



# Boundary observability and null-controllability for non-autonomous degenerate hyperbolic equations via energy estimates

MOHAMMAD AKIL, GENNI FRAGNELLI  AND SARAH ISMAIL

*Abstract.* Motivated by the broad applications of wave propagation in non-uniform and time-varying environments, such as in acoustics, elasticity, and seismology, we investigate the controllability of degenerate wave equations with time-dependent wave speeds. In this work, we examine non-autonomous degenerate wave equations in a one-dimensional spatial domain, addressing both divergence and non-divergence forms. A control function is applied at the non-degeneracy boundary point, while Dirichlet or Neumann conditions are imposed at the degeneracy one. Using the generalized energy conservation law, we first establish boundary observability for the homogeneous problem, a key step in our analysis. Building on this, we prove the null-controllability of the non-autonomous degenerate wave systems. To achieve this, we construct solutions using the transposition method, which accommodates low regularity requirements, enabling us to handle weaker smoothness assumptions while maintaining rigorous control over the system's behavior. We conclude by presenting some insightful observations and potential avenues for future work, which could further advance the understanding of this problem.

## 1. Introduction

From vibrating strings and membranes in acoustics [1] to the propagation of seismic waves in geophysics [2], the wave equation constitutes a cornerstone in the study of wave phenomena across disciplines. Mainly, in Engineering, the wave equation models the behavior of electromagnetic waves in communication systems [3], while in Oceanography, it governs the dynamics of water waves [4]. The basic one-dimensional wave equation is given by

$$u_{tt} = c^2 u_{xx},$$

where the wave function  $u(t, x)$  represents the displacement or amplitude of the wave at position  $x$  and time  $t$ . The second-order derivative  $u_{tt}$  measures the acceleration of the medium at point  $x$ , while  $u_{xx}$  measures the curvature of the wave and determines local bending, and  $c^2$  represents the wave propagation speed.

---

*Mathematics Subject Classification:* 93B05, 93C20, 93B07, 35K65

*Keywords:* Non-autonomous degenerate hyperbolic equations, Null-controllability, Boundary observability, Energy estimates, Hardy-Poincaré inequality, Solution by transposition.

Waves can travel through complex or heterogeneous media, such as strings with non-uniform density. For example, if a medium grows denser or more rigid near a specific point, the wave may decelerate or come to a halt at that location. To accurately model wave propagation in such environments, we turn to degenerate wave equations, which incorporate variable coefficients to account for the added complexity of the medium. This equation can be written in non-divergence form as

$$u_{tt} - a(x)u_{xx} = 0,$$

where  $a(x)$  is the tension/density ratio. Thus, if the density is extremely large at some point  $x_0$ , we can assume that  $a$  degenerates at this point, i.e.,  $a(x_0) = 0$ . In other words, we can say that  $a(x)$  controls the wave speed variation. This equation can also be written in divergence form as

$$u_{tt} - (a(x)u_x)_x = 0, \tag{1.1}$$

where the term  $a(x)u_x$  is the flux term representing the energy flow through the medium. This makes the divergence form more suitable for conservation law formulation since the derivative  $(a(x)u_x)_x$  enables easier application of boundary conditions and highlights conservation properties. While the non-divergence form is more intuitive physically, the divergence form is better suited for analysis and numerical methods; the choice of the most convenient representation depends on solving specific problems.

The function  $a(x)$  can exhibit either weak or strong degeneracy at 0. To formalize these two types of degeneracy, we define them for a general function  $g$  as follows:

**Definition 1.** We say that a function  $g : [0, 1] \rightarrow \mathbb{R}_+$ ,  $g(0) = 0$ ,  $g > 0$  in  $(0, 1]$ , is

- weakly degenerate (WD) at 0, if  $g \in C([0, 1]) \cap C^1((0, 1])$  and  $K < 1$  where

$$K := \sup_{x \in (0, 1]} \frac{x|g'(x)|}{g(x)}; \tag{1.2}$$

- strongly degenerate (SD) at 0, if  $g \in C^1([0, 1])$  and  $K \in [1, 2)$ , where  $K$  is defined as above.

For non-degenerate wave equations, it is well known that null-controllability can be achieved at a fixed time  $T > 0$  sufficiently large under appropriate geometric conditions on the control region and sufficient regularity of the coefficients [5,6]. However, when the wave speed vanishes at certain points or regions, the problem becomes more complex, since the degeneracy can lead to a loss of energy propagation, making it difficult to control the system using standard techniques.

The foundation for studying controllability of *degenerate* wave equations is established through several seminal works where the initial studies focused on one-dimensional problems. One of the first papers to consider a degenerate wave equation is [7], where Gueye investigates a certain class of weak degeneracy for the equation in

divergence form, with the particular diffusion term  $a(x) = x^\alpha$ ,  $\alpha \in (0, 1)$ . The author obtains an exact controllability result with Dirichlet boundary control located at the degeneracy point 0 and then extends these results to degenerate parabolic equations, giving a first answer to a question opened for quite some time. In [8] (see also the arxiv version of 2015), Alabau-Boussouira et al. investigate the same system from a different viewpoint: considering (1.1) with  $a(x)$  allowed to degenerate either weakly or strongly on a part of the domain, they obtain observability or exact controllability from the non-degenerate part of the boundary using direct techniques such as multipliers and sharp trace results. They also prove a negative result when the diffusion coefficient degenerates too violently, i.e., when  $K > 2$ . An interesting system is considered by Bai and Chai in [9] where they study one-dimensional linear degenerate wave equations in domains having moving boundary. They prove the exact controllability by Hilbert Uniqueness Method (HUM) when the control acts on the moving boundary, and they provide an explicit expression for the controllability time as well.

Furthermore, the controllability of modified degenerate wave equations is studied by many researchers. In [10], Zhang and Gao prove the existence of interior controls for one-dimensional semi-linear degenerate wave equations, when  $a(x) = x^\alpha$ . By a duality argument, they reduce the problem to an observability estimate for the linear degenerate wave equation. They establish the unique continuation for the degenerate wave equation, obtaining the observability estimate by the multiplier method. In a recent work, Boutaayamou et al. [11] consider a degenerate wave equation with drift in the presence of a leading operator in non-divergence form. They establish the boundary controllability for the solution of the associated Cauchy problem at a sufficiently large time under some necessary conditions. Moreover, Allal et al. [12] are the first to consider the exact boundary controllability for a degenerate wave equation with a singular potential. In this case, the degenerate and the singular functions are the power functions. The authors prove the exact boundary controllability in the range of both subcritical and critical potentials and for sufficiently large time, through a boundary controller acting away from the degenerate/singular point. In [13], the authors investigate a degenerate wave equation in non-divergence form with drift, they impose an interior control acting on a bounded interval and prove the controllability for such a problem using some conditions by a standard technique. Recently, in [14] the authors study the controllability for a general degenerate/singular equations via a control acting on the non-degenerate point.

On the other hand, when the behavior of waves is affected by varying environmental or material conditions, the scenario necessitates a *time-dependent* wave propagation speed. To the best of our knowledge, Reissig and Yagdjian [15] are the first to consider a non-autonomous wave equation. They investigate the influence of the time-dependent coefficient on  $L_p$ - $L_q$  decay estimates for the solution of the Cauchy problem; they find that there exist no  $L_p$ - $L_q$  decay estimates and that the energy estimate from Gronwall's inequality is near to an optimal one. Later, in [16], Reissig and Smith give a

classification of the oscillating behavior of the coefficient that determines in an essential way the possibility of deriving  $L_p$ - $L_q$  decay estimates. As control problem, Chambolle and Santosa [17] study an initial boundary-value problem for a wave equation with time-dependent sound speed. In particular, they consider the case where the sound speed can take on only two values. The authors propose a simple control law that they prove to lead to energy decay when the number of modes in the vibration is finite and none of the eigenfrequencies are repeated. In [18], Hirosawa explores the relation between the total energy behavior of solutions to wave equations featuring time-dependent propagation speeds and the smoothness of the coefficient. The research reveals that the asymptotic behavior of the total energy is governed by three main aspects of the coefficient: its degree of differentiability, the asymptotic behavior of its derivatives as  $t \rightarrow \infty$ , and the stabilization of the amplitude, characterized by an integral. The optimality of these properties is then ensured through explicit examples. In [19], D’Abbicco and Ebert study the long time behavior of the energy for wave-type equations with time-dependent speed imposing also a time-dependent damping and they investigate their interaction. The authors show how to describe the dissipative effect on the energy and study a class of dissipations for which the equation keeps its hyperbolic structure and properties. Recently, Zhou et al. [20] investigate the long-term behavior of non-autonomous wave equations with critical nonlinearity imposing a new-type nonlocal weak damping. The authors show that the solutions are exponentially approaching a more regular bounded subset, and based on this regularity result, they obtain the existence and smoothness of uniform exponential attractors.

It is observed that the literature lacks a thorough analysis of wave equations featuring both *degenerate* and *time-dependent* wave propagation speeds. Such equations model wave propagation in media with time-varying and spatially inhomogeneous properties—for instance, waves traveling through the Earth’s crust, where material properties change due to tectonic activity or temperature gradients. Controlling such systems is critically important, as the interaction between time dependence and degeneracy can give rise to new wave phenomena, such as dispersion or resonance effects.

Given its significance, in this paper we will investigate, for the first time, the null-controllability of *non-autonomous degenerate* wave equations, represented by the following systems:

$$\begin{cases} u_{tt} - \mathcal{A}_i(t)u = 0, & (t, x) \in Q_T, \\ B_i u(t, 0) = 0, & t \in (0, T), \\ u(t, 1) = f(t), & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, 1), \\ u_t(0, x) = u_1(x), & x \in (0, 1), \end{cases} \tag{P_i}$$

$i = 1, 2$ , where  $Q_T := (0, T) \times (0, 1)$ ,  $T > 0$  fixed, and  $B_i u(t, 0) = 0$ ,  $i = 1, 2$ , are suitable boundary conditions related to the operators  $\mathcal{A}_i$ ,  $i = 1, 2$ . In particular

$$\mathcal{A}_i(t)u := \begin{cases} b(t)a(x)u_{xx}, & i = 1, \\ b(t)(a(x)u_x)_x, & i = 2, \end{cases} \tag{1.3}$$

where  $b \in W^{1,\infty}(0, T)$  is a strictly positive function,  $a$  is (WD) or (SD) at 0 and

$$B_i u(t, 0) = \begin{cases} u(t, 0) = 0, & i = 1, \\ \begin{cases} u(t, 0) = 0, & \text{if } a \text{ is (WD) at } 0, \\ \lim_{x \rightarrow 0} (a u_x)(t, 0) = 0, & \text{if } a \text{ is (SD) at } 0, \end{cases} & i = 2. \end{cases} \tag{1.4}$$

We consider the control function  $f$  to act as a boundary control used to drive the solution to 0 at a sufficiently large time  $T > 0$ . In particular for  $(P_i)$ , we aim to prove global null-controllability in the sense of finding a control  $f \in L^2(0, T)$  such that

$$u(T, x) = u_t(T, x) = 0, \quad \forall x \in (0, 1), \tag{1.5}$$

for a given initial data  $(u_0, u_1)$  in a suitable space.

Further complications arise in  $(P_i)$  in comparison with the autonomous degenerate case. Indeed, we cannot use the standard Lümer-Phillips technique for the well-posedness, but, for the non-autonomous Cauchy problems, we have to use the Kato perturbation Theorem. Moreover, the energies of the associated homogeneous adjoint problems depend on the function  $b(t)$ ; this means that the conservation law that holds for the case  $b(t) = 1$  for all  $t \in [0, T]$ , does not hold in this case. Finally, due to the presence of  $b(t)$ , additional terms have to be estimated in order to prove that a *generalized conservation law* holds.

The paper is structured as follows: In Sect. 2, we introduce the Hilbert spaces required to address the problems  $(P_i)$ , establish key Hardy-Poincaré inequalities, and present essential hypotheses that underpin the subsequent analysis. Section 3 focuses on proving the well-posedness of  $(P_i)$  using Kato’s Perturbation Theorem. In Sect. 4, we derive energy estimates, providing both upper and lower bounds, and demonstrate that the *generalized energy conservation law* holds for non-autonomous degenerate wave equations in both divergence and non-divergence forms. Finally, in Sect. 5, we show that the adjoint problems  $(AP_i)$  are observable in time  $T > 0$ , and we establish the null-controllability of our systems by constructing a solution via transposition for  $(P_i)$ , associated to the control function  $f$ . Further extensions and broader applications are presented in Section 6.

Throughout this paper,  $(\cdot)'$  denotes the derivative of a function depending only on the real space variable  $x$ ;  $(\cdot)^\cdot$  denotes the derivative of a function depending only on the real time variable  $t$ . Moreover, we will denote by  $\|\cdot\| := \|\cdot\|_{L^2(0,1)}$  and by  $\|\cdot\|_{\infty,T} := \|\cdot\|_{L^\infty(0,T)}$ .

## 2. Functional spaces and preliminary findings

In the following, the next assumption will be crucial.

**Hypothesis 1.** *Let  $a \in C([0, 1])$  be such that  $a(0) = 0$ ,  $a > 0$  on  $(0, 1]$  and there exists  $K \in (0, 2)$  such that the function  $x \mapsto \frac{x^K}{a(x)}$  is non-decreasing in a right neighborhood of  $x = 0$ .*

Observe that if  $a$  is (WD) or (SD) at 0 then (1.2) implies that the function

$$x \mapsto \frac{x^\gamma}{a(x)} \tag{2.1}$$

is non-decreasing in  $(0, 1]$  for all  $\gamma \geq K$  and

$$\lim_{x \rightarrow 0} \frac{x^\gamma}{a(x)} = 0 \tag{2.2}$$

for all  $\gamma > K$ . These fact will play a crucial role in the next sections. Observe that (2.1) holds thanks to the definition of  $K$  in (1).

### 2.1. Framework for the non-divergence form

One of the main differences between the systems  $(P_i)$  with  $i = 1$  (i.e., in non-divergence form) and  $(P_i)$  with  $i = 2$  (i.e., in divergence form) arises in the natural spaces: while the second problem is studied in  $L^2(0, 1)$ , the first problem has to be studied in the weighted space

$$L^2_{\frac{1}{a}}(0, 1) := \left\{ u \in L^2(0, 1) \mid \|u\|_{L^2_{\frac{1}{a}}(0,1)}^2 < \infty \right\},$$

where

$$\|u\|_{\frac{1}{a}}^2 := \int_0^1 \frac{u^2}{a} dx \quad \text{and} \quad \langle u, v \rangle_{\frac{1}{a}} := \int_0^1 \frac{uv}{a} dx, \quad \forall u, v \in L^2_{\frac{1}{a}}(0, 1). \tag{2.3}$$

As in [21] or [22], we also consider the following Hilbert spaces

$$H^1_{\frac{1}{a}}(0, 1) := L^2_{\frac{1}{a}}(0, 1) \cap H^1_0(0, 1), \quad \langle u, v \rangle_{1, \frac{1}{a}} := \langle u, v \rangle_{\frac{1}{a}} + \langle u', v' \rangle_{L^2(0,1)}, \tag{2.4}$$

for all  $\forall u, v \in H^1_{\frac{1}{a}}(0, 1)$ , and

$$H^2_{\frac{1}{a}}(0, 1) := \left\{ u \in H^1_{\frac{1}{a}}(0, 1) \mid au'' \in L^2_{\frac{1}{a}}(0, 1) \right\}, \tag{2.5}$$

$$\langle u, v \rangle_{2, \frac{1}{a}} := \langle u, v \rangle_{1, \frac{1}{a}} + \langle \sqrt{a}u'', \sqrt{a}v'' \rangle_{L^2(0,1)},$$

for all  $u, v \in H^2_{\frac{1}{a}}(0, 1)$ . Obviously, the previous inner products induce the following norms

$$\|u\|_{1, \frac{1}{a}}^2 := \|u\|_{\frac{1}{a}}^2 + \|u'\|^2 \quad \text{and} \quad \|u\|_{2, \frac{1}{a}}^2 := \|u\|_{1, \frac{1}{a}}^2 + \|\sqrt{a}u''\|^2,$$

respectively. Moreover, the following Gauss-Green formula holds

$$\int_0^1 u'' v dx = - \int_0^1 u' v' dx, \quad \forall (u, v) \in H^2_{\frac{1}{a}}(0, 1) \times H^1_{\frac{1}{a}}(0, 1), \tag{2.6}$$

(see [22, Lemma 2.1]).

In particular, in [22, Proposition 2.6] the following Hardy-Poincaré inequality is proved:

**Proposition 1** (Hardy-Poincaré Inequality). *Assume Hypothesis 1. Then, there exists  $C > 0$  such that*

$$\int_0^1 \frac{v^2}{a} dx \leq C \int_0^1 (v')^2 dx, \tag{HP}$$

for all  $v \in H_0^1(0, 1)$ .

Observe that the previous inequality is proved in [22, Proposition 2.6] under the condition  $a(0) = a(1) = 0$ , but actually it holds requiring only  $a(0) = 0$ . Indeed, by assumptions there exists  $\epsilon > 0$  such that  $x \mapsto \frac{x^K}{a(x)}$  is non-decreasing in  $(0, \epsilon]$ ; thus

$$\begin{aligned} \int_0^1 \frac{v^2}{a} dx &\leq \int_0^\epsilon \frac{v^2}{a} dx + \frac{1}{\min_{[\epsilon, 1]} a} \int_\epsilon^1 v^2 dx \\ &\leq \frac{1}{a(\epsilon)} \int_0^\epsilon \frac{v^2}{x^2} dx + \frac{1}{\min_{[\epsilon, 1]} a} \int_\epsilon^1 v^2 dx \leq C \int_0^1 (v')^2 dx, \end{aligned}$$

by the Hardy and the Poincaré inequalities, for a positive constant  $C$ .

Denoting by  $C_{HP}$  the smallest constant that satisfies (HP) and assuming that  $a$  is (WD) or (SD), one has that  $C_{HP}$  is smaller than  $\frac{4}{a(1)}$ . Indeed, by (2.1) we get that

$$\int_0^1 \frac{v^2}{a} dx \leq \frac{1}{a(1)} \int_0^1 \frac{v^2}{x^2} dx \leq \frac{4}{a(1)} \int_0^1 (v')^2 dx,$$

for all  $v \in H_0^1(0, 1)$ .

Moreover as a consequence of Proposition 1, if Hypothesis 1 holds we get that  $H_{\frac{1}{a}}^1(0, 1)$  and  $H_0^1(0, 1)$  coincide and the norm  $\|\cdot\|_{1, \frac{1}{a}}$  is equivalent to  $\|\cdot\|_1$  in  $H_{\frac{1}{a}}^1(0, 1)$ , where

$$\|u\|_1 := \|u'\|, \quad \forall u \in H_{\frac{1}{a}}^1(0, 1),$$

is induced by the inner product

$$\langle u, v \rangle_1 := \int_0^1 u'v' dx,$$

for all  $u, v \in H_{\frac{1}{a}}^1(0, 1)$ . In particular, we have

$$\|u\|_1^2 \leq \|u\|_{1, \frac{1}{a}}^2 \leq (1 + C_{HP}) \|u\|_1^2,$$

for all  $u \in H_{\frac{1}{a}}^1(0, 1)$ . In the following, we will use  $\|\cdot\|_1$  and  $\langle \cdot, \cdot \rangle_1$  since they are more convenient.

Now, consider the operator  $(A_1, D(A_1))$ , where  $A_1u := au''$  with

$$D(A_1) := H_{\frac{1}{a}}^2(0, 1). \tag{ND}$$

$(A_1, D(A_1))$  is m-dissipative and self-adjoint in  $L_{\frac{1}{a}}^2(0, 1)$  (see [22, Theorem 2.3]). Thus, by [23, Corollary 3.20], the operator  $(A_1, D(A_1))$  is densely defined and generates a contraction semigroup.

Moreover, the next properties hold.

**Lemma 1.** *Assume Hypothesis 1.*

1. If  $u \in H^1_{\frac{1}{a}}(0, 1)$ , then  $\lim_{x \rightarrow 0} \frac{x}{a} u^2(x) = 0$ .
2. If  $u \in D(A_1)$ , then  $\lim_{x \rightarrow 0} x(u'(x))^2 = 0$ .
3. If  $y \in D(A_1)$  and  $u \in H^1_{\frac{1}{a}}(0, 1)$ , then  $\lim_{x \rightarrow 0} u(x)y'(x) = 0$ .

*Proof.* It is sufficient to prove only the second point if  $K > 1$ , being the other case and the other points proved in [11, Lemma 3.2].

Fixed  $u \in D(A_1)$ , setting  $z := x(u')^2$ . We have  $z' = (u')^2 + 2xu'u''$ . Clearly,  $(u')^2 \in L^1(0, 1)$ ; moreover, by assumption  $\left| 2\frac{x}{\sqrt{a}}u'u''\sqrt{a} \right| \leq \frac{2}{\sqrt{a(1)}}|u'\sqrt{a}u''| \in L^1(0, 1)$  by Hölder's inequality. Hence  $z \in W^{1,1}(0, 1)$  and  $\lim_{x \rightarrow 0} z(x) = L \in [0, +\infty)$ . If  $L \neq 0$  there would exist a neighborhood  $\mathcal{I}$  of 0 such that  $\frac{|L|}{2} \leq x(u')^2$ , for all  $x \in \mathcal{I}$ ; in particular,  $\frac{|L|}{2x} \leq (u')^2$  for all  $x \in \mathcal{I}$ , in contrast to the fact that  $u' \in L^2(0, 1)$ . Hence  $L = 0$  and the conclusion follows. □

The previous result will be crucial for the rest of the paper.

### 2.2. Framework for the divergence form

Following [24], for the system in divergence form, we take the following weighted Hilbert spaces. In the **weakly degenerate** case, we consider

$$H^1_a(0, 1) := \{u \in L^2(0, 1) \mid u \text{ absolutely continuous in } [0, 1], \\ \sqrt{a}u' \in L^2(0, 1) \text{ and } u(1) = u(0) = 0\}$$

and

$$H^2_a(0, 1) := \{u \in H^1_a(0, 1) \mid au' \in H^1(0, 1)\}.$$

On the other hand, in the **strongly degenerate** case, we consider

$$H^1_a(0, 1) := \{u \in L^2(0, 1) \mid u \text{ locally absolutely continuous in } (0, 1], \\ \sqrt{a}u' \in L^2(0, 1) \text{ and } u(1) = 0\}$$

and

$$H^2_a(0, 1) := \{u \in H^1_a(0, 1) \mid au' \in H^1(0, 1)\} \\ = \{u \in L^2(0, 1) \mid u \text{ locally absolutely continuous in } (0, 1], \\ au \in H^1_0(0, 1), au' \in H^1(0, 1), u(1) = 0 \text{ and } (au')(0) = 0\}.$$

In both cases, we consider inner products and norms given by

$$\langle u, v \rangle_{1,a} := \langle u, v \rangle_{L^2(0,1)} + \langle \sqrt{a}u', \sqrt{a}v' \rangle_{L^2(0,1)}, \quad \|u\|^2_{1,a} := \|u\|^2 + \|\sqrt{a}u'\|^2,$$

for all  $u, v \in H_a^1(0, 1)$ , and

$$\langle u, v \rangle_{2,a} := \langle u, v \rangle_{1,a} + \langle (au)', (av)' \rangle_{L^2(0,1)}, \quad \|u\|_{2,a}^2 := \|u\|_{1,a}^2 + \|(au)'\|^2,$$

for all  $u, v \in H_a^2(0, 1)$ .

**Proposition 2** [8, Proposition 2.2]. *Assume Hypothesis 1. Then*

$$\|w\|^2 \leq C_a \|\sqrt{a}w'\|^2 \quad \forall w \in H_a^1(0, 1),$$

where

$$C_a := \frac{1}{a(1)} \min \left\{ 4, \frac{1}{2 - K} \right\}. \tag{2.7}$$

Thanks to Proposition 2, one has that the norms  $\|\cdot\|_{1,a}$  and  $\|\cdot\|_a$  are equivalent in  $H_a^1(0, 1)$ , where

$$\|u\|_a := \left( \int_0^1 a(u')^2 dx \right)^{\frac{1}{2}}, \quad \forall u \in H_a^1(0, 1),$$

is induced by the inner product

$$\langle u, v \rangle_a := \langle \sqrt{a}u', \sqrt{a}v' \rangle_{L^2(0,1)} = \int_0^1 au'v'dx,$$

for all  $u, v \in H_a^1(0, 1)$ . In particular,

$$\|u\|_a^2 \leq \|u\|_{1,a}^2 \leq (1 + C_a)\|u\|_a^2,$$

for all  $u \in H_a^1(0, 1)$ . In the following we will use  $\|\cdot\|_a$  and  $\langle \cdot, \cdot \rangle_a$  since they are more convenient.

Also, in this case, the operator  $(A_2, D(A_2))$ , where  $A_2u := (au)'$  with

$$D(A_2) := H_a^2(0, 1), \tag{D}$$

is m-dissipative and self-adjoint in  $L^2(0, 1)$  (see [24, Proposition 2.5]); thus,  $(A_2, D(A_2))$  is densely defined and generates a contraction semigroup.

For the following, the next lemma proved in [24, Proposition 2.4] or in [8, Proposition 2.5] will be crucial.

**Lemma 2.** *Assume Hypothesis 1. Then*

$$\lim_{x \rightarrow 0} xu^2 = 0,$$

for all  $u \in H_a^1(0, 1)$ .

### 3. Well-posedness for the homogeneous adjoint problems

Let us consider the homogeneous adjoint problems of  $(P_i)$  given by

$$\begin{cases} y_{tt} - \mathcal{A}_i(t)y = 0, & (t, x) \in Q_T, \\ B_i y(t, 0) = 0, & t \in (0, T), \\ y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1), \\ y_i(0, x) = y_1(x), & x \in (0, 1), \end{cases} \tag{AP}_i$$

where  $\mathcal{A}_i$  and  $B_i$  are defined in (1.3) and (1.4), respectively. Clearly, thanks to the definition of the operators,

$$D(\mathcal{A}_i(t)) = D(A_i), \quad \forall t \in [0, T]. \tag{3.1}$$

Let us define

$$H_i = \begin{cases} L^2_{\frac{1}{a}}(0, 1), & i = 1, \\ L^2_{\frac{a}{a}}(0, 1), & i = 2, \end{cases} \quad \text{and} \quad K_i = \begin{cases} H^1_{\frac{1}{a}}(0, 1), & i = 1, \\ H^1_a(0, 1), & i = 2. \end{cases}$$

In order to study the well-posedness of  $(AP_i)$ , we introduce the following Hilbert space

$$\mathcal{H}_0 := K_i \times H_i, \tag{3.2}$$

with inner product and norm given by

$$\langle (u, v), (\tilde{u}, \tilde{v}) \rangle_{\mathcal{H}_0} := \langle u, \tilde{u} \rangle_{K_i} + \langle v, \tilde{v} \rangle_{H_i}$$

and

$$\|(u, v)\|_{\mathcal{H}_0}^2 := \|u\|_{K_i}^2 + \|v\|_{H_i}^2,$$

for all  $(u, v), (\tilde{u}, \tilde{v}) \in \mathcal{H}_0$  and for  $i = 1, 2$ . In particular

$$\begin{aligned} \langle (u, v), (\tilde{u}, \tilde{v}) \rangle_{\mathcal{H}_0} &= \int_0^1 u' \tilde{u}' dx + \int_0^1 \frac{v \tilde{v}}{a} dx \quad \text{and} \\ \|(u, v)\|_{\mathcal{H}_0}^2 &= \int_0^1 (u')^2 dx + \int_0^1 \frac{v^2}{a} dx, \end{aligned}$$

if  $i = 1$  and

$$\begin{aligned} \langle (u, v), (\tilde{u}, \tilde{v}) \rangle_{\mathcal{H}_0} &= \int_0^1 a u' \tilde{u}' dx + \int_0^1 v \tilde{v} dx \quad \text{and} \\ \|(u, v)\|_{\mathcal{H}_0}^2 &= \int_0^1 a (u')^2 dx + \int_0^1 v^2 dx, \end{aligned}$$

if  $i = 2$ . Moreover, we consider the matrix operator  $\mathcal{B}_i : D(\mathcal{B}_i) \subset \mathcal{H}_0 \rightarrow \mathcal{H}_0$ ,  $i = 1, 2$ , given by

$$\mathcal{B}_i(t) := \begin{pmatrix} 0 & Id \\ \mathcal{A}_i(t) & 0 \end{pmatrix},$$

with domain

$$D(\mathcal{B}_i(t)) := D(\mathcal{A}_i(t)) \times K_i \subset \mathcal{H}_0. \tag{3.3}$$

Now, we can rewrite the problem (AP<sub>i</sub>) as a Cauchy problem

$$\begin{cases} \dot{\mathcal{Y}}(t) = \mathcal{B}_i(t)\mathcal{Y}(t), & t \geq 0, \\ \mathcal{Y}(0) = \mathcal{Y}_0, \end{cases} \tag{CP}_i$$

where  $\mathcal{Y}(t) = \begin{pmatrix} y \\ y_t \end{pmatrix}$  and  $\mathcal{Y}_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \in \mathcal{H}_0$ .

Then, we prove the following existence and uniqueness result.

**Theorem 2.** *Assume a (WD) or (SD) at 0, and  $b \in W^{1,\infty}(0, T)$  a strictly positive function. Then, for all  $(y_0, y_1) \in K_i \times H_i$ ,  $i = 1, 2$ , (AP<sub>i</sub>) has a unique solution  $y \in C^1([0, T]; H_i) \cap C([0, T]; K_i) \hookrightarrow C([0, T]; H_i) \cap L^2(0, T; K_i)$ .*

*Moreover, if  $(y_0, y_1) \in D(\mathcal{A}_i) \times K_i$ , then  $y \in C^2([0, T]; H_i) \cap C^1([0, T]; K_i) \cap C([0, T]; D(\mathcal{A}_i))$ ,  $i = 1, 2$ .*

Since the Cauchy problem (CP<sub>i</sub>) is non-autonomous, the idea to prove the above theorem is to apply the following Kato Theorem (see [25, Theorem 1.2], [26, Theorem 1.9] or [27, Theorem 2.1]):

**Theorem 3.** *Let  $T > 0$  be fixed,  $\mathcal{H}$  a Hilbert space and  $\mathcal{A}(t) : D(\mathcal{A}(t)) \subset \mathcal{H} \rightarrow \mathcal{H}$  a linear operator satisfying the following hypotheses:*

- (H<sub>1</sub>)  $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$ ,  $\forall t \in [0, T]$ ;
- (H<sub>2</sub>)  $D(\mathcal{A}(0))$  is a dense subset of  $\mathcal{H}$ ;
- (H<sub>3</sub>)  $\forall t \in [0, T]$ ,  $\mathcal{A}(t)$  generates a strongly continuous semigroup on  $\mathcal{H}$ , and the family  $\mathcal{B} = \{\mathcal{A}(t)\}_{t \in [0, T]}$  is stable with stability constants  $C$  and  $p$  independent of  $t$

(i.e., the semigroup  $(S_t(s))_{s \geq 0}$  generated by  $\mathcal{A}(t)$  satisfies  $\left\| \prod_{j=1}^k S_{t_j}(s_j)u \right\|_{\mathcal{H}} \leq$

$Ce^{p \sum_{j=1}^k s_j} \|u\|_{\mathcal{H}}$ , for all  $u \in \mathcal{H}$ ,  $s_j \geq 0$ ,  $0 \leq t_1 \leq \dots \leq t_k \leq T$ ,  $k \in \mathbb{N}$ );

- (H<sub>4</sub>)  $\frac{\partial}{\partial t} \mathcal{A} \in L_*^\infty(0, T; B(D(\mathcal{A}(0)), \mathcal{H}))$  which is the space of equivalent classes of essentially bounded, strongly measurable functions from  $[0, T]$  into the set of bounded operators  $B(D(\mathcal{A}(0)), \mathcal{H})$ .

Then

$$\begin{cases} \dot{u}(t) = \mathcal{A}(t)u(t), & t \in (0, T), \\ u(0) = u_0 \in \mathcal{H}, \end{cases}$$

has a unique solution  $u \in C([0, T], \mathcal{H})$  for all  $u_0 \in \mathcal{H}$ . Moreover, if  $u_0 \in D(\mathcal{A}(0))$ , then  $u \in C([0, T]; D(\mathcal{A}(0))) \cap C^1([0, T]; \mathcal{H})$ .

In order to apply the previous theorem, we need to verify that the conditions  $(H_1) - (H_4)$  hold.

The condition  $(H_1)$  follows immediately by (3.1). Moreover, as written before,  $D(A_i)$  is dense in  $H_i, i = 1, 2$ ; thus,  $(H_2)$  clearly holds in both cases since  $D(A_i(0)) = D(A_i), i = 1, 2$ . In order to prove  $(H_3)$  and  $(H_4)$ , we need to introduce another inner product in  $\mathcal{H}_0$  that takes into account the term  $b(t)$ . Just to fix the idea, we assume that  $i = 1$ . Thus, for all  $t \geq 0$ , we consider

$$\langle \mathcal{U}, \tilde{\mathcal{U}} \rangle_t = \langle (u, v), (\tilde{u}, \tilde{v}) \rangle_t := b(t)\langle u, \tilde{u} \rangle_1 + \langle v, \tilde{v} \rangle_{\frac{1}{a}}, \quad \forall \mathcal{U} = (u, v), \tilde{\mathcal{U}} = (\tilde{u}, \tilde{v}) \in \mathcal{H}_0,$$

inducing the norm

$$\|\mathcal{U}\|_t^2 = \|(u, v)\|_t^2 = b(t)\|u\|_1^2 + \|v\|_{\frac{1}{a}}^2, \quad \forall \mathcal{U} = (u, v) \in \mathcal{H}_0,$$

which is equivalent to the norm  $\|\cdot\|_{\mathcal{H}_0}$  in  $\mathcal{H}_0$ , being

$$\frac{1}{\max\{\frac{1}{m}, 1\}} \|(u, v)\|_{\mathcal{H}_0}^2 \leq \|(u, v)\|_t^2 \leq \max\{M, 1\} \|(u, v)\|_{\mathcal{H}_0}^2, \quad (3.4)$$

for all  $\mathcal{U} = (u, v) \in \mathcal{H}_0$ . Here

$$m := \min_{t \in [0, T]} b(t) \quad \text{and} \quad M := \max_{t \in [0, T]} b(t) \quad (3.5)$$

(we recall that  $b \in W^{1,\infty}(0, T)$ , hence  $b \in C[0, T]$ ). The first part of  $(H_3)$  follows by the next proposition.

**Proposition 3.** *Assume a (WD) or (SD) at 0 and  $b \in W^{1,\infty}(0, T)$  a strictly positive function. For all fixed  $t \geq 0$ , the operator  $(\mathcal{B}_1(t), D(\mathcal{B}_1(t)))$  is  $m$ -dissipative and generates a strongly continuous semigroup.*

For the second part of  $(H_3)$ , it remains to prove that, for any  $t \geq 0, \mathcal{B}_1 = \{\mathcal{B}_1(t)\}_{t \in [0, T]}$  is a stable family of  $C_0$ -semigroup generators with stability constants  $C$  and  $p$ . Indeed, since  $\mathcal{B}_1(t)$  is the infinitesimal generator of a contraction semigroup on  $\mathcal{H}_0$  with respect to  $\|\cdot\|_t$ , this means that  $\|e^{s\mathcal{B}_1(t)}\mathcal{Y}\|_t \leq \|\mathcal{Y}\|_t, \forall \mathcal{Y} \in \mathcal{H}_0$  and for all  $s \geq 0$ . Now, for all  $\mathcal{Y} = (u, v) \in \mathcal{H}_0$  we have

$$\frac{d}{dt} \|\mathcal{Y}\|_t^2 = \frac{d}{dt} \left( b(t)\|u\|_1^2 + \|v\|_{\frac{1}{a}}^2 \right) = \dot{b}(t)\|u\|_1^2 \leq \frac{\|\dot{b}\|_{\infty, T}}{m} \|\mathcal{Y}\|_t^2, \quad \forall \mathcal{Y} \in \mathcal{H}_0.$$

By the Gronwall inequality, one has

$$\|\mathcal{Y}\|_t^2 \leq \|\mathcal{Y}\|_s^2 e^{\int_s^t C_b d\tau} = \|\mathcal{Y}\|_s^2 e^{(t-s)C_b}, \quad \forall t \geq s \geq 0 \quad \text{and} \quad \forall \mathcal{Y} \in \mathcal{H}_0,$$

where  $C_b = \frac{\|\dot{b}\|_{\infty, T}}{m}$ . Then, we get  $\|\mathcal{Y}\|_t \leq \|\mathcal{Y}\|_s e^{\frac{C_b}{2}(t-s)}, \forall t \geq s \geq 0$  and  $\forall \mathcal{Y} \in \mathcal{H}_0$ . Thanks to [26, Proposition 1.1], we obtain the desired result.

Now, it remains to prove  $(H_4)$ . Since  $\mathcal{A}_1(t)u = b(t)A_1u$ , we have that  $\frac{\partial \mathcal{A}_1(t)}{\partial t}u = \dot{b}(t)A_1u, \forall u \in D(\mathcal{A}_1(t))$ . Hence, for all  $\mathcal{Y} = (u, v) \in \mathcal{H}_0$

$$\left\| \frac{\partial \mathcal{B}_1(t)}{\partial t} \mathcal{Y} \right\|_{\mathcal{H}_0}^2 = \left\| \frac{\partial \mathcal{A}_1(t)}{\partial t} u \right\|_{\frac{1}{a}}^2 = \|\dot{b}(t)A_1u\|_{\frac{1}{a}}^2 \leq \|\dot{b}\|_{\infty, T}^2 \|A_1u\|_{\frac{1}{a}}^2;$$

thus  $\left\| \frac{\partial \mathcal{B}_1(t)}{\partial t} \mathcal{Y} \right\|_{\mathcal{H}_0} \leq C \|\mathcal{B}_1(t)\mathcal{Y}\|_{\mathcal{H}_0}$ , for all  $\mathcal{Y} \in D(\mathcal{B}_1(t))$ , for a positive constant  $C$ .

This shows that  $\frac{\partial \mathcal{B}_1(t)}{\partial t}u \in L_*^\infty(0, T; B(D(\mathcal{B}_1(0)), L_{\frac{1}{a}}^2(0, 1)))$ .

Hence, we can apply the Kato theorem, obtaining the thesis if  $i = 1$ . The computations are similar if  $i = 2$ . In this case, we consider as inner product

$$\langle \mathcal{Y}, \tilde{\mathcal{Y}} \rangle_t = \langle (u, v), (\tilde{u}, \tilde{v}) \rangle_t := b(t)\langle u, \tilde{u} \rangle_a + \langle v, \tilde{v} \rangle_{L^2(0,1)},$$

$\forall \mathcal{Y} = (u, v), \tilde{\mathcal{Y}} = (\tilde{u}, \tilde{v}) \in \mathcal{H}_0$ , which induces the norm

$$\|\mathcal{Y}\|_t^2 = \|(u, v)\|_t^2 = b(t)\|u\|_a^2 + \|v\|^2, \forall \mathcal{Y} = (u, v) \in \mathcal{H}_0.$$

Also in this case the norms  $\|\cdot\|_t$  and  $\|\cdot\|_{\mathcal{H}_0}$  are equivalent in  $\mathcal{H}_0$  since (3.4) still holds.

We conclude this section with an estimate on the solutions of  $(P_i), i = 1, 2$ .

**Theorem 4.** Assume a (WD) or (SD) at 0 and  $b \in W^{1,\infty}(0, T)$  a strictly positive function. Then, if  $(y_0, y_1) \in D(A_i) \times K_i$ , the unique solution  $y \in C^2([0, T]; H_i) \cap C^1([0, T]; K_i) \cap C([0, T]; D(A_i)), i = 1, 2$ , of  $(AP_i)$  satisfies the following estimate

$$\begin{aligned} & \sup_{t \in [0, T]} \left( \|y_t(t)\|_{\frac{1}{a}}^2 + \|y_x(t)\|^2 \right) \\ & \leq \left( \frac{\|\dot{b}\|_{\infty, T} T + 1}{m} e^{T \frac{\|\dot{b}\|_{\infty, T}}{m}} + 1 \right) \max\{1, b(0)\} \|(y_0, y_1)\|_{\mathcal{H}_t}^2, \end{aligned}$$

if  $i = 1$  and

$$\begin{aligned} & \sup_{t \in [0, T]} \left( \|y_t(t)\|^2 + \|\sqrt{a}y_x(t)\|^2 \right) \\ & \leq \left( \frac{\|\dot{b}\|_{\infty, T} T + 1}{m} e^{T \frac{\|\dot{b}\|_{\infty, T}}{m}} + 1 \right) \max\{1, b(0)\} \|(y_0, y_1)\|_{\mathcal{H}_t}^2, \end{aligned}$$

if  $i = 2$ .

*Proof.* Assume that  $y$  is the classical solution of  $(AP_i)$  and assume for simplicity that  $i = 1$ . Then, multiplying the equation of  $(AP_i)$  (with  $i = 1$ ) by  $\frac{y_t}{a}$  and integrating over  $(0, 1)$ , one has

$$\frac{1}{2} \frac{d}{dt} \|y_t(t)\|_{\frac{1}{a}}^2 + \frac{1}{2} b(t) \frac{d}{dt} \|y_x(t)\|^2 = 0.$$

Thus

$$\frac{d}{dt} \left( \|y_t(t)\|_{\frac{1}{a}}^2 + b(t)\|y_x(t)\|^2 \right) = \dot{b}(t)\|y_x(t)\|^2.$$

Integrating over  $(0, t)$ , one has, i.e.,

$$\begin{aligned} \|y_t(t)\|_{\frac{1}{a}}^2 + b(t)\|y_x(t)\|^2 &\leq \|\dot{b}\|_{\infty, T} \int_0^t \|y_x(\tau)\|^2 d\tau + \|y_1\|_{\frac{1}{a}}^2 + b(0)\|(y_0)_x\|^2 \\ &\leq \|\dot{b}\|_{\infty, T} \int_0^t \|y_x(\tau)\|^2 d\tau + \max\{1, b(0)\}\|(y_0, y_1)\|_{\mathcal{H}_t}^2. \end{aligned} \tag{3.6}$$

In particular, one has

$$m\|y_x(t)\|^2 \leq b(t)\|y_x(t)\|^2 \leq \|\dot{b}\|_{\infty, T} \int_0^t \|y_x(\tau)\|^2 d\tau + \max\{1, b(0)\}\|(y_0, y_1)\|_{\mathcal{H}_0}^2.$$

Hence

$$\|y_x(t)\|^2 \leq \frac{\|\dot{b}\|_{\infty, T}}{m} \int_0^t \|y_x(\tau)\|^2 d\tau + \frac{\max\{1, b(0)\}}{m}\|(y_0, y_1)\|_{\mathcal{H}_t}^2,$$

and, by the Gronwall inequality, we obtain

$$\|y_x(t)\|^2 \leq \frac{\max\{1, b(0)\}}{m}\|(y_0, y_1)\|_{\mathcal{H}_0}^2 e^{T \frac{\|\dot{b}\|_{\infty, T}}{m}},$$

for all  $t \in [0, T]$ . Coming back to (3.6), one has

$$\begin{aligned} \|y_t(t)\|_{\frac{1}{a}}^2 &\leq \|\dot{b}\|_{\infty, T} \int_0^t \|y_x(\tau)\|^2 d\tau + \max\{1, b(0)\}\|(y_0, y_1)\|_{\mathcal{H}_t}^2 \\ &\leq \left( \frac{\|\dot{b}\|_{\infty, T} T}{m} e^{T \frac{\|\dot{b}\|_{\infty, T}}{m}} + 1 \right) \max\{1, b(0)\}\|(y_0, y_1)\|_{\mathcal{H}_t}^2, \end{aligned}$$

for all  $t \in [0, T]$ . In particular,

$$\begin{aligned} &\sup_{t \in [0, T]} \left( \|y_t(t)\|_{\frac{1}{a}}^2 + \|y_x(t)\|^2 \right) \\ &\leq \left( \frac{\|\dot{b}\|_{\infty, T} T + 1}{m} e^{T \frac{\|\dot{b}\|_{\infty, T}}{m}} + 1 \right) \max\{1, b(0)\}\|(y_0, y_1)\|_{\mathcal{H}_0}^2. \end{aligned}$$

□

#### 4. Energy estimates: upper and lower bounds

In this section, we prove two estimates of the energy associated to the solution of  $(AP_i)$ : one from below and the other one from above. If the propagation speed  $b(t)$  is constant, for example  $b(t) = 1$  for all  $t \in [0, T]$ , then the total energy at the

initial time is conserved with respect to  $t$  (see [24] for the divergence case and [11] for the non-divergence one). On the other hand, the variable propagation speed with respect to  $t$  describes a change of the quantity of the total energy, precisely,  $\dot{b}(t) > 0$  and  $\dot{b}(t) < 0$  imply the increasing and the decreasing behavior of the total energy, respectively. Hence, oscillating propagation speed makes the behavior of the energy more complicate. In this paper, we consider the general case  $\dot{b}(t) \neq 0$ . Clearly, the case  $\dot{b}(t) = 0$  a.e.  $t \in [0, T]$  implies that the function  $b$  is a constant and thus we have again the *autonomous case*.

In the next subsection, we will prove that in the non-autonomous degenerate case, as in the non-degenerate case, the so-called *generalized energy conservation law* holds. In particular, setting  $E_y(t)$  the energy associate to  $(AP_i)$ , one has that there exists  $C_1, C_2 > 0$  such that

$$C_2 E_y(0) \leq E_y(t) \leq C_1 E_y(0), \tag{4.1}$$

for all  $t \in [0, T]$ .

In the following, we make the next assumption:

**Hypothesis 5.** Assume a (WD) or (SD) at 0 and  $b \in W^{1,\infty}(0, +\infty)$  a strictly positive function in  $(0, +\infty)$ .

#### 4.1. Energy bounds in the non-divergence case

Consider the degenerate hyperbolic problem with Dirichlet boundary conditions

$$\begin{cases} y_{tt} - b(t)a(x)y_{xx} = 0, & (t, x) \in Q_T, \\ y(t, 1) = y(t, 0) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1), \\ y_t(0, x) = y_1(x), & x \in (0, 1). \end{cases} \tag{4.2}$$

Let  $y$  be a mild solution of (4.2) and consider its energy given by the continuous function defined as

$$E_y(t) = \frac{1}{2} \int_0^1 \left( \frac{1}{a} y_t^2(t, x) + b(t) y_x^2(t, x) \right) dx, \quad \forall t \geq 0. \tag{4.3}$$

**Theorem 6.** Assume Hypothesis 5, fix  $T > 0$  and let  $y$  be a classical solution of (4.2). Then, (4.1) holds in  $[0, T]$ , with

$$C_1 := e^{\frac{\|\dot{b}\|_{\infty, T} T}{m}} \quad \text{and} \quad C_2 := e^{-\frac{\|\dot{b}\|_{\infty, T} T}{m}}. \tag{4.4}$$

In addition, if  $b$  is non-increasing, then  $E_y$  is non-increasing; if  $b$  is increasing then  $E_y$  is increasing. In particular, (4.1) holds in  $[0, T]$ , with

$$C_1 := \begin{cases} 1, & \text{if } b \text{ is non-increasing,} \\ e^{\frac{\|\dot{b}\|_{\infty, T} T}{b(0)}}, & \text{if } b \text{ is non-decreasing,} \end{cases} \tag{4.5}$$

and

$$C_2 := \begin{cases} e^{-\frac{\|\dot{b}\|_{\infty,T}}{b(T)}T}, & \text{if } b \text{ is non-increasing,} \\ 1, & \text{if } b \text{ is non-decreasing.} \end{cases} \tag{4.6}$$

*Proof.* Multiplying the equation by  $\frac{y_t}{a}$  (recall that  $y_t \in H^1_{\frac{1}{a}}(0, 1)$  by Theorem 2), integrating over  $(0, 1)$  and using the boundary conditions one has

$$\begin{aligned} 0 &= \frac{1}{2} \int_0^1 \frac{d}{dt} \left( \frac{y_t^2}{a} \right) dx + \frac{1}{2} \int_0^1 b(t) \frac{d}{dt} \left( y_x^2 \right) dx \\ &\iff \frac{d}{dt} E_y(t) = \frac{1}{2} \int_0^1 \frac{d}{dt} \left( \frac{y_t^2}{a} + b(t)y_x^2 \right) dx = \frac{1}{2} \int_0^1 \dot{b}(t)y_x^2 dx. \end{aligned} \tag{4.7}$$

Hence,

$$\begin{aligned} \left| \frac{d}{dt} E_y(t) \right| &\leq \frac{1}{2} \|\dot{b}\|_{\infty,T} \int_0^1 y_x^2(t, x) dx \\ &\leq \frac{\|\dot{b}\|_{\infty,T}}{m} \frac{1}{2} \int_0^1 b(t)y_x^2(t, x) dx \leq \frac{\|\dot{b}\|_{\infty,T}}{m} E_y(t), \end{aligned} \tag{4.8}$$

and

$$\frac{d}{dt} \left( E_y(t) e^{-\frac{\|\dot{b}\|_{\infty,T}}{m}t} \right) \leq 0$$

for all  $t \geq 0$ . Hence, the function  $t \in [0, T] \mapsto E_y(t) e^{-\frac{\|\dot{b}\|_{\infty,T}}{m}t}$  is non-increasing; thus  $E_y(t) e^{-\frac{\|\dot{b}\|_{\infty,T}}{m}t} \leq E_y(0)$ , for all  $t \in [0, T]$ , i.e.,  $E_y(t) \leq E_y(0) e^{\frac{\|\dot{b}\|_{\infty,T}}{m}t} \leq E_y(0) e^{\frac{\|\dot{b}\|_{\infty,T}}{m}T}$ , for all  $t \in [0, T]$ . Setting  $C_1 := e^{\frac{\|\dot{b}\|_{\infty,T}}{m}T}$ , one has  $E_y(t) \leq C_1 E_y(0)$ , for all  $t \in [0, T]$ . Moreover, by (4.8),  $\frac{d}{dt} E_y(t) \geq -\frac{\|\dot{b}\|_{\infty,T}}{m} E_y(t)$ , then

$$\frac{d}{dt} \left( E_y(t) e^{\frac{\|\dot{b}\|_{\infty,T}}{m}t} \right) \geq 0,$$

for all  $t \geq 0$ . Hence, the function  $t \in [0, T] \mapsto E_y(t) e^{\frac{\|\dot{b}\|_{\infty,T}}{m}t}$  is non-decreasing; thus  $E_y(0) \leq E_y(t) e^{\frac{\|\dot{b}\|_{\infty,T}}{m}t} \leq E_y(t) e^{\frac{\|\dot{b}\|_{\infty,T}}{m}T}$ , for all  $t \in [0, T]$ . Setting  $C_2 := e^{-\frac{\|\dot{b}\|_{\infty,T}}{m}T}$ , one has  $C_2 E_y(0) \leq E_y(t)$ , for all  $t \in [0, T]$  and the thesis follows.

Moreover, if  $b$  is non-increasing, then the energy  $E_y$  is non-increasing. In particular  $E_y(t) \leq E_y(0)$ , for all  $t \geq 0$  and (4.1) holds with  $C_1 = 1$ .

On the other hand, assuming  $b$  non-decreasing, one has that  $E_y$  is non-decreasing; in particular  $E_y(0) \leq E_y(t)$ , for all  $t \geq 0$  and (4.1) holds with  $C_2 = 1$ .

**Theorem 7.** Assume Hypothesis 5 and fix  $T > 0$ . If  $y$  is a classical solution of (4.2), then we have

$$\begin{aligned} \frac{1}{2} \int_0^T b(t)y_x^2(t, 1)dt &= \int_0^1 \left[ \frac{x^2 y_x y_t}{a} \right]_{t=0}^{t=T} dx + \int_{Q_T} x b y_x^2 dx dt \\ &+ \frac{1}{2} \int_{Q_T} \left( 2 - \frac{x a'}{a} \right) \frac{1}{a} x y_t^2 dx dt. \end{aligned} \tag{4.9}$$

*Proof.* Multiplying the equation in (4.2) by  $\frac{x^2 y_x}{a}$ , integrating over  $Q_T$  and proceeding as in [11, Theorem 3.4], we have

$$\begin{aligned} 0 &= \int_0^1 \left[ \frac{x^2 y_x y_t}{a} \right]_{t=0}^{t=T} dx - \frac{1}{2} \int_{Q_T} \frac{x^2}{a} (y_t^2)_x dx dt - \int_{Q_T} x^2 b y_x y_{xx} dx dt \\ &= \int_0^1 \left[ \frac{x^2 y_x y_t}{a} \right]_{t=0}^{t=T} dx - \frac{1}{2} \int_0^T \left[ x^2 b y_x^2 \right]_{x=0}^{x=1} dt \\ &+ \int_{Q_T} x b y_x^2 dx dt + \frac{1}{2} \int_{Q_T} \frac{x y_t^2}{a} \left( 2 - \frac{x a'}{a} \right) dx dt. \end{aligned} \tag{4.10}$$

In addition, thanks to Lemma 1,  $\lim_{x \rightarrow 0} x^2 y_x^2(t, x) = 0$ , hence

$$\int_0^T \left[ x^2 b y_x^2 \right]_{x=0}^{x=1} dt = \int_0^T b(t)y_x^2(t, 1)dt. \tag{4.11}$$

Thanks to (4.10) and (4.11), the thesis follows. □

**Corollary 1.** Assume Hypothesis 5 and fix  $T > 0$ . If  $y$  is a mild solution (4.2), then

$$\int_0^T b(t)y_x^2(t, 1)dt \leq M_1 E_y(0), \tag{4.12}$$

where  $M_1 := 2 \max \left\{ \frac{1}{a(1)m}, 1 \right\} (1 + C_1) + (2 + K)C_1 T$ .

*Proof.* As a first step, assume that  $y$  is a classical solution of (4.2), hence (4.9) holds. Proceeding as in the proof of [11, Theorem 3.4], we estimate the boundary terms in (4.9). In particular, by (2.1), one has

$$\begin{aligned} \left| \int_0^1 \frac{x^2 y_x(\tau, x) y_t(\tau, x)}{a} dx \right| &\leq \frac{1}{2a(1)} \int_0^1 y_x^2(\tau, x) dx + \frac{1}{2} \int_0^1 \frac{y_t^2(\tau, x)}{a} dx \\ &\leq \frac{1}{2a(1)m} \int_0^1 b(\tau) y_x^2(\tau, x) dx + \frac{1}{2} \int_0^1 \frac{y_t^2(\tau, x)}{a} dx \leq \max \left\{ \frac{1}{a(1)m}, 1 \right\} E_y(\tau), \end{aligned} \tag{4.13}$$

for all  $\tau \in [0, T]$ . By the previous inequality and Theorem 6, we get

$$\int_0^1 \left[ \frac{x^2 y_x(\tau, x) y_t(\tau, x)}{a} \right]_{\tau=0}^{\tau=T} dx \leq 2 \max \left\{ \frac{1}{a(1)m}, 1 \right\} (1 + C_1) E_y(0). \tag{4.14}$$

Now, using the fact that  $x|a'| \leq Ka$ , we find

$$\left| \frac{1}{2} \int_{Q_T} \frac{xy_t^2}{a} \left( 2 - \frac{xa'}{a} \right) dxdt \right| \leq \frac{2+K}{2} \int_{Q_T} \frac{y_t^2}{a} dxdt.$$

Finally, we clearly have  $\int_{Q_T} xby_x^2 dxdt \leq \int_{Q_T} by_x^2 dxdt$ . Hence, by (4.9) and Theorem 6,

$$\begin{aligned} \frac{1}{2} \int_0^T b(t)y_x^2(t, 1)dt &\leq 2 \max \left\{ \frac{1}{a(1)m}, 1 \right\} (1 + C_1)E_y(0) \\ &\quad + \int_{Q_T} by_x^2 dxdt + \frac{2+K}{2} \int_{Q_T} \frac{y_t^2}{a} dxdt \\ &\leq 2 \max \left\{ \frac{1}{a(1)m}, 1 \right\} (1 + C_1)E_y(0) + (2+K) \int_0^T E_y(t)dt \\ &\leq 2 \max \left\{ \frac{1}{a(1)m}, 1 \right\} (1 + C_1)E_y(0) + (2+K)C_1TE_y(0). \end{aligned}$$

Hence, the inequality (4.12) holds true for all  $T \geq 0$  if  $y$  is a classical solution. Now, let  $y$  be the mild solution of (4.2) associated to the initial data  $(y_0, y_1) \in \mathcal{H}_0$ , and consider a sequence  $\{(y_0^n, y_1^n)\}_{n \in \mathbb{N}} \subset D(\mathcal{B}_1(t))$  that approximates  $(y_0, y_1)$ . Let  $y^n$  be the classical solution of (4.2) associated to  $(y_0^n, y_1^n)$ . Clearly  $y^n$  satisfies (4.12), and by linearity  $(y^n)_x$  is a Cauchy sequence in  $L^2(0, T)$ , then we can conclude by passing to the limit.  $\square$

We shall also prove an estimate from above for the energy. To this aim the next equality is crucial.

**Theorem 8.** *Assume Hypothesis 5 and fix  $T > 0$ . If  $y$  is a classical solution of (4.2), then we have*

$$\begin{aligned} \frac{1}{2} \int_0^T b(t)y_x^2(t, 1)dt &= \int_0^1 \left[ \frac{xy_x y_t}{a} \right]_{t=0}^{t=T} dx + \frac{1}{2} \int_{Q_T} by_x^2 dxdt \\ &\quad + \frac{1}{2} \int_{Q_T} \left( 1 - \frac{xa'}{a} \right) \frac{1}{a} y_t^2 dxdt. \end{aligned} \tag{4.15}$$

*Proof.* We proceed as in [11, Theorem 3.3]. Thus, multiplying the equation in (4.2) by  $\frac{xy_x}{a}$  and integrating over  $Q_T$ , we have

$$\begin{aligned} 0 &= \int_0^1 \left[ \frac{xy_x y_t}{a} \right]_{t=0}^{t=T} dx - \frac{1}{2} \int_{Q_T} \frac{x}{a} (y_t^2)_x dxdt - \frac{1}{2} \int_{Q_T} xb(y_x^2)_x dxdt \\ &= \int_0^1 \left[ \frac{xy_x y_t}{a} \right]_{t=0}^{t=T} dx - \frac{1}{2} \int_0^T \left[ \frac{x}{a} y_t^2 \right]_{x=0}^{x=1} dt + \frac{1}{2} \int_{Q_T} \left( 1 - \frac{xa'}{a} \right) \frac{y_t^2}{a} dxdt \\ &\quad - \frac{1}{2} \int_0^T \left[ xby_x^2 \right]_{x=0}^{x=1} dt + \frac{1}{2} \int_{Q_T} by_x^2 dxdt. \end{aligned} \tag{4.16}$$

Now, thanks to the boundary conditions on  $y$  and Lemma 1, we have  $\int_0^T [xby_x^2]_{x=0}^{x=1} dt = \int_0^T b(t)y_x^2(t, 1)dt$ ,  $\frac{1}{a(1)}y_t^2(t, 1) = 0$  and  $\lim_{x \rightarrow 0} \frac{x}{a}y_t^2(t, x) = 0$  (recall that  $y$  is a classical solution, thus  $y_t(t, \cdot) \in H^1_{\frac{1}{a}}(0, 1)$ ). Hence, (4.16) reads

$$\begin{aligned} \frac{1}{2} \int_0^T b(t)y_x^2(t, 1)dt &= \int_0^1 \left[ \frac{xy_x y_t}{a} \right]_{t=0}^{t=T} dx + \frac{1}{2} \int_{Q_T} \left( 1 - \frac{xa'}{a} \right) \frac{1}{a} y_t^2 dxdt \\ &\quad + \frac{1}{2} \int_{Q_T} by_x^2 dxdt \end{aligned} \tag{4.17}$$

and the thesis holds. □

As a consequence, we have the next estimate on the energy.

**Corollary 2.** *Assume Hypothesis 5 and fix  $T > 0$ . If  $y$  is a mild solution (4.2), then*

$$\int_0^T b(t)y_x^2(t, 1)dt \geq M_2 E_y(0), \tag{4.18}$$

where

$$M_2 := T(2 - K)C_2 - 2(1 + C_1) \max \left\{ 1, \frac{1}{a(1)} \right\} - \frac{K}{2}(1 + C_1) \max \left\{ 1, \frac{C_{HP}}{m} \right\}.$$

*Proof.* As in the proof of Corollary 1, assume that  $y$  is a classical solution. First of all, observe that by the boundary condition on  $y$ , we have  $(by_x y)(t, 1) = (by_x y)(t, 0) = 0$ . Hence, multiplying the equation in (4.2) by  $\frac{-Ky}{2a}$  and integrating on  $Q_T$ , we obtain

$$\begin{aligned} 0 &= -\frac{K}{2} \int_0^1 \left[ \frac{yy_t}{a} \right]_{t=0}^{t=T} dx + \frac{K}{2} \int_{Q_T} \frac{y_t^2}{a} dxdt + \frac{K}{2} \int_0^T [by_x y]_{x=0}^{x=1} dt - \frac{K}{2} \int_{Q_T} by_x^2 dxdt \\ &= -\frac{K}{2} \int_0^1 \left[ \frac{yy_t}{a} \right]_{t=0}^{t=T} dx + \frac{K}{2} \int_{Q_T} \frac{y_t^2}{a} dxdt - \frac{K}{2} \int_{Q_T} by_x^2 dxdt. \end{aligned}$$

Summing the previous equality to (4.15) multiplied by 2 and using (1.2), we have

$$\begin{aligned} \int_0^T b(t)y_x^2(t, 1)dt &= 2 \int_0^1 \left[ \frac{xy_x y_t}{a} \right]_{t=0}^{t=T} dx - \frac{K}{2} \int_0^1 \left[ \frac{yy_t}{a} \right]_{t=0}^{t=T} dx + \frac{K}{2} \int_{Q_T} \frac{y_t^2}{a} dxdt \\ &\quad - \frac{K}{2} \int_{Q_T} by_x^2 dxdt + \int_{Q_T} \frac{y_t^2}{a} dxdt + \int_{Q_T} by_x^2 dxdt - \int_{Q_T} \frac{xa'}{a} \frac{y_t^2}{a} dxdt \\ &\geq 2 \int_0^1 \left[ \frac{xy_x y_t}{a} \right]_{t=0}^{t=T} dx - \frac{K}{2} \int_0^1 \left[ \frac{y_t y}{a} \right]_{t=0}^{t=T} dx + \left( 1 - \frac{K}{2} \right) \int_{Q_T} by_x^2 dxdt \\ &\quad + \left( 1 + \frac{K}{2} \right) \int_{Q_T} \frac{y_t^2}{a} dxdt - K \int_{Q_T} \frac{y_t^2}{a} dxdt \end{aligned}$$

$$= 2 \int_0^1 \left[ \frac{xy_x y_t}{a} \right]_{t=0}^{t=T} dx - \frac{K}{2} \int_0^1 \left[ \frac{y_t y}{a} \right]_{t=0}^{t=T} dx + (2 - K) \int_0^T E_y(t) dt.$$

By Theorem 6, it follows that

$$\begin{aligned} \int_0^T b(t) y_x^2(t, 1) dt &\geq 2 \int_0^1 \left[ \frac{xy_x y_t}{a} \right]_{t=0}^{t=T} dx \\ &\quad - \frac{K}{2} \int_0^1 \left[ \frac{y_t y}{a} \right]_{t=0}^{t=T} dx + (2 - K) C_2 T E_y(0). \end{aligned}$$

Now, we move to the boundary terms. As in [11, Theorem 3.7]

$$2 \left| \left( \frac{xy_x y_t}{a} \right) (\tau, x) \right| \leq \left( \frac{y_t^2}{a} + \frac{y_x^2}{a(1)} \right) (\tau, x)$$

for all  $\tau \in [0, T]$ . Thus, by Theorem 6

$$\begin{aligned} 2 \left| \int_0^1 \left[ \frac{xy_x y_t}{a} \right]_{t=0}^{t=T} dx \right| &\leq \int_0^1 \frac{y_t^2}{a} (T, x) dx + \frac{1}{a(1)m} \int_0^1 (by_x^2)(T, x) dx \\ &\quad + \int_0^1 \frac{y_t^2}{a} (0, x) dx + \frac{1}{a(1)m} \int_0^1 (by_x^2)(0, x) dx \\ &\leq 2 \max \left\{ 1, \frac{1}{a(1)} \right\} E_y(T) + 2 \max \left\{ 1, \frac{1}{a(1)} \right\} E_y(0) \\ &\leq 2(1 + C_1) \max \left\{ 1, \frac{1}{a(1)} \right\} E_y(0). \end{aligned} \tag{4.19}$$

Moreover, as in [11, Theorem 3.7],  $\left| \left( \frac{yy_t}{a} \right) (\tau, x) \right| \leq \frac{1}{2} \left( \frac{y_t^2}{a} + \frac{y_x^2}{a} \right) (\tau, x)$ , for all  $\tau \geq 0$ . Thus, by Proposition 1 and Theorem 6, we get

$$\begin{aligned} \frac{K}{2} \left| \int_0^1 \left[ \frac{yy_t}{a} \right]_{t=0}^{t=T} dx \right| &\leq \frac{K}{4} \int_0^1 \frac{y_t^2}{a} (T, x) dx + \frac{K}{4} \frac{C_{HP}}{m} \int_0^1 b(T) y_x^2(T, x) dx \\ &\quad + \frac{K}{4} \int_0^1 \frac{y_t^2}{a} (0, x) dx + \frac{K}{4} \frac{C_{HP}}{m} \int_0^1 b(0) y_x^2(0, x) dx \\ &\leq \frac{K}{2} \max \left\{ 1, \frac{C_{HP}}{m} \right\} E_y(T) + \frac{K}{2} \max \left\{ 1, \frac{C_{HP}}{m} \right\} E_y(0) \\ &\leq \frac{K}{2} (1 + C_1) \max \left\{ 1, \frac{C_{HP}}{m} \right\} E_y(0). \end{aligned}$$

Hence  $\int_0^T b(t) y_x^2(t, 1) dt \geq M_2 E_y(0)$ . Thus the conclusion holds true if  $y$  is a classical solution. Now, let  $y$  be the mild solution associated to the initial data  $(y_{0,T}, y_{1,T}) \in \mathcal{H}_0$ . By approximation, as in the proof of Corollary 1, we get the same estimates for mild solutions. □

### 4.2. Energy bounds in the divergence case

Consider the following degenerate hyperbolic problem

$$\begin{cases} y_{tt} - b(t)(a(x)y_x)_x = 0, & (t, x) \in Q_T, \\ y(t, 0) = \begin{cases} 0, & \text{if } a \text{ is (WD),} \\ \lim_{x \rightarrow 0} a(x)u_x(t, x) = 0, & \text{if } a \text{ is (SD),} \end{cases} & t \in (0, T), \\ y(t, 1) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, 1), \\ y_t(0, x) = y_1(x), & x \in (0, 1). \end{cases} \tag{4.20}$$

Let  $y$  be a mild solution of (4.20) and consider its energy given by

$$E_y(t) = \frac{1}{2} \int_0^1 \left( y_t^2(t, x) + b(t)a(x)y_x^2(t, x) \right) dx, \quad \forall t \geq 0. \tag{4.21}$$

**Theorem 9.** *Assume Hypothesis 5 and let  $y$  be a classical solution of (4.20). Then, (4.1) holds in  $[0, T]$ , with  $C_1$  and  $C_2$  given in (4.4). In addition, if  $b$  is non-increasing, then  $E_y$  is non-increasing; if  $b$  is increasing then  $E_y$  is increasing. In particular, for fixed  $T > 0$ , (4.1) holds in  $[0, T]$  with  $C_1$  and  $C_2$  defined as in (4.5) and (4.6), respectively.*

*Proof.* The proof is similar to the one of Theorem 6, thus we omit it. We only observe that, in this case, if  $y$  is a classical solution of (4.20), multiplying the equation by  $y_t$  (recall that  $y_t \in H_a^1(0, 1)$  by Theorem 2), integrating over  $(0, 1)$  and using the boundary conditions one has

$$\begin{aligned} 0 &= \frac{1}{2} \int_0^1 \frac{d}{dt} \left( y_t^2 \right) dx + \frac{1}{2} \int_0^1 b(t)a(x) \frac{d}{dt} \left( y_x^2 \right) dx \\ \iff \frac{d}{dt} E_y(t) &= \frac{1}{2} \int_0^1 \frac{d}{dt} \left( y_t^2 + b(t)a(x)y_x^2 \right) dx = \frac{1}{2} \int_0^1 \dot{b}(t)a(x)y_x^2 dx; \end{aligned} \tag{4.22}$$

hence

$$\begin{aligned} \left| \frac{d}{dt} E_y(t) \right| &\leq \frac{1}{2} \|\dot{b}\|_{\infty, T} \int_0^1 a(x)y_x^2(t, x) dx \\ &\leq \frac{1}{2} \frac{\|\dot{b}\|_{\infty, T}}{m} \int_0^1 b(t)a(x)y_x^2(t, x) dx \leq \frac{\|\dot{b}\|_{\infty, T}}{m} E_y(t), \end{aligned} \tag{4.23}$$

for all  $t \geq 0$ .

In the divergence case, the analogous of Theorem 7 is the next result.

**Theorem 10.** *Assume Hypothesis 5 and fix  $T > 0$ . If  $y$  is a classical solution of (4.2), then we have*

$$\begin{aligned} \frac{a(1)}{2} \int_0^T b(t)y_x^2(t, 1)dt &= \int_0^1 [xy_x y_t]_{t=0}^{t=T} dx \\ &+ \frac{1}{2} \int_{Q_T} y_t^2 dxdt + \frac{1}{2} \int_{Q_T} (a - xa')by_x^2 dxdt. \end{aligned} \tag{4.24}$$

For the proof of the previous result, Lemma 2 is crucial, but we omit it since it is similar to the one of [24, Lemma 3.2]; indeed if  $b(t) \equiv 1$  for all  $t \in [0, T]$ , then we have exactly the same result as in the autonomous case ([24, Lemma 3.2]).

Thanks to the previous theorem we can obtain the analog of Corollary 1.

**Corollary 3.** *Assume Hypothesis 5 and fix  $T > 0$ . If  $y$  is a mild solution (4.2), then*

$$\int_0^T b(t)y_x^2(t, 1)dt \leq \tilde{M}_1 E_y(0), \tag{4.25}$$

where  $\tilde{M}_1 := \frac{1}{a(1)} \left( 2(1 + C_1) \max \left\{ 1, \frac{1}{a(1)m} \right\} + 2(1 + K)C_1 T \right)$ .

*Proof.* As a first step, assume that  $y$  is a classical solution of (4.20). Then (4.24) holds and, as in [8, Lemma3.2],

$$2 |(xy_x y_t)(\tau, x)| \leq (y_t^2 + x^2 y_x^2)(\tau, x) \leq \left( y_t^2 + \frac{1}{a(1)} a y_x^2 \right)(\tau, x)$$

for all  $\tau \in [0, T]$ . Thus, by Theorem 9

$$\begin{aligned} 2 \left| \int_0^1 [xy_x y_t]_{t=0}^{t=T} dx \right| &\leq \int_0^1 y_t^2(T, x)dx + \frac{1}{a(1)m} \int_0^1 (aby_x^2)(T, x)dx \\ &+ \int_0^1 y_t^2(0, x)dx + \frac{1}{a(1)m} \int_0^1 (aby_x^2)(0, x)dx \\ &\leq 2 \max \left\{ 1, \frac{1}{a(1)m} \right\} E_y(T) + 2 \max \left\{ 1, \frac{1}{a(1)m} \right\} E_y(0) \\ &\leq 2(1 + C_1) \max \left\{ 1, \frac{1}{a(1)m} \right\} E_y(0). \end{aligned} \tag{4.26}$$

By (4.24) multiplied by 2, we have

$$\begin{aligned} a(1) \int_0^T b(t)y_x^2(t, 1)dt &\leq 2(1 + C_1) \max \left\{ 1, \frac{1}{a(1)m} \right\} E_y(0) \\ &+ \int_{Q_T} y_t^2 dxdt + (1 + K) \int_{Q_T} aby_x^2 dxdt \\ &\leq 2(1 + C_1) \max \left\{ 1, \frac{1}{a(1)m} \right\} E_y(0) \\ &+ 2(1 + K) \int_0^T E_y(t)dt \leq 2(1 + C_1) \max \left\{ 1, \frac{1}{a(1)m} \right\} E_y(0) \\ &+ 2(1 + K)C_1 T E_y(0). \end{aligned}$$

If  $y$  is a mild solution, then we can proceed as in Corollary 1 obtaining the thesis.

Thus, Corollary 2 becomes the following

**Corollary 4.** *Assume Hypothesis 5 and fix  $T > 0$ . If  $y$  is a mild solution (4.20), then*

$$\int_0^T b(t)y_x^2(t, 1)dt \geq \tilde{M}_2 E_y(0), \tag{4.27}$$

where

$$\begin{aligned} \tilde{M}_2 := & \frac{1}{a(1)} \left( (2 - K)C_2T - 2(1 + C_1) \max \left\{ 1, \frac{1}{a(1)m} \right\} \right. \\ & \left. - \frac{K}{2}(1 + C_1) \max \left\{ 1, \frac{C_a}{m} \right\} \right). \end{aligned}$$

*Proof.* We proceed as in Corollary 1; thus assume, as a first step, that  $y$  is a classical solution. As in [8, Theorem 3.4], multiplying the equation in (4.20) by  $\frac{Ky}{2}$  and integrating on  $Q_T$ , we obtain

$$\begin{aligned} 0 &= \frac{K}{2} \int_{Q_T} yy_t dx dt - \frac{K}{2} \int_{Q_T} by(ay_x)_x dx dt \\ &= \frac{K}{2} \int_0^1 [yy_t]_{t=0}^{t=T} dx - \frac{K}{2} \int_{Q_T} y_t^2 dx dt + \frac{K}{2} \int_{Q_T} aby_x^2 dx dt, \end{aligned}$$

being  $\int_0^T [aby_x y]_{x=0}^{x=1} dt = 0$ . Summing the previous equality to (4.24) multiplied by 2 and using (1.2), we have

$$\begin{aligned} a(1) \int_0^T b(t)y_x^2(t, 1)dt &= 2 \int_0^1 [xy_x y_t]_{t=0}^{t=T} dx \\ &+ \frac{K}{2} \int_0^1 [y_t y]_{t=0}^{t=T} dx + (2 - K) \int_0^T E_y(t)dt. \end{aligned}$$

By Theorem 9, it follows

$$\begin{aligned} a(1) \int_0^T b(t)y_x^2(t, 1)dt &\geq 2 \int_0^1 [xy_x y_t]_{t=0}^{t=T} dx \\ &+ \frac{K}{2} \int_0^1 [y_t y_x]_{t=0}^{t=T} dx + (2 - K)C_2T E_y(0). \end{aligned} \tag{4.28}$$

Now, we consider the boundary terms. As in Corollary 3

$$2 \left| \int_0^1 [xy_x y_t]_{t=0}^{t=T} dx \right| \leq 2(1 + C_1) \max \left\{ 1, \frac{1}{a(1)m} \right\} E_y(0). \tag{4.29}$$

Moreover, as in [8, Theorem 3.4], for all  $\tau \in [0, T]$  one has

$$\begin{aligned} \int_0^1 |(yy_t)(\tau, x)| dx &\leq \frac{1}{2} \int_0^1 (y_t^2 + y^2)(\tau, x) dx \\ &\leq \frac{1}{2} \int_0^1 y_t^2(\tau, x) dx + \frac{C_a}{2} \int_0^1 ay_x^2(\tau, x) dx, \end{aligned}$$

by Proposition 2, where  $C_a$  is defined in (2.7).

Thus, we get

$$\begin{aligned} \frac{K}{2} \left| \int_0^1 [yy_t]_{t=0}^{t=T} dx \right| &\leq \frac{K}{4} \int_0^1 y_t^2(T, x) dx + \frac{K}{4} \frac{C_a}{m} \int_0^1 a(x)b(T)y^2(T, x) dx \\ &\quad + \frac{K}{4} \int_0^1 \frac{y_t^2}{a}(0, x) dx + \frac{K}{4} \frac{C_a}{m} \int_0^1 a(x)b(0)y^2(0, x) dx \\ &\leq \frac{K}{2} \max \left\{ 1, \frac{C_a}{m} \right\} E_y(T) + \frac{K}{2} \max \left\{ 1, \frac{C_a}{m} \right\} E_y(0) \\ &\leq \frac{K}{2} (1 + C_1) \max \left\{ 1, \frac{C_a}{m} \right\} E_y(0). \end{aligned}$$

Hence, by (4.28),  $\int_0^T b(t)y_x^2(t, 1)dt \geq \tilde{M}_2 E_y(0)$ . , Thus the conclusion holds true if  $y$  is a classical solution. If  $y$  is a mild solution, we can proceed as in Corollary 1 obtaining the thesis.

### 5. Boundary observability and null-controllability

Following [24] for the divergence case and [11] for the non-divergence case, we give the next definition:

**Definition 2.** Problem (4.2) (or (4.20)) is said to be *observable in time*  $T > 0$  via the normal derivative at  $x = 1$  if there exists a constant  $C > 0$  such that, for any  $(y_T^0, y_T^1) \in \mathcal{H}_0$ , the mild solution  $y$  of (4.2) satisfies

$$C E_y(0) \leq \int_0^T b(t)y_x^2(t, 1)dt. \tag{5.1}$$

Moreover, any constant satisfying (5.1) is called *observability constant* for (4.2) (or for (4.20)) in time  $T$ .

Setting

$$C_T := \sup\{C > 0 : C \text{ satisfies (5.1)}\},$$

we have that (4.2) is observable if and only if

$$C_T = \inf_{(y_0, y_1) \neq (0,0)} \frac{\int_0^T b(t)y_x^2(t, 1)dt}{E_y(0)} > 0.$$

The inverse of  $C_T$ ,  $c_t := \frac{1}{C_T}$ , is called *the cost of observability* (or *the cost of control*) in time  $T$ .

Thus, as immediate consequences of Corollaries 2 and 4, one has the next results.

**Proposition 4.** *Assume Hypothesis 5 and set*

$$T_0 := \frac{1}{(2 - K)C_2} \left( 2(1 + C_1) \max \left\{ 1, \frac{1}{a(1)} \right\} + \frac{K}{2}(1 + C_1) \max \left\{ 1, \frac{C_{HP}}{m} \right\} \right). \tag{5.2}$$

If  $T > T_0$ , then (4.2) is observable in time  $T$ . Moreover,

$$T(2 - K)C_2 - 2(1 + C_1) \max \left\{ 1, \frac{1}{a(1)} \right\} - \frac{K}{2}(1 + C_1) \max \left\{ 1, \frac{C_{HP}}{m} \right\} \leq C_T.$$

The analogous statement of Proposition 4 in the divergence case is the following:

**Proposition 5.** *Assume Hypothesis 5 and set*

$$\tilde{T}_0 := \frac{1}{(2 - K)C_2} \left( 2(1 + C_1) \max \left\{ 1, \frac{1}{a(1)m} \right\} + \frac{K}{2}(1 + C_1) \max \left\{ 1, \frac{C_a}{m} \right\} \right). \tag{5.3}$$

If  $T > \tilde{T}_0$  then (4.20) is observable in time  $T$ . Moreover,

$$\frac{1}{a(1)} \left( (2 - K)C_2T - 2(1 + C_1) \max \left\{ 1, \frac{1}{a(1)m} \right\} - \frac{K}{2}(1 + C_1) \max \left\{ 1, \frac{C_a}{m} \right\} \right) \leq C_T.$$

In the following, we will study the problem of null-controllability for  $(P_i)$ . More precisely, denoting with  $K_i^*$  the dual space of  $K_i$ ,  $i = 1, 2$ , and given  $(u_0, u_1) \in H_i \times K_i^*$ , we look for a control  $f \in L^2(0, T)$  such that the solution of  $(P_i)$  satisfies (1.5).

To this aim, we give the definition of a solution for  $(P_i)$  by *transposition* which permits low regularity on the solution itself. Such a definition is formally obtained by multiplying the equation of  $(P_i)$  by  $\frac{v}{a}$ , in the non-divergence case, or by  $v$ , in the divergence case, and integrating by parts. In particular, we have the following definition:

**Definition 3.** Fix  $T > 0$ ; let  $f \in L^2(0, T)$  and  $(u_0, u_1) \in H_i \times K_i^*$ ,  $i = 1, 2$ . We say that  $u$  is a solution by transposition of  $(P_i)$  if

$$u \in C^1([0, T]; K_i^*) \cap C([0, T]; H_i)$$

and

$$\begin{aligned} \langle u_t(T), v_{0,T} \rangle_{H_a^{-1}(0,1), H_a^1(0,1)} - \int_0^1 \frac{1}{a} u(T) v_{1,T} dx &= \langle u_1, v(0) \rangle_{H_a^{-1}(0,1), H_a^1(0,1)} \\ &- \int_0^1 \frac{1}{a} u_0 v_t(0, x) dx - \int_0^T b(t) f(t) v_x(t, 1) dt, \end{aligned} \tag{5.4}$$

if  $i = 1$ , and

$$\begin{aligned} \langle u_t(T), v_{0,T} \rangle_{H_a^{-1}(0,1), H_a^1(0,1)} - \int_0^1 u(T) v_{1,T} dx &= \langle u_1, v(0) \rangle_{H_a^{-1}(0,1), H_a^1(0,1)} \\ &- \int_0^1 u_0 v_t(0, x) dx - \int_0^T b(t) f(t) v_x(t, 1) dt, \end{aligned} \tag{5.5}$$

if  $i = 2$ , for all  $(v_{0,T}, v_{1,T}) \in K_i \times H_i$ , where  $v$  solves the backward problem

$$\begin{cases} v_{tt} - \mathcal{A}_i(t)v = 0, & (t, x) \in Q_T, \\ v(t, 1) = v(t, 0) = 0, & t \in (0, T), \\ v(T, x) = v_{0,T}(x), & x \in (0, 1), \\ v_t(T, x) = v_{1,T}(x), & x \in (0, 1), \end{cases} \tag{5.6}$$

$i = 1, 2$ .

Observe that, setting  $y(t, x) := v(T - t, x)$ , one has that  $y$  satisfies (4.2) with  $y_0(x) = v_{0,T}(x)$  and  $y_1(x) = -v_{1,T}(x)$ . Hence, we can apply Theorem 2 obtaining that there exists a unique mild solution  $y$  of (4.2) in  $[0, T]$ . In particular, there exists a unique mild solution  $v$  of (5.6)

$$v \in C^1([0, T]; H_i) \cap C([0, T]; K_i),$$

$i = 1, 2$ , which depends continuously on the initial data  $V_T := (v_{0,T}, v_{1,T}) \in \mathcal{H}_0$ . Using Theorems 6 and 9, one can prove the next result.

**Proposition 6.** *Fix  $T > 0$  and assume Hypothesis 5. Then, there exists a unique solution  $u$  by transposition of  $(P_i)$ ,  $i = 1, 2$ .*

*Proof.* For simplicity, assume  $i = 1$ , being the case  $i = 2$  similar. Thus, consider the functional  $\mathcal{G} : \mathcal{H}_0 \rightarrow \mathbb{R}$  given by

$$\begin{aligned} \mathcal{G}(v_{0,T}, v_{1,T}) &:= \langle u_1, v(0) \rangle_{H^{-1}_a(0,1), H^1_a(0,1)} \\ &\quad - \int_0^1 \frac{1}{a} u_0 v_t(0, x) dx - \int_0^T b(t) f(t) v_x(t, 1) dt, \end{aligned} \tag{5.7}$$

where  $v$  satisfies (5.6). Clearly,  $\mathcal{G}$  is linear and continuous. Being  $\mathcal{G} \in (\mathcal{H}_0)^*$  (where  $(\mathcal{H}_0)^*$  is the dual space of  $\mathcal{H}_0$ ), we can use the Riesz Theorem obtaining that there exists a unique  $(\tilde{u}_{0,T}, \tilde{u}_{1,T}) \in \mathcal{H}_0$ , which can be identified with  $\mathcal{H}_0^*$ , such that

$$\begin{aligned} \mathcal{G}(v_{0,T}, v_{1,T}) &= \langle (\tilde{u}_{0,T}, \tilde{u}_{1,T}), (v_{0,T}, v_{1,T}) \rangle_{\mathcal{H}_0} \\ &= \langle \tilde{u}_{0,T}, v_{0,T} \rangle_{H^{-1}_a(0,1), H^1_a(0,1)} + \int_0^1 \frac{1}{a} \tilde{u}_{1,T} v_{1,T} dx. \end{aligned} \tag{5.8}$$

Moreover,  $\tilde{u}_{0,T}, \tilde{u}_{1,T}$  depend continuously on  $T$ , so there exists a unique

$$u \in C([0, T]; L^2_{\frac{1}{a}}(0, 1)) \cap C^1([0, T]; H^{-1}_{\frac{1}{a}}(0, 1))$$

such that  $u(T) = -\tilde{u}_{1,T}$  and  $u_t(T) = \tilde{u}_{0,T}$ . By (5.7) and (5.8), we can conclude that  $u$  is the unique solution by transposition of  $(P_i)$ . □

Now, we can prove the null-controllability result. To this aim, we consider the bilinear form  $\Lambda : \mathcal{H}_0 \times \mathcal{H}_0 \rightarrow \mathbb{R}$  defined as

$$\Lambda(V_T, W_T) := \int_0^T b(t) v_x(t, 1) w_x(t, 1) dt,$$

where  $v$  and  $w$  are the solutions of (5.6) associated to  $V_T := (v_{0,T}, v_{1,T})$  and  $W_T := (w_{0,T}, w_{1,T})$ , respectively. The following lemma holds.

**Lemma 3.** *Assume Hypothesis 5 and fix  $T > T_0$ , in the non-divergence case, or  $T > \tilde{T}_0$ , in the divergence one. Then, the bilinear form  $\Lambda$  is continuous and coercive.*

*Proof.* Also in this case we consider  $i = 1$ , being  $i = 2$  similar.

For  $i = 1$ , the proof is similar to the one given in [11], but we repeat here for the reader's convenience since in this case the energy is non-constant as in [11] and we have the term  $b(t)$ . By Theorem 6 and Corollary 1, one has

$$\begin{aligned} |\Lambda(V_T, W_T)| &\leq \int_0^T b(t) |v_x(t, 1)w_x(t, 1)| dt \\ &\leq \left( \int_0^T b(t)v_x^2(t, 1)dt \right)^{\frac{1}{2}} \left( \int_0^T b(t)w_x^2(t, 1)dt \right)^{\frac{1}{2}} \\ &\leq M_1 E_v^{\frac{1}{2}}(0) E_w^{\frac{1}{2}}(0) \leq \frac{M_1}{C_2} E_v^{\frac{1}{2}}(T) E_w^{\frac{1}{2}}(T) \\ &\leq \frac{M_1}{2C_2} \max\{1, M\} \left( \int_0^1 \frac{(v_{1,T})^2(x)}{a} dx + \int_0^1 v_x^2(T, x) dx \right)^{\frac{1}{2}} \times \\ &\times \left( \int_0^1 \frac{(w_{1,T})^2(x)}{a} dx + \int_0^1 w_x^2(T, x) dx \right)^{\frac{1}{2}} \\ &= \frac{M_1}{2C_2} \max\{1, M\} \|(v(T), v_t(T))\|_{\mathcal{H}_t} \|(w(T), w_t(T))\|_{\mathcal{H}_t} \\ &= \frac{M_1}{2C_2} \max\{1, M\} \|V_T\|_{\mathcal{H}_t} \|W_T\|_{\mathcal{H}_t}, \end{aligned}$$

for all  $V_T, W_T \in \mathcal{H}_0$ . In a similar way, one can prove that  $\Lambda$  is coercive. Indeed, by Theorem 2 one has

$$\begin{aligned} |\Lambda(V_T, V_T)| &= \int_0^1 b(t)v_x^2(t, 1)dt \geq M_2 E_v(0) \geq \frac{M_2}{C_1} E_v(T) \\ &\geq \frac{M_2}{2C_1} \min\{1, m\} \|V_T\|_{\mathcal{H}_t}^2, \end{aligned}$$

for all  $V_T \in \mathcal{H}_0$ .

Thanks to  $\Lambda$ , the following null-controllability result holds.

**Theorem 11.** *Assume Hypothesis 5 and fix  $T > T_0$ , in the non-divergence case, or  $T > \tilde{T}_0$ , in the divergence one. Then, for every  $(u_0, u_1)$  in  $H_i \times K_i^*$ ,  $i = 1, 2$ , there exists a control  $f \in L^2(0, T)$  such that the solution  $u$  by transposition of  $(P_i)$  satisfies*

$$u(T, x) = u_t(T, x) = 0 \quad \text{a.e. } x \in (0, 1).$$

*Proof.* Again assume  $i = 1$ , since the proof for  $i = 2$  is similar. Consider the continuous linear map  $\mathcal{L} : \mathcal{H}_0 \rightarrow \mathbb{R}$  defined as

$$\mathcal{L}(V_T) := - \int_0^1 \frac{u_0 v_t(0, x)}{a} dx + \langle u_1, v(0) \rangle_{H_{\frac{1}{a}}^{-1}(0,1), H_{\frac{1}{a}}^1(0,1)},$$

where  $v$  is the solution of (5.6) associated to the final data  $V_T := (v_{0,T}, v_{1,T}) \in \mathcal{H}_0$ . Thanks to the previous lemma and by the Lax-Milgram Theorem, there exists a unique  $\bar{V}_T \in \mathcal{H}_0$  such that

$$\Lambda(\bar{V}_T, W_T) = \mathcal{L}(W_T), \quad \forall W_T \in \mathcal{H}_0. \tag{5.9}$$

Set  $f(t) := b(t)\bar{v}_x(t, 1)$ ,  $\bar{v}$  being the solution of (5.6) with initial data  $\bar{V}_T$ . Then, by (5.9) we get

$$\begin{aligned} \int_0^T f(t)w_x(t, 1)dt &= \int_0^T b(t)\bar{v}_x(t, 1)w_x(t, 1)dt = \Lambda(\bar{V}_T, W_T) = \mathcal{L}(W_T) \\ &= \langle u_1, w(0) \rangle_{H_{\frac{1}{a}}^{-1}(0,1), H_{\frac{1}{a}}^1(0,1)} - \int_0^1 \frac{1}{a} u_0 w_t(0, x) dx, \end{aligned} \tag{5.10}$$

for all  $W_T \in \mathcal{H}_0$ .

Finally, denote by  $u$  the solution by transposition of  $(P_i)$  (with  $i = 1$ ) associated to the function  $f$  just introduced above. Then, we have

$$\begin{aligned} \int_0^T f(t)w_x(t, 1)dt &= - \langle u_t(T), w_{0,T} \rangle_{H_{\frac{1}{a}}^{-1}(0,1), H_{\frac{1}{a}}^1(0,1)} + \int_0^1 \frac{1}{a} u(T)w_{1,T} dx \\ &\quad + \langle u_1, w(0) \rangle_{H_{\frac{1}{a}}^{-1}(0,1), H_{\frac{1}{a}}^1(0,1)} - \int_0^1 \frac{1}{a} u_0 w_t(0, x) dx. \end{aligned} \tag{5.11}$$

By (5.10) and (5.11), it follows that

$$\langle u_t(T), w_{0,T} \rangle_{H_{\frac{1}{a}}^{-1}(0,1), H_{\frac{1}{a}}^1(0,1)} - \int_0^1 \frac{1}{a} u(T)w_{1,T} dx = 0,$$

for all  $(w_{0,T}, w_{1,T}) \in \mathcal{H}_0$ . Hence, we have

$$u(T, x) = u_t(T, x) = 0 \quad \text{a.e. } x \in (0, 1).$$

### 6. Main results and future perspectives

In this work, we examine the null-controllability of a wave equation with a propagation speed which depends on the time variable through the term  $b(t)$  and degenerates at a boundary point, say  $x_0 = 0$ . At the degeneracy point, we impose either Dirichlet or Neumann boundary conditions, while at the non-degeneracy point, we applied a control function to study the boundary observability of the problems  $(P_i)$ . Assuming an oscillating wave propagation speed, we derive upper and lower bounds for the energy, thereby proving that the generalized energy conservation law holds. Furthermore, by establishing the estimation given in Definition 2, we demonstrate that the

adjoint problems  $(AP_i)$ ,  $i = 1, 2$ , are observable in time  $T > 0$ . Finally, via the solution by transposition (see Definition 3), we concluded the null-controllability for  $(P_i)$ ,  $i = 1, 2$ .

The present analysis is confined to *one-dimensional linear* non-autonomous degenerate wave equations. On one hand, extending this study to nonlinear wave equations would highlight the practical relevance of the theoretical framework in various applications, such as nonlinear acoustics, where sound waves propagate in media with nonlinear properties. Nonlinearities can manifest in diverse forms, such as power-type nonlinearities, nonlinear damping, or nonlinear wave speed. In such cases, the system may exhibit finite-time blow-up or singularities, where the solution becomes unbounded. This necessitates an understanding of how control mechanisms can prevent or mitigate such phenomena, requiring new analytical tools to address the complex dynamics introduced by nonlinearities.

On the other hand, extending this work to higher dimensions introduces additional complexities related to the domain's geometry, the location of degeneracy points, and the placement of control functions. Additionally, the generalized energy conservation law would need to account for energy dissipation or concentration in different spatial directions, particularly near regions of degeneracy. Addressing these challenges would provide deeper insights into the controllability and observability of higher-dimensional systems, paving the way for broader applications in physics and engineering.

## Acknowledgements

G. Fragnelli and S. Ismail are members of *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA)* of the Istituto Nazionale di Alta Matematica (INdAM) and UMI "CliMath". They are partially supported by INdAM GNAMPA Project *Mathematical Methods for Climate Impacts: Control, Game Theory, and Inverse Problems for EBCM* (CUP E5324001950001).

G. Fragnelli is also a member of UMI "Modellistica Socio-Epidemiologica (MSE)". She is partially supported by the FFABR Fondo per il finanziamento delle attività base di ricerca 2017.

**Funding** Open access funding provided by Università degli Studi di Siena within the CRUI-CARE Agreement.

**Data availability statement** No datasets were generated or analyzed during the current study.

## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

**Open Access.** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long

as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

## REFERENCES

- [1] F. Friedlander, *The Wave Equation on a Curved Space-Time*. Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1975). <https://books.google.it/books?id=RDmpajLTw1oC>
- [2] A. Love, *A Treatise on the Mathematical Theory of Elasticity*. Dover Books on Engineering Series (Dover Publications, 1944). [https://books.google.it/books?id=z8rCbeD\\_MrYC](https://books.google.it/books?id=z8rCbeD_MrYC)
- [3] J.D. Jackson, *Classical Electrodynamics* (Wiley, 1998)
- [4] G.B. Whitham, *Linear and Nonlinear Waves* (Wiley-Interscience, 1974)
- [5] C. Bardos, G. Lebeau, J. Rauch, *SIAM J. Control Optim.* **30**(5), 1024 (1992). <https://doi.org/10.1137/0330055>
- [6] F. Conrad, B. Rao, in *Analysis and Optimization of Systems: State and Frequency Domain Approaches for Infinite-Dimensional Systems* (Springer Berlin Heidelberg, Berlin, Heidelberg, 1993), pp. 512–523
- [7] M. Gueye, *SIAM J. Control and Optim.* **52**(4), 2037 (2014). <https://doi.org/10.1137/120901374>
- [8] F. Alabau-Boussouira, P. Cannarsa, G. Leugering, *SIAM J. Control Optim.* **55**, 2052–2087 (2017)
- [9] J. Bai, S. Chai, *Appl. Math. Lett.* **119**, 107235 (2021). <https://doi.org/10.1016/j.aml.2021.107235>
- [10] M. Zhang, H. Gao, *J. Math. Anal. Appl.* **457**(1), 10 (2018). <https://doi.org/10.1016/j.jmaa.2017.07.057>
- [11] I. Boutaayamou, G. Fragnelli, D. Mugnai, *SIAM J. Control Optim.* **61**, 1934 (2023)
- [12] B. Allal, A. Mounmi, J. Salhi, *Math. Methods Appl. Sci.* **45**(17), 11526 (2022). <https://doi.org/10.1002/mma.8464>
- [13] L. Cao, S. Chai, *Evolution Equations and Control Theory* **14**(2), 399 (2025). <https://doi.org/10.3934/eect.2024060>
- [14] G. Fragnelli, D. Mugnai, A. Sbair, *Appl. Math. Optim.* p. 32 pp (2025). <https://doi.org/10.1007/s00245-025-10236-8>
- [15] M. Reissig, K. Yagdjian, *Math. Methods Appl. Sci.* **22**(11), 937 (1999). [https://doi.org/10.1002/\(SICI\)1099-1476\(19990725\)22:11<937::AID-MMA28>3.0.CO;2-O](https://doi.org/10.1002/(SICI)1099-1476(19990725)22:11<937::AID-MMA28>3.0.CO;2-O)
- [16] M. Reissig, J. Smith, *Hokkaido Mathematical Journal* **34**, 541 (2005)
- [17] A. Chambolle, F. Santosa, *ESAIM: COCV* **8**, 375 (2002). <https://doi.org/10.1051/cocv:2002029>
- [18] F. Hirose, *Math. Ann.* **339**, 819 (2007). <https://doi.org/10.1007/s00208-007-0132-0>
- [19] M. D'Abicco, M.R. Ebert, *J. Math. Anal. Appl.* **399**(1), 315 (2013). <https://doi.org/10.1016/j.jmaa.2012.10.017>
- [20] F. Zhou, K. Zhu, Y. Xie, *Discrete Contin. Dyn. Syst. Ser. B* **30**(1), 1 (2025). <https://doi.org/10.3934/dcdsb.2024076>
- [21] P. Cannarsa, G. Fragnelli, D. Rocchetti, *Netw. Heterog. Media* **2**, 695–715 (2007). <https://doi.org/10.3934/nhm.2007.2.695>
- [22] P. Cannarsa, G. Fragnelli, D. Rocchetti, *J. Evol. Equ.* **8**, 583–616 (2008). <https://doi.org/10.1007/s00028-008-0353-34>
- [23] K.J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations* (Springer-Verlag, New York, Berlin, Heidelberg, 1999)
- [24] F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli, *J. Evol. Equ.* **6**, 161–204 (2006). <https://doi.org/10.1007/s00028-006-0222-6>

- [25] T. Kato, Abstract differential equations and nonlinear mixed problems. Lezione fermiane (Scuola normale superiore, Pisa, 1985)
- [26] T. Kato, Linear and Quasi-Linear Equations of Evolution of Hyperbolic Type (Springer Berlin Heidelberg, Berlin, Heidelberg, 2011), pp. 125–191. [https://doi.org/10.1007/978-3-642-11105-1\\_4](https://doi.org/10.1007/978-3-642-11105-1_4)
- [27] D. Saba, G. Bayili, S. Nicaise, J. Math. Anal. Appl. **538**(1), 128441 (2024). <https://doi.org/10.1016/j.jmaa.2024.128441>

*Mohammad Akil*  
*Université Polytechnique Hauts-de-France*  
*le Mont Houy*  
*59313 Valenciennes Cedex 9*  
*France*  
*E-mail: mohammad.akil@uphf.fr*

*Genni Fragnelli*  
*Department of Information Engineering and Mathematics*  
*Università degli Studi di Siena*  
*Via Roma 56*  
*53100 Siena*  
*Italy*  
*E-mail: genni.fragnelli@unisi.it*

*Sarah Ismail*  
*Dipartimento di Matematica*  
*Università degli Studi di Bari Aldo Moro*  
*Via E. Orbona 4*  
*70125 Bari*  
*Italy*  
*E-mail: sarah.ismail@uniba.it*

Accepted: 4 August 2025