Annals of Pure and Applied Logic 175 (2024) 103391

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

journal homepage: www.elsevier.com/locate/apal

Full Length Article

Structural and universal completeness in algebra and logic

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ARTICLE INFO

Article history:
Received 17 April 2023
Received in revised form 23 October 2023
Accepted 28 November 2023
Available online 16 December 2023

MSC:
03G27
03G10
08C15
03B47

Keywords:
Structural completeness
Admissibility
Unification
Projectivity
Substructural logics
MTL

ABSTRACT

In this work we study the notions of structural and universal completeness both from the algebraic and logical point of view. In particular, we provide new algebraic characterizations of quasivarieties that are actively and passively universally complete, and passively structurally complete. We apply these general results to varieties of bounded lattices and to quasivarieties related to substructural logics. In particular we show that a substructural logic satisfying weakening is passively structurally complete if and only if every classical contradiction is explosive in it. Moreover, we fully characterize the passively structurally complete varieties of MTL-algebras, i.e., bounded commutative integral residuated lattices generated by chains.

1. Introduction

The main aim of this paper is to explore some connections between algebra and logic; mainly, we try to produce some *bridge theorems*. A bridge theorem is a statement connecting logical (and mostly syntactic) features of deductive systems and properties of classes of algebras; this connection is usually performed using the tools of general algebra and the rich theory that is behind it. The main reason behind this kind of exploration is in the further understanding one can gain by connecting two apparently distant fields. In this way, we can explore logical properties in purely algebraic terms; at the same time statements can be imported from logic that have an important and often new algebraic meaning.

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https://doi.org/10.1016/j.apal.2023.103391

The set of logical problems we want to explore is connected with the concept of *structural completeness* of a logic, the latter seen as a single-conclusion consequence relation that is substitution invariant and finitary. For a logic, being structurally complete means that each of its proper extensions admits new theorems. This notion can be formalized in a more rigorous way, using the concept of *admissible rule*. A rule is admissible in a logic if, whenever there is a substitution making its premises a theorem, such substitution also makes the conclusion a theorem. A logic is then structurally complete if all its admissible rules are derivable in the system. It is well-known that classical logic is structurally complete; intuitionistic logic is not but it satisfies a weaker although still interesting notion: it is *passively* structurally complete. We will see that this is not just a feature of intuitionism but it can be explained in a much more general framework, and it is connected to the way the contradictions of classical logic are treated. In more details, passive structural completeness means that all rules that do not apply to theorems are derivable. Naturally, the dual notion of *active* structural completeness also arises, which instead isolates the derivability of those admissible rules for which there exists a substitution making their premises a theorem. The latter notion has been explored in generality in [35]. Structural completeness and its hereditary version have been deeply studied in the literature: e.g., in general algebraic terms in [13], in substructural logics in [71], in fuzzy logics in [29], in intermediate logics in [30].

A natural extension of this kind of problems is to consider *multiple-conclusion* rules, i.e. formal pairs $\Sigma \Rightarrow \Delta$, where both Σ and Δ are finite sets of formulas over a suitable language. We say that a multiple-conclusion rule is *derivable* in a logic if at least one of the formulas δ in Δ is derivable from Σ in the logic (i.e. the pair (Σ, δ) belongs to the consequence relation defining the logic); likewise a multiple-conclusion rule $\Sigma \Rightarrow \Delta$ is *admissible* if a substitution making all the formulas in Σ into theorems makes at least one of the formulas in Δ a theorem. We mention that while the previous definition of admissibility for a (single-conclusion) rule corresponds to saying that one can add it to the consequence relation without obtaining new theorems, and both these definitions are widely used in the literature, the analogous correspondence does not hold for multiple-conclusion rules (and multiple-conclusion consequence relations, see [53,66] and the discussion in Subsection 2.4).

Now, a logic is *universally complete* if every admissible multiple-conclusion rule is derivable in it. It is then also possible to investigate the situation in which admissible multiple-conclusion rules are active or passive in a logic, and thus the corresponding notions of universal completeness. Universal completeness in connection to admissibility has been studied in [22].

The way in which our bridge theorems will be created exploits the machinery of the so-called Blok-Pigozzi connection [18]. Without going into details, this machinery allows one to express purely logical concepts in an algebraic language. The advantage of doing so is evident: on one hand we can use the entire wealth of results about classes of algebras and various algebraic operators. On the other hand, very often by means of this translation one ends up with algebraic results that are interesting in their own nature, irregardless of their logical origin. While not every logical system admits this translation, many interesting and/or classical systems do: classical and intuitionistic logic, relevance logics, substructural logics, many-valued logics, many modal logics and so on. In this framework, one can translate the previously described notions of structural and universal completeness into properties of the quasiequational or universal theory of the ω-generated free algebra in a quasivariety. In this setting, we will rephrase the notions of interest not in terms of formulas, but in terms of equations in a suitable language.

In this manuscript our aim is twofold; on one side we will try to describe in a complete and organic way (as much as it is possible) the phenomena mentioned above and the relations among them. In particular, we will recall the existing results trying to put them in a coherent perspective, which we believe is currently lacking, and we will provide many examples. On the other side, we will provide new results and novel characterizations of those notions that are missing an effective algebraic description. More specifically, we will first show how the characterization of active structural completeness in [35] can be extended to describe active universal completeness. Moreover, we will give algebraic descriptions of the notions of passive universal and structural completeness and the latter will result in a useful characterization. As a particularly interesting consequence, we show that a substructural logic satisfying the weakening rule is passively structurally complete if and only if every contradiction of classical logic is explosive in it. This generalizes and explains the passive structural completeness of intuitionistic logic. Moreover, it entails that all substructural logics (with weakening) with the *Glivenko property* with respect to classical logic are passively structurally complete. Further specializing the general result, we build on it to provide a clear characterization (and an axiomatization) of the minimal passively structurally complete logic that is an axiomatic extension of the t-norm based logic $\mathcal{MTL}$. From the algebraic side, this means that we characterize the passively structurally complete varieties of bounded commutative integral residuated lattices generated by chains.

The techniques we will employ in our study are the ones proper of general algebra. In particular, we will use the understanding of algebraic objects such as projective and exact algebras. The same objects are known to be relevant for the algebraic study of unification problems in algebraizable logics [47]. In fact, we will show how the notion of unifiability of a set of formulas (or, equivalently, a set of equations) plays a major role in our results.

The structure of this manuscript is as follows. In the next section we will discuss the needed preliminary notions. In particular, the Blok-Pigozzi connection, projective and exact algebras, algebraic unification, and finally, we define the notions of structural and universal completeness. Section 3 is devoted to universal completeness, and Section 4 to structural completeness, both in their various declinations. The last section is devoted to a deeper understanding of some relevant examples from the realms of algebra and (algebraic) logic respectively. In particular, in Subsection 5.1 we apply our results to the variety of (bounded) lattices; finally, in Subsection 5.2, we prove the aforementioned results and more about substructural logics.

2. Preliminaries

2.1. Universal algebra and the Blok-Pigozzi connection

Let K be a class of algebras; we denote by $\mathbf{I}, \mathbf{H}, \mathbf{P}, \mathbf{S}, \mathbf{P}_u$ the class operators sending K to the class of all isomorphic copies, homomorphic images, direct products, subalgebras and ultraproducts of members of K. The operators can be composed in the obvious way; for instance $\mathbf{SP}(\mathsf{K})$ denotes all algebras that are embeddable in a direct product of members of K; moreover there are relations among the classes resulting from applying the operators in a specific orders, for instance $\mathbf{PS}(\mathsf{K}) \subseteq \mathbf{SP}(\mathsf{K})$ and $\mathbf{HSP}(\mathsf{K})$ is the largest class we can obtain composing the operators. We will use all the known relations without further notice, but the reader can consult [72] or [21] for a textbook treatment.

If ρ is a type of algebras, an *equation* is a pair p, q of ρ-terms (i.e. elements of the absolutely free algebra $\mathbf{T}_\rho(\omega)$) that we write suggestively as $p \approx q$; a *clause* in ρ is a formal pair (Σ, Γ) that we write as $\Sigma \Rightarrow \Gamma$, where Σ, Γ are finite sets of equations; a clause is a *quasiequation* if $|\Gamma| = 1$ and it is *negative* if $\Gamma = \emptyset$. Clearly an equation is a quasiequation in which $\Sigma = \emptyset$.

Given any set of variables X, an assignment of X into an algebra $\mathbf{A}$ of type ρ is a function h mapping each variable $x \in X$ to an element of $\mathbf{A}$, that extends (uniquely) to a homomorphism (that we shall also call h) from the term algebra $\mathbf{T}_\rho(\omega)$ to $\mathbf{A}$. An algebra $\mathbf{A}$ satisfies an equation $p \approx q$ with an assignment h (and we write $\mathbf{A}, h \vDash p \approx q$) if $h(p) = h(q)$ in $\mathbf{A}$. An equation $p \approx q$ is *valid* in $\mathbf{A}$ (and we write $\mathbf{A} \vDash p \approx q$) if for all assignments h in $\mathbf{A}$, $\mathbf{A}, h \vDash p \approx q$; if Σ is a set of equations then $\mathbf{A} \vDash \Sigma$ if $\mathbf{A} \vDash \sigma$ for all $\sigma \in \Sigma$. A clause is *valid* in $\mathbf{A}$ (and we write $\mathbf{A} \vDash \Sigma \Rightarrow \Delta$) if for all assignments h to $\mathbf{A}$, $h(p) = h(q)$ for all $p \approx q \in \Sigma$ implies that there is an identity $s \approx t \in \Delta$ with $h(s) = h(t)$; in other words a clause $\Sigma \Rightarrow \Delta$ can be understood as the universal sentence $\forall \mathbf{x}(\bigwedge \Sigma \to \bigvee \Delta)$, where $\mathbf{x}$ are the variables occurring in $\Sigma \cup \Delta$ and $\bigwedge \emptyset = 1$, $\bigvee \emptyset = 0$. Conversely, note that an arbitrary universal formula of the language may be associated (by putting the quantified formula into conjunctive normal form) with a finite set of clauses. A clause $\Sigma \Rightarrow \Delta$ is *valid* in a class K if it is valid in all algebras in K, and we write $\mathsf{K} \vDash \Sigma \Rightarrow \Delta$ or $\vDash_{\mathsf{K}} \Sigma \Rightarrow \Delta$.

A class of algebras is a variety if it is closed under $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$, a quasivariety if it is closed under $\mathbf{I}$, $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{P}_u$ and a universal class if it is closed under $\mathbf{I}$, $\mathbf{S}$, and $\mathbf{P}_u$. The following facts were essentially discovered by A. Tarski and J. Łòs in the pioneering phase of model theory; for proof of this and similar statements the reader can consult [26].

Lemma 2.1. *Let* K *be any class of algebras. Then:*

1. K *is a universal class if and only if* $\mathbf{ISP}_u(\mathsf{K}) = \mathsf{K}$ *if and only if it is the class of algebras in which a set of universal sentences is valid;*
2. K *is a quasivariety if and only if* $\mathbf{ISPP}_u(\mathsf{K}) = \mathsf{K}$ *if and only if it is the class of algebras in which a set of quasiequations is valid;*
3. K *is a variety if and only if* $\mathbf{HSP}(\mathsf{K}) = \mathsf{K}$ *if and only if it is the class of algebras in which a set of equations is valid.*

Notation 1. We will often write $\mathbf{V}$ for $\mathbf{HSP}$ and $\mathbf{Q}$ for $\mathbf{ISPP}_u$.

For the definition of free algebras in a class K on a set X of generators, in symbols $\mathbf{F}_{\mathsf{K}}(X)$, we refer to [21]; we merely observe that every free algebra on a class K belongs to $\mathbf{ISP}(\mathsf{K})$. It follows that every free algebra in K is free in $\mathbf{ISP}(\mathsf{K})$ and therefore for any quasivariety Q, $\mathbf{F}_{\mathsf{Q}}(X) = \mathbf{F}_{\mathsf{V}(\mathsf{Q})}(X)$.

There are two fundamental results that we will be using many times and deserve a spotlight. Let $\mathbf{B}$, $(\mathbf{A}_i)_{i\in I}$ be algebras in the same signature; we say that $\mathbf{B}$ *embeds* in $\prod_{i\in I}\mathbf{A}_i$ if $\mathbf{B} \in \mathbf{IS}(\prod_{i\in I}\mathbf{A}_i)$. Let p_i be the i-th projection, or better, the composition of the embedding and the i-th projection, from $\mathbf{B}$ to $\mathbf{A}_i$; the embedding is *subdirect* if for all $i \in I$, $p_i(\mathbf{B}) = \mathbf{A}_i$ and in this case we will write

$$\mathbf{B} \leqslant_{sd} \prod_{i\in I}\mathbf{A}_i.$$

An algebra $\mathbf{B}$ is *subdirectly irreducible* if it is nontrivial and for any subdirect embedding

$$\mathbf{B} \leqslant_{sd} \prod_{i\in I}\mathbf{A}_i$$

there is an $i \in I$ such that $\mathbf{B}$ and $\mathbf{A}_i$ are isomorphic. It can be shown that $\mathbf{A}$ is *subdirectly irreducible* if and only if the congruence lattice $\mathrm{Con}(\mathbf{A})$ of $\mathbf{A}$ has a unique minimal element different from the trivial congruence. If V is a variety we denote by V_{si} the class of subdirectly irreducible algebras in V.

Theorem 2.2.

1. *(Birkhoff [16]) Every algebra can be subdirectly embedded in a product of subdirectly irreducible algebras. So if* $\mathbf{A} \in \mathsf{V}$*, then* $\mathbf{A}$ *can be subdirectly embedded in a product of members of* V_{si}.
2. *(Jónsson's Lemma [57]) Suppose that* K *is a class of algebras such that* $\mathsf{V}(\mathsf{K})$ *is congruence distributive; then* $\mathsf{V}_{si} \subseteq \mathbf{HSP}_u(\mathsf{K})$.

If Q is a quasivariety and $\mathbf{A} \in \mathsf{Q}$, a *relative congruence* of $\mathbf{A}$ is a congruence θ such that $\mathbf{A}/\theta \in \mathsf{Q}$; relative congruences form an algebraic lattice $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$. Moreover, for an algebra $\mathbf{A}$ and a set $H \subseteq A \times A$ there exists the least relative congruence $\theta_{\mathsf{Q}}(H)$ on $\mathbf{A}$ containing H. When $H = \{(a, b)\}$, we just write $\theta_{\mathsf{Q}}(a, b)$. When Q is a variety we simplify the notation by dropping the subscript Q.

For any congruence lattice property P we say that $\mathbf{A} \in \mathsf{Q}$ is *relative* P if $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ satisfies P. So for instance $\mathbf{A}$ is *relative subdirectly irreducible* if $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ has a unique minimal element; since clearly $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ is a meet subsemilattice of $\mathrm{Con}(\mathbf{A})$, any subdirectly irreducible algebra is relative subdirectly irreducible for any quasivariety to which it belongs. For a quasivariety Q we denote by Q_{rsi} the class of relative subdirectly irreducible algebras in Q. We have the equivalent of Birkhoff's and Jónsson's results for quasivarieties:

Theorem 2.3. *Let* Q *be any quasivariety.*

1. *(Mal'cev [64]) Every* $\mathbf{A} \in \mathsf{Q}$ *is subdirectly embeddable in a product of algebras in* Q_{rsi}*.*
2. *(Czelakowski-Dziobiak [32]) If* $\mathsf{Q} = \mathbf{Q}(\mathsf{K})$*, then* $\mathsf{Q}_{rsi} \subseteq \mathbf{ISP}_u(\mathsf{K})$*.*

The following fact will be used in the sequel.

Lemma 2.4. *Let* $\mathbf{A}$ *be an algebra such that* $\mathbf{V}(\mathbf{A})$ *is congruence distributive. Then* $\mathbf{Q}(\mathbf{A}) = \mathbf{V}(\mathbf{A})$ *if and only if every subdirectly irreducible algebra in* $\mathbf{HSP}_u(\mathbf{A})$ *is in* $\mathbf{ISP}_u(\mathbf{A})$*.*

Proof. Suppose first that $\mathbf{Q}(\mathbf{A}) = \mathbf{V}(\mathbf{A})$, and let $\mathbf{A}$ be a subdirectly irreducible algebra in $\mathbf{HSP}_u(\mathbf{A})$. Thus $\mathbf{A}$ is subdirectly irreducible in $\mathbf{V}(\mathbf{A}) = \mathbf{Q}(\mathbf{A})$, and by Theorem 2.3 $\mathbf{A} \in \mathbf{ISP}_u(\mathbf{A})$.

Conversely assume that every subdirectly irreducible algebra in $\mathbf{HSP}_u(\mathbf{A})$ is in $\mathbf{ISP}_u(\mathbf{A})$. Since $\mathbf{V}(\mathbf{A})$ is congruence distributive, by Theorem 2.2(2) every subdirectly irreducible algebra in $\mathbf{V}(\mathbf{A})$ is in $\mathbf{HSP}_u(\mathbf{A})$, thus in $\mathbf{ISP}_u(\mathbf{A})$. Now every algebra in $\mathbf{V}(\mathbf{A})$ is subdirectly embeddable in a product of subdirectly irreducible algebras in $\mathbf{V}(\mathbf{A})$ (Theorem 2.2(1)). Therefore, $\mathbf{V}(\mathbf{A}) \subseteq \mathbf{ISPISP}_u(\mathbf{A}) \subseteq \mathbf{ISPP}_u(\mathbf{A}) = \mathbf{Q}(\mathbf{A})$ and thus equality holds. □

In this work we are particularly interested in quasivarieties that are the equivalent algebraic semantics of a logic in the sense of Blok-Pigozzi [18]. We will spend some time illustrating the machinery of *Abstract Algebraic Logic* that establishes a Galois connection between *algebraizable logics* and *quasivarieties of logic*, since it is relevant to understand our results. For the omitted details we refer the reader to [18,40,43].

A (single-conclusion) *consequence relation* on the set of terms $\mathbf{T}_\rho(\omega)$ (also called *algebra of formulas*) of some algebraic language ρ is a relation $\vdash \subseteq \mathcal{P}(\mathbf{T}_\rho(\omega)) \times \mathbf{T}_\rho(\omega)$ (and we write $\Sigma \vdash \gamma$ for $(\Sigma, \gamma) \in \vdash$) such that:

1. if $\alpha \in \Gamma$ then $\Gamma \vdash \alpha$;
2. if $\Gamma \vdash \delta$ for all $\delta \in \Delta$ and $\Delta \vdash \beta$, then $\Gamma \vdash \beta$.

We call *substitution* any endomorphism of $\mathbf{T}_\rho(\omega)$; $\vdash$ is *substitution invariant* (also called *structural*) if $\Gamma \vdash \alpha$ implies $\{\sigma(\gamma) : \gamma \in \Gamma\} \vdash \sigma(\alpha)$ for each substitution σ. Finally, $\vdash$ is *finitary* if $\Gamma \vdash \alpha$ implies that there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \alpha$. By a *logic* $\mathcal{L}$ in what follows we mean a substitution-invariant finitary consequence relation $\vdash_\mathcal{L}$ on $\mathbf{T}_\rho(\omega)$ for some algebraic language ρ, $\vdash_\mathcal{L} \subseteq \mathcal{P}(\mathbf{T}_\rho(\omega)) \times \mathbf{T}_\rho(\omega)$.

Now, a *theorem* of a logic $\mathcal{L}$ (given by $\vdash_\mathcal{L}$) is a formula φ such that $\varnothing \vdash_\mathcal{L} \varphi$; in this case we will usually omit $\varnothing$ and just write $\vdash_\mathcal{L} \varphi$. A *multiple-conclusion rule* of $\mathcal{L}$ is an ordered pair (Σ, Γ) where Σ, Γ are finite sets of formulas. We usually write a multiple-conclusion rule as $\Sigma \Rightarrow \Gamma$. We will call single-conclusion rules just *rules*. We say that a multiple-conclusion rule $\Sigma \Rightarrow \Delta$ is *derivable* in $\mathcal{L}$ if $\Sigma \vdash_\mathcal{L} \delta$ for some $\delta \in \Delta$. A (single-conclusion) rule $\Sigma \Rightarrow \delta$ is then derivable in $\mathcal{L}$ if and only if $\Sigma \vdash_\mathcal{L} \delta$. If $\vdash_1$ and $\vdash_2$ are two logics over the same language, $\vdash_2$ is an *extension* of $\vdash_1$ if $\vdash_1 \subseteq \vdash_2$; $\vdash_2$ is an *axiomatic extension* of $\vdash_1$ if there is a set of formulas Γ such that $\vdash_2$ is the smallest logic that is an extension of $\vdash_1$ satisfying $\vdash_2 \gamma$ for all $\gamma \in \Gamma$.

In loose terms, to establish the algebraizability of a logic $\mathcal{L}$ with respect to a quasivariety of algebras $\mathsf{Q}_\mathcal{L}$ over the same language ρ, one needs a finite set of one-variable equations

$$\tau(x) = \{\delta_i(x) \approx \varepsilon_i(x) : i = 1, \ldots, n\}$$

over terms of type ρ and a finite set of formulas of $\mathcal{L}$ in two variables

$$\Delta(x, y) = \{\varphi_1(x, y), \ldots, \varphi_m(x, y)\}$$

that allow to transform equations, quasiequations and clauses in $\mathsf{Q}_{\mathcal{L}}$ into formulas, single- and multiple-conclusion rules of $\mathcal{L}$; moreover this transformation must intuitively respect both the consequence relation of the logic and the semantical consequence of the quasivariety; more precisely, for all sets of formulas Γ of $\mathcal{L}$ and formulas $\varphi \in \mathbf{T}_\rho(\omega)$

$$\Gamma \vdash_{\mathcal{L}} \varphi \quad \text{iff} \quad \tau(\Gamma) \vDash_{\mathsf{Q}_{\mathcal{L}}} \tau(\varphi)$$

where $\tau(\Gamma)$ is a shorthand for $\{\tau(\gamma) : \gamma \in \Gamma\}$, and also

$$(x \approx y) =\!\!\mid\vDash_{\mathsf{Q}_{\mathcal{L}}} \tau(\Delta(x, y)).$$

If a logic has a variety as its equivalent algebraic semantics it is said to be *strongly algebraizable*. We mention that algebraizability is preserved by extensions (and strong algebraizability by axiomatic extensions).

A quasivariety Q is a *quasivariety of logic* if it is the equivalent algebraic semantics for some logic $\mathcal{L}_{\mathsf{Q}}$; the Galois connection between algebraizable logics and quasivarieties of logic is given by

$$\mathcal{L}_{\mathsf{Q}_{\mathcal{L}}} = \mathcal{L} \qquad\qquad \mathsf{Q}_{\mathcal{L}_{\mathsf{Q}}} = \mathsf{Q}.$$

Not every quasivariety is a quasivariety of logic; for instance no *idempotent quasivariety*, such as any quasivariety of lattices, can be a quasivariety of logics. Nonetheless quasivarieties of logic are plentiful. In fact any ideal determined variety is such, as well as any quasivariety coming from a congruential variety with normal ideals (see [9] for details). Moreover, every quasivariety is *categorically equivalent* to a quasivariety of logic [68]. This means that if an algebraic concept is expressible through notions that are invariant under categorical equivalence, and it holds for a quasivariety Q, then it holds for its categorically equivalent quasivariety of logic Q'; and hence it can be transformed into a logical concept in $\mathcal{L}_{\mathsf{Q}'}$ using the Blok-Pigozzi connection. The following result hints at what kind of properties can be transferred by categorical equivalence.

Theorem 2.5 *([12]). Let* K *be a class closed under subalgebras and direct products. If* K *is categorically equivalent to a quasivariety* Q*, then* K *is a quasivariety.*

Suppose now that Q and R are quasivarieties and suppose that $F : \mathsf{Q} \longrightarrow \mathsf{R}$ is a functor between the two algebraic categories witnessing the categorical equivalence. Now, F preserves all the so-called *categorical properties*, i.e., those notions that can be expressed as properties of morphisms. In particular, embeddings are mapped to embeddings (since in algebraic categories they are exactly the categorical monomorphisms), surjective homomorphisms are mapped to surjective homomorphisms (since they correspond to *regular* epimorphisms in the categories). Moreover, we observe that direct products are preserved as well, since they can be expressed via families of surjective homomorphisms (see e.g. [21]). Therefore, if Q' is a subquasivariety of Q, then the restriction of F to Q' witnesses a categorical equivalence between Q' and

$$\mathsf{R}' = \{\mathbf{B} \in \mathsf{R} : \mathbf{B} = F(\mathbf{A}) \text{ for some } \mathbf{A} \in \mathsf{Q}'\}.$$

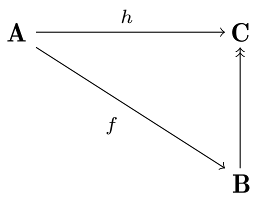
It follows from Theorem 2.5 that R' is a subquasivariety of R, and that R' is a variety whenever Q' is such. Given a quasivariety Q, we denote by $\Lambda_q(\mathsf{Q})$ the lattice of subquasivarieties of Q. Hence the correspondence sending $\mathsf{Q}' \longmapsto \mathsf{R}'$ is a lattice isomorphism between $\Lambda_q(\mathsf{Q})$ in $\Lambda_q(\mathsf{R})$ that preserves all the categorical properties. Moreover, we observe that, since ultraproducts in an algebraic category admit a categorical definition which turns out to be equivalent to the algebraic one (see for instance [37]), the functor F also maps universal subclasses to universal subclasses; more precisely, $\mathsf{U} \subseteq \mathsf{Q}$ is a universal class if and only if $F(\mathsf{U}) \subseteq \mathsf{R}$ is a universal class.

Let us show an example of how we can use these correspondences, that is also a preview of what we will see in the coming sections; if Q is a quasivariety, a subquasivariety Q' is *equational* in Q if $\mathsf{Q}' = \mathbf{H}(\mathsf{Q}') \cap \mathsf{Q}$. A quasivariety is *primitive* if every subquasivariety of Q is equational in Q. It is clear from the discussion above that this concept is preserved by categorical equivalence and that the lattice isomorphism described above sends primitive subquasivarieties in primitive subquasivarieties.

2.2. Projectivity, weak projectivity and exactness

We now introduce the algebraic notions that will be the key tools for our investigation: projective, weakly projective, exact, and finitely presented algebras.

Definition 2.6. Given a class K of algebras, an algebra $\mathbf{A} \in \mathsf{K}$ is *projective* in K if for all $\mathbf{B}, \mathbf{C} \in \mathsf{K}$, any homomorphism $h : \mathbf{A} \longmapsto \mathbf{C}$, and any surjective homomorphism $g : \mathbf{B} \longmapsto \mathbf{C}$, there is a homomorphism $f : \mathbf{A} \longmapsto \mathbf{B}$ such that $h = gf$.



Determining the projective algebras in a class is usually a challenging problem, especially in a general setting. If however K contains all the free algebras in K (in particular, if K is a quasivariety), projectivity admits a simpler formulation. We call an algebra $\mathbf{B}$ a *retract* of an algebra $\mathbf{A}$ if there is a homomorphism $g : \mathbf{A} \longmapsto \mathbf{B}$ and a homomorphism $f : \mathbf{B} \longmapsto \mathbf{A}$ with $gf = \mathrm{id}_{\mathbf{B}}$ (and thus, necessarily, f is injective and g is surjective). The following theorem was proved first by Whitman for lattices [78] but it is well-known to hold for any class of algebras.

Theorem 2.7. *Let Q be a quasivariety. Then the following are equivalent:*

1. *$\mathbf{A}$ is projective in Q;*
2. *$\mathbf{A}$ is a retract of a free algebra in Q;*
3. *$\mathbf{A}$ is a retract of a projective algebra in Q.*

In particular every free algebra in Q is projective in Q.

Definition 2.8. Given a quasivariety Q we say that an algebra $\mathbf{A}$ is *finitely presented in* Q if there exists a finite set X and a finite set H of pairs of terms over X such that $\mathbf{A} \cong \mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(H)$.

The proof of the following theorem is standard (but see [47]).

Theorem 2.9. *For a finitely presented algebra $\mathbf{A} \in \mathsf{Q}$ the following are equivalent:*

1. *$\mathbf{A}$ is projective in Q;*
2. *$\mathbf{A}$ is projective in the class of all finitely presented algebras in Q;*
3. *$\mathbf{A}$ is a retract of a finitely generated free algebra in Q.*

As a consequence we stress that if Q is a quasivariety and $\mathsf{V} = \mathbf{V}(\mathsf{Q})$ then all the algebras that are projective in Q are also projective in V (and vice versa). Moreover, all the finitely generated projective algebras in Q lie inside $\mathbf{Q}(\mathbf{F}_{\mathsf{Q}}(\omega))$.

Definition 2.10. An algebra $\mathbf{A}$ is *weakly projective in an algebra* $\mathbf{B}$ if $\mathbf{A} \in \mathbf{H}(\mathbf{B})$ implies $\mathbf{A} \in \mathbf{S}(\mathbf{B})$; an algebra is *weakly projective in a class* K if it is weakly projective in any algebra $\mathbf{B} \in \mathsf{K}$.

Definition 2.11. If Q is a quasivariety of algebras and $\mathbf{A} \in \mathsf{Q}$, $\mathbf{A}$ is *exact* in Q if it is embeddable into some free algebra in Q.

Clearly any projective algebra in Q is weakly projective in Q and any weakly projective algebra in Q is exact in Q. Observe also the following consequence of the definition.

Lemma 2.12. *Let Q be a quasivariety and let $\mathbf{A}$ be a κ-generated algebra in Q, for some cardinal κ; then the following are equivalent:*

1. *$\mathbf{A}$ is exact in Q;*
2. *$\mathbf{A} \in \mathbf{IS}(\mathbf{F}_{\mathsf{Q}}(\kappa))$.*

Remark 2.13. For finitely generated algebras our definition of exactness coincides with the one in [23], i.e. a finitely generated algebra $\mathbf{A}$ is exact in Q if and only if $\mathbf{A} \in \mathbf{IS}(\mathbf{F}_{\mathsf{Q}}(\omega))$.

We close this subsection with a couple of results connecting projectivity and weak projectivity.

Proposition 2.14. *Let $\mathbf{A}$ be a finite subdirectly irreducible algebra; if $\mathbf{A}$ is weakly projective in $\mathbf{Q}(\mathbf{A})$, then it is projective in $\mathbf{Q}(\mathbf{A})$.*

Proof. Let $\mathsf{Q} = \mathbf{Q}(\mathbf{A})$; since $\mathbf{A}$ is finite, Q is locally finite. Let $\mathbf{F}$ be a finitely generated (hence finite) free algebra in Q such that $\mathbf{A} \in \mathbf{H}(\mathbf{F})$; since $\mathbf{A}$ is weakly projective, $\mathbf{A}$ is embeddable in $\mathbf{F}$ and without loss of generality we may assume that $\mathbf{A} \leqslant \mathbf{F}$. Consider the set

$$V = \{\alpha \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{F}) : \alpha \cap A^2 = 0_{\mathbf{A}}\},$$

where we denote by $0_{\mathbf{A}}$ the minimal congruence of $\mathbf{A}$. It is easy to see that V is an inductive poset so we may apply Zorn's Lemma to find a maximal congruence $\theta \in V$. Clearly $a \longmapsto a/\theta$ is an embedding of $\mathbf{A}$ into $\mathbf{F}/\theta$. We claim that $\mathbf{F}/\theta$ is relative subdirectly irreducible and to prove so, since everything is finite, it is enough to show that θ is meet irreducible in $\mathrm{Con}_{\mathsf{Q}}(\mathbf{F})$; so let $\alpha, \beta \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ such that $\alpha \wedge \beta = \theta$. Then

$$0_{\mathbf{A}} = \theta \cap A^2 = (\alpha \wedge \beta) \cap A^2 = (\alpha \cap A^2) \wedge (\beta \cap A^2);$$

But $\mathbf{A}$ is subdirectly irreducible, so 0_A is meet irreducible in $\mathrm{Con}(\mathbf{A})$; hence either $\alpha \cap A^2 = 0_{\mathbf{A}}$ or $\beta \cap A^2 = 0_{\mathbf{A}}$, so either $\alpha \in V$ or $\beta \in V$. Since θ is maximal in V, either $\alpha = \theta$ or $\beta = \theta$, which proves that $\mathbf{F}/\theta$ is relative subdirectly irreducible. Therefore, by Theorem 2.3(2), $\mathbf{F}/\theta \in \mathbf{IS}(\mathbf{A})$; since $\mathbf{F}/\theta$ and $\mathbf{A}$ are both finite and each one is embeddable in the other, they are in fact isomorphic. Thus $\mathbf{A} \leqslant \mathbf{F}$, and there is a homomorphism from $\mathbf{F}$ onto $\mathbf{A}$ that maps each $a \in A$ to itself. This shows that $\mathbf{A}$ is a retract of $\mathbf{F}$, and therefore $\mathbf{A}$ is projective in $\mathbf{Q}(\mathbf{A})$. □

For varieties we have to add the hypothesis of congruence distributivity, since the use of Theorem 2.2(2) is paramount; for the very similar proof see [54, Theorem 9].

Proposition 2.15. *Let* $\mathbf{A}$ *be a finite subdirectly irreducible algebra such that* $\mathbf{V}(\mathbf{A})$ *is congruence distributive; if* $\mathbf{A}$ *is weakly projective in* $\mathbf{V}(\mathbf{A})$*, then it is projective in* $\mathbf{V}(\mathbf{A})$.

We observe that in algebraic categories projectivity is a property preserved by categorical equivalence and the same holds for weak projectivity and exactness. Finally by [42] being finitely presented and being finitely generated are also categorical properties preserved by equivalences.

2.3. Algebraic unification

The main objects of our study, i.e., the notions of universal and structural completeness, are closely related to unification problems. The classical syntactic unification problem given two terms s, t finds a *unifier* for them; that is, a uniform replacement of the variables occurring in s and t by other terms that makes s and t identical. When the latter syntactical identity is replaced by equality modulo a given equational theory E, one speaks of *E-unification*. S. Ghilardi [47] proved that there is a completely algebraic way of studying (E-)unification problems in varieties of logic, which makes use of finitely presented and projective algebras and thus is invariant under categorical equivalence.

Let us discuss Ghilardi's idea in some detail showing how it can be applied to quasivarieties. If Q is a quasivariety and Σ is a finite set of equations in the variables $X = \{x_1, \dots, x_n\}$ we can identify a substitution σ with an assignment from X to $\mathbf{F}_{\mathsf{Q}}(\omega)$, extending to a homomorphism from $\mathbf{F}_{\mathsf{Q}}(X)$ to $\mathbf{F}_{\mathsf{Q}}(\omega)$.

Definition 2.16. A *unification problem* for a quasivariety Q is a finite set of identities Σ in the language of Q; Σ is *unifiable* in Q if there is a substitution σ such that $\mathsf{Q} \vDash \sigma(\Sigma)$, i.e.

$$\mathsf{Q} \vDash p(\sigma(x_1), \dots, \sigma(x_n)) \approx q(\sigma(x_1), \dots, \sigma(x_n))$$

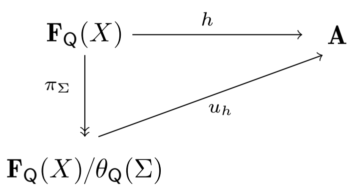
for all $p \approx q \in \Sigma$. The substitution σ is called a *unifier* for Σ.

Observe that Σ is *unifiable* in Q if and only if it is unifiable in $\mathbf{V}(\mathsf{Q})$. Let us now present the algebraic approach, where a unification problem can be represented by a finitely presented algebra in Q.

Definition 2.17. If $\mathbf{A}$ is in Q, a *unifier* for $\mathbf{A}$ is a homomorphism $u : \mathbf{A} \longrightarrow \mathbf{P}$ where $\mathbf{P}$ is a projective algebra in Q; we say that an algebra is *unifiable in* Q if at least one such homomorphism exists. A quasivariety Q is *unifiable* if every finitely presented algebra in Q is unifiable.

Notation 2. When we write $\mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma)$, $\theta_{\mathsf{Q}}(\Sigma)$ is the relative congruence generated in $\mathbf{F}_{\mathsf{Q}}(X)$ by the set $\{(p, q) : p \approx q \in \Sigma\}$.

Remark 2.18. Let Q be a quasivariety. Consider a set of identities Σ over variables in a set X, in the language of Q, and an assignment $h : \mathbf{F}_{\mathsf{Q}}(X) \to \mathbf{A} \in \mathsf{Q}$ such that $\mathbf{A}, h \vDash \Sigma$; i.e. such that the kernel $\ker(h)$ contains $\theta_{\mathsf{Q}}(\Sigma)$ (or equivalently its set of generators $\{(p, q) : p \approx q \in \Sigma\}$). Then the Second Homomorphism Theorem yields directly that h can be lifted to a homomorphism $u_h : \mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma) \to \mathbf{A}$ that closes the following diagram:



where $\pi_\Sigma : \mathbf{F}_\mathsf{Q}(X) \to \mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma)$ is the natural epimorphism; i.e. $h = u_h \circ \pi_\Sigma$. We will often use this fact in proofs.

The following summarizes the needed results of [47] applied to quasivarieties.

Theorem 2.19. *Let Q be a quasivariety, and let Σ be a finite set of equations in the language of Q with variables in a (finite) set X; then:*

1. *if Σ is unifiable via $\sigma : \mathbf{F}_\mathsf{Q}(X) \to \mathbf{F}_\mathsf{Q}(Y)$ then $u_\sigma : \mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma) \to \mathbf{F}_\mathsf{Q}(Y)$ defined by*
$$u_\sigma(t/\theta_\mathsf{Q}(\Sigma)) = \sigma(t)$$
is a unifier for $\mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma)$;
2. *conversely let $\mathbf{A} = \mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma)$. If there is a unifier $u : \mathbf{A} \to \mathbf{P}$, where $\mathbf{P}$ is projective and a retract of $\mathbf{F}_\mathsf{Q}(Y)$ witnessed by an embedding $i : \mathbf{P} \to \mathbf{F}_\mathsf{Q}(Y)$, the substitution*
$$\sigma_u : x \longmapsto i(u(x/\theta_\mathsf{Q}(\Sigma)))$$
is a unifier for Σ in Q.

Proof. For the first claim, consider $\sigma : \mathbf{F}_\mathsf{Q}(X) \to \mathbf{F}_\mathsf{Q}(Y)$; then since $\theta_\mathsf{Q}(\Sigma)$ is the least congruence of $\mathbf{F}_\mathsf{Q}(X)$ containing the set of pairs $S = \{(p, q) : p \approx q \in \Sigma\}$, and given that $S \subseteq \ker(\sigma)$, by Remark 2.18 we can obtain the map u_σ that is indeed a unifier for $\mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma)$.

The second claim is easily seen, since σ_u is defined by a composition of homomorphism and as above the set of pairs $S = \{(p, q) : p \approx q \in \Sigma\}$ is contained in its kernel, which yields that σ_u is a unifier for Σ in Q. □

Corollary 2.20. *A finite set of identities Σ is unifiable in Q if and only if the finitely presented algebra $\mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma)$ is unifiable in Q.*

The following observation shows how to characterize unifiability in quasivarieties.

Definition 2.21. For a quasivariety Q, we let $\mathbf{F}_\mathsf{Q}$ be the *smallest free algebra*, i.e. $\mathbf{F}_\mathsf{Q}(\varnothing)$ (if there are constant operations) or else $\mathbf{F}_\mathsf{Q}(x)$.

We have the following observation:

Lemma 2.22. *Let Q be a quasivariety and let $\mathbf{A} \in \mathsf{Q}$. Then the following are equivalent:*

1. *$\mathbf{A}$ is unifiable in Q;*
2. *there is a homomorphism from $\mathbf{A}$ to $\mathbf{F}_\mathsf{Q}$.*

Proof. Note that (2) trivially implies (1), since $\mathbf{F}_\mathsf{Q}$ is projective. Vice versa, if $\mathbf{A}$ is unifiable, there is a homomorphism from $\mathbf{A}$ to some projective algebra $\mathbf{P}$. Since $\mathbf{P}$ is a retract of some free algebra in Q, and $\mathbf{F}_\mathsf{Q}$ is a homomorphic image of every free algebra in Q, the claim follows. □

Remark 2.23. Note that being unifiable for an algebra $\mathbf{A}$ in a quasivariety Q really means having a homomorphism to some free algebra; now, given any two free algebras $\mathbf{F}, \mathbf{G}$, there always is a homomorphism from $\mathbf{F}$ to $\mathbf{G}$. The reader can then observe that one could rewrite the previous lemma substituting to $\mathbf{F}_\mathsf{Q}$

$$\mathbf{F}_{\mathsf{V}}(X)/\theta \twoheadrightarrow \mathbf{F}_{\mathsf{V}}(X)/\theta' \rightarrowtail \mathbf{P} \rightarrowtail \mathbf{F}_{\mathsf{V}}(\omega)$$

Fig. 1. $\mathbf{F}_{\mathsf{V}}(X)/\theta'$ is exact.

any factor of a free algebra that is also a subalgebra, so in particular any weakly projective or exact algebra. We stated the previous lemma for convenience, since we will use that specific instance often in the proofs.

The above lemma implies for instance that if $\mathbf{F}_{\mathsf{Q}}$ is trivial, then Q is unifiable since every algebra admits a homomorphism onto a trivial algebra. Hence, examples of unifiable algebras include lattices, groups, lattice-ordered abelian groups, residuated lattices. On the other hand, both bounded lattices and bounded residuated lattices (explored in Subsection 5.1 and 5.2 respectively) are unifiable if and only if they admit a homomorphism to the algebra with two elements 0 and 1 (over the appropriate signature), which is the smallest (0-generated) free algebra.

We observe in passing that if $\mathbf{A} \cong \mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}$ is a finitely presented unifiable algebra in Q, witnessed by a unifier $u : \mathbf{A} \longrightarrow \mathbf{P}$, then u can be split into a homomorphism onto its image $u(\mathbf{A})$, and an embedding from $u(\mathbf{A})$ to $\mathbf{P}$. By the Third Homomorphism Theorem there is a $\theta' \in \mathrm{Con}(\mathbf{F}_{\mathsf{Q}}(X))$ corresponding to the kernel of the onto homomorphism $u : \mathbf{A} \longrightarrow u(\mathbf{A})$, $\theta' \geqslant \theta$, such that $\mathbf{F}_{\mathsf{V}}(X)/\theta'$ embeds in $\mathbf{P}$; note that $\theta' \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{F}_{\mathsf{Q}}(X))$, since $\mathbf{P} \in \mathsf{Q}$. The diagram in Fig. 1 shows that indeed $\mathbf{F}_{\mathsf{V}}(X)/\theta'$ is exact.

Let us now introduce the usual notion of order among unifiers. Given two unifiers u_1, u_2 for $\mathbf{A}$ we say that u_1 is *less general then* u_2 (and we write $u_1 \preceq u_2$), if there is a homomorphism h that makes the following diagram commute.

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{u_2} & \mathbf{P}_2 \\ & \searrow^{u_1} & \downarrow h \\ & & \mathbf{P}_1 \end{array}$$

Clearly $\preceq$ is a preordering and so the equivalence classes of the associated equivalence relation (i.e. the unifiers that are *equally general*) form a poset $U_{\mathbf{A}}$; using the maximal sets of that poset it is possible to define a hierarchy of unification types (see [47]). In particular, the unification type is *unitary* if there is one maximal element, that is called *the most general unifier* or *mgu*, and is thought of as a *best solution* to the unification problem.

Definition 2.24. We say that a quasivariety Q has *projective unifiers* if every finitely presented unifiable algebra in Q is projective, and that it has *exact unifiers* if every finitely presented unifiable algebra in Q is exact.

If Q has projective unifiers, then (from the algebraic perspective) the identity map is a unifier, and it is also the most general unifier.

Next we have a lemma whose proof is straightforward (modulo Lemma 2.22).

Lemma 2.25. *Let Q be a quasivariety; then the following are equivalent:*

1. *Q has projective (exact) unifiers;*
2. *for any finitely presented $\mathbf{A} \in \mathsf{Q}$, $\mathbf{A}$ admits a homomorphism to $\mathbf{F}_{\mathsf{Q}}$ if and only if $\mathbf{A}$ is projective (exact).*

If Q is locally finite, then we have a necessary and sufficient condition.

Lemma 2.26. *Let* Q *be a locally finite quasivariety of finite type, then the following are equivalent:*

1. Q *has projective unifiers;*
2. *every finite unifiable algebra in* Q *is projective in the class of finite algebras in* Q*.*

Proof. (1) implies (2) is obvious. Assume (2), let $\mathbf{A}$ be unifiable and finite and let $\mathbf{B} \in \mathsf{Q}$ such that $f : \mathbf{B} \longrightarrow \mathbf{A}$ is a onto homomorphism. Let $a_1, \dots, a_n$ be the generators of $\mathbf{A}$ and let $b_1, \dots, b_n \in B$ with $f(b_i) = a_i$ for $i = 1 \dots n$; if $\mathbf{B}'$ is the subalgebra generated by $b_1, \dots, b_n$ then f restricted to $\mathbf{B}'$ is onto. Clearly $\mathbf{B}'$ is finite. Hence by hypothesis there exists a $g : \mathbf{A} \longrightarrow \mathbf{B}'$ such that fg is the identity on $\mathbf{A}$. This shows that $\mathbf{A}$ is projective in $\mathbf{B}$ and hence in Q. Thus (1) holds. □

Having exact unifiers is weaker than having projective unifiers:

Example 2.27. The variety D of distributive lattices is unifiable, indeed it has no constants and it is idempotent; hence its least free algebra is trivial. But D does not have projective unifiers: a distributive lattice is projective if and only if the meet of join irreducible elements is again join irreducible [11], so there are finite non projective distributive lattices. However every finitely presented (i.e. finite) distributive lattice is exact [22].

Example 2.28. A different example is the variety ST of Stone algebras; a Stone algebra is a pseudocomplemented bounded distributive lattice in the signature $(\wedge, \vee, *, 0, 1)$ such that $x^* \vee x^{**} \approx 1$ holds. A Stone algebra is unifiable if and only if it has a homomorphism to the two element Boolean algebra if and only if it is nontrivial. While there are nontrivial Stone algebras that are not projective, any nontrivial finitely presented Stone algebra is exact ([22, Lemma 17]). Hence ST has exact unifiers.

Moreover, there are examples of varieties with a most general unifier that do not have projective unifiers.

Example 2.29. From the results in [48], the variety SH of Stonean Heyting algebras (that is, Heyting algebras such that $\neg x \vee \neg\neg x \approx 1$ holds) is such that every unifiable algebra $\mathbf{A} \in \mathsf{SH}$ has a most general unifier. However, SH does not have projective unifiers. The algebra $\mathbf{F}_{\mathsf{SH}}(x, y, z)/\theta$, where θ is the congruence generated by the pair $(\neg x \to (y \vee z), 1)$, is unifiable but not projective. We observe that Ghilardi's argument relies heavily on some properties of Heyting algebras and uses Kripke models, making it difficult to generalize.

Trivial examples show that having projective or exact unifiers is not inherited in general by subvarieties (see for instance [35, Example 7.2]). The following lemma (that we extract from [35, Lemma 5.4]) gives a sufficient condition for having projective unifiers. We write a detailed proof for the reader's convenience.

Lemma 2.30 *([35]). Let* Q *be a quasivariety and let* Q' *be a subquasivariety of* Q *such that if* $\mathbf{B} = \mathbf{F}_{\mathsf{Q}'}(X)/\theta_{\mathsf{Q}'}(\Sigma)$ *is finitely presented and unifiable in* Q'*, then* $\mathbf{A} = \mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma)$ *is unifiable in* Q*. If* Q *has projective unifiers then* Q' *has projective unifiers.*

Proof. It is an easy exercise in general algebra to show that if $\Theta = \bigcap\{\theta \in \mathrm{Con}(\mathbf{F}_{\mathsf{Q}}(X)) : \mathbf{F}_{\mathsf{Q}}(X)/\theta \in \mathsf{Q}'\}$ then

$$\mathbf{F}_{\mathsf{Q}'}(X)/\theta_{\mathsf{Q}'}(\Sigma) \cong \mathbf{F}_{\mathsf{Q}}(X)/(\theta_{\mathsf{Q}}(\Sigma) \vee \Theta).$$

It follows that $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$ via the natural surjection

$$p : a/\theta_{\mathsf{Q}}(\Sigma) \longmapsto a/(\theta_{\mathsf{Q}}(\Sigma) \vee \Theta)$$

composed with the isomorphism. Moreover if $f : \mathbf{A} \longrightarrow \mathbf{C}$ is a homomorphism and $\mathbf{C} \in \mathsf{Q}'$, then $\ker(p) \leqslant \ker(f)$ and by the Second Homomorphism Theorem there is a $f' : \mathbf{B} \longrightarrow \mathbf{C}$ with $f'p = f$.

Now let $\mathbf{B} = \mathbf{F}_{\mathsf{Q}'}(X)/\theta_{\mathsf{Q}'}(\Sigma)$ be finitely presented and unifiable and let $\mathbf{A} = \mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma)$; then $\mathbf{A}$ is finitely presented and unifiable as well, so, since Q has projective unifiers, $\mathbf{A}$ is projective in Q. We now show that $\mathbf{B}$ is projective. Suppose there are algebras $\mathbf{C}, \mathbf{D} \in \mathsf{Q}' \subseteq \mathsf{Q}$ and homomorphisms $h : \mathbf{B} \to \mathbf{D}, g : \mathbf{C} \to \mathbf{D}$ with g surjective. Then, there is a homomorphism $hp : \mathbf{A} \to \mathbf{D}$, and since $\mathbf{A}$ is projective by the definition of projectivity there is a homomorphism $f : \mathbf{A} \to \mathbf{C}$ such that $gf = hp$. Factoring f as above, there is f' such that $f'p = f$. Therefore since $gf'p = gf = hp$ and p is surjective, we get that $gf' = h$ which means that $\mathbf{B}$ is projective in Q'. □

We will see later in Section 3.2 (Example 3.30) that Lemma 2.30 does not hold with "projective unifiers" replaced by "exact unifiers". We can build on the previous lemma and obtain the following.

Lemma 2.31. *Suppose that Q is a quasivariety such that $\mathbf{F}_{\mathsf{Q}} = \mathbf{F}_{\mathsf{Q}'}$ for all nontrivial $\mathsf{Q}' \subseteq \mathsf{Q}$. If Q has projective unifiers, then every subquasivariety Q' has projective unifiers.*

Proof. First we observe that the trivial quasivariety has projective unifiers. Now, let Q' be a nontrivial subquasivariety of Q, let $\mathbf{B} = \mathbf{F}_{\mathsf{Q}'}(X)/\theta_{\mathsf{Q}'}(\Sigma)$ be finitely presented and unifiable in Q' and let $\mathbf{A} = \mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma)$. Then $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$ and, since $\mathbf{B}$ is unifiable there is a homomorphism from $\mathbf{B}$ to $\mathbf{F}_{\mathsf{Q}'} = \mathbf{F}_{\mathsf{Q}}$. Hence $\mathbf{A}$ is unifiable as well; thus the hypothesis of Lemma 2.30 is satisfied, and so Q' has projective unifiers. Given that the trivial quasivariety clearly has projective unifiers, the thesis holds. □

We close this subsection with a corollary appearing also in [35] that is useful to some examples we will explore in what follows. We reproduce the easy proof for the reader's convenience.

Corollary 2.32. *Let Q be a quasivariety and let $\mathbf{V}(\mathsf{Q}) = \mathsf{V}$; if V has exact (projective) unifiers, then so does Q.*

Proof. First recall that Q and V have the same free algebras. Consider the two finitely presented algebras $\mathbf{A} = \mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{V}}(\Sigma)$ and $\mathbf{B} = \mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma)$; if $\mathbf{B}$ is unifiable then, as $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$ via the epimorphism p described in the proof of Lemma 2.30, $\mathbf{A}$ is unifiable as well hence it is exact. Therefore there is an embedding $u : \mathbf{A} \longrightarrow \mathbf{F}_{\mathsf{Q}}(\omega)$ by Lemma 2.12; then by (the proof of) Lemma 2.30 there is a $g : \mathbf{B} \longrightarrow \mathbf{F}_{\mathsf{Q}}(\omega)$ with $gp = u$. Since u is injective, so is p and hence $\mathbf{A}$ and $\mathbf{B}$ are isomorphic. This proves the thesis. □

2.4. Structural and universal completeness

We now introduce the main notions of interest of this work, that is, structural and universal completeness.

Definition 2.33. Let $\mathcal{L}$ be a logic over an algebraic language ρ, given by a substitution invariant finitary consequence relation $\vdash \subseteq \mathcal{P}(\mathbf{T}_\rho(\omega)) \times \mathbf{T}_\rho(\omega)$. A multiple-conclusion rule $\Sigma \Rightarrow \Gamma$ is *passive* in $\mathcal{L}$ if there is no substitution σ such that $\vdash \sigma(\alpha)$ for all $\alpha \in \Sigma$; a multiple-conclusion rule is *active* otherwise. Finally, $\Sigma \Rightarrow \Delta$ is *negative* if $\Delta = \emptyset$.

The following is a key notion for the rest of the work.

Definition 2.34. A multiple-conclusion rule $\Sigma \Rightarrow \Delta$ is *admissible* in a logic $\mathcal{L}$ if for every substitution σ:

$$\vdash \sigma(\alpha) \text{ for all } \alpha \in \Sigma \text{ implies } \vdash \sigma(\beta) \text{ for some } \beta \in \Delta;$$

in particular, a negative multiple-conclusion rule $\Sigma \Rightarrow \emptyset$ is admissible if and only if there is no substitution σ making all the premises in Σ a theorem of $\mathcal{L}$.

The notion of admissible rule was first introduced by Lorenzen in the 1950s in the context of intuitionistic logic [63]; there a rule is considered to be admissible in a logic if, when added to its consequence relation, it does not produce new theorems. We mention that the latter notion can be used interchangeably with the one of Definition 2.34 for single-conclusion rules, and both definitions are used in the literature. However, interestingly, the equivalence between these two definitions does not hold if one considers multiple-conclusion rules (and multiple-conclusion consequence relations), the interested reader can check [66] and [53] for a detailed discussion.

We also observe that every passive multiple-conclusion rule is necessarily admissible, and that admissible negative multiple-conclusion rules are necessarily passive. Importantly, an admissible clause is not necessarily derivable; a popular example is Harrop's rule for intuitionistic logic

$$\{\neg p \to (q \vee r)\} \;\Rightarrow\; \{(\neg p \to q) \vee (\neg p \to r)\}$$

which is admissible but not derivable.

Definition 2.35. A logic is said to be

- *universally complete* if every admissible multiple-conclusion rule is derivable;
- *structurally complete* if every admissible rule is derivable;
- *actively universally complete* if every active admissible multiple-conclusion rule is derivable;
- *actively structurally complete* if every active admissible rule is derivable[1];
- *passively universally complete* if every passive admissible multiple-conclusion rule is derivable;
- *passively structurally complete* if every passive admissible rule is derivable;
- *non negatively universally complete* if every non negative admissible multiple-conclusion rule is derivable.

Modulo algebraizability, one obtains the corresponding notions for a quasivariety. In particular, we can express admissibility and derivability of multiple-conclusion rules in $\mathcal{L}_{\mathsf{Q}}$ using the (quasi)equational logic of Q; this is because the Blok-Pigozzi Galois connection transforms (sets of) formulas in $\mathcal{L}_{\mathsf{Q}}$ into (sets of) equations in Q in a uniform way. The obtained notions make sense for quasivarieties that do not necessarily correspond to a logic.

Definition 2.36. Let Q be a quasivariety. A clause $\Sigma \Rightarrow \Delta$ is *admissible* in Q if for every substitution σ:

$$\vDash_{\mathsf{Q}} \sigma(\alpha) \text{ for all } \alpha \in \Sigma \;\text{ implies }\; \vDash_{\mathsf{Q}} \sigma(\beta) \text{ for some } \beta \in \Delta;$$

in particular, a negative clause $\Sigma \Rightarrow \emptyset$ is admissible if and only if there is no substitution σ making all the equations in Σ valid in Q. A clause is *passive* if there is no substitution unifying its premises, *active* otherwise. Q is *(actively/passively/non-negatively) universally/structurally complete* if every (active/passive/non-negative) admissible clause/quasiequation is valid in Q.

If P is one of those properties, then we say that a logic (or a quasivariety) is *hereditarily* P if the logic and all its extensions (or the quasivariety and all its subquasivarieties) have the property P. Some of these properties are well-known to be distinct: for instance classical logic is non-negatively universally complete

[1] Logics with this property have been more often called *almost structurally complete* but here we follow A. Citkin's advice (see [35, footnote 2, page 8]).

but not universally complete, while intuitionistic logic is not structurally complete (thanks to Harrop's example) but it is passively structurally complete (as reported by Wroński in 2005, see [29]). The following is a consequence of algebraizability.

Theorem 2.37. *Let* Q *be a quasivariety of logic,* Σ, Δ *sets of equations in the language of* Q *and* Σ', Δ' *the corresponding sets of formulas in* $\mathcal{L}_{\mathsf{Q}}$*. Then:*

1. $\Sigma' \Rightarrow \Delta'$ *is admissible in* $\mathcal{L}_{\mathsf{Q}}$ *if and only if* $\Sigma \Rightarrow \Delta$ *is admissible in* Q*;*
2. $\Sigma' \Rightarrow \Delta'$ *is derivable in* $\mathcal{L}_{\mathsf{Q}}$ *if and only if* $\mathsf{Q} \vDash \Sigma \Rightarrow \Delta$*.*

Moreover, by Corollary 2.20 we get the following.

Proposition 2.38. *Let* Q *be a quasivariety of logic,* Σ, Δ *sets of equations in the language of* Q *and* Σ', Δ' *the corresponding sets of formulas in* $\mathcal{L}_{\mathsf{Q}}$*. Then:*

1. $\Sigma' \Rightarrow \Delta'$ *is active in* $\mathcal{L}_{\mathsf{Q}}$ *if and only if* $\mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma)$ *is unifiable in* Q*;*
2. $\Sigma' \Rightarrow \Delta'$ *is passive in* $\mathcal{L}_{\mathsf{Q}}$ *if and only if* $\mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma)$ *is not unifiable in* Q*.*

The next lemma (also derivable from [22, Theorem 2]) characterizes admissibility of clauses. Given a quasivariety Q, and $\Sigma \Rightarrow \Delta$ a clause in the language of Q, let

$$\mathsf{Q}_{\Sigma\Rightarrow\Delta} = \{\mathbf{A} \in \mathsf{Q} : \mathbf{A} \vDash \Sigma \Rightarrow \Delta\}. \tag{2.1}$$

Lemma 2.39. *Let* Q *be any quasivariety, and let* $\Sigma \Rightarrow \Delta$ *be a clause in the language of* Q*. Then the following are equivalent:*

1. $\Sigma \Rightarrow \Delta$ *is admissible in* Q*;*
2. $\mathbf{F}_{\mathsf{Q}}(\omega) \vDash \Sigma \Rightarrow \Delta$*;*
3. $\mathbf{H}(\mathsf{Q}) = \mathbf{H}(\mathsf{Q}_{\Sigma\Rightarrow\Delta})$*.*

Proof. The equivalence between (1) and (2) follows directly from the definition of admissibility. Assume now $\mathbf{F}_{\mathsf{Q}}(\omega) \vDash \Sigma \Rightarrow \Delta$, then $\mathbf{F}_{\mathsf{Q}}(\omega) \in \mathsf{Q}_{\Sigma\Rightarrow\Delta}$. Clearly $\mathbf{HSP}_u(\mathbf{F}_{\mathsf{Q}}(\omega)) \subseteq \mathbf{H}(\mathsf{Q}_{\Sigma\Rightarrow\Delta}) \subseteq \mathbf{H}(\mathsf{Q})$. Now every algebra is embeddable in an ultraproduct of its finitely generated subalgebras and every finitely generated algebra is a homomorphic image of $\mathbf{F}_{\mathsf{Q}}(\omega)$. Therefore if $\mathbf{A} \in \mathsf{Q}$, then $\mathbf{A} \in \mathbf{SP}_u\mathbf{H}(\mathbf{F}_{\mathsf{Q}}(\omega)) \subseteq \mathbf{HSP}_u(\mathbf{F}_{\mathsf{Q}}(\omega))$. So $\mathbf{HSP}_u(\mathbf{F}_{\mathsf{Q}}(\omega)) = \mathbf{H}(\mathsf{Q})$ and thus (3) holds.

Conversely assume (3). Since $\mathbf{F}_{\mathsf{Q}}(\omega) \in \mathbf{H}(\mathsf{Q}) = \mathbf{H}(\mathsf{Q}_{\Sigma\Rightarrow\Delta})$, there is $\mathbf{A} \in \mathsf{Q}_{\Sigma\Rightarrow\Delta}$ such that $\mathbf{F}_{\mathsf{Q}}(\omega) \in \mathbf{H}(\mathbf{A})$. Since $\mathbf{F}_{\mathsf{Q}}(\omega)$ is projective in Q, it follows that $\mathbf{F}_{\mathsf{Q}}(\omega) \in \mathbf{S}(\mathbf{A}) \subseteq \mathbf{S}(\mathsf{Q}_{\Sigma\Rightarrow\Delta}) \subseteq \mathsf{Q}_{\Sigma\Rightarrow\Delta}$. Therefore, $\mathbf{F}_{\mathsf{Q}}(\omega) \vDash \Sigma \Rightarrow \Delta$ and (2) holds. □

To conclude the preliminaries, we present the following lemma which will be particularly useful in our proofs.

Lemma 2.40. *Let* Q *be a quasivariety, and* Σ, Δ *be finite sets of equations over variables in a finite set* X*. The following are equivalent:*

1. $\mathsf{Q} \vDash \Sigma \Rightarrow \Delta$*;*
2. $\mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma) \vDash \Sigma \Rightarrow \Delta$*;*
3. *there is* $p \approx q \in \Delta$ *such that* $p/\theta_{\mathsf{Q}}(\Sigma) = q/\theta_{\mathsf{Q}}(\Sigma)$ *in* $\mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma)$*.*

Proof. It is clear that (1) implies (2) and (2) implies (3). We now show that (3) implies (1).

Let $\mathbf{A} \in \mathsf{Q}$. If there is no assignment of the variables in X to $\mathbf{A}$ that models Σ, then $\mathbf{A} \models \Sigma \Rightarrow \Delta$. Otherwise, suppose there is an assignment h such that $\mathbf{A}, h \models \Sigma$. Then, by Remark 2.18 there is a homomorphism $u_h : \mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma) \to \mathbf{A}$ such that $h = u_h\pi_\Sigma$, where π_Σ is the natural epimorphism from $\mathbf{F}_{\mathsf{Q}}(X)$ to $\mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma)$. Now by (3) there is at least an identity $p \approx q \in \Delta$ such that $(p, q) \in \ker(\pi_\Sigma)$. Since $h = u_h\pi_\Sigma$, $(p, q) \in \ker(h)$, which means that $\mathbf{A}, h \models p \approx q$ and therefore $\mathbf{A} \models \Sigma \Rightarrow \Delta$. Since $\mathbf{A}$ is an arbitrary algebra of Q this shows that $\mathsf{Q} \models \Sigma \Rightarrow \Delta$. □

3. Universal completeness

In this section we study from the algebraic perspective the notion of universal completeness and its variations: active, non-negative, passive universal completeness, together with their hereditary versions. That is, we shall see which algebraic properties correspond to the notions coming from the logical perspective (detailed in the preliminaries Subsection 2.4). For each notion, we will present a characterization theorem and some examples. While the characterizations of active and passive universal completeness (to the best of our knowledge) are fully original, we build on existing ones for the other notions, presenting some new results and a coherent presentation in our framework.

3.1. Universal quasivarieties

We start with universal completeness. The following expands [22, Proposition 6] with point (4).

Theorem 3.1. *For any quasivariety* Q *the following are equivalent:*

1. Q *is universally complete;*
2. *for every universal class* $\mathsf{U} \subseteq \mathsf{Q}$, $\mathbf{H}(\mathsf{U}) = \mathbf{H}(\mathsf{Q})$ *implies* $\mathsf{U} = \mathsf{Q}$.
3. $\mathsf{Q} = \mathbf{ISP}_u(\mathbf{F}_{\mathsf{Q}}(\omega))$*;*
4. *every finitely presented algebra in* Q *is in* $\mathbf{ISP}_u(\mathbf{F}_{\mathsf{Q}}(\omega))$.

Proof. (2) implies (1) via Lemma 2.39. We show that (1) implies (2). Let $\mathsf{U} \subseteq \mathsf{Q}$ be a universal class such that $\mathbf{H}(\mathsf{U}) = \mathbf{H}(\mathsf{Q})$ and suppose that $\mathsf{U} \models \Sigma \Rightarrow \Delta$; then

$$\mathbf{H}(\mathsf{Q}) = \mathbf{H}(\mathsf{U}) \subseteq \mathbf{H}(\mathsf{Q}_{\Sigma\Rightarrow\Delta}) \subseteq \mathbf{H}(\mathsf{Q}).$$

So $\mathbf{H}(\mathsf{Q}_{\Sigma\Rightarrow\Delta}) = \mathbf{H}(\mathsf{Q})$ and by Lemma 2.39 $\Sigma \Rightarrow \Delta$ is admissible in Q. By (1), $\mathsf{Q} \models \Sigma \Rightarrow \Delta$; therefore U and Q are two universal classes in which exactly the same clauses are valid, thus they are equal. Hence (2) holds, and thus (2) and (1) are equivalent.

(1) implies (3) follows by Lemma 2.39. Moreover, (3) clearly implies (4). We now show that (4) implies (1), which completes the proof. Consider a clause $\Sigma \Rightarrow \Delta$ that is admissible in Q, or equivalently (by Lemma 2.39), such that $\mathbf{F}_{\mathsf{Q}}(\omega) \models \Sigma \Rightarrow \Delta$. The finitely presented algebra $\mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma) \in \mathbf{ISP}_u(\mathbf{F}_{\mathsf{Q}}(\omega))$ by (4), and thus $\mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma) \models \Sigma \Rightarrow \Delta$. By Lemma 2.40, $\mathsf{Q} \models \Sigma \Rightarrow \Delta$ and thus Q is universally complete. □

Remark 3.2. Note that condition (2) of Theorem 3.1 does not hold for any quasivariety Q in which the trivial algebra is not embedded in any nontrivial algebra. Thus any such quasivariety is not universally complete.[2]

By algebraizability we get at once:

[2] We thank an anonymous reviewer for this remark.

Corollary 3.3. *For a quasivariety of logic* Q *the following are equivalent:*

1. Q *is universally complete;*
2. $\mathcal{L}_{\mathsf{Q}}$ *is universally complete.*

The following theorem and lemma show a sufficient and a necessary condition respectively for a quasivariety to be universally complete.

Theorem 3.4. *If every finitely presented algebra in* Q *is exact then* Q *is universally complete.*

Proof. If every finitely presented algebra in Q is exact, it is in $\mathbf{IS}(\mathbf{F}_{\mathsf{Q}}(\omega))$, and thus also in $\mathbf{ISP}_u(\mathbf{F}_{\mathsf{Q}}(\omega))$. The claim then follows from Theorem 3.1. □

Theorem 3.5. *If* Q *is universally complete, then* Q *is unifiable.*

Proof. Suppose by counterpositive that there is a finite set of identities Σ that is not unifiable in Q. Then $\Sigma \Rightarrow \emptyset$ is (passively) admissible but not derivable; indeed it does not hold in the trivial algebra. This implies that Q is not universally complete, and the claim is proved. □

Since projectivity implies exactness, we observe the following immediate consequence of Theorem 3.4.

Corollary 3.6. *If every finitely presented algebra in* Q *is projective then* Q *is universally complete.*

For locally finite varieties there is a stronger result, observed in [22].

Lemma 3.7. *[22] Let* Q *be a locally finite quasivariety; then* $\mathbf{A} \in \mathbf{ISP}_u(\mathbf{F}_{\mathsf{Q}}(\omega))$ *if and only if every finite subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$ *is in* $\mathbf{IS}(\mathbf{F}_{\mathsf{Q}}(\omega))$.

Theorem 3.8 *([22]). Let* Q *be a locally finite variety of finite type. Then* Q *is universally complete if and only if* Q *is unifiable and has exact unifiers.*

Proof. Suppose that Q is universally complete; then, by Theorem 3.5, Q is unifiable. Since it is universally complete, every finite algebra in Q is in $\mathbf{ISP}_u(\mathbf{F}_{\mathsf{Q}}(\omega))$, hence in $\mathbf{IS}(\mathbf{F}_{\mathsf{Q}}(\omega))$ (by Lemma 3.7). Thus every finite unifiable algebra in Q is exact and Q has exact unifiers. The converse claim follows from Theorem 3.4. □

Remark 3.9. We observe that Theorem 3.5 limits greatly the examples of universally complete quasivarieties. In particular, in quasivarieties with finite type the trivial algebra is finitely presented, and thus if Q is universally complete, it must be unifiable. This means that a quasivariety with more than one constant in its finite type cannot be universally complete if there are nontrivial models where the constants are distinct; similarly if there is only one constant, then it must generate the trivial algebra in nontrivial models, or equivalently, in $\mathbf{F}_{\mathsf{Q}}$. If there are no constants, then $\mathbf{F}_{\mathsf{Q}} = \mathbf{F}_{\mathsf{Q}}(x)$ and, in order to be able to embed the trivial algebra, there has to be an idempotent term.

Let us now discuss some different examples of universally complete (quasi)varieties.

Example 3.10. Let us consider *lattice-ordered abelian groups* (or abelian ℓ-groups for short). These are algebras $\mathbf{G} = (G, \wedge, \vee, \cdot, ^{-1}, 1)$ where $(G, \cdot, ^{-1}, 1)$ is an abelian group, $(G, \wedge, \vee)$ is a lattice, and the group operation distributes over the lattice operations. Every finitely presented abelian ℓ-groups is projective [15]; thus, the variety of abelian ℓ-groups is universally complete by Corollary 3.6.

The same holds for the variety of *negative cones* of abelian ℓ-groups. Given an ℓ-group $\mathbf{G}$, the set of elements $G^- = \{x \in G : x \leqslant 1\}$ can be seen as a *residuated lattice* (see Section 5.2) $\mathbf{G}^- = (G^-, \cdot, \to, \wedge, \vee, 1)$ where $(\cdot, \wedge, \vee, 1)$ are inherited by the group and $x \to y = x^{-1} \cdot y \wedge 1$. The algebraic category of negative cones of abelian ℓ-groups is equivalent to the one of abelian ℓ-groups [46], thus every finitely presented algebra is projective and the variety of negative cones of ℓ-groups LG^- is universally complete. Observe that in all these cases the unique constant 1 is absorbing w.r.t. any basic operation, and it generates the trivial algebra.

Example 3.11. *Hoops* are a particular variety of residuated monoids related to logic which were defined in an unpublished manuscript by Büchi and Owens, inspired by the work of Bosbach on partially ordered monoids (see [17] for details on the theory of hoops). Hoops have a constant which is absorbing w.r.t. any basic operation; hence the least free algebra is trivial in any variety of hoops and any variety of hoops is unifiable. In [8] it was shown that every finite hoop is projective in the class of finite hoops which via Lemma 2.26 entails that every locally finite variety of hoops has projective unifiers. Since any locally finite quasivariety is contained in a locally finite variety, every locally finite quasivariety of hoops is universally complete. The same holds in the variety of $\to$-subreducts of hoops, usually denoted by HBCK; again locally finite varieties of HBCK-algebras have projective unifiers [8] and hence they are universally complete. For a non-locally finite example, we say that a hoop is *cancellative* if the underlying monoid is cancellative; cancellative hoops form a variety C that is categorically equivalent to the one of abelian ℓ-groups [17]. Hence C is a non locally finite variety of hoops which is universally complete.

The classes of algebras in the above examples all have projective unifiers. However:

Example 3.12. In lattices there are no constants but any variety of lattices is idempotent; hence the least free algebra is trivial and every lattice is unifiable. Every finite distributive lattice is exact [22] and distributive lattices are locally finite, so distributive lattices are universally complete by Theorem 3.8. Moreover, as we have already observed in Example 2.27, distributive lattices do not have projective unifiers.

We now consider the hereditary version of universal completeness.

Definition 3.13. A quasivariety Q is *primitive universal* if all its subquasivarieties are universally complete.

All the above examples of universally complete varieties are primitive universal and this is not entirely coincidental. Distributive lattices are trivially primitive universal, since they do not have any trivial subquasivariety. For all the other examples, we have a general result.

Theorem 3.14. *Let Q be a quasivariety with projective unifiers and such that $\mathbf{F}_\mathsf{Q}$ is trivial; then Q is primitive universal.*

Proof. Observe that for any subquasivariety $\mathsf{Q}' \subseteq \mathsf{Q}$, $\mathbf{F}_{\mathsf{Q}'}$ is trivial as well. Hence every algebra in Q is unifiable in any subvariety to which it belongs. Let $\mathbf{B} = \mathbf{F}_{\mathsf{Q}'}(X)/\theta_{\mathsf{Q}'}(\Sigma)$ be finitely presented in Q'; then $\mathbf{A} = \mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma)$ is finitely presented in Q and thus it is projective in Q. But then Lemma 2.30 applies and $\mathbf{B}$ is projective; thus Q' has projective unifiers and thus it is universally complete by Corollary 3.6. □

Is the same conclusion true if we replace "projective unifiers" with "unifiable, locally finite with exact unifiers"? We do not know, but we know that we cannot use an improved version of Lemma 2.30 since it cannot be improved to account for exact unifiers (see Example 3.30).

3.2. Non-negative and active universal quasivarieties

The situation in which universal completeness fails due *only* to the trivial algebras has been first investigated in [22]. Given Q a quasivariety, let Q^+ be the class of nontrivial algebras in Q; the following expands [22, Proposition 8] by (4).

Theorem 3.15. *For a quasivariety Q the following are equivalent:*

1. *Q is non-negatively universally complete;*
2. *every admissible clause is valid in Q^+;*
3. *every nontrivial algebra is in $\mathbf{ISP}_u(\mathbf{F}_\mathsf{Q}(\omega))$;*
4. *every nontrivial finitely presented algebra is in $\mathbf{ISP}_u(\mathbf{F}_\mathsf{Q}(\omega))$.*

Proof. The equivalence of the first three points is in [22, Proposition 8], and (3) clearly implies (4). Assume now that (4) holds, we show (1). Let $\Sigma \Rightarrow \Delta$ be a non-negative admissible clause with variables in a finite set X, we show that $\mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma) \models \Sigma \Rightarrow \Delta$. If $\mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma)$ is trivial, then it models $\Sigma \Rightarrow \Delta$ (given that Δ is not $\varnothing$). Suppose now that $\mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma)$ is nontrivial, then it is in $\mathbf{ISP}_u(\mathbf{F}_\mathsf{Q}(\omega))$ by hypothesis and then it models $\Sigma \Rightarrow \Delta$ since the latter is admissible, and thus $\mathbf{F}_\mathsf{Q}(\omega) \models \Sigma \Rightarrow \Delta$ by Lemma 2.39. By Lemma 2.40, Q models $\Sigma \Rightarrow \Delta$ and (1) holds. □

Moreover we observe again:

Theorem 3.16. *If Q is a quasivariety of logic the following are equivalent:*

1. *Q is non-negatively universally complete;*
2. *$\mathcal{L}_\mathsf{Q}$ is non-negatively universally complete.*

We can also obtain an analogue of Theorem 3.4.

Theorem 3.17. *If every nontrivial finitely presented algebra in Q is exact (or projective), then Q is non-negatively universally complete.*

Proof. If every nontrivial finitely presented algebra is exact (or projective), then it is in $\mathbf{IS}(\mathbf{F}_\mathsf{Q}(\omega))$, and therefore in $\mathbf{ISP}_u(\mathbf{F}_\mathsf{Q}(\omega))$. The claim then follows from Theorem 3.15. □

Analogously to the case of universal completeness, we get a stronger result for locally finite quasivarieties.

Theorem 3.18. *Let Q be a locally finite quasivariety. Then Q is non-negatively universally complete if and only if every nontrivial finitely presented algebra is exact.*

Proof. Suppose that Q is locally finite and there is a finite nontrivial algebra $\mathbf{A} \in \mathsf{Q}$ that is not exact. Then $\mathbf{A} \notin \mathbf{IS}(\mathbf{F}_\mathsf{Q}(\omega))$ and thus, by Lemma 3.7, $\mathbf{A} \notin \mathbf{ISP}_u(\mathbf{F}_\mathsf{Q}(\omega))$. Therefore Q cannot be non-negatively universally complete by Theorem 3.15. The other direction follows from Theorem 3.17. □

Example 3.19. Boolean algebras are an example of a non-negatively universally complete variety that is not universally complete. It is easily seen that every nontrivial finite Boolean algebra is exact (indeed, projective), which shows that Boolean algebras are non-negatively universally complete by Theorem 3.18. However, there are negative admissible clauses: e.g., the ones with premises given by the presentation of the trivial algebra, which is finitely presented but not unifiable. Thus Boolean algebras are not universally complete.

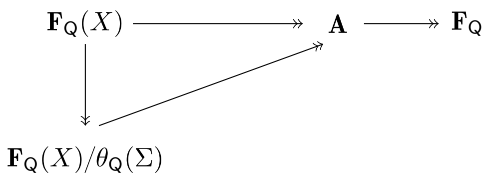
Example 3.20. Stone algebras are a different example; in [22] the authors proved, using the duality between Stone algebras and particular Priestley spaces, that every finite nontrivial Stone algebra is exact; hence Stone algebras are non-negatively universally complete.

We now move on to describe active universal completeness from the algebraic perspective.

Theorem 3.21. *Let* Q *be a quasivariety. The following are equivalent:*

1. Q *is actively universally complete;*
2. *every unifiable algebra in* Q *is in* $\mathbf{ISP}_u(\mathbf{F}_\mathsf{Q}(\omega))$*;*
3. *every finitely presented and unifiable algebra in* Q *is in* $\mathbf{ISP}_u(\mathbf{F}_\mathsf{Q}(\omega))$*;*
4. *every clause admissible in* Q *is satisfied by all finitely presented unifiable algebras in* Q*;*
5. *for every* $\mathbf{A} \in \mathsf{Q}$*,* $\mathbf{A} \times \mathbf{F}_\mathsf{Q} \in \mathbf{ISP}_u(\mathbf{F}_\mathsf{Q}(\omega))$*.*

Proof. We start by showing that (1) implies (2). Assume (1), and let $\Sigma \Rightarrow \Delta$ be such that $\mathbf{F}_\mathsf{Q}(\omega) \models \Sigma \Rightarrow \Delta$; equivalently, by Lemma 2.39, $\Sigma \Rightarrow \Delta$ is an admissible clause in Q. If Σ is unifiable, by hypothesis $\Sigma \Rightarrow \Delta$ is valid in Q. Suppose now that Σ has variables in a finite set X and it is not unifiable, that is, via Corollary 2.20 there is no homomorphism from $\mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma)$ to $\mathbf{F}_\mathsf{Q}$. Let $\mathbf{A}$ be a unifiable algebra in Q; we argue that there is no assignment of the variables in Σ that validates Σ in $\mathbf{A}$. Indeed otherwise the following diagram would commute and $\mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma)$ would be unifiable, yielding a contradiction.



Therefore, $\Sigma \Rightarrow \Delta$ is vacuously satisfied in $\mathbf{A}$, which is any unifiable algebra in Q, thus (2) holds. Now, clearly (2) implies (3), and (3) and (4) are equivalent by the definitions.

Let us show that (4) implies (1). Let $\Sigma \Rightarrow \Delta$ be an active admissible clause in Q with variables in a finite set X; we want to show that it is also valid in Q. Since by hypothesis $\Sigma \Rightarrow \Delta$ is active admissible, Σ is unifiable, and therefore so is $\mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma)$ by Corollary 2.20. Then by (4), $\mathbf{F}_\mathsf{Q}(X)/\theta_\mathsf{Q}(\Sigma) \models \Sigma \Rightarrow \Delta$, which implies that $\mathsf{Q} \models \Sigma \Rightarrow \Delta$ by Lemma 2.40. Therefore the first four points are equivalent.

Finally, we show that (1) implies (5) and (5) implies (2), which completes the proof. We start with (1) $\Rightarrow$ (5). Let $\mathbf{A} \in \mathsf{Q}$, and consider a clause $\Sigma \Rightarrow \Delta$ valid in $\mathbf{F}_\mathsf{Q}(\omega)$. We show that $\mathbf{A} \times \mathbf{F}_\mathsf{Q} \models \Sigma \Rightarrow \Delta$. Now, if $\mathsf{Q} \models \Sigma \Rightarrow \Delta$, in particular $\mathbf{A} \times \mathbf{F}_\mathsf{Q} \models \Sigma \Rightarrow \Delta$. Suppose that $\mathsf{Q} \not\models \Sigma \Rightarrow \Delta$. Since Q is actively universally complete, $\Sigma \Rightarrow \Delta$ must be a passive rule, thus Σ is not unifiable. Equivalently, there is no assignment h of the variables in Σ such that $\mathbf{F}_\mathsf{Q}, h \models \Sigma$. Thus, there is also no assignment h' of the variables in Σ such that $\mathbf{A} \times \mathbf{F}_\mathsf{Q}, h' \models \Sigma$, thus $\mathbf{A} \times \mathbf{F}_\mathsf{Q} \models \Sigma \Rightarrow \Delta$.

It is left to prove (5) $\Rightarrow$ (2). Let $\mathbf{A}$ be a unifiable algebra in Q, then there is a homomorphism $h : \mathbf{A} \to \mathbf{F}_\mathsf{Q}$ (Lemma 2.22). Consider the map $h' : \mathbf{A} \to \mathbf{A} \times \mathbf{F}_\mathsf{Q}$ be defined as $h'(a) = (a, h(a))$. Clearly, h' is an embedding of $\mathbf{A}$ into $\mathbf{A} \times \mathbf{F}_\mathsf{Q} \in \mathbf{ISP}_u(\mathbf{F}_\mathsf{Q}(\omega))$ (by (5)). Thus also $\mathbf{A} \in \mathbf{ISP}_u(\mathbf{F}_\mathsf{Q}(\omega))$, which completes the proof. □

We observe that the previous characterization extends to clauses some of the results in [35] about active structural completeness. We also get the usual result.

Theorem 3.22. *For a quasivariety of logic* Q *the following are equivalent:*

1. Q *is actively universally complete;*
2. $\mathcal{L}_{\mathsf{Q}}$ *is actively universally complete.*

Moreover, we have the following lemma whose proof is the same as the one of Theorems 3.4 and 3.8.

Theorem 3.23. *If* Q *has exact (or projective) unifiers, then* Q *is actively universally complete. If* Q *is also locally finite then it is actively universally complete if and only if it has exact unifiers.*

Example 3.24. A *discriminator* on a set A is a ternary operation t on A defined by

$$t(a,b,c) = \begin{cases} a, & \text{if } a \neq b; \\ c, & \text{if } a = b. \end{cases}$$

A variety V is a *discriminator variety* [73] if there is a ternary term that is the discriminator on all the subdirectly irreducible members of V. Discriminator varieties have many strong properties: for instance they are congruence permutable and congruence distributive.

In [20, Theorem 3.1] it has been essentially shown that discriminator varieties have projective unifiers, and therefore they are all actively universally complete by Theorem 3.23.

Example 3.25. Let us now consider some examples within the algebraic semantics of many-valued logics; in [8] it has been shown that in any locally finite variety of bounded hoops or BL-algebras (the equivalent algebraic semantics of Hájek Basic Logic [51]), the finite unifiable algebras are exactly the finite projective algebras. It follows that any of such varieties has projective unifiers and hence it is actively universally complete. This holds also for any locally finite quasivariety of bounded hoops or BL-algebras, or their reducts, i.e., bounded HBCK-algebras.

In contrast with the case of (unbounded) hoops, not all of them are non-negatively universally complete, as we will now discuss. Let us call *chain* a totally ordered algebra. Every finite BL-chain is an ordinal sum of finite Wajsberg hoops, the first of which is an MV-algebra [4]. No finite MV-chain different from the 2-element Boolean algebra $\mathbf{2}$ is unifiable (they are all simple and the least free algebra is $\mathbf{2}$), and thus not exact. It follows by basic facts about ordinal sums that if a locally finite quasivariety Q of BL-algebras contains a chain whose first component is different from $\mathbf{2}$, Q is not non-negatively universally complete. The same holds, mutatis mutandis, for bounded hoops and bounded HBCK-algebras. In Section 5.2 we shall see a different class of (discriminator) varieties coming from many-valued logics that are actively universally complete.

Definition 3.26. We call a quasivariety Q *active primitive universal* if every subquasivariety of Q is actively universally complete.

It is evident from the characterization theorem of actively universally complete quasivarieties that a quasivariety of logic Q is active primitive universal if and only if $\mathcal{L}_{\mathsf{Q}}$ is hereditarily actively universally complete. We have the following fact:

Theorem 3.27. *Suppose that* Q *is a quasivariety such that* $\mathbf{F}_{\mathsf{Q}} = \mathbf{F}_{\mathsf{Q}'}$ *for all nontrivial* $\mathsf{Q}' \subseteq \mathsf{Q}$. *If* Q *has projective unifiers then it is active primitive universal.*

Proof. The proof follows from Theorem 3.23 and Lemma 2.31. □

All varieties in Example 3.25 satisfy the hypotheses of Theorem 3.27 (as the reader can easily check). For discriminator varieties all the examples of lattice-based varieties in Section 5.2 of this paper have the same property (but see also [20] or [31] for more examples); hence they are all active primitive universal.

Now, a variety is *q-minimal* if it does not have any proper nontrivial subquasivariety; so a q-minimal variety is necessarily equationally complete. We have this result by Bergman and McKenzie:

Theorem 3.28. *[14] A locally finite equationally complete variety is q-minimal if and only if it has exactly one subdirectly irreducible algebra that is embeddable in any nontrivial member of the variety. Moreover, this is always the case if the variety is congruence modular.*

It follows immediately that every actively universally complete q-minimal variety is active primitive universal.

Example 3.29. Discriminator varieties are actively universally complete as seen in Example 3.24. Now, given a finitely generated discriminator variety V, it is generated by a finite algebra $\mathbf{A}$ having a discriminator term on it, also called a *quasi-primal* algebra. By [77] V is equationally complete and, since it is congruence modular, it is q-minimal; hence V is active primitive universal.

Finally, we observe that Lemma 2.31 cannot be improved to "having exact unifiers" and the counterexample is given by *De Morgan lattices*; we will see below that they form an actively universally complete variety that is not active primitive universal.

Example 3.30. A De Morgan lattice is a distributive lattice with a unary operation $\neg$ which is involutive and satisfies the De Morgan Laws. It is well-known that the variety DM of De Morgan lattices is locally finite and has exactly two proper non trivial subvarieties, i.e. the variety BLa of Boolean lattices (axiomatized by $x \leqslant y \vee \neg y$) and the variety KL of Kleene lattices (axiomatized by $x \wedge \neg x \leqslant y \vee \neg y$). It is easily seen that all these nontrivial varieties have the same one-generated free algebra whose universe is $\{x, \neg x, x \vee \neg x, x \wedge \neg x\}$. It follows that all the nontrivial subquasivarieties of De Morgan lattices have the same least free algebra and DM satisfies the hypotheses of Theorem 3.27. Admissibility in De Morgan lattices has been investigated in [67] and [22]. Now for a finite algebra $\mathbf{A} \in \mathsf{DM}$ the following are equivalent:

1. $\mathbf{A}$ is unifiable;
2. the clause $\{x \approx \neg x\} \Rightarrow \varnothing$ is valid in $\mathbf{A}$;
3. $\mathbf{A} \in \mathbf{IS}(\mathbf{F}_{\mathsf{DM}}(\omega))$.

The equivalence of (2) and (3) has been proved in [22, Lemma 28], while (3) implies (1) trivially. If we assume that (2) does not hold for $\mathbf{A}$, then there is an $a \in A$ with $\neg a = a$; so if $f : \mathbf{A} \longrightarrow \mathbf{F}_{\mathsf{DM}}(x)$ is a homomorphism and $f(a) = \varphi$, then $\varphi = \neg\varphi$. But there is no element in $\mathbf{F}_{\mathsf{DM}}(x)$ with that property, so $\mathbf{A}$ cannot be unifiable. This concludes the proof of the equivalence of the three statements.

Therefore DM has exact unifiers and thus it is actively universally complete by Theorem 3.23. Now consider the subvariety of DM of Kleene lattices. In [22] it is shown that the clause

$$\Phi := \{x \leqslant \neg x, x \wedge \neg y \leqslant \neg x \vee y\} \Rightarrow \neg y \leqslant y$$

is admissible in KL. It is also active, as the reader can easily check that the substitution $x \longmapsto z \wedge \neg z$, $y \longmapsto \neg z$ unifies the premises of Φ. However it fails in the three element Kleene lattice $\mathbf{K}_3$ in Fig. 2, with the assignment $x = a$, $y = \neg a$; hence KL is not actively universally complete. So DM is a variety that is actively universally complete but not active primitive universal.

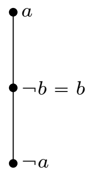


Fig. 2. The lattice $\mathbf{K}_3$.

Note that in KL there must be a finite unifiable algebra that is not exact (since KL cannot have exact unifiers). Now a finite Kleene lattice $\mathbf{A}$ is exact if and only if both $\{x \approx \neg x\} \Rightarrow \varnothing$ and Φ are valid in $\mathbf{A}$ [22, Lemma 38]. Let $\mathbf{A} = \mathbf{K}_3 \times \mathbf{2}$; the reader can easily check that $\mathbf{A}$ is unifiable in KL (since it satisfies $\{x \approx \neg x\} \Rightarrow \varnothing$ and hence it is unifiable in DM) but does not satisfy Φ. This shows (as promised) that Lemma 2.30 cannot be improved.

3.3. Passive universal quasivarieties

We will now see that passive universal completeness in a quasivariety corresponds to an algebraic notion we have already introduced: unifiability. Moreover, we shall see that it corresponds to the apparently weaker notion of negative universal completeness, that is, every (passive) admissible negative clause is derivable. We recall that a quasivariety Q is unifiable if every finitely presented algebra in Q is unifiable.

Theorem 3.31. *For every quasivariety* Q *the following are equivalent:*

1. Q *is passively universally complete;*
2. Q *is negatively universally complete;*
3. Q *is unifiable.*

Proof. Assume (1) and let $\Sigma \Rightarrow \varnothing$ be a negative admissible clause; then it is necessarily passive, since there is no substitution that unifies $\varnothing$. Thus, by (1), $\Sigma \Rightarrow \varnothing$ is valid in Q.

We prove that (2) implies (3) by contrapositive. Suppose that Q is not unifiable, that is, there exists a finite set of identities Σ that is not unifiable. Then the negative clause $\Sigma \Rightarrow \varnothing$ is (passively) admissible, but it is not derivable (in particular, it fails in the trivial algebra).

Finally, if (3) holds, then (1) trivially holds, since if every set of identities is unifiable there is no passive admissible clause. □

In some cases, we can improve the previous result.

Lemma 3.32. *Let* Q *be a quasivariety such that* $\mathbf{I}(\mathbf{F}_{\mathsf{Q}}) = \mathbf{IP}_u(\mathbf{F}_{\mathsf{Q}})$*, then the following are equivalent.*

1. Q *is unifiable;*
2. *every algebra in* Q *is unifiable.*

Proof. Notice that if $\mathbf{F}_{\mathsf{Q}}$ is trivial all algebras are unifiable and then the two statements are verified; let us then assume $\mathbf{F}_{\mathsf{Q}}$ nontrivial. We prove the nontrivial direction by contraposition. Consider an arbitrary algebra $\mathbf{A} \in \mathsf{Q}$ and assume that it is not unifiable; without loss of generality we let $\mathbf{A} = \mathbf{F}_{\mathsf{Q}}(X)/\theta$ for some set X and some relative congruence θ. Since $\mathbf{A}$ is not unifiable, there is no assignment $h : \mathbf{F}_{\mathsf{Q}}(X) \to \mathbf{F}_{\mathsf{Q}}$ such that $\mathbf{F}_{\mathsf{Q}}, h \vDash \Sigma_\theta$, where $\Sigma_\theta = \{t \approx u : (t, u) \in \theta\}$. Equivalently, iff $\mathbf{F}_{\mathsf{Q}} \vDash \Sigma_\theta \Rightarrow \varnothing$. Now, the equational consequence relation relative to a class of algebras K is finitary if and only if K is closed under ultraproducts (see for instance [74]); thus by hypothesis the equational consequence relation relative to $\mathbf{F}_{\mathsf{Q}}$ is finitary, and we obtain that $\mathbf{F}_{\mathsf{Q}} \models \Sigma'_\theta \Rightarrow \emptyset$, for Σ'_θ some finite subset of Σ_θ. That is, Σ' is finite and not unifiable, thus Q is not unifiable and the proof is complete. □

Observe that a quasivariety Q such that $\mathbf{F}_{\mathsf{Q}}$ is finite satisfies the hypothesis of the previous lemma. We do not know whether the condition $\mathbf{I}(\mathbf{F}_{\mathsf{Q}}) = \mathbf{IP}_u(\mathbf{F}_{\mathsf{Q}})$ is necessary.

Corollary 3.33. *Let Q be a quasivariety such that $\mathbf{I}(\mathbf{F}_{\mathsf{Q}}) = \mathbf{IP}_u(\mathbf{F}_{\mathsf{Q}})$, then the following are equivalent.*

1. *Q is passively universally complete;*
2. *Q is negatively universally complete;*
3. *Q is unifiable;*
4. *every algebra in Q is unifiable.*

We also have the following.

Corollary 3.34. *A quasivariety of logic Q is passively universally complete if and only if $\mathcal{L}_{\mathsf{Q}}$ is passively universally complete.*

4. Structural completeness

In this section we investigate the algebraic counterparts of structural completeness and its variations. The main new results are about the characterization of passively structurally complete quasivarieties; moreover, we also show a characterization of primitive quasivarieties grounding on the results in [50].

4.1. Structural quasivarieties

The bridge theorems for structural completeness have been first established by Bergman [13]. We present the proof for the sake of the reader, expanding with point (6).

Theorem 4.1 *([13]). For a quasivariety Q the following are equivalent:*

1. *Q is structurally complete;*
2. *$\mathsf{Q} = \mathbf{Q}(\mathbf{F}_{\mathsf{Q}}(\omega))$;*
3. *any proper subquasivariety of Q generates a proper subvariety of $\mathbf{H}(\mathsf{Q})$;*
4. *for all $\mathsf{Q}' \subseteq \mathsf{Q}$ if $\mathbf{H}(\mathsf{Q}') = \mathbf{H}(\mathsf{Q})$, then $\mathsf{Q} = \mathsf{Q}'$;*
5. *for all $\mathbf{A} \in \mathsf{Q}$ if $\mathbf{V}(\mathbf{A}) = \mathbf{H}(\mathsf{Q})$, then $\mathbf{Q}(\mathbf{A}) = \mathsf{Q}$;*
6. *every finitely presented algebra in Q is in $\mathbf{Q}(\mathbf{F}_{\mathsf{Q}}(\omega))$.*

Proof. First, (1) is equivalent to (2) via Lemma 2.39. The implications (3) ⇔ (4) ⇒ (5) ⇒ (2) are straightforward. (2) implies (4) since if $\mathsf{Q}' \subseteq \mathsf{Q}$ and $\mathbf{H}(\mathsf{Q}') = \mathbf{H}(\mathsf{Q})$, we get that $\mathbf{F}_{\mathsf{Q}'}(\omega) = \mathbf{F}_{\mathbf{H}(\mathsf{Q}')}(\omega) = \mathbf{F}_{\mathbf{H}(\mathsf{Q})}(\omega) = \mathbf{F}_{\mathsf{Q}}(\omega)$; thus $\mathsf{Q} = \mathbf{Q}(\mathbf{F}_{\mathsf{Q}}(\omega)) = \mathbf{Q}(\mathbf{F}'_{\mathsf{Q}}(\omega)) \subseteq \mathsf{Q}'$ and then equality holds. Thus the first five points are equivalent; Finally, clearly (2) implies (6), and (6) implies (2) since a quasivariety is generated by its finitely presented algebras ([50, Proposition 2.1.18]). □

Corollary 4.2. *A variety V is structurally complete if and only if every proper subquasivariety of V generates a proper subvariety; therefore if $\mathbf{A}$ is such that $\mathbf{V}(\mathbf{A})$ is structurally complete, then $\mathbf{V}(\mathbf{A}) = \mathbf{Q}(\mathbf{A})$.*

As usual we get also:

Corollary 4.3. *Let* Q *be a quasivariety of logic; then* Q *is structurally complete if and only if* $\mathcal{L}_{\mathsf{Q}}$ *is structurally complete.*

Let us extract some sufficient conditions for structural completeness.

Lemma 4.4. *Let* Q *be a quasivariety; if*

1. $\mathsf{Q} = \mathbf{Q}(\mathsf{K})$ *and every* $\mathbf{A} \in \mathsf{K}$ *is finitely generated and exact in* Q*, or*
2. *every finitely generated algebra in* Q *is exact, or*
3. *every finitely presented algebra in* Q *is exact, or*
4. *every finitely generated relative subdirectly irreducible in* Q *is exact,*

then Q *is structurally complete. Moreover if every* $\mathbf{A} \in \mathsf{K}$ *is exact in* $\mathbf{V}(\mathsf{K})$ *and every subdirectly irreducible member of* $\mathbf{V}(\mathsf{K})$ *is in* $\mathbf{IS}(\mathsf{K})$*, then* $\mathbf{V}(\mathsf{K})$ *is structurally complete.*

Proof. Assume first that $\mathsf{Q} = \mathbf{Q}(\mathsf{K})$, with K a class of finitely generated and exact algebras in Q; then $\mathsf{K} \subseteq \mathbf{IS}(\mathbf{F}_{\mathsf{Q}}(\omega))$; therefore $\mathsf{Q} = \mathbf{Q}(\mathsf{K}) \subseteq \mathbf{Q}(\mathbf{F}_{\mathsf{Q}}(\omega))$ and thus equality holds. Hence Q is structurally complete by the characterization theorem (Theorem 4.1).

The other points follow; in particular, note that any algebra in a quasivariety Q is embeddable in an ultraproduct of its finitely generated subalgebras, thus $\mathsf{Q} = \mathbf{Q}(\mathsf{K})$ where K is the class of finitely generated algebras in Q. Moreover, every quasivariety is generated by its finitely presented algebras, and also by its finitely generated relative subdirectly irreducible algebras.

For the last claim, every subdirectly irreducible member of $\mathbf{V}(\mathsf{K})$ lies in $\mathbf{IS}(\mathsf{K})$ and thus is exact in $\mathbf{V}(\mathsf{K})$. Since any variety is generated as a quasivariety by its finitely generated subdirectly irreducible members, $\mathbf{V}(\mathsf{K})$ is structurally complete. □

We observe that none of the previous conditions is necessary. For locally finite quasivarieties we have a necessary and sufficient condition for structural completeness because of the following:

Lemma 4.5 *([22]). Let* Q *be a locally finite quasivariety and* $\mathbf{A}$ *a finite algebra in* Q*. Then* $\mathbf{A} \in \mathbf{Q}(\mathbf{F}_{\mathsf{Q}}(\omega))$ *if and only if* $\mathbf{A} \in \mathbf{ISP}(\mathbf{F}_{\mathsf{Q}}(\omega))$*.*

The following theorem improves [22, Corollary 11].

Theorem 4.6. *For a locally finite quasivariety* Q *of finite type the following are equivalent:*

1. Q *is structurally complete;*
2. *each finite algebra in* Q *is in* $\mathbf{ISP}(\mathbf{F}_{\mathsf{Q}}(\omega))$*;*
3. *every finite relative subdirectly irreducible in* Q *is exact.*

Proof. Assume (1); then each finite algebra in Q is in $\mathbf{Q}(\mathbf{F}_{\mathsf{Q}}(\omega))$ and thus, by Lemma 4.5, is in $\mathbf{ISP}(\mathbf{F}_{\mathsf{Q}}(\omega))$ and (2) holds. If (2) holds and $\mathbf{A}$ is finite relative subdirectly irreducible, then it is in $\mathbf{IS}(\mathbf{F}_{\mathsf{Q}}(\omega))$, i.e. it is exact. Finally if (3) holds, then Q is structurally complete by Lemma 4.4. □

4.2. Primitive quasivarieties

We now consider the hereditary notion of structural completeness.

Definition 4.7. A class of algebras K in a quasivariety Q is *equational relative to* Q if $\mathsf{K} = \mathbf{V}(\mathsf{K}) \cap \mathsf{Q}$. In particular, a subquasivariety Q' of Q is *equational relative to* Q if $\mathsf{Q}' = \mathbf{H}(\mathsf{Q}') \cap \mathsf{Q}$; a quasivariety Q is *primitive* if every subquasivariety of Q is equational relative to Q.

Clearly primitivity is downward hereditary and a variety V is primitive if and only if every subquasivariety of V is a variety. We can show the following.

Theorem 4.8. *For a quasivariety* Q *the following are equivalent:*

1. Q *is primitive;*
2. *every subquasivariety of* Q *is structurally complete;*
3. *for all subdirectly irreducible* $\mathbf{A} \in \mathbf{H}(\mathsf{Q})$ *and for any* $\mathbf{B} \in \mathsf{Q}$*, if* $\mathbf{A} \in \mathbf{H}(\mathbf{B})$*, then* $\mathbf{A} \in \mathbf{ISP}_u(\mathbf{B})$*.*

Proof. We first show the equivalence between (1) and (2). Suppose that Q is primitive and let $\mathsf{Q}' \subseteq \mathsf{Q}$; if $\mathsf{Q}'' \subseteq \mathsf{Q}'$ and $\mathbf{H}(\mathsf{Q}'') = \mathbf{H}(\mathsf{Q}')$ then

$$\mathsf{Q}' = \mathbf{H}(\mathsf{Q}') \cap \mathsf{Q} = \mathbf{H}(\mathsf{Q}'') \cap \mathsf{Q} = \mathsf{Q}''$$

so Q' is structurally complete by Theorem 4.1.

Conversely assume (2), let $\mathsf{Q}' \subseteq \mathsf{Q}$ and let $\mathsf{Q}'' = \mathbf{H}(\mathsf{Q}') \cap \mathsf{Q}$ (it is clearly a quasivariety); then $\mathbf{H}(\mathsf{Q}'') = \mathbf{H}(\mathsf{Q}')$ and thus $\mathsf{Q}'' = \mathsf{Q}'$, again using the characterization of Theorem 4.1. So Q' is equational in Q and Q is primitive.

Assume (1) again, and let $\mathbf{A}, \mathbf{B} \in \mathsf{Q}$ with $\mathbf{A}$ subdirectly irreducible and $\mathbf{A} \in \mathbf{H}(\mathbf{B})$. Since Q is primitive we have

$$\mathbf{Q}(\mathbf{B}) = \mathbf{H}(\mathbf{Q}(\mathbf{B})) \cap \mathsf{Q}$$

and hence $\mathbf{A} \in \mathbf{Q}(\mathbf{B})$. Since $\mathbf{A}$ is subdirectly irreducible, $\mathbf{A} \in \mathbf{ISP}_u(\mathbf{B})$ by Theorem 2.3 and (3) holds.

Conversely, assume (3) and let Q' be a subquasivariety of Q; if $\mathbf{B} \in \mathbf{H}(\mathsf{Q}') \cap \mathsf{Q}$, observe that $\mathbf{B} \in \mathbf{H}(\mathsf{Q})$ and hence $\mathbf{B} \leqslant_{sd} \prod \mathbf{A}_i$ where the $\mathbf{A}_i$ are subdirectly irreducible in $\mathbf{H}(\mathsf{Q}) \cap \mathbf{H}(\mathsf{Q}')$. Then for all i there is $\mathbf{B}_i \in \mathsf{Q}'$ such that $\mathbf{A}_i \in \mathbf{H}(\mathbf{B}_i)$ and hence by hypothesis $\mathbf{A}_i \in \mathbf{SP}_u(\mathbf{B}_i)$ and so $\mathbf{A}_i \in \mathsf{Q}'$ for all i. Therefore $\mathbf{B} \in \mathsf{Q}'$, so $\mathbf{H}(\mathsf{Q}') \cap \mathsf{Q} = \mathsf{Q}'$ and Q' is equational in Q. Therefore Q is primitive and (1) holds. □

As commented in the preliminary section (Subsection 2.1):

Corollary 4.9. *Let* Q *be a quasivariety of logic;* Q *is primitive if and only if* $\mathcal{L}_\mathsf{Q}$ *is hereditarily structurally complete.*

We will see how Theorem 4.8 can be improved in the locally finite case. Let Q be a quasivariety and let $\mathbf{A} \in \mathsf{Q}$; we define

$$[\mathsf{Q} : \mathbf{A}] = \{\mathbf{B} \in \mathsf{Q} : \mathbf{A} \notin \mathbf{IS}(\mathbf{B})\}.$$

The following lemma describes some properties of $[\mathsf{Q} : \mathbf{A}]$; the proofs are quite standard with the exception of point (3). As a matter of fact a proof of the forward implication of (3) appears in [50, Corollary 2.1.17]. However the proof is somewhat buried into generality and it is not easy to follow; so we felt that a suitable translation would make it easier for the readers.

Lemma 4.10. *Let* Q *be a quasivariety; then*

1. *if* $\mathbf{A} \in \mathsf{Q}$ *is finite and* Q *has finite type, then* $[\mathsf{Q} : \mathbf{A}]$ *is a universal class;*
2. *if* $\mathbf{A}$ *is relative subdirectly irreducible and finitely presented, then* $[\mathsf{Q} : \mathbf{A}]$ *is a quasivariety;*
3. $\mathbf{A}$ *is weakly projective in* Q *if and only if* $[\mathsf{Q} : \mathbf{A}]$ *is closed under* $\mathbf{H}$ *if and only if* $[\mathsf{Q} : \mathbf{A}]$ *is equational relative to* Q*;*
4. *if* $\mathbf{A}$ *is relative subdirectly irreducible, finitely presented and weakly projective in* Q*, then* $[\mathsf{Q} : \mathbf{A}]$ *is a variety.*

Moreover if Q *is locally finite of finite type, the converse implications in (1), (2) and (4) hold.*

Proof. For (1), if $\mathbf{A}$ is finite, then there is a first order universal sentence Ψ such that, for all $\mathbf{B} \in \mathsf{Q}$, $\mathbf{B} \models \Psi$ if and only if $\mathbf{A} \in \mathbf{IS}(\mathbf{B})$. More in detail, if $|A| = n$,

$$\Psi := \exists x_1 \dots \exists x_n (\bigwedge \{x_i \neq x_j : i, j \leqslant n, i \neq j\} \bigwedge \mathbf{D}(\mathbf{A})),$$

where $\mathbf{D}(\mathbf{A})$ is the diagram of $\mathbf{A}$, that is, a conjunction of universal sentences that describe the operation tables of $\mathbf{A}$ (identifying each element of $\mathbf{A}$ with a different x_i), and $\bigwedge$ is first order logic conjunction. Now consider $\mathbf{B} \in \mathbf{ISP}_u([\mathsf{Q} : \mathbf{A}])$, we show that $\mathbf{A} \notin \mathbf{IS}(\mathbf{B})$; if $\mathbf{A} \in \mathbf{IS}(\mathbf{B})$, then $\mathbf{A} \in \mathbf{ISP}_u([\mathsf{Q} : \mathbf{A}])$. Hence there exists a family $(\mathbf{A}_i)_{i\in I} \subseteq [\mathsf{Q} : \mathbf{A}]$ and an ultrafilter U on I such that $\mathbf{C} = \Pi_{i\in I}\mathbf{A}_i/U$ and $\mathbf{A} \in \mathbf{IS}(\mathbf{C})$. So $\mathbf{C} \models \Psi$; but then by Łòs Lemma there is a (necessarily nonempty) set of indexes $I' \in U$ such that Ψ is valid in each $\mathbf{A}_i$ with $i \in I'$, which is clearly a contradiction, since each $\mathbf{A}_i \in [\mathsf{Q} : \mathbf{A}]$. Thus $\mathbf{A} \notin \mathbf{IS}(\mathbf{B})$, $\mathbf{B} \in [\mathsf{Q} : \mathbf{A}]$, and therefore $\mathbf{ISP}_u([\mathsf{Q} : \mathbf{A}]) = [\mathsf{Q} : \mathbf{A}]$ which is a universal class by Lemma 2.1.

Conversely let Q be locally finite of finite type; every algebra in Q is embeddable in an ultraproduct of its finitely generated (i.e. finite) subalgebras, say $\mathbf{A} \in \mathbf{ISP}_u(\{\mathbf{B}_i : i \in I\})$. If $\mathbf{A}$ is not finite, then $\mathbf{A} \notin \mathbf{S}(\mathbf{B}_i)$ for all i, so $\mathbf{B}_i \in [\mathsf{Q} : \mathbf{A}]$ for all i. Since $[\mathsf{Q} : \mathbf{A}]$ is universal, we would have that $\mathbf{A} \in [\mathsf{Q} : \mathbf{A}]$, a clear contradiction. So $\mathbf{A} \in \mathbf{IS}(\mathbf{B}_i)$ for some i and hence it is finite.

For (2), suppose that $\mathbf{A}$ is relative subdirectly irreducible and finitely presented, i.e. $\mathbf{A} \cong \mathbf{F}_{\mathsf{Q}}(\mathbf{x})/\theta_{\mathsf{Q}}(\Sigma)$ where $\mathbf{x} = (x_1, \dots, x_n)$ and $\Sigma = \{p_i(\mathbf{x}) \approx q_i(\mathbf{x}) : i = 1, \dots, m\}$. We set $a_i = x_i/\theta_{\mathsf{Q}}(\Sigma)$; since $\mathbf{A}$ is relative subdirectly irreducible, it has a relative monolith μ, i.e. a minimal nontrivial relative congruence. Since μ is minimal, there are $c, d \in A$ such that μ is the relative congruence generated by the pair (c, d). Now let t_c, t_d terms in $\mathbf{F}_{\mathsf{Q}}(\mathbf{x})$ such that $t_c(a_1, \dots, a_n) = c$ and $t_d(a_1, \dots, a_n) = d$ and let Φ be the quasiequation

$$\bigwedge_{i=1}^{m} p_i(\mathbf{x}) \approx q_i(\mathbf{x}) \ \longrightarrow \ t_c(\mathbf{x}) \approx t_d(\mathbf{x}).$$

Then $\mathbf{A} \not\models \Phi$; moreover if $\mathbf{C} \in \mathsf{Q}$ is a homomorphic image of $\mathbf{A}$ which is not isomorphic with $\mathbf{A}$, then $\mathbf{C} \models \Phi$. We claim that $[\mathsf{Q} : \mathbf{A}] = \{\mathbf{B} \in \mathsf{Q} : \mathbf{B} \models \Phi\}$ and since Φ is a quasiequation this implies that $[\mathsf{Q} : \mathbf{A}]$ is a quasivariety. Clearly if $\mathbf{B} \models \Phi$, then $\mathbf{A} \notin \mathbf{IS}(\mathbf{B})$; conversely assume that $\mathbf{B} \not\models \Phi$. Then there are $b_1, \dots, b_n \in B$ such that $p_i(b_1, \dots, b_n) = q_i(b_1, \dots, b_n)$ but $t_c(b_1, \dots, b_n) \neq t_d(b_1, \dots, b_n)$. Let g be the homomorphism extending the assignment $x_i \longmapsto b_i$; then $\theta_{\mathsf{Q}}(\Sigma) \subseteq \ker(g)$ so by the Second Homomorphism Theorem there is a homomorphism $f : \mathbf{A} \longrightarrow \mathbf{B}$ such that $f(a_i) = b_i$. Observe that $f(\mathbf{A}) \in \mathsf{Q}$ (since it is a subalgebra of $\mathbf{B} \in \mathsf{Q}$) and $f(\mathbf{A}) \not\models \Phi$, so by what we said above $f(\mathbf{A}) \cong \mathbf{A}$; this clearly implies $\mathbf{A} \in \mathbf{IS}(\mathbf{B})$, so $\mathbf{B} \notin [\mathsf{Q} : \mathbf{A}]$ as wished.

For the converse, let Q be locally finite of finite type; by (1) $\mathbf{A}$ is finite. Suppose that $\mathbf{A} \leqslant_{sd} \prod_{i\in I} \mathbf{B}_i$ where each $\mathbf{B}_i$ is relative subdirectly irreducible in Q. Since $\mathbf{A}$ is finite, each $\mathbf{B}_i$ can be taken to be finite; if $\mathbf{A} \notin \mathbf{IS}(\mathbf{B}_i)$ for all i, then $\mathbf{B}_i \in [\mathsf{Q} : \mathbf{A}]$ for all i and hence, being $[\mathsf{Q} : \mathbf{A}]$ a quasivariety we have $\mathbf{A} \in [\mathsf{Q} : \mathbf{A}]$ which is impossible. Hence there is an i such that $\mathbf{A} \in \mathbf{IS}(\mathbf{B}_i)$, so that $|A| \leqslant |B_i|$; on the other hand $\mathbf{B}_i \in \mathbf{H}(\mathbf{A})$, so $|B_i| \leqslant |A|$. Since everything is finite we have $\mathbf{A} \cong \mathbf{B}_i$ and then $\mathbf{A}$ is relative subdirectly irreducible.

For the first forward direction of (3), suppose that $\mathbf{B} \in \mathbf{H}([\mathsf{Q} : \mathbf{A}])$. If $\mathbf{A} \in \mathbf{IS}(\mathbf{B})$, then $\mathbf{A} \in \mathbf{SH}([\mathsf{Q} : \mathbf{A}]) \subseteq \mathbf{HS}([\mathsf{Q} : \mathbf{A}])$. Now $[\mathsf{Q} : \mathbf{A}] \subseteq \mathsf{Q}$ and $\mathbf{A}$ is weakly projective in Q; so $\mathbf{A} \in \mathbf{S}([\mathsf{Q} : \mathbf{A}])$ which is impossible. It follows that $\mathbf{A} \notin \mathbf{IS}(\mathbf{B})$ and $\mathbf{B} \in [\mathsf{Q} : \mathbf{A}]$; thus $[\mathsf{Q} : \mathbf{A}]$ is closed under $\mathbf{H}$. For the second forward direction, it is easy to see that if $[\mathsf{Q} : \mathbf{A}]$ is closed under $\mathbf{H}$ then $[\mathsf{Q} : \mathbf{A}]$ is equational relative to Q. Assume now that $[\mathsf{Q} : \mathbf{A}]$ is closed under $\mathbf{H}$, we show that $\mathbf{A}$ is weakly projective in Q. Suppose that $\mathbf{A} \in \mathbf{H}(\mathbf{B})$ for some $\mathbf{B} \in \mathsf{Q}$; if $\mathbf{A} \notin \mathbf{IS}(\mathbf{B})$, then $\mathbf{B} \in [\mathsf{Q} : \mathbf{A}]$ and, since $[\mathsf{Q} : \mathbf{A}]$ is closed under $\mathbf{H}$, $\mathbf{A} \in [\mathsf{Q} : \mathbf{A}]$, again a contradiction. Hence $\mathbf{A} \in \mathbf{IS}(\mathbf{B})$ and $\mathbf{A}$ is weakly projective in Q. A completely analogous proof shows that if $[\mathsf{Q} : \mathbf{A}]$ is equational relative to Q then $\mathbf{A}$ is weakly projective, which completes the proof of (3).

(4) follows directly from (1), (2) and (3). □

Thus if $\mathbf{A}$ is relative subdirectly irreducible and finitely presented, $[\mathsf{Q} : \mathbf{A}]$ is a quasivariety; this is the key to prove the following result, appearing in [50, Proposition 5.1.24]. We present a self-contained proof for the sake of the reader.

Theorem 4.11 *([50]). If Q is a locally finite quasivariety of finite type, then the following are equivalent.*

1. *Q is primitive;*
2. *for all finite relative subdirectly irreducible $\mathbf{A} \in \mathsf{Q}$, $[\mathsf{Q} : \mathbf{A}]$ is equational relative to Q;*
3. *every finite relative subdirectly irreducible $\mathbf{A} \in \mathsf{Q}$ is weakly projective in Q;*
4. *every finite relative subdirectly irreducible $\mathbf{A} \in \mathsf{Q}$ is weakly projective in the class of finite algebras in Q.*

Proof. (2) and (3) are equivalent by Lemma 4.10, and (3) and (4) are equivalent in locally finite quasivarieties.

Now, (1) implies (2) by Lemma 4.10, since if $\mathbf{A}$ is a finite relative subdirectly irreducible algebra then $[\mathsf{Q} : \mathbf{A}]$ is a quasivariety, and if Q is primitive every subquasivariety is equational relative to Q by definition.

Finally, assume (3) and let Q' be a subquasivariety of Q; consider a finite algebra $\mathbf{B} \in \mathbf{H}(\mathsf{Q}') \cap \mathsf{Q}$, then $\mathbf{B}$ is a subdirect product of finite relative subdirectly irreducible algebras in Q, that is, $\mathbf{B} \leqslant_{sd} \prod_{i\in I} \mathbf{A}_i$ where each $\mathbf{A}_i$ is finite relative subdirectly irreducible in Q, and thus it is also weakly projective in Q by hypothesis. Since $\mathbf{B} \in \mathbf{H}(\mathsf{Q}')$, there is $\mathbf{A} \in \mathsf{Q}'$ such that $\mathbf{B} \in \mathbf{H}(\mathbf{A})$. But then for each $i \in I$, $\mathbf{A}_i \in \mathbf{H}(\mathbf{A})$; since each $\mathbf{A}_i$ is weakly projective in Q, it is also isomorphic to a subalgebra of $\mathbf{A}$. Thus, $\mathbf{B} \in \mathbf{ISP}(\mathbf{A}) \subseteq \mathsf{Q}'$, and therefore $\mathsf{Q}' = \mathbf{H}(\mathsf{Q}') \cap \mathsf{Q}$, which means that Q is primitive and (1) holds. □

Most results in the literature are about structurally complete and primitive *varieties* of algebras and the reason is quite obvious; first the two concepts are easier to formulate for varieties. Secondly being subdirectly irreducible is an absolute concept (every subdirectly irreducible algebra is relative subdirectly irreducible in any quasivariety to which it belongs) while being relative subdirectly irreducible depends on the subquasivariety we are considering. Of course when a quasivariety is generated by a “simple” class (e.g. by finitely many finite algebras), then Theorem 2.3(2) gives a simple solution, but in general describing the relative subdirectly irreducible algebras in a quasivariety is not an easy task.

Now, it is clear that if Q is non-negatively universally complete, then it is structurally complete. Finding examples of (quasi)varieties that are structurally complete but not primitive is not easy; in [31] an example is given of an intermediate logic (*Medvedev's logic*) which is structurally complete but not hereditarily structurally complete. Through the Blok-Pigozzi connection this translates into a variety of Heyting algebras that is structurally complete but not primitive. A different idea is to find a finite algebra $\mathbf{A}$ such that $\mathbf{A}$ satisfies the hypotheses of Lemma 4.4, but $\mathbf{V}(\mathbf{A})$ contains some strict (i.e. not a variety) subquasivariety. We will see an example of this in Section 5.1. Let us now show some different kinds of examples of primitive (quasi)varieties.

Example 4.12. The variety of bounded distributive lattices is primitive (as we will discuss in Section 5.1), since it is equationally complete and congruence modular and so is q-minimal by Theorem 3.28.

It is well-known (and easy to check) that the variety of distributive lattices is a *dual discriminator variety*; a *dual discriminator* on a set A is a ternary operation d on A defined by

$$d(a,b,c) = \begin{cases} c, & \text{if } a \neq b; \\ a, & \text{if } a = b. \end{cases}$$

A variety V is a dual discriminator variety [41] if there is a ternary term that is the dual discriminator on all the subdirectly irreducible members of V. Dual discriminator varieties, as opposed to discriminator varieties, do not necessarily have projective unifiers. However, recently in [24] the authors have extended the results in [14] (such as Theorem 3.28) in two directions: every minimal dual discriminator variety is q-minimal, hence primitive and, if the variety is also idempotent, then minimality can be dropped and the variety is primitive. This last fact gives raise to different examples of primitive varieties.

Example 4.13. A *weakly associative lattice* is an algebra $\langle A, \vee, \wedge \rangle$ where $\vee$ and $\wedge$ are idempotent, commutative and satisfy the absorption laws but (as the name reveals) only a weak form of associativity. In [41] the authors proved that there is a largest dual discriminator variety U of weakly associative lattices; since weakly associative lattices are idempotent, U is a primitive variety of weakly associative lattices.

Example 4.14. The *pure dual discriminator variety* D (see [41, Theorem 3.2]) is a variety with a single ternary operation $d(x,y,z)$ satisfying

$$\begin{aligned} &d(x,y,y) \approx y \\ &d(x,y,x) \approx x \\ &d(x,x,y) \approx x \\ &d(x,y,d(x,y,z)) \approx d(x,y,z) \\ &d(u,v,d(x,y,z)) \approx d(d(u,v,x), d(u,v,y), d(u,v,z)) \end{aligned}$$

which is enough to prove that D is a dual discriminator variety. Since d is idempotent D is an idempotent dual discriminator variety and so it is primitive.

A different example is given by the following.

Example 4.15. A *modal algebra* is a Boolean algebra with a modal operator $\Box$, that we take as a basic unary operation, satisfying $\Box 1 \approx 1$ and $\Box(x \wedge y) \approx \Box x \wedge \Box y$; there is an extensive literature on modal algebras (see for instance [80] and the bibliography therein). A modal algebra is a *K4-algebra* if it satisfies $\Box x \leqslant \Box\Box x$; in [75] V.V. Rybakov classified all the primitive varieties of K4-algebras. However very recently [25] J. Carr discovered a mistake in Rybakov's proof; namely Rybakov in his description missed some varieties that all have the properties of containing a finitely presented unifiable weakly projective algebra that is not projective. So any of such varieties, though primitive, does not have projective unifiers.

We now present some examples of (quasi)varieties that are the equivalent algebraic semantics of axiomatic extensions (and their fragments) of Basic Logic; in particular, of infinite-valued Łukasiewicz logic.

Example 4.16. Wajsberg algebras are the equivalent algebraic semantics of infinite-valued Łukasiewicz logic in the signature of bounded commutative residuated lattices $(\cdot, \to, \wedge, \vee, 0, 1)$ and they are term-equivalent to the better known MV-algebras [27]; Wajsberg hoops are their 0-free subreducts. There are some recent results about structural completeness and primitivity for these algebras. There are exactly two structurally complete *varieties* of Wajsberg algebras: the variety of Boolean algebras and the variety generated by perfect MV-algebras. They are also the only primitive varieties of Wajsberg algebras (this is implicit in [49], but see also [1, Section 8]). Whereas a variety of Wajsberg hoops is structurally complete if and only if it is primitive if and only if every subdirectly irreducible algebra is either finite or perfect [1].

If we look instead at quasivarieties, the structurally complete quasivarieties of Wajsberg algebras have been characterized in [49], while the structurally complete quasivarieties of Wajsberg hoops have been characterized in [2]. Moreover, it has been shown that there are nontrivial primitive quasivarieties of Wajsberg algebras [1] and of Wajsberg hoops [2].

Let us now consider the variety of →-subreducts of Wajsberg hoops, that is a subvariety of BCK-algebras usually denoted by LBCK; every locally finite subvariety of LBCK-algebras is a variety of HBCK-algebras, so it is universally complete. However: the only non locally finite subvariety is the entire variety LBCK [60]; LBCK is generated as a quasivariety by its finite chains [3], and every infinite chain contains all the finite chains as subalgebras [60]. So if Q is a quasivariety which contains only finitely many chains, then $\mathbf{V}(\mathsf{Q})$ is locally finite, hence universally complete and so $\mathsf{Q} = \mathbf{V}(\mathsf{Q})$; otherwise Q contains infinitely many chains and so $\mathbf{V}(\mathsf{Q}) = \mathsf{Q} = \mathsf{LBCK}$. Hence every subquasivariety of LBCK is a variety and LBCK is primitive.

Another related class of examples is given by *basic hoops* [3], i.e. the equivalent algebraic semantics of the 0-free fragment of Basic Logic [51]; they are hoops and hence by Example 3.11 any locally finite variety of basic hoops is primitive universal, hence primitive. Cancellative hoops are basic hoops and so (again by Example 3.11) they form a non-locally finite variety of hoops that is primitive. We note that Wajsberg hoops are basic hoops and Wajsberg algebras are BL-algebras. In general, outside the cases outlined above, the status of (locally finite) (quasi)varieties of BL-algebras, of non locally finite varieties of basic hoops, and of quasivarieties of basic hoops is unclear and it is the subject of a current investigation.

4.3. Actively structurally complete quasivarieties

The problem of active structural completeness has been tackled in [35]; it is an extensive and profound paper touching many aspects and there is no need to reproduce it here. We will only state the main result and display an example.

Theorem 4.17 *(Theorem 8.1 in [35]). For a quasivariety Q the following are equivalent:*

1. *Q is actively structurally complete;*
2. *every unifiable algebra of Q is in $\mathbf{Q}(\mathbf{F}_\mathsf{Q}(\omega))$;*
3. *every finitely presented unifiable algebra in Q is in $\mathbf{Q}(\mathbf{F}_\mathsf{Q}(\omega))$;*
4. *every admissible quasiequation in Q is valid in all the finitely presented unifiable algebras in Q;*
5. *for every $\mathbf{A} \in \mathsf{Q}$, $\mathbf{A} \times \mathbf{F}_\mathsf{Q} \in \mathbf{Q}(\mathbf{F}_\mathsf{Q}(\omega))$.*
6. *for every $\mathbf{A} \in \mathsf{Q}_{rsi}$, $\mathbf{A} \times \mathbf{F}_\mathsf{Q} \in \mathbf{ISP}_u(\mathbf{F}_\mathsf{Q}(\omega))$.*

We have as usual:

Corollary 4.18. *Let Q be a quasivariety of logic; Q is actively structurally complete if and only if $\mathcal{L}_\mathsf{Q}$ is actively structurally complete.*

Example 4.19. An $S4$-algebra is a $K4$-algebra satisfying $\Box x \leqslant x$; if we define $\Diamond x := \neg\Box\neg x$, then a *monadic* algebra is an $S4$-algebras satisfying $\Diamond x \leqslant \Box\Diamond x$. We extract this example by the results in [35, Section 8]. Let $\mathbf{A}$, $\mathbf{B}$ be the monadic algebra and the $S4$-algebra in Fig. 3 and let $\mathsf{V} = \mathbf{V}(\mathbf{A})$ and $\mathsf{W} = \mathbf{V}(\mathbf{B})$.

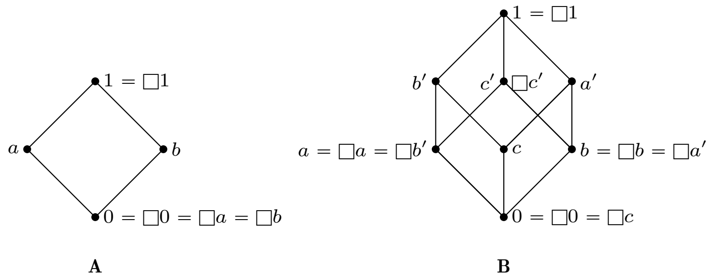


Fig. 3. **A** and **B**.

Let $\mathsf{U} = \mathsf{V} \vee \mathsf{W} = \mathbf{V}(\{\mathbf{A}, \mathbf{B}\})$, then by [35, Theorem 8.12] U is actively structurally complete but not structurally complete, and U does not have exact unifiers. Since U is generated by two finite algebras it is locally finite, and by Theorem 3.23 it cannot be actively universally complete; so U is an example of a variety that is actively structurally complete but not actively universally complete.

4.4. Passive quasivarieties

Passively structurally complete quasivarieties have been studied in [69] in relation to the joint embedding property (which is shown to be a consequence of passive structural completeness). Moreover, it is shown that if Q is a quasivariety of finite type, with a finite nontrivial member, Q is PSC if and only if the nontrivial members of Q have a common retract (see [69, Theorem 7.6]). We here take a different path, and we start with the following observation.

Proposition 4.20. *A quasivariety* Q *is passively structurally complete if and only if every non-negative passive admissible clause is derivable in* Q.

Proof. For the non-trivial direction, suppose Q is passively structurally complete, and let $\Sigma \Rightarrow \Delta$ be a non-negative ($\Delta \neq \emptyset$) passive admissible clause. This means that Σ is not unifiable, and thus, each quasiequation $\Sigma \Rightarrow \delta$, for any $\delta \in \Delta$, is passive admissible. By hypothesis, each such $\Sigma \Rightarrow \delta$ is valid in Q, thus so is $\Sigma \Rightarrow \Delta$ and the conclusion holds. □

It is clear that a key concept to study passive clauses is understanding the unifiability of the premises. In order to do so, we introduce the following notion.

Definition 4.21. We say that a finite set of identities Σ is *trivializing* in a class of algebras K if the quasiequation $\Sigma \Rightarrow (x \approx y)$ is valid in K, where the variables x, y do not appear in Σ.

Notice that such a quasiequation $\Sigma \Rightarrow (x \approx y)$ is valid in an algebra $\mathbf{A}$ if and only if either $\mathbf{A}$ is trivial, or there is no assignment of the variables of Σ in $\mathbf{A}$ that makes Σ valid in $\mathbf{A}$.

Lemma 4.22. *Let* Q *be a quasivariety, and let* Σ *be a finite set of equations in its language. The following are equivalent:*

1. Σ *is not unifiable in* Q*;*
2. $\mathbf{F}_{\mathsf{Q}}$ *is nontrivial and* Σ *is trivializing in* $\mathbf{Q}(\mathbf{F}_{\mathsf{Q}})$*;*
3. $\mathbf{F}_{\mathsf{Q}} \models \Sigma \Rightarrow \emptyset$.

Proof. It is easy to see that (2) and (3) are equivalent, modulo the fact that a set of identities is trivializing in $\mathbf{Q}(\mathbf{F}_{\mathsf{Q}})$ if and only if it is trivializing in $\mathbf{F}_{\mathsf{Q}}$.

Let us now assume that the identities in Σ are on a (finite) set of variables X. Then, given Lemma 2.22, Σ is not unifiable in Q if and only if there is no homomorphism $h : \mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma) \to \mathbf{F}_{\mathsf{Q}}$. We show that the latter holds if and only if there is no homomorphism $k : \mathbf{F}_{\mathsf{Q}}(X) \to \mathbf{F}_{\mathsf{Q}}$ such that $k(t) = k(u)$ for each $t \approx u \in \Sigma$. Indeed, for the non-trivial direction, suppose that there is a homomorphism $k : \mathbf{F}_{\mathsf{Q}}(X) \to \mathbf{F}_{\mathsf{Q}}$ with the above property. Then the following diagram commutes, i.e., there is a homomorphism $h : \mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma) \to \mathbf{F}_{\mathsf{Q}}$:

$$\begin{array}{ccc} \mathbf{F}_{\mathsf{Q}}(X) & \xrightarrow{k} & \mathbf{F}_{\mathsf{Q}} \\ {\scriptstyle \pi}\downarrow & \nearrow{\scriptstyle h} & \\ \mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma) & & \end{array}$$

Notice that there is no homomorphism $k : \mathbf{F}_{\mathsf{Q}}(X) \to \mathbf{F}_{\mathsf{Q}}$ such that $k(t) = k(u)$ for each $t \approx u \in \Sigma$ if and only if there is no assignment of variables in X validating Σ in $\mathbf{F}_{\mathsf{Q}}$. The latter is equivalent to $\mathbf{F}_{\mathsf{Q}} \models \Sigma \Rightarrow \emptyset$, and we have proved that (1) and (3) are equivalent. $\square$

We are now ready to prove the characterization theorem for passive structural completeness.

Theorem 4.23. *Let Q be a quasivariety, then the following are equivalent.*

1. *Q is passively structurally complete;*
2. *$\mathbf{F}_{\mathsf{Q}} \models \Sigma \Rightarrow \emptyset$ implies Σ is trivializing in Q;*
3. *either $\mathbf{F}_{\mathsf{Q}}$ is trivial, or Σ is trivializing in $\mathbf{Q}(\mathbf{F}_{\mathsf{Q}})$ implies Σ is trivializing in Q;*
4. *every nontrivial finitely presented algebra is unifiable.*

Proof. We first show that (1) and (2) are equivalent. By definition, Q is passively structurally complete if and only if each quasiequation $\Sigma \Rightarrow \delta$ where Σ is not unifiable in Q is valid in Q. That is, Σ not unifiable in Q implies $\mathsf{Q} \models \Sigma \Rightarrow \delta$, for all identities δ. By Proposition 4.22, the latter is equivalent to: $\mathbf{F}_{\mathsf{Q}} \models \Sigma \Rightarrow \emptyset$ implies $\mathsf{Q} \models \Sigma \Rightarrow \delta$, for all identities δ. From this it follows the particular case where $\delta = \{x \approx y\}$, with x, y not appearing in Σ. In turn, if $\mathbf{F}_{\mathsf{Q}} \models \Sigma \Rightarrow \emptyset$ implies $\mathsf{Q} \models \Sigma \Rightarrow (x \approx y)$, then clearly it also implies that $\mathsf{Q} \models \Sigma \Rightarrow \delta$ for any δ, and thus (1) $\Leftrightarrow$ (2).

Now, (2) and (3) are equivalent by Lemma 4.22, thus the first three points are equivalent. Let us now assume (2) and prove (4). We consider a nontrivial finitely presented algebra in Q, $\mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma)$. If it is not unifiable, by Lemma 4.22 $\mathbf{F}_{\mathsf{Q}} \models \Sigma \Rightarrow \emptyset$. By (2) this implies that Σ is trivializing in Q, that is, $\mathsf{Q} \models \Sigma \Rightarrow (x \approx y)$ (with x, y new variables). This clearly implies that $\mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma)$ is trivial, a contradiction. Thus $\mathbf{F}_{\mathsf{Q}}(X)/\theta_{\mathsf{Q}}(\Sigma)$ is unifiable and (4) holds.

Finally, we prove that (4) implies (1). Suppose $\Sigma \Rightarrow \delta$ is a passive quasiequation over variables in X, that is, Σ is not unifiable in Q. By Lemma 4.22 $\mathbf{F}_{\mathsf{Q}} \models \Sigma \Rightarrow \emptyset$. Let x, y be variables not in X, $X' = X \cup \{x, y\}$, and consider the finitely presented algebra $\mathbf{F}_{\mathsf{Q}}(X')/\theta_{\mathsf{Q}}(\Sigma)$; suppose by way of contradiction that it is not trivial. By (4) it is unifiable, that is, there is a homomorphism $h : \mathbf{F}_{\mathsf{Q}}(X')/\theta_{\mathsf{Q}}(\Sigma) \to \mathbf{F}_{\mathsf{Q}}$. Then, considering the natural epimorphism $\pi_\Sigma : \mathbf{F}_{\mathsf{Q}}(X') \to \mathbf{F}_{\mathsf{Q}}(X')/\theta_{\mathsf{Q}}(\Sigma)$, the composition $h\pi_\Sigma$ is an assignment from X' to $\mathbf{F}_{\mathsf{Q}}$ satisfying Σ; but $\mathbf{F}_{\mathsf{Q}} \models \Sigma \Rightarrow \emptyset$, a contradiction. Thus $\mathbf{F}_{\mathsf{Q}}(X')/\theta_{\mathsf{Q}}(\Sigma)$ is trivial, and therefore $x/\theta_{\mathsf{Q}}(\Sigma) = y/\theta_{\mathsf{Q}}(\Sigma)$. By Lemma 2.40 $\mathsf{Q} \models \Sigma \Rightarrow (x \approx y)$, and thus $\mathsf{Q} \models \Sigma \Rightarrow \delta$ and (1) holds. $\square$

Analogously to the case of passive universal completeness, if the smallest free algebra is isomorphic to all its ultraproducts we can improve the previous result.

Lemma 4.24. *Let Q be a quasivariety such that $\mathbf{I}(\mathbf{F}_{\mathsf{Q}}) = \mathbf{IP}_u(\mathbf{F}_{\mathsf{Q}})$, then the following are equivalent.*

1. *every nontrivial finitely presented algebra in* Q *is unifiable;*
2. *every nontrivial algebra in* Q *is unifiable.*

Proof. Notice that if $\mathbf{F}_{\mathsf{Q}}$ is trivial all algebras are unifiable and then the two statements are verified; let us then assume $\mathbf{F}_{\mathsf{Q}}$ nontrivial. The proof is analogous to the one of Lemma 3.32; we prove the nontrivial direction by contraposition. Consider an arbitrary algebra $\mathbf{A} = \mathbf{F}_{\mathsf{Q}}(X)/\theta \in \mathsf{Q}$ and assume that it is not unifiable. Then there is no assignment $h : \mathbf{F}_{\mathsf{Q}}(X) \to \mathbf{F}_{\mathsf{Q}}$ such that $\mathbf{F}_{\mathsf{Q}}, h \models \Sigma_\theta$, where $\Sigma_\theta = \{t \approx u : (t, u) \in \theta\}$. Equivalently, iff $\mathbf{F}_{\mathsf{Q}} \models \Sigma_\theta \Rightarrow \emptyset$. Now, the equational consequence relation relative to $\mathbf{F}_{\mathsf{Q}}$ is finitary (since all ultraproducts of $\mathbf{F}_{\mathsf{Q}}$ are isomorphic to $\mathbf{F}_{\mathsf{Q}}$); thus we obtain that $\mathbf{F}_{\mathsf{Q}} \models \Sigma'_\theta \Rightarrow \emptyset$, for Σ'_θ some finite subset of Σ_θ. But $\mathbf{F}_{\mathsf{Q}}(X)/\theta \not\models \Sigma'_\theta \Rightarrow (x \approx y)$ (with $x, y \notin X$), since it is nontrivial, which contradicts (2) of Theorem 4.23; equivalently it contradicts (1) and thus the proof is complete. □

Corollary 4.25. *Let* Q *be a quasivariety such that* $\mathbf{I}(\mathbf{F}_{\mathsf{Q}}) = \mathbf{IP}_u(\mathbf{F}_{\mathsf{Q}})$*, then the following are equivalent.*

1. Q *is passively structurally complete;*
2. $\mathbf{F}_{\mathsf{Q}} \models \Sigma \Rightarrow \emptyset$ *implies* Σ *is trivializing in* Q*;*
3. *either* $\mathbf{F}_{\mathsf{Q}}$ *is trivial, or* Σ *is trivializing in* $\mathbf{Q}(\mathbf{F}_{\mathsf{Q}})$ *implies* Σ *is trivializing in* Q*;*
4. *every nontrivial finitely presented algebra is unifiable;*
5. *every nontrivial algebra in* Q *is unifiable.*

Remark 4.26. The previous corollary can be applied whenever $\mathbf{F}_{\mathsf{Q}}$ is finite, therefore to all locally finite quasivarieties, but also to more complex classes of algebras, e.g., all subquasivarieties of FL_w (see Subsection 5.2). We observe that we do not know whether the condition $\mathbf{I}(\mathbf{F}_{\mathsf{Q}}) = \mathbf{IP}_u(\mathbf{F}_{\mathsf{Q}})$ is necessary to add the last point to Corollary 4.25.

Moreover as usual a quasivariety of logic Q is passively structurally complete if and only if $\mathcal{L}_{\mathsf{Q}}$ is passively structurally complete. We will see an interesting application of Theorem 4.23 (or Corollary 4.25) in substructural logics in Subsection 5.2; let us now show some other consequences. Given a quasivariety Q, let us consider the following set:

$$\mathcal{P}_{\mathsf{Q}} = \{\Sigma \Rightarrow \{x \approx y\} : \mathbf{F}_{\mathsf{Q}} \models \Sigma \Rightarrow \emptyset, \text{ and } x, y \notin \Sigma\}.$$

$\mathcal{P}_{\mathsf{Q}}$ axiomatizes a subquasivariety of Q, that we denote with P_{Q}. From Theorem 4.23 we get the following.

Corollary 4.27. *Let* Q *be a quasivariety such that the smallest free algebra* $\mathbf{F}_{\mathsf{Q}'}$ *of every nontrivial subquasivariety* $\mathsf{Q}' \subseteq \mathsf{Q}$ *is isomorphic to* $\mathbf{F}_{\mathsf{Q}}$*. Then every passively structurally complete subquasivariety of* Q *is contained in* P_{Q}*, which is the largest subquasivariety of* Q *that is passively structurally complete.*

Moreover, for locally finite quasivarieties the characterization theorem reads as follows.

Corollary 4.28. *Let* Q *be a locally finite quasivariety, then the following are equivalent.*

1. Q *is passively structurally complete;*
2. *every nontrivial algebra in* Q *is unifiable;*
3. *every nontrivial finite algebra in* Q *is unifiable;*
4. *every finite subdirectly irreducible in* Q *is unifiable.*

We call a nontrivial algebra $\mathbf{A}$ *Kollár* if it has no trivial subalgebras, and a quasivariety Q is a *Kollár quasivariety* if all nontrivial algebras in Q are Kollár. By [59] if $\mathbf{A}$ belongs to a Kollár quasivariety, $1_{\mathbf{A}}$, the largest congruence of $\mathbf{A}$, is compact in $\mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$; from there a straightforward application of Zorn's Lemma yields that if $\mathbf{A}$ is nontrivial there is at least one maximal congruence $\theta \in \mathrm{Con}_{\mathsf{Q}}(\mathbf{A})$ below $1_{\mathbf{A}}$ (i.e. $\mathbf{A}/\theta$ is relative simple).

Theorem 4.29. *If* Q *is a Kollár quasivariety and* $\mathbf{F}_{\mathsf{Q}}$ *is the only finitely generated relative simple algebra in* Q*, then* Q *is passively structurally complete.*

Proof. Let $\mathbf{A}$ be a nontrivial finitely presented algebra in Q; since Q is a Kollár quasivariety, $\mathbf{A}$ has a relative simple homomorphic image, that must be finitely generated. Hence it must be equal to $\mathbf{F}_{\mathsf{Q}}$, so $\mathbf{A}$ is unifiable; by Theorem 4.23 Q is passively structurally complete. □

Corollary 4.30. *For a locally finite Kollár quasivariety* Q *such that* $\mathbf{F}_{\mathsf{Q}}$ *has no proper subalgebra the following are equivalent:*

1. $\mathbf{F}_{\mathsf{Q}}$ *is the only finite relative simple algebra in* Q*;*
2. Q *is passively structurally complete.*

Proof. If (1) holds, than (2) holds by Theorem 4.29. Conversely assume (2); then every nontrivial finitely presented algebra in Q is unifiable. Since Q is locally finite $\mathbf{F}_{\mathsf{Q}}$ is finite and nontrivial since Q is Kollár and $\mathbf{F}_{\mathsf{Q}}$ is a subalgebra of all free algebras; now since $\mathbf{F}_{\mathsf{Q}}$ has no proper subalgebra no finite relative simple algebra different from $\mathbf{F}_{\mathsf{Q}}$ can be unifiable, but Q must contain at least a relative simple algebra [50, Theorem 3.1.8]. Hence $\mathbf{F}_{\mathsf{Q}}$ must be relative simple and (1) holds. □

The next results will allow us to find interesting applications in varieties of bounded lattices, which we will explore in Section 5.1. We say that an algebra $\mathbf{A}$ in a variety V is *flat* if $\mathbf{HS}(\mathbf{A})$ does not contain any simple algebra different from $\mathbf{F}_{\mathsf{V}}$.

Theorem 4.31. *Let* V *be a Kollár variety; if every finitely generated algebra in* V *is flat then* V *is passively structurally complete. If* V *is locally finite and* $\mathbf{F}_{\mathsf{V}}$ *has no proper subalgebras, then the converse holds as well.*

Proof. First, if $\mathbf{F}_{\mathsf{V}}$ is trivial then V is vacuously passively structurally complete. If $\mathbf{F}_{\mathsf{V}}$ is nontrivial and every finitely generated algebra is flat, then the only finitely generated simple lattice in V must be $\mathbf{F}_{\mathsf{V}}$; since V is Kollár, V is passively structurally complete by Theorem 4.29.

For the converse if V is locally finite and passively structurally complete, then $\mathbf{F}_{\mathsf{V}}$ is the only finite simple algebra in V by Corollary 4.30. It follows that no finite simple algebra different from $\mathbf{F}_{\mathsf{V}}$ can appear in $\mathbf{HS}(\mathbf{A})$ for any finite $\mathbf{A} \in \mathsf{V}$. So every finite algebra in V must be flat. □

Theorem 4.32. *Let* V *be a congruence distributive Kollár variety and let* W *be a finitely generated subvariety of* V*. Then:*

1. *if each generating algebra is flat, then* W *is passively structurally complete;*
2. *if* W *is passively structurally complete and* $\mathbf{F}_{\mathsf{W}}$ *has no proper subalgebras, then each generating algebra is flat.*

Proof. For (1), suppose that $\mathsf{W} = \mathbf{V}(K)$ where K is a finite set of finite algebras; by Jónsson Lemma any simple algebra in V is in $\mathbf{HS}(K)$. If K consists entirely of flat algebras, then there cannot be any simple algebra in V different from $\mathbf{F}_{\mathsf{W}}$, so W is passively structurally complete by Theorem 4.29.

For (2) if $\mathbf{A} \in K$ is not flat, then there is an algebra $\mathbf{B} \in \mathbf{HS}(K)$ which is simple and different from $\mathbf{F}_{\mathsf{V}}$. Clearly $\mathbf{B} \in \mathsf{W}$, so by Corollary 4.30 W is not passively structurally complete. □

5. Applications to algebra and logic

In this last section we will see some relevant examples and applications of our results in the realm of algebra and (algebraic) logic that deserve a deeper exploration than the examples already presented in the previous sections. We will start with focusing on varieties of lattices and bounded lattices, and then move to *residuated lattices*; these are the equivalent algebraic semantics of substructural logics, the latter seen as extensions of the Full Lambek Calculus $\mathcal{FL}$ (see [43]).

As a main result, in the last subsection we present the logical counterpart of the characterization of passive structural completeness in substructural logics with weakening, that is, such a logic is passively structurally complete if and only if every classical contradiction is explosive in it; building on this, from the algebraic perspective, we are able to axiomatize the largest variety of semilinear bounded commutative integral residuated lattices that is passively structurally complete (and such that all of its quasivarieties have this property). Notice that this characterization establishes negative results as well: if a logic (or a quasivariety) is not passively structurally complete, a fortiori it is not structurally complete either.

5.1. (Bounded) lattices

In this subsection we start with some results about primitive (quasi)varieties of lattices, and then move to bounded lattices, where in particular we obtain some new results about passively structurally complete varieties.

5.1.1. Primitivity in lattices

Many examples of quasivarieties that are primitive can be found in lattices satisfying *Whitman's condition* (W); Whitman's condition is a clause that holds in free lattices:

$$\{x \wedge y \leqslant u \vee v\} \Rightarrow \{x \leqslant u \vee v, y \leqslant u \vee v, x \wedge y \leqslant u, x \wedge y \leqslant v\}. \tag{W}$$

We say that a finite algebra is *finitely projective* in a class K if it is projective in the subclass of the finite algebras in K. Now a finite lattice is finitely projective in the variety of all lattices if and only if it satisfies (W) ([33]), which implies:

Lemma 5.1. *Let* K *be a finite set of finite lattices. If every lattice in* K *satisfies (W) then* $\mathbf{Q}(\mathsf{K})$ *is primitive.*

Proof. $\mathbf{Q}(\mathsf{K})$ is locally finite and by Theorem 2.3(2) every relative subdirectly irreducible lies in $\mathbf{IS}(\mathsf{K})$; as (W) is a universal sentence it is preserved under subalgebras, thus they all satisfy (W) and hence they are all finitely projective in the variety of lattices and then also in $\mathbf{Q}(\mathsf{K})$. By Theorem 4.11(4), $\mathbf{Q}(\mathsf{K})$ is primitive. □

Luckily finite lattices satisfying (W) abound, so there is no shortage of primitive quasivarieties of lattices. For varieties of lattices the situation is slightly different; in particular, Lemma 4.4 may suggest that it is not enough that all lattices in K are weakly projective in $\mathbf{V}(\mathsf{K})$ to guarantee that $\mathbf{V}(\mathsf{K})$ is structurally complete.

First we introduce some lattices: $\mathbf{M}_n$ for $3 \leqslant n \leqslant \omega$ are the modular lattices consisting of a top, a bottom, and n atoms while the lattices $\mathbf{M}_{3,3}$ and $\mathbf{M}^+_{3,3}$ are displayed in Fig. 4.

Observe that all the above lattices, with the exception of $\mathbf{M}_{3,3}$, satisfy (W). Now Gorbunov ([50], Theorem 5.1.29) showed that $\mathbf{M}^+_{3,3}$ is *splitting* in the lattice of subquasivarieties of modular lattices. This means that for any quasivariety Q of modular lattices, either $\mathbf{M}^+_{3,3} \in \mathsf{Q}$ or else $\mathsf{Q} = \mathbf{Q}(\mathbf{M}_n)$ for some $n \leqslant \omega$. Observe that, for $n < \omega$, $\mathsf{Q}(\mathbf{M}_n)$ is primitive by Lemma 5.1 and $\mathbf{V}(\mathbf{M}_n) = \mathbf{Q}(\mathbf{M}_n)$ by Lemma 2.4; then the only thing left to show is that $\mathbf{V}(\mathbf{M}_\omega)$ is a primitive variety and Gorbunov did exactly that. On the other hand

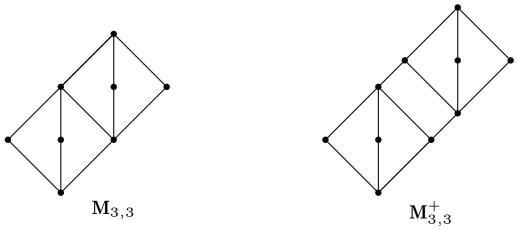


Fig. 4. $\mathbf{M}_{3,3}$ and $\mathbf{M}_{3,3}^{+}$.

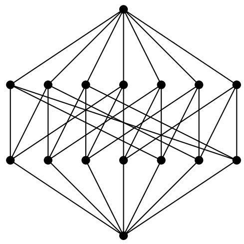


Fig. 5. The Fano lattice.

no variety V of lattices containing $\mathbf{M}_{3,3}^{+}$ can be primitive; in fact $\mathbf{M}_{3,3}$ is a simple homomorphic image of $\mathbf{M}_{3,3}^{+}$ that cannot be embedded in $\mathbf{M}_{3,3}^{+}$. By Lemma 2.4, $\mathbf{Q}(\mathbf{M}_{3,3}^{+}) \subsetneq \mathbf{V}(\mathbf{M}_{3,3}^{+})$, so V contains a strict (i.e. not a variety) subquasivariety and cannot be primitive. Thus Gorbunov's result can be formulated as: *a variety of modular lattices is primitive if and only if it does not contain* $\mathbf{M}_{3,3}^{+}$. Note that it cannot be improved to quasivarieties: since $\mathbf{M}_{3,3}^{+}$ satisfies (W), $\mathbf{Q}(\mathbf{M}_{3,3}^{+})$ is primitive by Lemma 5.1. However we observe:

Lemma 5.2. *If* Q *is a quasivariety of modular lattices and* $\mathbf{M}_{3,3} \in \mathsf{Q}$, *then* Q *is not primitive.*

Proof. Clearly the two element lattice $\mathbf{2} \in \mathsf{Q}$ and it is easy to check that $\mathbf{M}_{3,3}^{+} \leqslant_{sd} \mathbf{2} \times \mathbf{M}_{3,3}$ so $\mathbf{M}_{3,3}^{+} \in \mathsf{Q}$ and $\mathbf{M}_{3,3} \in \mathbf{H}(\mathbf{M}_{3,3}^{+})$. Since $\mathbf{M}_{3,3}$ cannot be embedded in $\mathbf{M}_{3,3}^{+}$, in Q there is a simple finite (so finitely presented, since lattices have finite type) algebra that is not weakly projective. By Theorem 4.11, Q is not primitive. □

Therefore to find a variety of modular lattices that is structurally complete but not primitive it is enough to find a finite lattice $\mathbf{F}$ such that $\mathbf{M}_{3,3}^{+} \in \mathbf{V}(\mathbf{F})$ but $\mathsf{K} = \{\mathbf{F}\}$ satisfies the hypotheses of Lemma 4.4. Bergman in [13, Example 2.14.4] observed that the *Fano lattice* $\mathbf{F}$ has exactly those characteristics; the Fano lattice is the (modular) lattice of subspaces of $(\mathbb{Z}_2)^3$ seen as a vector space on $\mathbb{Z}_2$ and it is displayed in Fig. 5.

Now: $\mathbf{F}$ is projective in $\mathbf{V}(\mathbf{F})$ ([52, Theorem 6.2]), and the subdirectly irreducible members of $\mathbf{V}(\mathbf{F})$ are exactly $\mathbf{2}, \mathbf{M}_3, \mathbf{M}_{3,3}, \mathbf{F}$, which are all subalgebras of $\mathbf{F}$. It follows that $\mathbf{F}$ does not satisfy (W) (since $\mathbf{M}_{3,3}$ does not), $\mathbf{V}(\mathbf{F})$ is structurally complete by Lemma 4.4 and not primitive by Lemma 5.2 (since $\mathbf{M}_{3,3} \in \mathbf{V}(\mathbf{F})$). Also $\mathbf{Q}(\mathbf{F})$ is structurally complete (again by Lemma 4.4) but, since $\mathbf{M}_{3,3} \in \mathbf{Q}(\mathbf{F})$, it cannot be primitive as well.

Primitive varieties of lattices have been studied in depth in [54]; there the authors proved the following theorem that explains the behavior we have seen above.

Theorem 5.3 *([54]). If* $\mathbf{A}$ *is a lattice satisfying (W), then* $\mathbf{V}(\mathbf{A})$ *is primitive if and only if every subdirectly irreducible lattice in* $\mathbf{HS}(\mathbf{A})$ *satisfies (W).*

We believe that many of the techniques in [54] could be adapted to gain more understanding of primitive quasivarieties of lattices, but proceeding along this path would make this part too close to being a paper in

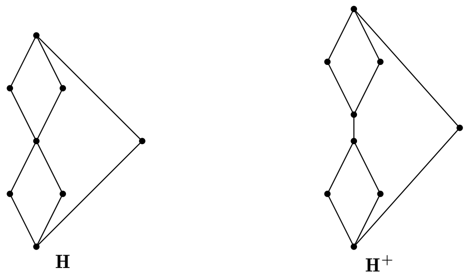


Fig. 6. $\mathbf{H}$ and $\mathbf{H}^+$.

lattice theory, and we have chosen a different focus. We only borrow an example from [54] that shows that Lemma 5.1 cannot be inverted for quasivarieties. Let $\mathbf{H}^+, \mathbf{H}$ be the lattices in Fig. 6.

It is easily seen that the pair $\mathbf{H}^+, \mathbf{H}$ behaves almost like the pair $\mathbf{M}^+_{3,3}, \mathbf{M}_{3,3}$: $\mathbf{H}^+$ satisfies (W) (so $\mathbf{Q}(\mathbf{H}^+)$ is primitive), $\mathbf{H}$ does not satisfy (W) and $\mathbf{H}^+ \leqslant_{sd} \mathbf{2} \times \mathbf{H}$. As above we can conclude that $\mathbf{V}(\mathbf{H}^+)$ is not primitive. However $\mathbf{V}(\mathbf{H})$ is primitive [54] so $\mathbf{Q}(\mathbf{H})$ is a primitive quasivariety generated by a finite lattice not satisfying (W).

5.1.2. Bounded lattices

We now focus on applications of our results in varieties of bounded lattices. A *bounded* lattice is a lattice with two constants, 1 and 0, that represent the top and the bottom of the lattice respectively. Bounded lattices form a variety L^b that shares many features with variety of lattices. In particular, let $\mathbf{2}^b$ be the two element bounded lattice, then the variety of bounded distributive lattices is $\mathsf{D}^b = \mathbf{ISP}(\mathbf{2}^b)$. Therefore

$$\mathbf{Q}(\mathbf{F}_{\mathsf{D}_b}(\omega)) \subseteq \mathsf{D}^b = \mathbf{ISP}(\mathbf{2}^b) \subseteq \mathbf{Q}(\mathbf{F}_{\mathsf{D}_b}(\omega))$$

and by Theorem 4.1, the variety of bounded distributive lattices D^b is structurally complete, as shown in [35]. In [14] it is shown that locally finite, congruence modular, minimal varieties are q-minimal; since these hypotheses apply to D^b, the latter is also primitive. However, it is not non-negatively universally complete; it is a nice exercise in general algebra to show that for any variety V of bounded lattices, 1 is join irreducible in $\mathbf{F}_{\mathsf{V}}(\omega)$. It follows that

$$\{x \vee y \approx 1\} \Rightarrow \{x \approx 1, y \approx 1\}$$

is an active clause that is admissible in V. But it is clearly not derivable, since any nontrivial variety of bounded lattices contains $\mathbf{2}^b \times \mathbf{2}^b$ which does not satisfy the clause.

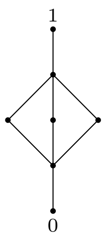
Proposition 5.4. *No nontrivial variety of bounded lattices is actively universally complete.*

Actually something more is true; if V is a variety of bounded lattices that is structurally complete, then by Theorem 4.6, each finite subdirectly irreducible algebra $\mathbf{A} \in \mathsf{V}$ must satisfy the above clause, i.e. 1 must be join irreducible in $\mathbf{A}$. But the bounded lattices $\mathbf{N}^b_5$ and $\mathbf{M}^b_3$ do not satisfy that, so any structurally complete variety of bounded lattice must omit them both. As in the unbounded case, this means that the variety must be the variety of bounded distributive lattices. Thus:

Proposition 5.5 *([35]). The variety of bounded distributive lattices is the only (active) structurally complete variety of bounded lattices.*

We have seen that active structural completeness does not have much meaning in bounded lattices. Passive structural completeness has more content, as we are now going to show. Notice that any variety of bounded lattices is Kollár and $\mathbf{F}_{\mathsf{V}} = \mathbf{2}^b$ for any variety V of bounded lattices. Since $\mathbf{2}^b$ is simple and has no proper subalgebras, any simple bounded lattice not isomorphic with $\mathbf{2}^b$ is not unifiable; in particular if a variety V contains a finite simple lattice $\mathbf{L}$ different from $\mathbf{2}^b$, then $\mathbf{V}(\mathbf{L})$ cannot be passively structurally complete by Corollary 4.30, and hence neither can V.

We will use this fact to show that the only variety of bounded modular lattices that is passively structurally complete is the one we already know to possess that property, i.e. the variety D^b of bounded distributive lattices. We will use the notion of *splitting* algebra in a variety. An algebra $\mathbf{A}$ splitting in a variety V if $\mathbf{A} \in \mathsf{V}$ and there is a subvariety $\mathsf{W}_{\mathbf{A}} \subseteq \mathsf{V}$ such that $\mathbf{A} \notin \mathsf{W}_{\mathbf{A}}$ and for any variety $\mathsf{U} \subseteq \mathsf{V}$ either $\mathbf{A} \in \mathsf{U}$ or $\mathsf{U} \subseteq \mathsf{W}_{\mathbf{A}}$. This simply means that the lattice of subvarieties of V is the disjoint union of the filter generated by $\mathbf{V}(\mathbf{A})$ and the ideal generated by $\mathsf{W}_{\mathbf{A}}$. A key step is to show that $\mathbf{M}_3^b$ is splitting in the variety of bounded modular lattices; in the unbounded case, this follows from the fact that $\mathbf{M}_3$ is projective and subdirectly irreducible. However, $\mathbf{M}_3^b$ is not projective in the variety of bounded modular lattices. Indeed, the lattice in the figure below is a bounded modular lattice having $\mathbf{M}_3^b$ as homomorphic image, but it has no subalgebra isomorphic with $\mathbf{M}_3^b$, which hence cannot be a retract.



However we can use A. Day idea in [34]; a finite algebra $\mathbf{A}$ is *finitely projected* in a variety V if for any $\mathbf{B} \in \mathsf{V}$ if $f : \mathbf{B} \longrightarrow \mathbf{A}$ is surjective, then there is a finite subalgebra $\mathbf{C}$ of $\mathbf{B}$ with $f(\mathbf{C}) \cong \mathbf{A}$. Clearly any finite projective lattice is finitely projected. The key result is:

Theorem 5.6. *([34], Theorem 3.7) If V is a congruence distributive variety, then any finitely projected subdirectly irreducible algebra in V is splitting in V.*

Lemma 5.7. *Let V^b be a variety of bounded lattices and let V be the variety of lattice subreducts of V^b. If $\mathbf{L}$ is finitely projected in V, then $\mathbf{L}^b$ is finitely projected in V^b.*

Proof. The fact that V is indeed a variety is easy to check. Let now $\mathbf{A}^b \in \mathsf{V}^b$ and suppose that there is an onto homomorphism $f : \mathbf{A}^b \longrightarrow \mathbf{L}^b$; then f is onto from $\mathbf{A}$ to $\mathbf{L}$ and since $\mathbf{L}$ is finitely projected in V there is a subalgebra $\mathbf{B}$ of $\mathbf{A}$ with $f(\mathbf{B}) \cong \mathbf{L}$. But $B \cup \{0, 1\}$ is the universe of a finite subalgebra $\mathbf{C}$ of $\mathbf{A}^b$, and of course $f(0) = 0$, $f(1) = 1$; then $f(\mathbf{C}) \cong \mathbf{L}^b$ and so $\mathbf{L}^b$ is finitely projected in V^b. □

Theorem 5.8. *A variety of modular bounded lattices is passively structurally complete if and only if it is the variety of bounded distributive lattices.*

Proof. D^b is structurally complete, hence passively structurally complete. Conversely observe that $\mathbf{M}_3$ is projective in the variety of modular lattices, so $\mathbf{M}_3^b$ is finitely projected in the variety of bounded modular lattices. Hence, by Theorem 5.6, $\mathbf{M}_3^b$ is splitting in the variety, which means that for any variety V of bounded modular lattices, either $\mathbf{M}_3^b \in \mathsf{V}$ or V is D^b. But if $\mathbf{M}_3^b \in \mathsf{V}$ then V cannot be passively universally complete, since $\mathbf{M}_3^b$ is simple. The conclusion follows. □

In order to find other relevant varieties of bounded lattices that are passively structurally complete, we are going to take a closer look at flat lattices. Finding flat bounded lattices is not hard since the lattice

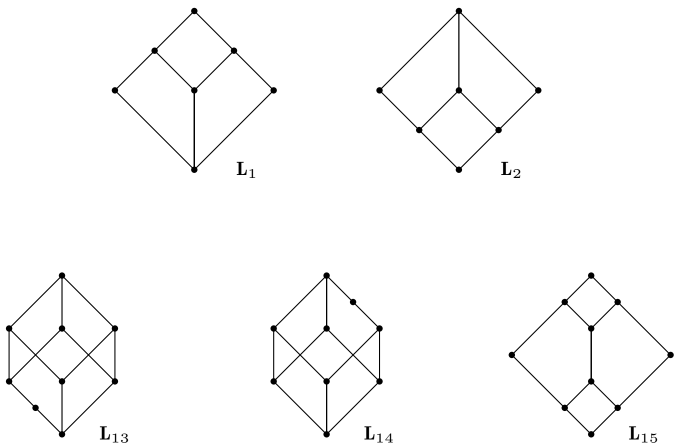


Fig. 7. $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_{13}, \mathbf{L}_{14}, \mathbf{L}_{15}$.

of subvarieties of lattices has been studied thoroughly and a lot is known about it (an excellent survey is [55]). Clearly $\mathbf{N}_5$ is flat and hence so is $\mathbf{N}_5^b$; however we know exactly all the covers of the minimal non-distributive varieties of lattices (which is of course $\mathbf{V}(\mathbf{N}_5)$). There are 15 finite subdirectly irreducible non-simple lattices, commonly called $\mathbf{L}_1, \dots, \mathbf{L}_{15}$ (some of them are in Fig. 7) that generate all the join irreducible (in the lattice of subvarieties) covers of $\mathbf{V}(\mathbf{N}_5)$. It is easy to see their bounded versions all are join irreducible covers of $\mathbf{V}(\mathbf{N}_5^b)$ in the lattice of subvarieties of bounded lattices. We suspect that they are also the only join irreducible covers; one needs only to check that the (rather long) proof for lattices [58] goes through for bounded lattices but we leave this simple but tedious task to the reader. In any case for $i = 1, \dots, 15$ the subdirectly irreducible algebras in $\mathbf{V}(\mathbf{L}_i^b)$ are exactly $\mathbf{2}^b, \mathbf{N}_5^b$ and $\mathbf{L}_i^b$ (via a straightforward application of Jónsson Lemma); so each $\mathbf{L}_i^b$ is flat and each $\mathbf{V}(\mathbf{L}_i^b)$ is passively structurally complete (by Theorem 4.32).

Let's make more progress: consider the rules

$$x \wedge y \approx x \wedge z \quad \Rightarrow \quad x \wedge y \approx x \wedge (y \vee z) \tag{$SD_\wedge$}$$

$$x \vee y \approx x \vee z \quad \Rightarrow \quad x \vee y \approx x \vee (y \wedge z). \tag{$SD_\vee$}$$

A lattice is *meet semidistributive* if it satisfies $SD_\wedge$, *join semidistributive* if it satisfies $SD_\vee$ and *semidistributive* if it satisfies both. Clearly (meet/join) semidistributive lattices form quasivarieties called $\mathsf{SD}_\wedge$, $\mathsf{SD}_\vee$ and SD respectively, and so do their bounded versions. It is a standard exercise to show that homomorphic images of a finite (meet/join) semidistributive lattices are (meet/join) semidistributive. It is also possible to show none of the three quasivariety (and their bounded versions) is a variety (see [55] p. 82 for an easy argument); they are also not locally finite since for instance $\mathbf{F} = \mathbf{F}_{\mathsf{SD}}(x, y, z)$ is infinite; hence $\mathbf{F}^b$ is a bounded infinite three-generated lattice and thus SD^b is not locally finite as well. A variety V of (bounded) lattices is (meet/join) semidistributive if $\mathsf{V} \subseteq \mathsf{SD}$ ($\mathsf{V} \subseteq \mathsf{SD}_\wedge$ / $\mathsf{V} \subseteq \mathsf{SD}_\vee$).

We need a little bit of lattice theory. A filter of $\mathbf{L}$ is an upset F of $\mathbf{L}$ that is closed under meet; a filter is *prime* if $a \vee b \in F$ implies $a \in F$ or $b \in F$. An *ideal* I of $\mathbf{L}$ is the dual concept, i.e. a downset that is closed under join; an ideal is *prime* if $a \wedge b \in I$ implies $a \in I$ or $b \in I$. The following lemma is straightforward.

Lemma 5.9. *If F is a prime filter of $\mathbf{L}$ (I is a prime ideal of $\mathbf{L}$), then $L \backslash F$ is a prime ideal of $\mathbf{L}$ ($\mathbf{L} \backslash I$ is a prime filter of $\mathbf{L}$).*

Lemma 5.10. *Any bounded (meet/join) semidistributive lattice is unifiable in the variety of bounded lattices.*

Proof. Let $\mathbf{L}$ be bounded and meet semidistributive. Since $\mathbf{L}$ is lower bounded by 0 a standard application of Zorn Lemma yields a maximal proper filter F of $\mathbf{L}$; we claim that F is also prime. Let $a, b \notin F$; then the filter generated by $F \cup \{a\}$ must be the entire lattice. Hence there must be a $c \in F$ with $c \wedge a = 0$; similarly there must be a $d \in F$ with $d \wedge b = 0$. Let $e = c \wedge d$; then $e \in F$ and $e \wedge a = e \wedge b = 0$ and by meet semidistributivity $e \wedge (a \vee b) = 0$. But if $a \vee b \in F$, then $0 \in F$, a clear contradiction. Hence $a \vee b \notin F$ and F is prime.

Let now $\varphi : \mathbf{L} \Longrightarrow \mathbf{2}^b$ defined by

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in F; \\ 0, & \text{if } x \notin F. \end{cases}$$

Using the fact that F is prime and $L \backslash F$ is prime it is straightforward to check that φ is a homomorphism. Therefore $\mathbf{L}$ is unifiable.

A dual proof shows that the conclusion holds for join semidistributivity and a fortiori for semidistributivity. □

Proposition 5.11. *Any bounded finite (meet/join) semidistributive lattice is flat.*

Proof. If $\mathbf{L}$ is finite and (meet/join) semidistributive, every lattice in $\mathbf{HS}(\mathbf{L})$ is finite and (meet/join) semidistributive. So it is unifiable and, if simple, it must be equal to $\mathbf{2}^b$; therefore $\mathbf{L}$ is flat. □

Corollary 5.12. *Every locally finite (meet/join) semidistributive variety of bounded lattices is passively structurally complete.*

In [62] several (complex) sets of equations implying semidistributivity are studied; one of them is useful to us, since it describes a class of locally finite varieties. The description is interesting in that involves some of the $\mathbf{L}_i's$ we have introduced before.

Theorem 5.13. *[62] There exists a finite set Γ of lattices equations such that, if V is any variety of lattices such that $\mathsf{V} \vDash \Gamma$, then the following hold:*

1. *V is semidistributive;*
2. *V is locally finite;*
3. *only $\mathbf{L}_{13}, \mathbf{L}_{14}, \mathbf{L}_{15} \in \mathsf{V}$.*

A variety satisfying Γ is called *almost distributive* and it is straightforward to check that a similar result holds for varieties of bounded lattices. Therefore:

Proposition 5.14. *Every almost distributive variety of bounded lattices is passively structurally complete.*

We close this subsection with a couple of observations; first $\mathbf{V}(\mathbf{L}_1^b, \mathbf{L}_2^b)$ is a variety of bounded lattices that is passively structurally complete (by Theorem 4.32) but neither meet nor join semidistributive. Next, what about infinite flat (bounded) lattices? We stress that in [65] there are several examples of this kind and we believe that a careful analysis of the proofs therein could give some insight on how to construct a non locally finite variety of bounded lattices that it is passively structurally complete. But again, this is not a paper in lattice theory; therefore we defer this investigation.

5.2. Substructural logics and residuated lattices

Originally, *substructural logics* were introduced as logics which, when formulated as Gentzen-style systems, lack some (including "none" as a special case) of the three basic *structural rules* (i.e. exchange, weakening and contraction) of classical logic. Nowadays, substructural logics are often intended as those logics whose equivalent algebraic semantics are residuated structures, and they encompass most of the interesting non-classical logics: intuitionistic logic, basic logic, fuzzy logics, relevance logics and many other systems. Precisely, by substructural logics we mean here the extensions of the Full Lambek Calculus $\mathcal{FL}$, whose equivalent algebraic semantics are given by quasivarieties of FL-algebras, particular residuated lattices that we shall now define (see [43] for details and a survey on substructural logics). We observe that $\mathcal{FL}$ and its axiomatic extensions are actually *strongly algebraizable*, i.e. their equivalent algebraic semantics are all *varieties* of FL-algebras.

A *residuated lattice* is an algebra $\mathbf{A} = \langle A, \vee, \wedge, \cdot, /, \backslash, 1\rangle$ where

1. $\langle A, \vee, \wedge\rangle$ is a lattice;
2. $\langle A, \cdot, 1\rangle$ is a monoid;
3. $/$ and $\backslash$ are the right and left divisions w.r.t. $\cdot$, i.e., $x \cdot y \leqslant z$ iff $y \leqslant x\backslash z$ iff $x \leqslant z/y$, where $\leqslant$ is given by the lattice ordering.

Residuated lattices form a variety RL and an equational axiomatization, together with many equations holding in these very rich structures, can be found in [19].

A residuated lattice $\mathbf{A}$ is *integral* if it satisfies the equation $x \leqslant 1$; it is *commutative* if $\cdot$ is commutative, and in this case the divisions coincide: $x\backslash y = y/x$, and they are usually denoted with $x \to y$. The classes of residuated lattices that satisfy any combination of integrality and commutativity are subvarieties of RL. We shall call the variety of integral residuated lattices IRL, commutative residuated lattices CRL, and their intersection CIRL.

Residuated lattices with an extra constant 0 in the language are called FL-algebras, since they are the equivalent algebraic semantics of the Full Lambek calculus $\mathcal{FL}$. Residuated lattices are then the equivalent algebraic semantics of 0-free fragment of $\mathcal{FL}$, $\mathcal{FL}^+$. An FL-algebra is 0-*bounded* if it satisfies the inequality $0 \leqslant x$ and the variety of zero-bounded FL-algebras is denoted by FL_o; integral and 0-bounded $\mathcal{FL}$-algebras are called $\mathcal{FL}_w$ algebras (since they are the equivalent algebraic semantics of the Full Lambek Calculus with weakening), and we call its commutative subvariety $\mathcal{FL}_{ew}$.

Restricting ourselves to the commutative case there is another interesting equation, prelinearity:

$$(x \to y) \vee (y \to x) \approx 1.$$

It can be shown (see [19] and [56]) that a subvariety of $\mathsf{FL}_{\mathsf{ew}}$ or CIRL satisfies the above equation if and only if any algebra therein is a subdirect product of totally ordered algebras, and this implies that all the subdirectly irreducible algebras are totally ordered. Such varieties are called *semilinear* (or *representable*) and the subvariety axiomatized by that equation is the largest subvariety of $\mathsf{FL}_{\mathsf{ew}}$ or CIRL that is semilinear. The semilinear subvariety of FL_{ew} is usually denoted by MTL, since it is the equivalent algebraic semantics of Esteva-Godo's *Monoidal t-norm based logic* [38].

5.2.1. Active universal completeness

We have already seen examples of subvarieties of FL_{ew}-algebras that are actively universally complete, but those were all locally finite subvarieties of BL-algebras, that is, MTL-algebras satisfying the divisibility equation: $x \wedge y = x(x \to y)$. In this section we will display a different class of examples. If $\mathbf{A}$ is any algebra a congruence $\theta \in \mathrm{Con}(\mathbf{A})$ is a *factor congruence* if there is a $\theta' \in \mathrm{Con}(\mathbf{A})$ such that $\theta \vee \theta' = 1_{\mathbf{A}}$, $\theta \wedge \theta' = 0_{\mathbf{A}}$

and θ, θ' permute. The pair (θ, θ') is called a *pair of factor congruences*. It is an easy exercise in general algebra to show that in this case $\mathbf{A} \cong \mathbf{A}/\theta \times \mathbf{A}/\theta'$; note that $1_\mathbf{A}$ and $0_\mathbf{A}$ are a pair of factor congruences that give a trivial decomposition. A less known fact (that appears in [31]) is:

Lemma 5.15. *Let* $\mathbf{A}$ *be any algebra and* (θ, θ') *a pair of factor congruences on* $\mathbf{A}$*; then* $\mathbf{A}/\theta$ *is a retract of* $\mathbf{A}$ *if and only if there is a homomorphism* $h : \mathbf{A}/\theta \longrightarrow \mathbf{A}/\theta'$.

Proof. Suppose first that there is a homomorphism $h : \mathbf{A}/\theta \longrightarrow \mathbf{A}/\theta'$. Since $\mathbf{A} \cong \mathbf{A}/\theta \times \mathbf{A}/\theta'$ for $u \in A$, $u = (a/\theta, b/\theta')$, we set $f(u) = a/\theta$; then $f : \mathbf{A} \longrightarrow \mathbf{A}/\theta$ is clearly an onto homomorphism, since $(a/\theta, a/\theta') \in A$ for all $a \in A$. Let

$$g(a/\theta) = (a/\theta, h(a/\theta)).$$

One can check that g is a homomorphism with standard calculations and clearly $fg = id_{\mathbf{A}/\theta}$. Hence $\mathbf{A}/\theta$ is a retract of $\mathbf{A}$.

Conversely suppose that f, g witness a retraction from $\mathbf{A}/\theta$ in $\mathbf{A}$; then if $g(a/\theta) = (u/\theta, v/\theta')$, set $h(a/\theta) = v/\theta'$. It is then easy to see that h is a homomorphism and the thesis holds. □

Observe that in any FL-algebra every compact (i.e., finitely generated) congruence is principal; as a matter of fact if $\mathbf{A}$ is in FL, $X = \{(a_1, b_1), \dots, (a_n, b_n)\}$ is a finite set of pairs from A and $p = \bigwedge_{i=1}^n [(a_i \backslash b_i) \wedge (b_i \backslash a_i) \wedge 1]$ then $\vartheta_\mathbf{A}(X) = \vartheta_\mathbf{A}(p, 1)$.

Theorem 5.16. *Let* Q *be a quasivariety of* FL_w*-algebras in which every principal congruence is a factor congruence; then* Q *has projective unifiers.*

Proof. Let $\mathbf{F}_\mathsf{Q}(X)/\theta$ be a finitely presented unifiable algebra in Q; then there is an onto homomorphism from $\mathbf{F}_\mathsf{Q}(X)/\theta(\Sigma)$ to $\mathbf{F}_\mathsf{Q} = \mathbf{2}$. Now $\theta = \theta(\Sigma)$ is a principal congruence, hence it is a factor congruence with witness θ', i.e. $\mathbf{F}_\mathsf{Q}(X) \cong \mathbf{F}_\mathsf{Q}(X)/\theta \times \mathbf{F}_\mathsf{Q}(X)/\theta'$. If $\theta' = 1_\mathbf{A}$, then $\mathbf{F}_\mathsf{Q}(X) = \mathbf{F}_\mathsf{Q}(X)/\theta$ and so it is projective. Otherwise $\mathbf{F}_\mathsf{Q} = \mathbf{2}$ is embeddable in $\mathbf{F}_\mathsf{Q}(X)/\theta'$; hence there is a homomorphism from $\mathbf{F}_\mathsf{Q}(X)/\theta$ to $\mathbf{F}_\mathsf{Q}(X)/\theta'$. By Lemma 5.15 $\mathbf{F}_\mathsf{Q}(X)/\theta$ is a retract of $\mathbf{F}_\mathsf{Q}(X)$, i.e. it is projective. □

So any quasivariety of FL_w-algebras with the property that every principal congruence is a factor congruence is actively universally complete (Theorem 3.23); really it is active primitive universally complete by Theorem 3.27, since $\mathbf{F}_\mathsf{Q}$ is the two-element Boolean algebra for any nontrivial quasivariety Q of FL_w-algebras. We observe in passing that for any FL_w algebra every factor congruence is principal; this is because every variety of FL_w-algebras is Kollár and congruence distributive. Now, discriminator varieties of FL_{ew}-algebras have been completely described in [61]; as a consequence we have:

Theorem 5.17. *For a variety* V *of* FL_{ew}*-algebras the following are equivalent:*

1. V *is a discriminator variety;*
2. V *is semisimple, i.e. all the subdirectly irreducible members of* V *are simple;*
3. *there is an* $n \in \mathbb{N}$ *such that* $\mathsf{V} \vDash x \vee \neg(x^n) \approx 1$*;*
4. *for any* $\mathbf{A} \in \mathsf{V}$ *every compact (i.e. principal) congruence is a factor congruence.*

Proof. The equivalence of (1), (2) and (3) has been proved in [61]. Assume then (1); it is well-known that in every discriminator variety every principal congruence is a factor congruence. In fact if V is a discriminator variety with discriminator term $t(x, y, z)$ let for any $\mathbf{A} \in \mathsf{V}$ and $a, b \in A$

$$\theta_{\mathbf{A}}(a,b) = \{(u,v) : t(a,b,u) = t(a,b,v)\}$$
$$\gamma_{\mathbf{A}}(a,b) = \{(u,v) : t(a,t(a,b,u),u) = t(a,t(a,b,v),v)\}.$$

Using the properties of the discriminator term it is easy to verify that they are congruences and the complement of each other; since discriminator varieties are congruence permutable they are factor congruences and (4) holds.

Conversely assume (4) and let $\mathbf{A}$ be a subdirectly irreducible member of V. Let $\mu_{\mathbf{A}}$ be the minimal nontrivial congruence of $\mathbf{A}$; then $\mu_{\mathbf{A}}$ is principal, so it must be a factor congruence. This is possible if and only if $\mu_{\mathbf{A}} = 1_{\mathbf{A}}$; therefore $\mathbf{A}$ is simple, and V is semisimple. □

Corollary 5.18. *Every discriminator (or, equivalently, semisimple) variety* V *of* FL_{ew}*-algebras is active primitive universal.*

We observe that Theorem 5.16 does not add anything as far as BL-algebras are concerned; in fact any discriminator variety of FL_{ew}-algebras must satisfy $x^n \approx x^{n+1}$ for some n ([61]) and the varieties of BL-algebras with that property are exactly the locally finite varieties, which we already pointed out are actively universally complete in Example 3.25.

5.2.2. Passive structural completeness

A particularly interesting application of our characterization of passively structurally complete varieties is in the subvariety of integral and 0-bounded FL-algebras. Let us rephrase Theorem 4.23 in this setting. First, using residuation it is easy to see that every finite set of identities in FL is equivalent to a single identity. Moreover, in every nontrivial subquasivariety Q of FL_w, the smallest free algebra $\mathbf{F}_{\mathsf{Q}}$ is the two-element Boolean algebra $\mathbf{2}$, and the quasivariety it generates is the variety of Boolean algebras.

Corollary 5.19. *Let* Q *be a quasivariety of* FL_w*-algebras, then the following are equivalent:*

1. Q *is passively structurally complete;*
2. *every trivializing identity in the variety of Boolean algebras is trivializing in* Q*;*
3. *every nontrivial finitely presented algebra is unifiable;*
4. *every nontrivial algebra is unifiable.*

The previous corollary has a possibly more transparent shape from the point of view of the logics. Let us call a formula φ in the language of FL-algebras *explosive in a logic* $\mathcal{L}$, with consequence relation $\vdash_{\mathcal{L}}$, if $\varphi \vdash_{\mathcal{L}} \delta$ for all formulas δ in the language of $\mathcal{L}$. Moreover, we call φ a *contradiction in* $\mathcal{L}$ if $\varphi \vdash_{\mathcal{L}} 0$. Since FL_w-algebras are 0-bounded, it is clear that contradictions coincide with explosive formulas in all extensions of $\mathcal{FL}_w$.

Corollary 5.20. *Let* $\mathcal{L}$ *be an extension of* $\mathcal{FL}_w$*, then the following are equivalent:*

1. $\mathcal{L}$ *is passively structurally complete.*
2. *Every contradiction of classical logic is explosive in* $\mathcal{L}$*.*
3. *Every passive rule of* $\mathcal{L}$ *has explosive premises.*

Let us first explore the consequences of the equivalence between (1) and (2) in Corollary 5.20. It is well known that intuitionistic logic is passively structurally complete (reported by Wroński at the 51st Conference on the History of Logic, Krakow, 2005). This can indeed be shown using the Glivenko Theorem for intuitionistic logic, and in our setting this is easily seen as a consequence of Corollary 5.20; indeed, observe that any contradiction of classical logic φ is such that its negation $\neg\varphi$ is a theorem of classical logic. Using the Glivenko translation and the deduction theorem, we obtain that φ is explosive in intuitionistic logic as well, which is then passively structurally complete. We will now show how this argument can be extended to a wide class of logics.

Let us write the negations corresponding to the two divisions as $\neg x = x\backslash 0$ and $\sim x = 0/x$. Following [44,45], we say that two logics $\mathcal{L}_1$ and $\mathcal{L}_2$ are *Glivenko equivalent* if for all formulas φ:

$$\vdash_{\mathcal{L}_1} \neg\varphi \quad \text{iff} \quad \vdash_{\mathcal{L}_2} \neg\varphi$$

(equivalently, $\vdash_{\mathcal{L}_1}\sim\varphi$ iff $\vdash_{\mathcal{L}_2}\sim\varphi$). Given a logic $\mathcal{L}$, we call *Glivenko logic of* $\mathcal{L}$ the smallest axiomatic extension of $\mathcal{FL}$ that is Glivenko equivalent to $\mathcal{L}$. Moreover, we call *Glivenko logic of* $\mathcal{L}$ *with respect to* $\mathcal{L}'$, and denote it with $\mathcal{G}_{\mathcal{L}'}(\mathcal{L})$ the smallest axiomatic extension of $\mathcal{L}'$ that is Glivenko equivalent to $\mathcal{L}$ (all these notions make sense by the results in [44,45]). $\mathcal{G}_{\mathcal{L}'}(\mathcal{L})$ is axiomatized relatively to $\mathcal{L}'$ by the set of axioms $\{\neg\sim\varphi : \vdash_{\mathcal{L}} \varphi\}$, or equivalently by the set $\{\sim\neg\varphi : \vdash_{\mathcal{L}} \varphi\}$.

Here we are interested in the Glivenko equivalent of classical logic with respect to $\mathcal{FL}_w$. From the algebraic perspective, this corresponds to the largest subvariety of FL_w that is Glivenko equivalent to Boolean algebras, $\mathsf{G}_{\mathsf{FL}_w}(\mathsf{B})$. The latter is axiomatized in [43, Corollary 8.33] as the subvariety of FL_w satisfying:

1. $\sim(x \wedge y) = \sim(xy)$
2. $\sim(x\backslash y) = \sim(\neg x \vee y)$
3. $\neg(x\backslash y) = \neg(\sim x \vee y)$
4. $\sim(x\backslash y) = \sim(\neg\sim x\backslash\neg\sim y)$
5. $\sim(x/y) = \sim(\neg\sim x/\neg\sim y)$.

Theorem 5.21. *Every axiomatic extension $\mathcal{L}$ of the Glivenko logic of classical logic with respect to $\mathcal{FL}_w$ is passively structurally complete.*

Proof. Consider a contradiction of classical logic φ, by the deduction theorem $\vdash_{\mathcal{CL}} \neg\varphi$ (where $\vdash_{\mathcal{CL}}$ is the consequence relation of classical logic). Since $\mathcal{L}$ is Glivenko equivalent to classical logic, $\vdash_{\mathcal{L}} \neg\varphi$. It can be easily checked that this implies that $\varphi \vdash_{\mathcal{L}} 0$ (it is a consequence of the parametrized local deduction theorem which holds in every extension of $\mathcal{FL}$ [43], but it is also straightforward to see in models). Thus φ is a contradiction of $\mathcal{L}$, or equivalently it is explosive in $\mathcal{L}$, which is then passively structurally complete by Corollary 5.20. □

Thus, every subvariety of $\mathsf{G}_{\mathsf{FL}_w}(\mathsf{B})$ is passively structurally complete. In particular, the commutative subvariety $\mathsf{G}_{\mathsf{FL}_{ew}}(\mathsf{B})$ is the variety of pseudocomplemented FL_{ew}-algebras ([39, Corollary 2.12]), i.e. the subvariety of FL_{ew} axiomatized by

$$x \wedge \neg x \approx 0.$$

Examples of passively structurally complete varieties then include Heyting algebras, Stonean MTL-algebras and as a consequence, e.g., product algebras and Gödel algebras.

We observe that these are not all of the passively structurally complete varieties of FL_w (nor of FL_{ew}). Let us indeed obtain a different kind of examples.

Definition 5.22. We say that a variety $\mathsf{V} \subseteq \mathsf{FL}_{ew}$ has a *Boolean retraction term* if there exists a term t in the language of V such that, for every $\mathbf{A} \in \mathsf{V}$, t defines an idempotent endomorphism on $\mathbf{A}$ whose image is the Boolean skeleton of $\mathbf{A}$, that is, the set of complemented elements of $\mathbf{A}$.

Varieties with a Boolean retraction term have been studied at length by Cignoli and Torrens in a series of papers, see in particular [28]. These are all varieties in which all nontrivial algebras retract onto a nontrivial Boolean algebra, thus they satisfy the hypotheses of Corollary 5.19 and they are passively structurally complete. Some of these varieties have been shown in [7] to have projective unifiers, thus they satisfy Theorem 3.17 and they are non-negatively universally complete. Among those we cite some varieties of interest in the realm of many-valued logics: the variety of product algebras, the variety generated by perfect MV-algebras, the variety NM^- of nilpotent minimum algebras without negation fixpoint and some varieties that have been called *nilpotent product* in [6] or [5].

We will see that in the semilinear variety of FL_{ew}, MTL, we can fully characterize passively structurally complete varieties as those with a Boolean retraction term. By [28], the largest subvariety of MTL with a Boolean retraction term is axiomatized relatively to MTL by the Di Nola-Lettieri equation:

$$(x+x)^2 = x^2 + x^2 \tag{DL}$$

where $x + y = \neg(\neg x \cdot \neg y)$. The latter identity has been introduced by Di Nola and Lettieri to axiomatize within MV-algebras the variety generated by the Chang algebra, or equivalently by perfect MV-algebras. This variety is called sDL in [76] (BP_0 in [70,10]), and it includes, for instance: pseudocomplemented MTL-algebras (also called SMTL-algebras), and thus Gödel algebras and product algebras; involutive BP_0-algebras and thus the variety generated by perfect MV-algebras and nilpotent minimum algebras without negation fixpoint.

Let us say that an element of an FL_{ew}-algebra $\mathbf{A}$ has *finite order* n if $x^n = 0$, and *infinite order* if there is no such n. We call *perfect* an algebra $\mathbf{A} \in \mathsf{FL}_{ew}$ such that, for all $a \in A$, a has finite order if and only if $\neg a$ has infinite order. Now, sDL turns out to be the variety generated by the perfect chains (see [76,10]).

Lemma 5.23. *A chain $\mathbf{A} \in \mathsf{FL}_{ew}$ is perfect if and only if there is no element with finite order $a \in A$ such that $a \geqslant \neg a$.*

Proof. By order preservation, if there is an element $a \in A, a \geqslant \neg a, a^n = 0$, then both a and its negation have finite order, thus the chain is not perfect. Suppose now a chain $\mathbf{A}$ is not perfect. Observing that for every element $x \in A$ it cannot be that both x and $\neg x$ have infinite order, we get that there is an element $a \in A$ such that both a and its negation $\neg a$ have finite order. If $a \ngeqslant \neg a$, since $\mathbf{A}$ is a chain, $a < \neg a$. Then $\neg\neg a \leqslant \neg a$, and they both have finite order. □

Theorem 5.24. *For a subvariety V of MTL the following are equivalent:*

1. *V is passively structurally complete;*
2. *V is a subvariety of sDL.*

Proof. Since subvarieties of sDL have a Boolean retraction term (2) implies (1) by Corollary 5.19. Suppose now that $\mathsf{V} \nsubseteq \mathsf{sDL}$. Then there is a chain $\mathbf{A}$ in V that is not perfect. By Lemma 5.23, there exists $a \in A, a \geqslant \neg a, a^n = 0$ for some $n \in \mathbb{N}$. Thus, $\neg(a \vee \neg a)^n = 1$. But the identity $\neg(x \vee \neg x)^n = 0$ holds in Boolean algebras. Thus $\neg(x \vee \neg x)^n \approx 1$ is trivializing in Boolean algebras but not in V. By Corollary 5.19, V is not passively structurally complete and thus (1) implies (2). □

Remark 5.25. Notice that the previous theorem also implies that a variety of MTL-algebras that is not a subvariety of sDL cannot be structurally complete.

We mention that structural completeness in subvarieties of MTL (or their logical counterparts) has been studied by several authors: e.g., [79] and [49] for Łukasiewicz logics, [36] Gödel logic, and [29] for

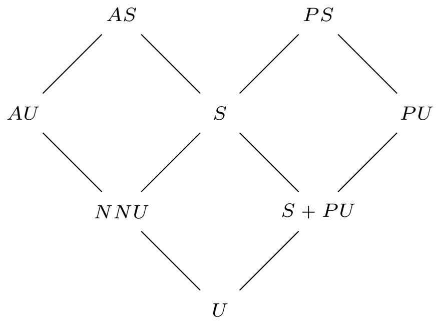


Fig. 8. The classes of universal and structural completeness.

fuzzy logics in the MTL framework; in the latter the authors show for instance that all subvarieties of pseudocomplemented MTL-algebras (SMTL) are passively structurally complete. This result is here obtained as a consequence of Theorem 5.24, since SMTL is a subvariety of sDL. From the results mentioned above and the characterization theorem, it also follows that the only varieties of MV-algebras (the equivalent algebraic semantics of infinite-valued Łukasiewicz logic) that are structurally complete are Boolean algebras and the variety generated by perfect MV-algebras (this result has been obtained following a different path in [49]).

We also remark that a nontrivial variety of FL_{ew}-algebras can be at most non-negatively universally complete since trivial algebras are finitely presented and not unifiable (unifiability is a necessary condition for universal completeness by Theorem 3.5); by Proposition 4.20 this happens if and only if the variety is actively universally complete and passively structurally complete. Thus, for instance, a semisimple variety of FL_{ew}-algebras satisfying the conditions in Corollary 5.19 would be non-negatively universally complete. We stress that this observation is not of particular interest in MTL-algebras, since the only discriminator variety in sDL is the variety of Boolean algebras. Indeed, consider a chain $\mathbf{A}$ in a discriminator variety V in sDL. Then there is some $n \in \mathbb{N}$ such that $\mathsf{V} \models x \vee \neg x^n \approx 1$. Let now $a \in A$; either a has infinite order, and then from $a \vee \neg a^n$ we obtain that $a = 1$, or a has finite order, and then $\neg a$ has infinite order. So by the analogous reasoning $\neg a = 1$. Therefore $\mathbf{A}$ is the two-element chain, and V is the variety of Boolean algebras.

6. Conclusions

In Fig. 8 we display several classes of varieties that we have considered in this paper (and the labels should be self explanatory); we are dropping the hereditary subclasses to avoid clutter. Observe that this is really a meet semilattice under inclusion.

Almost all the classes are provably distinct.

1. The variety of bounded distributive lattice is structurally complete (Proposition 5.5) but it is neither passively universally complete, since it is Kollár and the least free algebra is not trivial, nor non-negatively universally complete (Proposition 5.4). Hence $S \neq NNU, S + PU$.
2. The variety of Boolean algebras is non-negatively universally complete but not universally complete (Example 3.19) so $NNU \neq U$.
3. Any locally finite variety of BL-algebras is actively universally complete and some of them are not non-negatively universally complete (Example 3.25), so $AU \neq NNU$.
4. The variety in Example 4.19 is actively structurally complete but not actively universally complete, hence $AS \neq AU$.
5. Any locally finite variety of bounded semidistributive lattices different from the distributive variety is passively structurally complete (Corollary 5.12) but not structurally complete, since the only structurally

complete variety of bounded distributive lattices is the distributive variety (Proposition 5.5); as above it is also not passively universally complete. Hence $PS \neq S, PU$.

6. The variety $\mathbf{V}(\mathbf{M}^+_{3,3})$ (Section 5.1) is passively universally complete, as any variety of lattices, but it is not structurally complete since $\mathbf{Q}(\mathbf{M}^+_{3,3}) \subsetneq \mathbf{V}(\mathbf{M}^+_{3,3})$; hence $PU \neq S + PU$.
7. Example 4.19 shows that $AS \neq S$.
8. The intersection of AU and S is NNU as a consequence of Proposition 4.20.

Moreover for the primitive counterparts:

1. the variety $\mathbf{V}(\mathbf{F})$ generated by the Fano lattice is structurally complete and passively universally complete but not primitive (Section 5.1).
2. the variety of De Morgan lattices (Example 3.30) is actively universally complete but not active primitive universal.
3. the variety of pointed monounary algebras is actively structurally complete but not active primitive structural (Example 7.2 in [35]).

There are three examples that we were not able to find, which would guarantee total separation of all the classes we have considered:

1. A (quasi)variety that is structurally complete and passively universally complete, but not universally complete.
2. A non-negatively universally complete (quasi)variety such that not all subquasivarieties are non-negatively universally complete.
3. A universally complete variety which is not primitive universal.

The natural example for (3) would be a locally finite variety with exact unifiers having a subvariety without exact unifiers. However we are stuck because of lack of examples: we have only one unifiable locally finite variety with exact (non projective) unifiers, i.e. the variety of distributive lattices, which is trivially primitive universal. A similar situation happens for (2); all the examples of non-negatively universally complete varieties we have are either equationally complete and congruence distributive (so they do not have nontrivial subquasivarieties), or else are actively universally complete just by consequence of their characterization (such as the subvarieties of FL_{ew} in Section 5.2). Then we have Stone algebras that are not equationally complete but the only nontrivial subvariety is the variety of Boolean algebras, that is non-negatively universally complete. Now from Corollary 2.32 it is immediate that every subquasivariety of ST is non-negatively universally complete. In conclusion a deeper investigation of universally complete and non-negatively universally complete varieties is needed.

For (1) the situation is (slightly) easier to tackle: any primitive variety of lattices that is not universally complete gives a counterexample. While it seems impossible that all the primitive varieties in Section 5.1.1 are universally complete, actually proving that one it is not does not seem easy. This is due basically to the lack of information on free algebras in specific varieties of lattices, such as for instance $\mathbf{V}(\mathbf{M}_3)$; note that this variety is locally finite and hence all the finitely generated free algebras are finite. But we are not aware of any characterization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

We thank the anonymous reviewers who helped improve this work. The authors acknowledge partial support by the MOSAIC project (H2020-MSCA-RISE-2020 Project 101007627). Ugolini acknowledges support from the Ramón y Cajal programme RyC2021-032670-I.

References

[1] P. Aglianò, Quasivarieties of Wajsberg hoops, Fuzzy Sets Syst. 465 (2023) 108514.
[2] P. Aglianò, F. Manfucci, Structurally complete finitary extensions of positive Łukasiewicz logic, https://arxiv.org/pdf/2309.17297.pdf, 2023, submitted for publication.
[3] P. Aglianò, I.M.A. Ferreirim, F. Montagna, Basic hoops: an algebraic study of continuous t-norms, Stud. Log. 87 (1) (2007) 73–98.
[4] P. Aglianò, F. Montagna, Varieties of BL-algebras I: general properties, J. Pure Appl. Algebra 181 (2003) 105–129.
[5] P. Aglianò, S. Ugolini, Rotation logics, Fuzzy Sets Syst. 388 (2020) 1–25.
[6] P. Aglianò, S. Ugolini, MTL-algebras as rotations of basic hoops, J. Log. Comput. (2021) 763–784.
[7] P. Aglianò, S. Ugolini, Projectivity and unification in substructural logics of generalized rotations, Int. J. Approx. Reason. 153 (2023) 172–192.
[8] P. Aglianò, S. Ugolini, Projectivity in (bounded) commutative integral residuated lattices, Algebra Univers. 84 (2) (2023).
[9] P. Aglianò, A. Ursini, On subtractive varieties III: from ideals to congruences, Algebra Univers. 37 (1997) 296–333.
[10] S. Aguzzoli, T. Flaminio, S. Ugolini, Equivalences between subcategories of MTL-algebras via Boolean algebras and prelinear semihoops, J. Log. Comput. 27 (2017) 2525–2549.
[11] R. Balbes, Projective and injective distributive lattices, Pac. J. Math. 21 (1967) 405–420.
[12] P. Bankston, R. Fox, On categories of algebras equivalent to a quasivariety, Algebra Univers. 16 (1983) 153–158.
[13] C. Bergman, Structural completeness in algebra and logic, in: Algebraic Logic, in: Colloq. Soc. Math. J. Bolyai, vol. 54, North Holland, New York, 1988, pp. 59–73.
[14] C. Bergman, R. McKenzie, Minimal varieties and quasivarieties, J. Aust. Math. Soc. Ser. A 48 (1990) 133–147.
[15] W.M. Beynon, Applications of duality in the theory of finitely generated lattice-ordered abelian groups, Can. J. Math. 29 (1977) 243–254.
[16] G. Birkhoff, Subdirect unions in universal algebra, Bull. Am. Math. Soc. 50 (1944) 764–768.
[17] W.J. Blok, I.M.A. Ferreirim, On the structure of hoops, Algebra Univers. 43 (2000) 233–257.
[18] W.J. Blok, D. Pigozzi, Algebraizable Logics, Mem. Amer. Math. Soc., vol. 396, American Mathematical Society, Providence, Rhode Island, 1989.
[19] K. Blount, C. Tsinakis, The structure of residuated lattices, Int. J. Algebra Comput. 13 (4) (2003) 437–461.
[20] S. Burris, Discriminator varieties and symbolic computation, J. Symb. Comput. 13 (1992) 175–207.
[21] S. Burris, H.P. Sankappanavar, A Course in Universal Algebra, Graduate Texts in Mathematics, Springer, Berlin, 1981.
[22] L. Cabrer, G. Metcalfe, Admissibility via natural dualities, J. Pure Appl. Algebra 219 (2015) 4229–4253.
[23] L. Cabrer, G. Metcalfe, Exact unification and admissibility, Log. Methods Comput. Sci. 11 (2015) 1–15.
[24] X. Caicedo, M. Campercholi, K. Kearnes, P. Sanchèz Terraf, A. Szendrei, D. Vaggione, Every minimal dual discriminator variety is minimal as a quasivariety, Algebra Univers. 82 (2021) 36.
[25] J. Carr, Hereditary structural completeness over K4: Rybakov's Theorem revisited, Master's thesis, Universiteit von Amsterdam, 2022.
[26] C.C. Chang, H.J. Keisler, Model Theory, North Holland, 1990.
[27] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, Algebraic Foundations of Many-Valued Reasoning, Kluwer, 2000.
[28] R. Cignoli, A. Torrens, Varieties of commutative integral bounded residuated lattices admitting a Boolean retraction term, Stud. Log. 100 (2012) 1107–1136.
[29] P. Cintula, G. Metcalfe, Structural completeness in fuzzy logics, Notre Dame J. Form. Log. 50 (2009) 153–182.
[30] A. Citkin, On structurally complete superintuitionistic logics, Dokl. Akad. Nauk 241 (1) (1978) 40–43, Russian Academy of Sciences.
[31] A. Citkin, Hereditarily structurally complete superintuitionistic deductive systems, Stud. Log. 106 (4) (2018) 827–856.
[32] J. Czelakowski, W. Dziobiak, Congruence distributive quasivarieties whose finitely subdirectly irreducible members form a universal class, Algebra Univers. 27 (1990) 128–149.
[33] B. Davey, B. Sands, An application of Whitman's condition to lattices with no infinite chains, Ann. Pure Appl. Log. 7 (1977) 171–178.
[34] A. Day, Splitting algebras and a weak notion of projectivity, Algebra Univers. 5 (1975) 153–162.
[35] W. Dzik, M. Stronkowski, Almost structural completeness, an algebraic approach, Ann. Pure Appl. Log. 167 (2016) 525–556.
[36] W. Dzik, A. Wroński, Structural completeness of Gödel's and Dummett's propositional calculi, Stud. Log. 32 (1973) 69–73.

[37] P.C. Eklof, Ultraproduct for algebraists, in: J. Barwise (Ed.), Handbook of Mathematical Logic, in: Studies in Logic and the Foundation of Mathematics, vol. 90, Elsevier, Amsterdam, 1977, pp. 105–137.
[38] F. Esteva, L. Godo, Monoidal t-norm based logic: towards a logic for left-continuous t-norms, Fuzzy Sets Syst. 124 (2001) 271–288.
[39] D. Fazio, G.St. John, Connexive implications in substructural logics, Rev. Symb. Log. (2023) 1–36, https://doi.org/10.1017/S1755020323000254.
[40] Josep Maria Font, Abstract Algebraic Logic: An Introductory Textbook, Studies in Logic, vol. 60, College Publications, London, 2016.
[41] E. Fried, A. Pixley, The dual discriminator function in universal algebra, Acta Sci. Math. 41 (1979) 83–100.
[42] P. Gabriel, F. Ullmer, Lokar präsentiertbare Kategorien, Lecture Notes in Math., vol. 221, Springer Verlag, 1971.
[43] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Studies in Logics and the Foundations of Mathematics, vol. 151, Elsevier, Amsterdam, the Netherlands, 2007.
[44] N. Galatos, H. Ono, Glivenko theorems for substructural logics over FL, J. Symb. Log. 71 (2006) 1353–1384.
[45] N. Galatos, H. Ono, Glivenko theorems revisited, Ann. Pure Appl. Log. 181 (2009) 246–250.
[46] N. Galatos, C. Tsinakis, Generalized MV-algebras, J. Algebra 283 (2005) 254–291.
[47] S. Ghilardi, Unification through projectivity, J. Log. Comput. 7 (1997) 733–752.
[48] S. Ghilardi, Unification in intuitionistic logic, J. Symb. Log. 64 (1999) 859–890.
[49] J. Gispert, Least V-quasivarieties of MV-algebras, Fuzzy Sets Syst. 292 (2016) 274–284.
[50] V. Gorbunov, Algebraic Theory of Quasivarieties, Plenum, New York, 1998.
[51] P. Hájek, Metamathematics of Fuzzy Logics, Kluwer Academic Publisher, Dordrecht, the Netherlands, 1998.
[52] C. Herrman, A. Hühn, Lattices of normal subgroups which are generated by frames, in: A. Hün, E.T. Schmidt (Eds.), Lattice Theory, in: Colloq. Soc. Math. J. Bolyai, vol. 14, North Holland, New York, 1976, pp. 97–136.
[53] R. Iemhoff, Consequence relations and admissible rules, J. Philos. Log. 45 (2016) 327–348.
[54] P. Jipsen, J.B. Nation, Primitive lattice varieties, Int. J. Algebra Comput. 32 (2022) 717–752.
[55] P. Jipsen, H. Rose, Varieties of Lattices, Lecture Notes in Math., vol. 1533, Springer Verlag, 1992.
[56] P. Jipsen, C. Tsinakis, A survey of residuated lattices, in: J. Martinez (Ed.), Ordered Algebraic Structures, Kluwer Academic Publisher, 1982, pp. 19–56.
[57] B. Jónsson, Algebras whose congruence lattices are distributive, Math. Scand. 21 (1967) 110–121.
[58] B. Jónsson, I. Rival, Lattice varieties covering the smallest non-modular lattice variety, Pac. J. Math. 82 (1979) 463–478.
[59] J. Kollár, Congruences and one element subalgebras, Algebra Univers. 9 (1979) 266–267.
[60] Y. Komori, Super-Łukasiewicz implicational logics, Nagoya Math. J. 72 (1978) 127–133.
[61] T. Kowalski, Semisimplicity, EDPC and discriminator varieties of residuated lattices, Stud. Log. 77 (2005) 255–265.
[62] J.G. Lee, Almost distributive lattice varieties, Algebra Univers. 21 (1985) 280–304.
[63] P. Lorenzen, Einführung in die operative Logik und Mathematik, Grundlehren der mathematischen Wissenschaften, vol. 78, Springer, 1955.
[64] A.I. Mal'cev, Subdirect products of models, Dokl. Akad. Nauk SSSR 109 (1956) 264–266.
[65] R. McKenzie, On minimal lattice varieties, Algebra Univers. 32 (1994) 63–103.
[66] G. Metcalfe, Admissible rules: from characterizations to applications, in: L. Ong, R. de Queiroz (Eds.), Logic, Language, Information and Computation, WoLLIC 2012, in: Lecture Notes in Computer Science, vol. 7456, Springer, Berlin, Heidelberg, 2012.
[67] G. Metcalfe, C. Röthlisberger, Admissibility in De Morgan algebras, Soft Comput. 16 (2012) 1875–1882.
[68] T. Moraschini, J.G. Raftery, On prevarieties of logic, Algebra Univers. 80 (2019).
[69] T. Moraschini, J.G. Raftery, J.J. Wannenburg, Singly generated quasivarieties and residuated structures, Math. Log. Q. 66 (2020) 150–172.
[70] C. Noguera, Algebraic study of axiomatic extensions of triangular norm based fuzzy logics, Ph.D. thesis, University of Barcelona, 2007.
[71] J.S. Olson, J.G. Raftery, C.J. Van Alten, Structural completeness in substructural logics, Log. J. IGPL 16 (5) (2008) 453–495.
[72] D. Pigozzi, On some operations on classes of algebras, Algebra Univers. 2 (1972) 346–353.
[73] A. Pixley, The ternary discriminator function in universal algebra, Math. Ann. 191 (1971) 167–180.
[74] J. Raftery, Correspondences between Gentzen and Hilbert systems, J. Symb. Log. 71 (3) (2006) 903–957.
[75] V.V. Rybakov, Hereditarily structurally complete modal logics, J. Symb. Log. 60 (1995) 266–288.
[76] S. Ugolini, Varieties of residuated lattices with an MV-retract and an investigation into state theory, Ph.D. thesis, University of Pisa, 2018.
[77] H. Werner, Eine Charakterisierung funktional vollständiger Algebren, Arch. Math. (Basel) 21 (1970) 381–385.
[78] P.M. Whitman, Free lattices, Ann. Math. 42 (1941) 325–330.
[79] P. Wojtylak, On structural completeness of many-valued logics, Stud. Log. 37 (2) (1978) 139–147.
[80] F. Wolter, Superintuitionistic companions of classical modal logics, Stud. Log. 58 (1997) 229–259.