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Entanglement, Quantum Correlators, and Connectivity in Graph States

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This work presents a comprehensive exploration of the entanglement and graph connectivity properties of Graph States (GSs). Qubit entanglement in Pseudo Graph States (PGSs) is quantified using the Entanglement Distance (ED), a recently introduced measure of bipartite entanglement. In addition, a new approach is proposed for probing the underlying graph connectivity of genuine GSs, using Pauli matrix quantum correlators. These findings also reveal interesting implications for measurement processes, demonstrating the equivalence of some projective measurements. Finally, the emphasis is placed on the simplicity of data analysis in this framework. This work contributes to a deeper understanding of the entanglement and connectivity properties of GSs, offering valuable information for quantum information processing and quantum computing applications. The famous stabiliser formalism, which is the typically preferred framework for the study of this type of states, is not used in this work; on the contrary, this approach is based exclusively on the concepts of expectation values, quantum correlations, and projective measurement, which have the advantage of being very intuitive and fundamental tools of quantum theory.

1. Introduction

Besides being one of the most striking properties of quantum mechanics, entanglement is a primary resource for quantum cryptography, quantum computation, and for quantum-based technologies. In the last decades, the quantum information community has developed several approaches to characterize its abundant phenomenology and various properties.[1,2] The entanglement of multipartite states proves to be a more difficult concept to understand, than that of bipartite states.

In fact, entangled bipartite pure states are such that a measurement on a subsystem completely determines the state of the complementary subsystem. In this case, there exists a subsequent local measurement for the latter subsystem, for which the outcome is certain.

On the contrary, in the case of pure multipartite entangled states, a measure on a subsystem, although it modifies the rest of the system, can leave the latter in a maximally entangled state, so no local measurement on the latter has a certain outcome.

In light of the above considerations, the importance of understanding and characterizing the connectivity properties of multipartite quantum states is evident.

We will denote with σ_k^μ, where $k = x, y, z$, the Pauli matrices operating on the qubit μ, and with σ_k^Q the tensor product $\bigotimes_{\nu\in Q} \sigma_k^\nu$ for any set Q of qubits. We call Pauli observable any operator that can be written as a tensor product of Pauli matrices.

Graph States (GSs) constitute a class of maximally entangled pure quantum states that have emerged as a powerful resource for quantum information processing.[3–6] Indeed, they are valuable for realizing quantum gates and enabling fault-tolerant quantum computation. Additionally, GSs serve as the foundation for various quantum computing protocols, especially for the one-way quantum computer, also known as the measurement-based quantum computer.[4–7] It can be shown[7] that any quantum circuit can be efficiently simulated using a GS, on which appropriate measurements are performed; as such, GSs represent a universal resource for quantum computing. Therefore, any result obtained for the GS model can, in principle, be extrapolated to other models of quantum computation. GSs are complex high-dimensional superpositions of states of N qubits, prepared as follows.

Let V be the set of indices that identify a set of N qubits and let E be a set of pairs of indices (a, b), with $a, b \in V$. Let's start with the initial product state $|\Psi\rangle = |+\rangle^V := \bigotimes_{\mu\in V} |+\rangle^\mu$, where every $|+\rangle^\mu = \frac{1}{\sqrt{2}}(|0\rangle^\mu + |1\rangle^\mu)$ is the eigenstate of σ_x^μ with eigenvalue $+1$,

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for $\mu \in V$. For each pair $(a, b) \in E$, we consider the fine-tuned unitary operator

$$U_{ab}(\varphi_{ab}) = e^{-i\frac{\varphi_{ab}}{4}} e^{i\frac{\varphi_{ab}}{4}\sigma_z^a} e^{i\frac{\varphi_{ab}}{4}\sigma_z^b} e^{-i\frac{\varphi_{ab}}{4}\sigma_z^a\sigma_z^b} \tag{1}$$

where $\varphi_{ab} \in \mathbb{R}$. The Pseudo Graph States (PGS) is defined as

$$|G(\varphi)\rangle = \prod_{(a,b)\in E} U_{ab}(\varphi)|\Psi\rangle \tag{2}$$

while the genuine GS correspond to the case $\varphi = \pi$

$$|G\rangle = \prod_{(a,b)\in E} U_{ab}(\pi)|\Psi\rangle \tag{3}$$

Note that all the operators (1) commute with each other. For sake of simplicity we assume here $\forall (a, b),\ \varphi_{ab} = \varphi$.

Each of the operators (1) entangles a pair of qubits.[8]

It has been shown that a general GS does not correspond to the ground state of a physical system. However, a GS can be obtained artificially in a physical system that allows the activation of Ising-like interactions $\sigma_z^a\sigma_z^b$. In this case, the time duration of the interaction determines the value of φ (hereinafter referred to as interaction strength). Physical implementations of such systems were performed on some of the quantum computer prototypes developed by IBM.[9,10] In practice, any physical device for universal quantum computing, can be used to realise GS in the way described above.[5]

Since a graph-state is uniquely defined by a couple of sets (V, E), it is uniquely defined by a undirected graph $G(V, E)$, where each qubit (associated with an element of V) is a vertex, and each pair in E is an edge (or a link) of the graph. In most of the literature, the preferred terminology is to refer to GSs defined on lattices as cluster states. However, in the present work, we address the study of GS in the general case and therefore associated with generic graphs.

The genuine GS $|G(\pi)\rangle = |G\rangle$ of a given graph $G(V, E)$ is the unique common eigenvector with eigenvalue +1 of the operators

$$K_\mu = \sigma_x^\mu \sigma_z^{N(\mu)} \tag{4}$$

where $\mu \in V$ and $N(\mu)$ denotes the set of neighbors of μ. The group $\mathcal{S}$ generated by the set $\{K_\mu\}_{\mu\in V}$ is called the "stabilizer" of the graph-state. Clearly, $\forall g \in \mathcal{S},\ g|G\rangle = |G\rangle$, and the projector onto a GS can be expressed as $|G\rangle\langle G| = \frac{1}{2^N}\sum_{g\in\mathcal{S}} g$.

GS vectors are thus in one-to-one correspondence with their stabilizer $\mathcal{S}$, and any operation applied to $|G\rangle$ can be mapped to an operation applied to $\mathcal{S}$. For example, for any unitary operation U (i.e., any quantum gate), if $\mathcal{S}$ stabilizes $|G\rangle$, then $U\mathcal{S}U^\dagger$ stabilizes $U|G\rangle$.[11]

The group $\mathcal{S}$ is completely determined by its N generators $\{K_\mu\}_{\mu\in V}$ (which belong to the Pauli group and thus have a simple algebra). On the other hand, to write explicitly the corresponding state vector, it is necessary to determine the 2^N amplitudes. For this reason, the stabilizer formalism usually provides a significant computational advantage.

Furthermore, the stabilizer formalism is often used as a preferred framework to compare different models of quantum computation, for example, for implementing error-correcting codes or examining the effects of quantum gates and measurement processes.[7,11]

However, we believe that, while the stabilizer representation is more useful for studying known initial states and how they transform under the action of such operations, it proves to be an unnecessary complication in other contexts, such as the probing and tomography of unknown states. This is because the calculation of correlations and expectation values requires taking into account all elements of $\mathcal{S}$ rather than its mere generators.

Thus, in this work, we do not resort to the stabilizer formalism but rather employ the more intuitive notion of correlation and expectation values.

We start by quantifying the entanglement in the general case of PGS using the Entanglement Distance (ED), a measure of entanglement recently introduced in Ref. [12]. Subsequently, we explore a novel approach to investigate the underlying graph connectivity of genuine GS using correlators of Pauli matrices. In particular, we compute correlations between pairs of qubits (i.e., two-qubit correlators) and demonstrate that these quantities depend exclusively on the relation between their neighborhoods (i.e., in graph theory language, whether they are *twins*, *adjacent twins*, *leaf vertices*, etc.). We discuss the possibility of a more comprehensive exploration of graph properties through the use of higher-order correlators (involving more than two qubits). Furthermore, we show that our approach can highlight when two projective measurements are equivalent. Also, we emphasise the simplicity of data analysis offered by our approach in this context, as all correlators can only assume the values of −1, 0, or 1. We conclude this work by summarising the advantages of our method with respect to the stabilizer formalism, by showing that these two approaches offer a complementary characterization of GS.

2. Entanglement in Pseudo Graph States

The ways of quantifying entanglement in multipartite states are manifold.[1,2] In this work, we will solely refer to qubit-wise entanglement, that is entanglement of bipartitions (μ, μ^C), where μ is a qubit, and μ^C is its complement relative to the set of all qubits in the system.

The ED, first defined in Ref. [12], is an entanglement measure for general multipartite pure states; it has been adapted in Ref. [13] to the more general framework of multipartite mixed states. It has already found since then some interesting applications.[14–16] It finds its theoretical grounds on the Fubini-Study metric associated to the local-unitary invariant projective Hilbert space, called in this context the Entanglement Metric, of which deep geometric meaning has been further explored in Ref. [17].

The single-qubit ED is defined as

$$E_\mu(|s\rangle) := 1 - \sum_{j=x,y,z} |\langle s|\,\sigma_j^\mu\,|s\rangle|^2 \tag{5}$$

which equates 1 if μ is maximally entangled with the rest of the system, and 0 if it is fully factorizable. Equation (5) thus stems as a measure of bipartite entanglement on the bipartition (μ, μ^C).

We choose here to use the latter definition of entanglement, which possesses the advantage of being very easy to compute, relative to the von Neumann entropy. We further define the total entanglement of a state as $\sum_{\mu\in Q} E_\mu(|s\rangle)$.

From the anticommutation relations of the Pauli matrices

$$\{\sigma_i^\mu, \sigma_j^\nu\} = 2\mathbb{1}\delta_{ij}\delta_{\mu\nu} + 2\sigma_i^\mu\sigma_j^\nu(1-\delta_{\mu\nu}) \tag{6}$$

we straightforwardly derive

$$\begin{aligned}
\sigma_x^a U_{ab}(\varphi) &= e^{-i\frac{\varphi}{2}\sigma_z^a} e^{i\frac{\varphi}{2}\sigma_z^a\sigma_z^b} U_{ab}(\varphi)\sigma_x^a \\
\sigma_y^a U_{ab}(\varphi) &= e^{-i\frac{\varphi}{2}\sigma_z^a} e^{i\frac{\varphi}{2}\sigma_z^a\sigma_z^b} U_{ab}(\varphi)\sigma_y^a \\
\sigma_z^a U_{ab}(\varphi) &= U_{ab}(\varphi)\sigma_z^a \\
\sigma_j^\nu U_{ab}(\varphi) &= U_{ab}(\varphi)\sigma_j^\nu \ , \ \forall j = x,y,z, \ \forall \nu \neq a, b
\end{aligned} \tag{7}$$

Defining $U_G(\varphi) = \prod_{(a,b)\in E} U_{ab}(\varphi)$, we obtain

$$\begin{aligned}
\sigma_x^a U_G(\varphi) &= U_G(\varphi)\left(\prod_{b\in N(a)} e^{-i\frac{\varphi}{2}\sigma_z^a} e^{i\frac{\varphi}{2}\sigma_z^a\sigma_z^b}\right)\sigma_x^a \\
\sigma_y^a U_G(\varphi) &= U_G(\varphi)\left(\prod_{b\in N(a)} e^{-i\frac{\varphi}{2}\sigma_z^a} e^{i\frac{\varphi}{2}\sigma_z^a\sigma_z^b}\right)\sigma_y^a \\
\sigma_z^a U_G(\varphi) &= U_G(\varphi)\sigma_z^a
\end{aligned} \tag{8}$$

The expectation values of the first Pauli matrix hence write

$$\begin{aligned}
\langle G(\varphi)|\sigma_x^\nu|G(\varphi)\rangle &= \langle\Psi|U_G^\dagger(\varphi)\sigma_x^\nu U_G(\varphi)|\Psi\rangle \\
&= \langle\Psi|U_G^\dagger(\varphi)U_G(\varphi)\left(\prod_{\mu\in N(\nu)} e^{-i\frac{\varphi}{2}\sigma_z^\nu} e^{i\frac{\varphi}{2}\sigma_z^\nu\sigma_z^\mu}\right)\sigma_x^\nu|\Psi\rangle \\
&= \langle\Psi|e^{-i\frac{n_\nu\varphi}{2}\sigma_z^\nu}\left(\prod_{\mu\in N(\nu)} e^{i\frac{\varphi}{2}\sigma_z^\nu\sigma_z^\mu}\right)|\Psi\rangle \\
&= \cos(n_\nu\varphi/2)\cos^{n_\nu}(\varphi/2)
\end{aligned} \tag{9}$$

where $N(\nu)$ is the set of the first neighbors of ν, and $n_\nu = |N(\nu)|$ is its cardinality. We used the fact that all the terms including a Pauli matrix σ_z^μ acting on some $\mu \in N(\nu)$ vanish, since they appear only once and $\forall\mu$, $\langle\Psi|\sigma_z^\mu|\Psi\rangle = 0$.

The expectation values of the second Pauli matrix write

$$\begin{aligned}
\langle G(\varphi)|\sigma_y^\nu|G(\varphi)\rangle &= \langle\Psi|U_G^\dagger(\varphi)\sigma_y^\nu U_G(\varphi)|\Psi\rangle \\
&= -i\langle\Psi|\left(\prod_{\mu\in N(\nu)} e^{-i\frac{\varphi}{2}\sigma_z^\nu} e^{i\frac{\varphi}{2}\sigma_z^\nu\sigma_z^\mu}\right)\sigma_y^\nu|\Psi\rangle \\
&= -i\langle\Psi|e^{-i\frac{n_\nu\varphi}{2}\sigma_z^\nu}\left(\prod_{\mu\in N(\nu)} e^{i\frac{\varphi}{2}\sigma_z^\nu\sigma_z^\mu}\right)|\Psi_\nu^-\rangle \\
&= -\sin(n_\nu\varphi/2)\cos^{n_\nu}(\varphi/2)
\end{aligned} \tag{10}$$

where $|\Psi_\nu^-\rangle = |+\rangle^{V\setminus\{\nu\}} \otimes |-\rangle^\nu$, that is, the pure product state with every qubit in the state $|+\rangle$ except for qubit ν which is in the state $|-\rangle$. The final result stems from the fact that the only non-vanishing terms are the ones including one and only one Pauli matrix σ_z^ν acting on ν, since $\forall\mu$, $\langle\Psi|\sigma_z^\mu|\Psi_\nu^-\rangle = \delta_{\mu\nu}$.

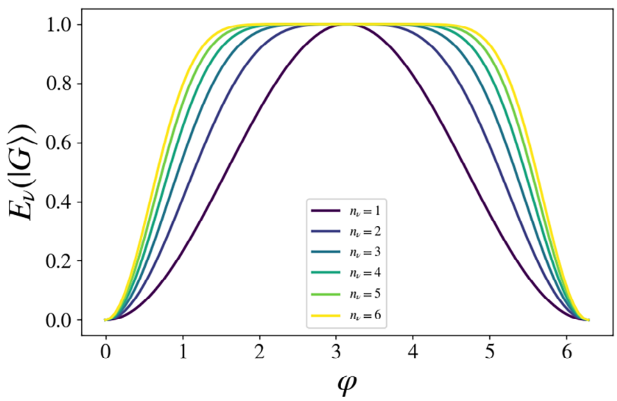

Figure 1. The ED of a single qubit, as a function of the interaction strength (or duration), for different numbers n_ν of nearest neighbors. The numerical results agree perfectly with the analytical one of Equation (12).

Finally, the commutation relations (8) trivially imply

$$\begin{aligned}
\langle G(\varphi)|\sigma_z^\nu|G(\varphi)\rangle &= \langle\Psi|U_G^\dagger(\varphi)\sigma_z^\nu U_G(\varphi)|\Psi\rangle \\
&= \langle\Psi|\sigma_z^\nu|\Psi\rangle = 0
\end{aligned} \tag{11}$$

It result that the single-qubit ED of a given qubit ν in a PGS depends on both the interaction strength φ and on the number n_ν of its nearest neighbors

$$E_\nu(|G(\varphi)\rangle) = 1 - \cos(\varphi/2)^{2n_\nu} \tag{12}$$

The numerical confirmation of this result is displayed in **Figure 1**.

As stated before, the value $\varphi = \pi$ corresponds to the genuine GS, in which every non isolated qubit is maximally entangled, regardless of the number of its neighbors. Consider a PGS close to the genuine GS, i.e., where this typical interaction strength is added with a small error $\delta\varphi$, we retrieve

$$E_\nu(|G(\pi+\delta\varphi)\rangle) \approx 1 - \left(\frac{\delta\varphi}{2}\right)^{2n_\nu} \tag{13}$$

hence the qubits in a quasi GS get exponentially closer to the maximal value of entanglement as the number of their nearest neighbors increases; this is in agreement with previous results presented in the literature, where it has been found that the entanglement of single qubits in GS depends on the degree of the corresponding vertex (i.e., on n_ν).[9,10] The only non trivial case where the small error could be relevant is the one of a qubit with only one link, where the correction is of $o(\delta\varphi^2)$.

It results, as **Figure 2** emphasizes, that the limit for a large number of bounds writes

$$E_\nu(|G(\varphi)\rangle) \underset{n_\nu\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if} \varphi = 2n\pi, \ \forall n \in \mathbb{N} \\ 1 & \text{else.} \end{cases} \tag{14}$$

i.e., up to a null measure set of values of φ, the ED of a single qubit approaches *one* when the number of its neighbors becomes

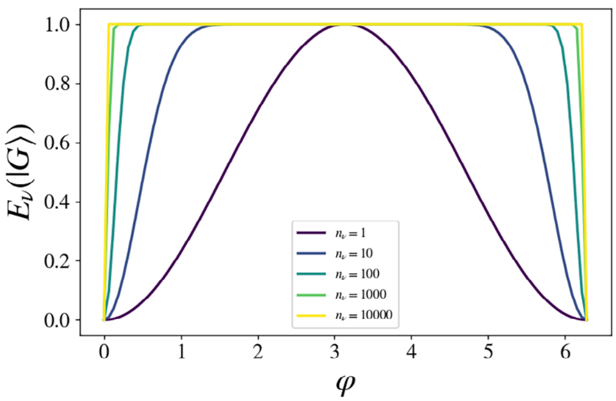

Figure 2. The ED of a single qubit, as a function of the interaction strength (or duration), for different numbers n_ν of nearest neighbors.

very large. In other words, even if the pairwise interaction is very weak, the qubit-wise entanglement, in the sense of (5), can be very close to its maximal value.

Note that, as can be seen in **Figure 3** the entropy of entanglement shows the same behavior and scaling as the ED, suggesting that the later stems as a valid alternative to the former as a measure of bipartite entanglement. It also has the benefit of being easier to compute, both numerically and analytically, as it only requires the calculation of expectation values, in contrast with the entropy of entanglement, which requires to compute partial trace and matrix logarithms.

3. Correlators and the Effects of Measurement in Graph States

We now focus on the case of *genuine* GS, i.e., when $\varphi = \pi$. In particular, we want to compute the various two-point correlators. We denote

$$U_G := \prod_{(a,b)\in E} U_{ab}(\varphi = \pi) = \prod_{(a,b)\in E} \frac{\mathbb{I} + \sigma_z^a + \sigma_z^b - \sigma_z^a\sigma_z^b}{2} \tag{15}$$

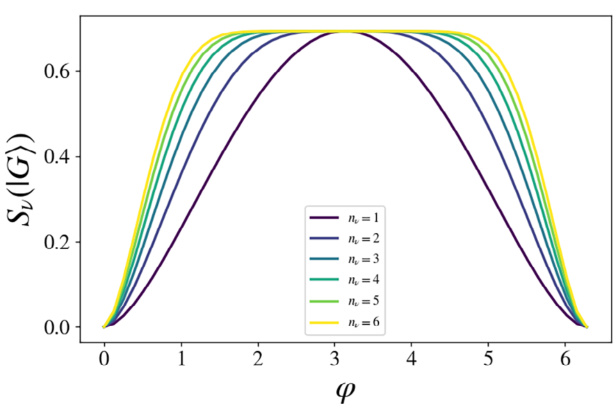

Figure 3. The entropy of entanglement for a bipartition (ν, ν^C), as a function of the interaction strength (or duration), for different numbers n_ν of nearest neighbors, numerically computed. The scaling and behavior of this well known measure of bipartite entanglement is evidently very similar to that of the ED.

From (8), we derive the commutation relations

$$\begin{aligned} \sigma_x^a U_G &= U_G \sigma_z^{N(a)} \sigma_x^a \\ \sigma_y^a U_G &= U_G \sigma_z^{N(a)} \sigma_y^a \\ \sigma_z^a U_G &= U_G \sigma_z^a \end{aligned} \tag{16}$$

Note that, for two ensembles A and B, we have

$$\sigma_z^A \sigma_z^B = \sigma_z^{A\cup B} = \sigma_z^{A\Delta B} \tag{17}$$

where $A\Delta B = (A\cup B) \setminus (A\cap B)$ is the symmetric difference between sets A and B.

This operation is commutative and associative. Remark that $A\Delta B = \emptyset$ if and only if $A = B$. Furthermore, $\underset{i}{\Delta} A_i := A_0 \Delta A_1 \Delta \cdots \Delta A_k \Delta \cdots = \emptyset$ if and only if $\forall \nu$, there is an even number k of sets A_i containing ν.

We can now calculate the correlators, taking advantage of the fact that $\forall A \neq \emptyset,\ \langle\Psi|\sigma_z^A|\Psi\rangle = 0$.

3.1. Two-Points Correlators

We start here by computing pairwise correlations.

$$\begin{aligned} \langle G|\sigma_x^\nu \sigma_x^\mu|G\rangle &= \langle\Psi|U_G \sigma_x^\nu \sigma_x^\mu U_G|\Psi\rangle \\ &= \langle\Psi|\sigma_z^{N(\nu)}\sigma_z^{N(\mu)}|\Psi\rangle \\ &= \langle\Psi|\sigma_z^{N(\nu)\Delta N(\mu)}|\Psi\rangle \\ &= \begin{cases} 1 & \text{if } N(\nu) = N(\mu), \\ 0 & \text{else} \end{cases} \end{aligned} \tag{18}$$

since $(N(\nu)\cup N(\mu)) \setminus (N(\nu)\cap N(\mu)) = \emptyset$ if and only if $N(\nu) = N(\mu)$. In terms of graph theory, $\langle G|\sigma_x^\nu\sigma_x^\mu|G\rangle = 1$ if and only if μ and ν are *twins* (see **Figure 4** for a visual example.).

$$\begin{aligned} \langle G|\sigma_x^\nu \sigma_y^\mu|G\rangle &= \langle\Psi|U_G \sigma_x^\nu(-i\sigma_z^\mu\sigma_x^\mu) U_G|\Psi\rangle \\ &= -i\langle\Psi|\sigma_z^{N(\nu)}\sigma_z^{N(\mu)}\sigma_z^\mu|\Psi\rangle \\ &= -i\langle\Psi|\sigma_z^{N(\nu)\Delta N(\mu)\Delta\{\mu\}}|\Psi\rangle \\ &= 0 \end{aligned} \tag{19}$$

because, the graph being undirected, if $\nu \in N(\mu)$ then also $\mu \in N(\nu)$, hence $N(\nu)\Delta N(\mu) \neq \{\mu\}$, where $\{\mu\}$ is the singleton set containing the qubit μ only.

$$\begin{aligned} \langle G|\sigma_x^\nu \sigma_z^\mu|G\rangle &= \langle\Psi|U_G \sigma_x^\nu\sigma_z^\mu U_G|\Psi\rangle \\ &= \langle\Psi|\sigma_z^{N(\nu)}\sigma_z^\mu|\Psi\rangle \\ &= \langle\Psi|\sigma_z^{N(\nu)\Delta\{\mu\}}|\Psi\rangle \\ &= \begin{cases} 1 & \text{if } N(\nu) = \{\mu\} \\ 0 & \text{else} \end{cases} \end{aligned} \tag{20}$$

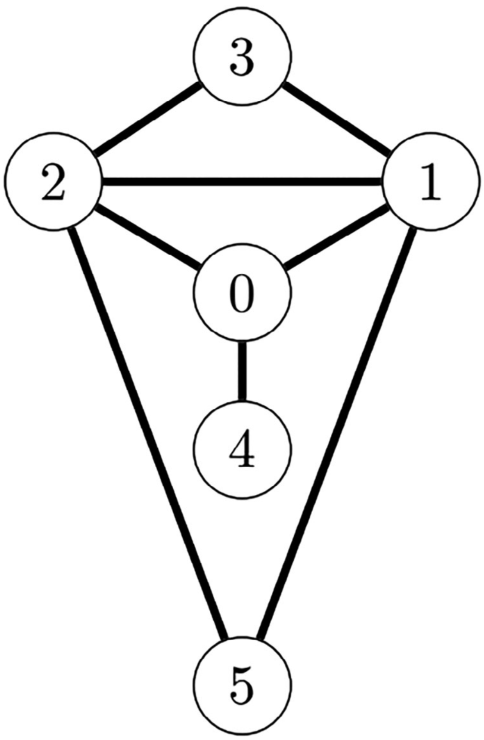

Figure 4. Example of a graph. Here, vertices 3 and 5 are *twins*, 1 and 2 are adjacent twins and 4 is a *leaf*.

In terms of graph theory, $\langle G|\sigma_x^\nu \sigma_z^\mu|G\rangle = 1$ if and only if ν is a *leaf vertex* (or *pendant vertex*) attached to G through μ (see Figure 4 for a visual example.).

$$\begin{aligned}
\langle G|\sigma_y^\nu \sigma_y^\mu|G\rangle &= \langle \Psi|U_G(i\sigma_x^\nu\sigma_z^\nu)(-i\sigma_z^\mu\sigma_x^\mu)U_G|\Psi\rangle \\
&= \langle \Psi|\sigma_z^{N(\nu)}\sigma_z^\nu\sigma_z^\mu\sigma_z^{N(\mu)}|\Psi\rangle \\
&= \langle \Psi|\sigma_z^{N(\nu)\Delta\{\nu\}\Delta N(\mu)\Delta\{\mu\}}|\Psi\rangle \\
&= \langle \Psi|\sigma_z^{\left(N(\nu)\cup\{\nu\}\right)\Delta\left(N(\mu)\cup\{\mu\}\right)}|\Psi\rangle \\
&= \begin{cases} 1 & \text{if } N(\nu)\cup\{\nu\} = N(\mu)\cup\{\mu\} \\ 0 & \text{else} \end{cases}
\end{aligned} \tag{21}$$

In terms of graph theory, $\langle G|\sigma_y^\nu \sigma_y^\mu|G\rangle = 1$ if and only if μ and ν are *adjacent twins* (see Figure 4 for a visual example.).

$$\begin{aligned}
\langle G|\sigma_y^\nu \sigma_z^\mu|G\rangle &= \langle \Psi|U_G(i\sigma_x^\nu\sigma_z^\nu)\sigma_z^\mu U_G|\Psi\rangle \\
&= i\langle \Psi|\sigma_z^{N(\nu)}\sigma_z^\nu\sigma_z^\mu|\Psi\rangle \\
&= i\langle \Psi|\sigma_z^{\left(N(\nu)\cup\{\nu\}\right)\Delta\{\mu\}}|\Psi\rangle \\
&= 0,
\end{aligned} \tag{22}$$

$$\langle G|\sigma_z^\nu \sigma_z^\mu|G\rangle = \langle \Psi|\sigma_z^\nu\sigma_z^\mu|\Psi\rangle = 0 \tag{23}$$

For two arbitrary measurements, performed in the directions determined by the unitary vectors $\boldsymbol{\nu}^\nu$ and $\boldsymbol{\nu}^\mu$, the correlation then writes

$$\begin{aligned}
\langle G|\sigma_\nu^\nu \sigma_\nu^\mu|G\rangle = \sum_{i,j=x,y,z} & \nu_i^\nu \nu_j^\mu \langle G|\sigma_i^\nu\sigma_j^\mu|G\rangle \\
&= \nu_x^\nu\nu_x^\mu \ \text{ if } N(\nu) = N(\mu) \\
&+ \nu_x^\nu\nu_z^\mu \ \text{ if } N(\nu) = \{\mu\} \\
&+ \nu_z^\nu\nu_x^\mu \ \text{ if } N(\mu) = \{\nu\} \\
&+ \nu_y^\nu\nu_y^\mu \ \text{ if } N(\nu)\cup\{\nu\} = N(\mu)\cup\{\mu\}
\end{aligned} \tag{24}$$

where we denoted $\sigma_\nu^\mu = \sum_{j=x,y,z} \nu_j^\mu \sigma_j^\mu$. It is fairly obvious that any such correlator can henceforth be fully determined by a quick inspection of the adjacency matrix A_G associated to G. For instance, the condition $N(\nu) = N(\mu)$ is equivalent to $A_\nu = A_\mu$.

This result makes it clear that non-vanishing pairwise correlations arise only for very specific connectivity properties of the sites being considered. More precisely, graphs, which contain neither twins, nor adjacent twins, nor leaf vertex, have only vanishing pairwise correlations. This is for instance the case for regular lattices.

Quite interestingly, this also implies that, in GS, most measurements that can be performed on one qubit yields no information on other qubits, and leaves the rest of the system entangled. Such entangled states hence contain persistent entanglement: a relatively big number of measurements are necessary to completely break their entanglement.

One can also exploit the properties of these correlators to probe the connectivity properties of a graph. Such a procedure could be for instance useful to check that, in a physical apparatus realizing the GS, the implementation of the link operators U_{ab} was successful and free of errors (that would be, the unwanted presence or absence of some of them).

From the above results, checking for twins, adjacent twins and leaf vertices will follow a fairly obvious measurement procedure. Yet it is possible to go further and check for instance for mere pairwise neighborhood, by removing irrelevant vertices from the graph. To do this, we can use the well-known fact that projective measurement of a single qubit in the direction z effectively removes it from the graph, i.e., isolates it.[5] Formally,

$$\begin{aligned}
P_{z\pm}^a|G\rangle &= P_{z\pm}^a U_G|\Psi\rangle = U_G P_{z\pm}^a|+\rangle^a \otimes |+\rangle^{V\setminus\{a\}} \\
&= \begin{cases} \frac{1}{\sqrt{2}} U_G|0\rangle^a \otimes |+\rangle^{V\setminus\{a\}} = \frac{1}{\sqrt{2}}|0\rangle^a \otimes |G\setminus\{a\}\rangle \\ \frac{1}{\sqrt{2}} U_G|1\rangle^a \otimes |+\rangle^{V\setminus\{a\}} = \frac{1}{\sqrt{2}}|1\rangle^a \otimes \sigma_z^{N(a)}|G\setminus\{a\}\rangle \end{cases}
\end{aligned} \tag{25}$$

Since $\sigma_z^{N(a)}|G\setminus\{a\}\rangle$ is local-unitary equivalent to $|G\setminus\{a\}\rangle$, such projective measurement results in an equivalent statistics as the desired GS with graph $G\setminus\{a\}$, up to some rotations of the measurement axis.

With a few computations, it can easily be checked that

$$\langle G|\left(\prod_{\mu\neq a,b} P^{\mu}_{z\pm}\right)\sigma^a_y\sigma^b_y\left(\prod_{\mu\neq a,b} P^{\mu}_{z\pm}\right)|G\rangle = \begin{cases} \pm 1 & \text{if } b\in N(a)\ (\leftrightarrow a\in N(b)) \\ 0 & \text{else}\end{cases} \tag{26}$$

It is hence enough, in order to examine the existence of a given link (a, b), to perform a projective measurement on the rest of the graph, or at least on the sites that may be linked to a or b, prior to measuring the correlator $\langle\sigma^a_y\sigma^b_y\rangle$.

3.2. Higher Order Correlators

The inspection of higher order correlators can be used to retrieve informations on more general properties of the graph.

3.2.1. Neighborhood Probing

Given an educated guess $\widetilde{N}(v)$ for the neighborhood of v, one can check its validity by computing the correlator

$$\langle G|\sigma^v_x\sigma^{\widetilde{N}(v)}_z|G\rangle \begin{cases} 1 & \text{if } \widetilde{N}(v)=N(v) \\ 0 & \text{else}\end{cases} \tag{27}$$

3.2.2. Topological Probing

The correlator

$$\langle G|\sigma^V_x|G\rangle = \begin{cases} 1 & \text{if } \Delta_{\mu\in V}N(\mu)=\emptyset \\ 0 & \text{else}\end{cases} \tag{28}$$

results in 1 if and only if every site has an even number of neighbors.

Furthermore,

$$\langle G|\sigma^V_y|G\rangle = \begin{cases} i^{|V|} & \text{if } \Delta_{\mu\in V}\left(N(\mu)\cup\{\mu\}\right)=\emptyset \\ 0 & \text{else}\end{cases} \tag{29}$$

results in ± 1 if and only if every site has an odd number of neighbors. It is 1 if $|V| \bmod 4 = 0$, -1 if $|V| \bmod 4 = 2$.

Euler's handshaking lemma states that, in any undirected graph, there is always an even number of vertices v such that n_v is odd. This guarantees that, as expected, this correlator never takes imaginary values.

In particular, if both (28) and (29) are null, G is not a regular graph (i.e., for which $\exists k\in\mathbb{N}$ such that $\forall v,\ n_v = k$). For instance, it can't be a lattice with periodic boundary conditions.

3.3. Relation to Measurement Processes

As already mentioned in the introduction, GS were proposed as a support for measurement-based quantum computation. To this aim, the system is first prepared in a GS of which the associated graph $G(V, E)$ is a regular lattice (usually, a finite square lattice). Then, a quantum circuit is built from this state by performing series of local projective measurements.

Hereafter, we thus investigate the effects of such measurements on the overall state, in the light shed by the above results.

As noticed in Ref. [16], if the expectation value of a product of Pauli observables (i.e., any product of Pauli matrices) on a given pure state $|s\rangle$, i.e., a generalized correlator, equates 1, then these observables are equivalent with respect to this state. Namely, they act on the state in the same fashion, and the associated projective measurements are themselves equivalent.

Formally, for any couple of observables $A,\ B$ such that $A^2 = B^2 = \mathbb{I}$, $\langle s|\,AB\,|s\rangle = 1$ implies

$$\begin{aligned} AB\,|s\rangle &= |s\rangle \\ B\,|s\rangle &= A\,|s\rangle \\ P_B\,|s\rangle &= P_A\,|s\rangle \\ P_B\,|s\rangle &= P_BP_A\,|s\rangle \end{aligned} \tag{30}$$

where $P_O = \frac{1}{2}(\mathbb{I}+O)$ are projectors onto the eigenstates of O of eigenvalue $+1$.

The projective measurement of A is thus equivalent to that of B.

For instance, Equation (18) implies that, if μ and v are twin vertices, the projective measure of σ^v_x is equivalent to that of σ^μ_x.

The case of higher order correlators leads to somewhat less trivial observations. Consider a measurement of σ^v_x with an outcome of $+1$. Formally, this corresponds to applying the projector $P^v_x = \frac{1}{2}(\mathbb{I}+\sigma^v_x)$ to the GS $|G\rangle$, up to renormalization. Yet Equation (27) together with Equation (30) tell us that this is in fact equivalent to applying $P^{N(v)}_z = \frac{1}{2}(\mathbb{I}+\sigma^{N(v)}_z)$. Notice that the latter projector is a non-local one, as it can't be written as the product of local single-qubit projectors; its effect is to project $|G\rangle$ onto the subspace $\{|\varphi\rangle \text{s.t. } \sigma^{N(v)}_z|\varphi\rangle = |\varphi\rangle\}$.

Non-locality implies that it does not correspond in itself to any physical measurement process, and rather stems as an entangling operation. It may indeed map a product state to an entangled state.

Let us examine further the effect of this projector on a GS. Omitting the renormalization factor, we obtain

$$\begin{aligned} P^v_x\,|G\rangle &= P^{N(v)}_z\,|G\rangle = P^{N(v)}_z U_G\,|\Psi\rangle = U_G P^{N(v)}_z\,|\Psi\rangle \\ &= \frac{1}{2}U_G\,|+\rangle^{V\setminus N(v)}\otimes\left(|+\rangle^{N(v)}+|-\rangle^{N(v)}\right) \end{aligned} \tag{31}$$

It results that, as can also be seen by considering the commutation relations (8), the operation $U_G P^v_x U_G$ effectively entangles every qubit $\mu\in N(v)$ in a state local-unitary equivalent to the Greenberger–Horne–Zeilinger state of n_v qubits, a prototypical case of maximally entangled state.

3.4. Remarks on the Simplicity of Data Analysis

In an ideal setting, relatively few measurements should, in principle, be enough to compute all of these correlators.

This is due to the fact that, for perfect GS, their outcomes can only be 1, −1, or 0. Yet the measurement of a Pauli observable can only result in outcomes of ±1, whether it is a single-qubit or a multi-qubit (i.e., correlator) observable.

Hence if the statistics yields, for a given Pauli observable P, an expectation value of $\langle G|P|G\rangle = 1$, we expect to measure only ones. It is thus enough to have measured a single −1 to conclude that $\langle G|P|G\rangle = 0$. The same reasoning obviously applies to the case of opposite value $\langle G|P|G\rangle = -1$.

Conversely, if the statistics yields an expectation value of 0, the probability of a measurement outcome ±1 is $\frac{1}{2}$, hence a uniform series of measurement outcomes becomes exponentially less likely as the number of measurement M grows. Precisely, if the value 1 has been measured M times in a row (and the value −1 has never been measured) the statistics yields $\langle G|P|G\rangle = 1$ with a probability of $1 - \frac{1}{2^M}$. Hence one would need at most $M = -\log_2(\epsilon)$ measurement samples to retrieve the true statistics with a confidence of $1 - \epsilon$.

In a quasi GS (i.e., $\varphi = \pi + \delta\varphi$), the link operators write $U_{ab}(\pi + \delta\varphi) = U_{ab}\,\delta U_{ab}$, with

$$\delta U_{ab} = \mathbb{I} - i\frac{\delta\varphi}{4}\left(\mathbb{I} - \sigma_z^a - \sigma_z^b + \sigma_z^a\sigma_z^b\right) \tag{32}$$

up to $o(\delta\varphi^2)$. The resulting commutation relations write

$$\sigma_k^a U_{ab}\,\delta U_{ab} = U_{ab}\,\delta U_{ab}\left(\sigma_z^b + i\frac{\delta\varphi}{2}\sigma_z^a - i\frac{\delta\varphi}{2}\sigma_z^a\sigma_z^b\right)\sigma_k^a \tag{33}$$

for $k = x, y$, while $[\sigma_z^a, U_{ab}\,\delta U_{ab}] = 0$.

Yet expectation values are always real, thus only even powers of $i\delta\varphi$ can appear in their final expression.

It results that the error on the correlators computed above is at most of order $o(\delta\varphi^2)$.

4. Conclusion

Throughout this work, we have developed a new approach to characterize GS. This approach is complementary to the stabilizer formalism widely used in the quantum computing community. While the stabilizer approach is a powerful tool for the analysis and construction of quantum algorithms with GS, the approach we propose relies on quantities, such as correlators, with a more straightforward interpretation.

Formally, a pure quantum state constitutes a statistical distribution for all possible measurement outcomes. As such, it is entirely determined by its statistical moments. In other words, knowing all possible expectation values and correlators of a state is equivalent to knowing the whole state. Although characterizing a pure state solely through expectation values may seem unreasonable from a computational point of view, a number of relevant partial informations can be obtained this way.

Correlators possess the desirable property of being both easily calculable and physically meaningful. In fact, they allow encoding the complexity of a graph state in terms of experimentally accessible quantities, revealing the structure of interactions between the composing qubits. Using this framework, we have been able to highlight simple relations between correlators and the connectivity properties of the graph defining a given graph state. The presented results offer a toolbox to investigate the topological structure of GS that can be used to verify the presence of local errors in their physical implementation. Moreover, since GS represents a universal resource for quantum computations, these results can be exported to any other universal resource, provided the appropriate mapping is carried out.

Furthermore, we have showed that correlators have the additional advantage of highlighting when pairs of projective measurements are equivalent with respect to a given state. This provides a new approach to understanding the effects of projective measurements on GS, revealing how multipartite entanglement emerges from simple binary interactions. Additionally, it could potentially enable determining simpler ways to implement quantum gates.

A follow-up to this work would be to thoroughly examine the formal connections between stabilizer-based and correlator-based approaches to improve the characterization of GSs and their structure.

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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connectivity, entanglement, graph state, measurement-based quantum computation, projective measurement, quantum correlator, quantum information

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[1] O. Gühne, G. Toth, *Phys. Rep.* **2009**, *474*, 1.
[2] R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, *Rev. Mod. Phys.* **2009**, *81*, 865.
[3] H. J. Briegel, R. Raussendorf, *Phys. Rev. Lett.* **2001**, *86*, 910.
[4] M. Hein, J. Eisert, H. J. Briegel, *Phys. Rev. A* **2004**, *69*, 062311.
[5] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. V. d. Nest, H.-J. Briegel, *Proc. International School of Physics "Enrico Fermi"* **2006**, *162*, 115.
[6] R. Raussendorf, T.-C. Wei, *Annu. Rev. Condens. Matter Phys.* **2012**, *3*, 239.
[7] M. A. Nielsen, *Rep. Math. Phys.* **2006**, *57*, 147.

[8] As an example to clarify this point, consider the simplest case $V = \{a, b\}$, $E = \{(a, b)\}$ and $\varphi_{ab} = \pi$. We have:$|G\rangle = U_{ab}(\varphi = \pi)| + +\rangle = \frac{1}{2}(| + +\rangle + | - +\rangle + | + -\rangle - | - -\rangle) = \frac{1}{\sqrt{2}}(| + 0\rangle + | - 1\rangle)$, a maximally entangled state of two qubits.
[9] K. P. Gnatenko, N. A. Susulovska, *EPL* **2022**, *136*, 40003.
[10] K. P. Gnatenko, V. M. Tkachuk, *Phys. Lett. A* **2021**, *396*, 127248.
[11] M. A. Nielsen, I. L. Chuang, *Quantum computation and quantum information: 10th anniversary edition*, Cambridge University Press, Cambridge **2010**.
[12] D. Cocchiarella, S. Scali, S. Ribisi, B. Nardi, G. Bel-Hadj-Aissa, R. Franzosi, *Phys. Rev. A* **2020**, *101*, 042129.
[13] A. Vesperini, G. Bel-Hadj-Aissa, R. Franzosi, *Sci. Rep.* **2023**, *13*, 2852.
[14] A. Vafafard, A. Nourmandipour, R. Franzosi, *Phys. Rev. A* **2022**, *105*, 052439.
[15] A. Nourmandipour, A. Vafafard, A. Mortezapour, R. Franzosi, *Sci. Rep.* **2021**, *11*, 16259.
[16] A. Vesperini, *Ann. Phys.* **2023**, *457*, 169406.
[17] A. Vesperini, G. Bel-Hadj-Aissa, L. Capra, R. Franzosi, Unveiling the geometric meaning of quantum entanglement, **2023**, http://arxiv.org/abs/2307.16835.