1 \, dx &= \langle u_1, v(0) \rangle_{\left(H^2_{\frac{1}{a}}(0,1)\right)^*, H^2_{\frac{1}{a}}(0,1)} \\ &- \int_0^1 \frac{1}{a} u_0 v_t(0,x) \, dx - \int_0^T f(t) v_{xx}(t,1) dt \end{aligned} \tag{4.32}$$

for all $\left(v_T^0, v_T^1\right) \in \mathcal{H}_0$, where v solves the backward problem

$$\begin{cases} v_{tt}(t,x) + a(x) v_{xxxx}(t,x) = 0, & (t,x) \in (0,+\infty) \times (0,1), \\ v(t,0) = 0, \ v_x(t,0) = 0, & t > 0, \\ v(t,1) = 0, \ v_x(t,1) = 0, & t > 0, \\ v(T,x) = v_T^0(x), \ v_t(T,x) = v_T^1(x), & x \in (0,1). \end{cases} \tag{4.33}$$

Observe that, by Theorem 2.2, there exists a unique mild solution of (4.33) in $[T,+\infty)$. Now, setting $y(t,x) := v(T-t,x)$, one has that y satisfies (2.5) with $y_T^0(x) = v_T^0(x)$ and $y_T^1(x) = -v_T^1(x)$. Hence, we can apply Theorem 2.2 to (2.5) obtaining that there exists a unique mild solution y of (2.5) in $[0,+\infty)$. In particular, there exists a unique mild solution v of (4.33) in $[0,T]$. Thus, we can conclude that there exists a unique mild solution

$$v \in C^1 \left([0,+\infty); L^2_{\frac{1}{a}}(0,1) \right) \cap C \left([0,+\infty); H^2_{\frac{1}{a}}(0,1) \right)$$

of (4.33) in $[0,+\infty)$ which depends continuously on the initial data $V_T := \left(v_T^0, v_T^1\right) \in \mathcal{H}_0$.

By Theorem 3.1, the energy is preserved in our setting, as well, so that the method of transposition done in Alabau-Boussouira et al. [13] continues to hold thanks to (3.23). Therefore, there exists a unique solution by transposition $u \in C^1 \left([0,+\infty); H^{-2}_{\frac{1}{a}}(0,1) \right) \cap C \left([0,+\infty); L^2_{\frac{1}{a}}(0,1) \right)$ of (1.1), that is, a solution of (4.32). To prove this fact, consider the functional $\mathcal{G} : \mathcal{H}_0 \to \mathbb{R}$ given by

$$\mathcal{G}\left(v_T^0, v_T^1\right) = \langle u_1, v(0)\rangle_{\left(H^2_{\frac{1}{a}}(0,1)\right)^*, H^2_{\frac{1}{a}}(0,1)} - \int_0^1 \frac{1}{a} u_0 v_t(0,x)\,dx - \int_0^T f(t) v_{xx}(t,1)\,dt, \tag{4.34}$$

for all $T > 0$, where v solves (4.33). Clearly, $\mathcal{G}$ is linear. Moreover, it is continuous. Indeed, for all $T > 0$, we have

$$\left|\mathcal{G}\left(v_T^0, v_T^1\right)\right| \le \|u_1\|_{\left(H^2_{\frac{1}{a}}(0,1)\right)^*} \|v(0)\|_{H^2_{\frac{1}{a}}(0,1)} + \|u_0\|_{L^2_{\frac{1}{a}}(0,1)} \|v_t(0)\|_{L^2_{\frac{1}{a}}(0,1)} + \|f\|_{L^2(0,T)} \left(\int_0^T v_{xx}^2(t,1)\,dt\right)^{\frac{1}{2}}.$$

By (3.23) and Theorem 3.1, there exists a positive constant C such that

$$\int_0^T v_{xx}^2(t,1)\,dt \le CE_v(0) = CE_v(T) = \frac{C}{2}\int_0^1 \left(\frac{v_t^2(T,x)}{a(x)} + v_{xx}^2(T,x)\right) dx = \frac{C}{2}\left\|\left(v_T^0, v_T^1\right)\right\|_{\mathcal{H}_0}.$$

Hence,

$$\left|\mathcal{G}\left(v_T^0, v_T^1\right)\right| \le \|u_1\|_{\left(H^2_{\frac{1}{a}}(0,1)\right)^*} \|v(0)\|_{H^2_{\frac{1}{a}}(0,1)} + \|u_0\|_{L^2_{\frac{1}{a}}(0,1)} \|v_t(0)\|_{L^2_{\frac{1}{a}}(0,1)} + \|f\|_{L^2(0,T)} \left\|\left(v_T^0, v_T^1\right)\right\|_{\mathcal{H}_0}.$$

Using again Theorem 3.1, we have $\|v_t(0)\|_{L^2_{\frac{1}{a}}(0,1)} \le E_v(T)$; thus, $\|v_t(0)\|_{L^2_{\frac{1}{a}}(0,1)} \le C\left\|\left(v_T^0, v_T^1\right)\right\|_{\mathcal{H}_0}$. Analogously, thanks to Proposition 2.1, there exists $C > 0$ such that $\|v(0)\|_{H^2_{\frac{1}{a}}(0,1)} \le C\left\|\left(v_T^0, v_T^1\right)\right\|_{\mathcal{H}_0}$. Thus, we can conclude that there exists $C > 0$ so that

$$|\mathcal{G}\left(v_T^0, v_T^1\right)| \le C\left\|\left(v_T^0, v_T^1\right)\right\|_{\mathcal{H}_0},$$

that is, $\mathcal{G}$ is continuous.

Being $\mathcal{G} \in (\mathcal{H}_0)^* = \left(H^2_{\frac{1}{a}}(0,1)\right)^* \times L^2_{\frac{1}{a}}(0,1)$, we can use the Riesz Theorem obtaining that for any $T > 0$, there exists a unique $\left(\tilde{u}_T^0, \tilde{u}_T^1\right) \in (\mathcal{H}_0)^*$ such that

$$\mathcal{G}\left(v_T^0, v_T^1\right) = \left\langle \left(\tilde{u}_T^0, \tilde{u}_T^1\right), \left(v_T^0, v_T^1\right)\right\rangle_{(\mathcal{H}_0)^*, \mathcal{H}_0} = \left\langle \tilde{u}_T^0, v_T^0\right\rangle_{\left(H^2_{\frac{1}{a}}(0,1)\right)^*, H^2_{\frac{1}{a}}(0,1)} + \int_0^1 \frac{\tilde{u}_T^1 v_T^1}{a}\,dx. \tag{4.35}$$

Moreover, $\tilde{u}_T^0, \tilde{u}_T^1$ depend continuously on T, so there exists a unique $u \in C\left([0,+\infty); L^2_{\frac{1}{a}}(0,1)\right) \cap C^1\left([0,+\infty); H^{-2}_{\frac{1}{a}}(0,1)\right)$ such that $u(T) = -\tilde{u}_T^1$ and $u_t(T) = \tilde{u}_T^0$. By (4.34) and (4.35), we can conclude that u is the unique solution by transposition of (1.1).

Now, we are ready to examine null controllability. To this aim, consider the bilinear form $\Lambda : \mathcal{H}_0 \times \mathcal{H}_0 \to \mathbb{R}$ defined as

$$\Lambda(V_T, W_T) := \int_0^T v_{xx}(t,1) w_{xx}(t,1)\,dt,$$

where v and w are the solutions of (4.33) associated to the data $V_T := \left(v_T^0, v_T^1\right)$ and $W_T := \left(w_T^0, w_T^1\right)$, respectively. The following lemma holds.

Lemma 4.1. *Assume a (WD) or (SD) at 0. The bilinear form Λ is continuous and coercive.*

Proof. By Theorem 3.1, E_v and E_w are constant in time, and due to (3.23), one has that Λ is continuous. Indeed, by Holder's inequality and (3.23),

$$|\Lambda(V_T, W_T)| \le \int_0^T |v_{xx}(t,1)w_{xx}(t,1)|\,dt \le \left(\int_0^T v_{xx}^2(t,1)dt\right)^{\frac{1}{2}}\left(\int_0^T w_{xx}^2(t,1)dt\right)^{\frac{1}{2}}$$
$$\le CE_v^{\frac{1}{2}}(T)E_w^{\frac{1}{2}}(T) = C\left(\int_0^1 \frac{(v_T^1)^2(x)}{a}dx + \int_0^1 [(v_T^0)_{xx}]^2(x)dx\right)^{\frac{1}{2}} \cdot$$
$$\left(\int_0^1 \frac{(w_T^1)^2(x)}{a}dx + \int_0^1 [(w_T^0)_{xx}]^2(x)dx\right)^{\frac{1}{2}}$$
$$= C\|(v(T), v_t(T))\|_{\mathcal{H}_0}\|(w(T), w_t(T))\|_{\mathcal{H}_0} = C\|V_T\|_{\mathcal{H}_0}\|W_T\|_{\mathcal{H}_0}$$

for a positive constant C independent of $(V_T, W_T) \in \mathcal{H}_0 \times \mathcal{H}_0$.

In a similar way one can prove that Λ is coercive. Indeed, by Theorem 3.5, one immediately has that there exists $C > 0$ such that

$$\Lambda(V_T, V_T) = \int_0^T v_{xx}^2(t,1)dt \ge C_T E_v(0) = C_T E_v(T) \ge C\|V_T\|^2_{\mathcal{H}_0}$$

for all $V_T \in \mathcal{H}_0$. □

Function Λ is used to prove the null controllability property for the original problem (1.1). To this aim, let us start defining T_0 as the lower bound found in Corollary 4.1, that is,

$$T_0 := \frac{4}{2-K}\max\left\{1, \frac{4}{a(1)}, \frac{4K}{a(1)}\right\}. \tag{4.36}$$

Theorem 4.1. *Assume a (WD) or (SD) at 0. Then, for all $T > T_0$ and for every $(u_0, u_1) \in L^2_{\frac{1}{a}}(0,1) \times H^{-2}_{\frac{1}{a}}(0,1)$, there exists a control $f \in L^2(0,T)$ such that the solution of (1.1) satisfies*

$$u(T,x) = u_t(T,x) = 0 \quad \forall\, x \in (0,1). \tag{4.37}$$

Proof. Consider the map $\mathcal{L} : \mathcal{H}_0 \to \mathbb{R}$ given by

$$\mathcal{L}(V_T) := \langle u_1, v(0)\rangle_{\left(H^2_{\frac{1}{a}}(0,1)\right)^*, H^2_{\frac{1}{a}}(0,1)} - \int_0^1 \frac{u_0 v_t(0,x)}{a}dx,$$

where v is the solution of (4.33) associated to the initial data $V_T := \left(v_T^0, v_T^1\right) \in \mathcal{H}_0$. Clearly, $\mathcal{L}$ is continuous and linear and, thanks to Lemma 4.1, we can apply the Lax–Milgram Theorem. Thus, there exists a unique $\overline{V}_T \in \mathcal{H}_0$ such that

$$\Lambda(\overline{V}_T, W_T) = \mathcal{L}(W_T) \quad \forall\, W_T \in \mathcal{H}_0.$$

Set $f(t) := \bar{v}_{xx}(t,1)$, where $\bar{v}$ is the unique solution of (4.33) associated to $\overline{V}_T$. Then

$$\int_0^T f(t)w_{xx}(t,1)dt = \int_0^T \bar{v}_{xx}(t,1)w_{xx}(t,1)dt = \Lambda\left(\overline{V}_T, W_T\right) = \mathcal{L}(W_T) = \langle u_1, w(0)\rangle_{\left(H^2_{\frac{1}{a}}(0,1)\right)^*, H^2_{\frac{1}{a}}(0,1)} - \int_0^1 \frac{u_0 w_t(0,x)}{a}dx \tag{4.38}$$

for all $W_T \in \mathcal{H}_0$.

Finally, denote by u the solution by transposition of (1.1) associated to the function f introduced above. We have that

$$\int_0^T f(t)w_{xx}(t,1)dt = -\left\langle u_t(T), w_T^0 \right\rangle_{\left(H^2_{\frac{1}{a}}(0,1)\right)^*, H^2_{\frac{1}{a}}(0,1)} + \int_0^1 \frac{u(T)w_T^1}{a}dx + \langle u_1, w(0)\rangle_{\left(H^2_{\frac{1}{a}}(0,1)\right)^*, H^2_{\frac{1}{a}}(0,1)} - \int_0^1 \frac{u_0 w_t(0,x)}{a}dx. \tag{4.39}$$

Combining (4.38) and (4.39), it follows that

$$\left\langle u_t(T), w_T^0 \right\rangle_{\left(H^2_{\frac{1}{a}}(0,1)\right)^*, H^2_{\frac{1}{a}}(0,1)} - \int_0^1 \frac{u(T)w_T^1}{a}dx = 0$$

for all $\left(w_T^0, w_T^1\right) \in \mathcal{H}_0$. Hence, we have (4.37). □

5 | CONCLUSIONS AND OPEN PROBLEMS

In this paper, we have proved that if the function a is weakly or strongly degenerate and $T > T_0$, where T_0 is given in (4.36), then (1.1) is boundary null controllable, that is,

$$u(T,x) = u_t(T,x) = 0$$

for all $x \in (0,1)$. However, in contrast to wave equations, nondegenerate beam equations can be generically controlled in arbitrary short times since there is no finite speed of propagation (see Komornik [16]). Hence, one would expect to obtain null controllability for (1.1) also for short times using the same technique proposed in Komornik [16], where *nondegenerate* variable coefficients are considered.

Another open problem is to prove null controllability for (1.1) or to show that it fails when $K \geq 2$. Indeed, for degenerate wave equations, we know that, if $K \geq 2$, null controllability fails; on the other hand, for degenerate beam equations, we do not know anything in this case: the assumption $K < 2$ is made here only for technical reasons.

AUTHOR CONTRIBUTIONS

Alessandro Camasta: validation, writing—review and editing, and investigation. **Genni Fragnelli:** investigation, validation, writing—review and editing, supervision, and methodology.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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APPENDIX A:

Proof of Lemma 3.1.1. If $K < 1$, where K is the constant of Hypothesis 2.1, then the assertion follows immediately by (2.7) with $\gamma = 1$. Thus, assume $K \geq 1$. Set $z(x) := \frac{x}{a(x)} y^2(x)$. Then $z \in L^1(0,1)$. Indeed,

$$\int_0^1 \frac{x}{a} y^2 dx \leq \int_0^1 \frac{y^2}{a} dx.$$

Moreover, $z' = \frac{y^2}{a} + 2\frac{xyy'}{a} - \frac{a'x}{a^2} y^2$; thus, for a suitable $\varepsilon > 0$ given by Hypothesis 2.1,

$$\int_0^\varepsilon |z'| dx \leq \int_0^1 \frac{y^2}{a} dx + 2\left(\int_0^\varepsilon \frac{x^2 (y')^2}{a} dx\right)^{\frac{1}{2}} \left(\int_0^1 \frac{y^2}{a} dx\right)^{\frac{1}{2}} + K \int_0^1 \frac{y^2}{a} dx$$
$$\leq (1+K) \int_0^1 \frac{y^2}{a} dx + \frac{2\varepsilon}{\sqrt{a(\varepsilon)}} \left(\int_0^1 (y')^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 \frac{y^2}{\sigma} dx\right)^{\frac{1}{2}}.$$

This is enough to conclude that $z \in W^{1,1}(0,1)$, and thus, there exists $\lim_{x\to 0} z(x) = L \in \mathbb{R}$. If $L \neq 0$, sufficiently close to $x = 0$ we would have that $\frac{y^2(x)}{a} \geq \frac{|L|}{2x} \notin L^1(0,1)$, while $\frac{y^2}{a} \in L^1(0,1)$.

Proof of Lemma 3.1.2. In order to prove the lemma, fixed $y \in D(A)$, we will establish that y'' is absolutely continuous in $[0,1]$. To this aim, let $\delta > 0$. Clearly,

$$y''(\delta) = y''(1) - \int_\delta^1 y'''(x) dx; \tag{A1}$$

thus,

$$\begin{aligned} y''(\delta) &= y''(1) - \int_\delta^1 \left(y'''(1) - \int_x^1 y''''(s) ds \right) dx = y''(1) - (1-\delta) y'''(1) + \int_\delta^1 \left(\int_x^1 y''''(s) ds \right) dx \\ &= y''(1) - y'''(1) + \delta y'''(1) + \int_\delta^1 \int_\delta^s y''''(s) dx\, ds = y''(1) - y'''(1) + \delta y'''(1) + \int_\delta^1 y''''(s)(s-\delta) ds. \end{aligned} \tag{A2}$$

Trivially, $\lim_{\delta\to 0} \delta y'''(1) = 0$ and, proceeding as in Theorem 3.2, we have

$$\lim_{\delta\to 0} \int_\delta^1 y''''(s)(s-\delta) ds = \int_0^1 s y''''(t,s) ds.$$

Thus, if we pass to the limit as $\delta \to 0$ in (A2), we conclude that

$$\exists \lim_{\delta\to 0} y''(\delta) = y''(1) - y'''(1) + \int_0^1 s y''''(s) ds \in \mathbb{R}.$$

By continuity, it is possible to define $y''(0) := \lim_{\delta\to 0} y''(\delta)$. In particular, by (A1),

$$\int_0^1 y'''(x) dx = y''(1) - y''(0)$$

and the thesis follows.