

## Preface to the third volume

Mathematical fuzzy logic (MFL) is a subdiscipline of mathematical logic. Comprising the mathematical study of a certain family of formal logical systems whose algebraic semantics involve some notion of *truth degree*. The main motivations for MFL, its historical origins, and its later evolution were briefly explained in the preface of the first volume of this handbook series.

Fuzzy logics can be seen as a particular kind of many-valued logical system where the intended semantics is typically based on algebras of linearly ordered truth values. As a subdiscipline of mathematical logic, MFL has acquired the typical core agenda of this field and is studied by many mathematically-minded researchers regardless of its original motivations, as witnessed by the plethora of works on a diversity of topics that have been accumulating over the years.

This handbook series aims to be an up-to-date, systematic presentation of the best developed areas of MFL. The first two volumes were released in 2011, also in *Studies in Logic, Mathematical Logic and Foundations* (vols. 37 and 38), College Publications. The first volume starts with a gentle introduction to MFL assuming only some basic knowledge of classical logic. The second chapter presents and develops a general and uniform framework for MFL based on the notions and methods of abstract algebraic logic. The third chapter is a presentation of the deeply developed proof theory of fuzzy logics. The fourth chapter presents the standard algebraic semantics for fuzzy logics based on classes of semilinear residuated lattices. The fifth chapter closes the first volume with the study of Hájek's BL logic and its algebraic counterpart.

The second volume of the series starts with the sixth chapter and is devoted to another widely studied fuzzy logic, Łukasiewicz logic  $L$ , and MV-algebras as its algebraic semantics. The seventh chapter deals with a third distinguished fuzzy logic, Gödel–Dummett logic, and its variants. The eighth chapter studies fuzzy logics in expanded languages providing greater expressive power. The ninth chapter collects results on functional representations of fuzzy logics and their free algebras. The last two chapters are dedicated to complexity issues: Chapter X studies the computational complexity of propositional fuzzy logics, while Chapter XI is devoted to the arithmetical hierarchy of first-order predicate fuzzy logics.

Intense research in MFL has continued in recent years, bringing to a higher level of maturity some of the topics that were omitted from the previous volumes of the handbook. This third volume collects chapters on seven such areas. It starts with Chapter XII, a systematic study of the most prominent part of the algebraic semantics for fuzzy logics: integral residuated chains. The thirteenth chapter presents a different type of semantics for some prominent fuzzy logics obtained by extending Hintikka's and Giles' semantic games. In a similar fashion, the fourteenth chapter is devoted to another game-theoretic interpretation of t-norm-based fuzzy logics, namely, that given by Ulam–Rényi games arising from the theory of error correcting codes. The fifteenth chapter surveys a series of works on fuzzy logics with evaluated syntax in which intermediate truth-values are incorporated as syntactical devices that accompany each formula as a lower bound for its truth-degree. Chapter XVI surveys another area that has recently been intensively developed: fuzzy description logics, understood as tractable fragments of first-order

fuzzy logics amenable to knowledge representation. The seventeenth chapter focuses on another application of MFL by presenting the theory of MV-algebras endowed with functions, called *states*, that allow one to model finitely additive probability measures. The volume is concluded by Chapter XVIII offering a philosophical discussion of the role of MFL in the study of vagueness.

It should be emphasized that this is not a handbook written by a single team of authors, but a collection of chapters prepared by distinguished experts in each area. Nevertheless, the editors have encouraged a reasonable level of homogeneity between the chapters, as regards their structure and notation. The series is conceived as a unit with consecutive page and chapters enumeration. The majority of chapters in this volume contain a purely theoretical (mathematical) presentation, making it mostly a book on mathematical logic, focusing on the study of a particular family of many-valued non-classical logics. However, as briefly outlined before, some chapters feature connections to other mathematical fields, to applications in computer science and, in the case of the last chapter, to developments in analytical philosophy. The intended audience of the book is quite wide, comprising at least the following groups of readers: (1) students of logic looking for a systematic presentation of MFL where they can study the discipline from scratch, (2) experts on MFL that may use it as a reference book for consultation, (3) readers interested in fuzzy set theory and its applications looking for the logical foundations of (some parts) of the area, and (4) readers interested in philosophical and linguistic issues related to reasoning in presence of vagueness, looking for a mathematical apparatus that can be applied to some aspects of those issues.

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**Dedication** This volume is dedicated to two great researchers who passed away during the preparation of the book: Siegfried Gottwald and Franco Montagna. Siegfried was one of the pioneers in the area of MFL, which he deeply influenced with his monograph *A Treatise on Many-Valued Logics*, and remained a constant contributor to the field and a close collaborator of this handbook series. Franco joined the MFL community later, but he soon became one of the field's major figures, as witnessed by the fact that he is the only author who contributed to each of the three volumes of this series.

Petr Cintula, Christian G. Fermüller, and Carles Noguera  
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## *Contents of Volume 1*

<b>I</b>	<b>Introduction to Mathematical Fuzzy Logic</b> Libor Běhounek, Petr Cintula, and Petr Hájek	1
<b>II</b>	<b>A General Framework for Mathematical Fuzzy Logic</b> Petr Cintula and Carles Noguera	103
<b>III</b>	<b>Proof Theory for Mathematical Fuzzy Logic</b> George Metcalfe	209
<b>IV</b>	<b>Algebraic Semantics: Semilinear FL-Algebras</b> Rostislav Horčík	283
<b>V</b>	<b>Hájek's Logic BL and BL-Algebras</b> Manuela Busaniche and Franco Montagna	355

## *Contents of Volume 2*

<b>VI</b>	<b>Łukasiewicz Logic and MV-Algebras</b> Antonio Di Nola and Ioana Leuştean	469
<b>VII</b>	<b>Gödel–Dummett Logics</b> Matthias Baaz and Norbert Preining	585
<b>VIII</b>	<b>Fuzzy Logics with Enriched Language</b> Francesc Esteva, Lluís Godo, and Enrico Marchioni	627
<b>IX</b>	<b>Free Algebras and Functional Representation for Fuzzy Logics</b> Stefano Aguzzoli, Simone Bova, and Brunella Gerla	713
<b>X</b>	<b>Computational Complexity of Propositional Fuzzy Logics</b> Zuzana Haniková	793
<b>XI</b>	<b>Arithmetical Complexity of First-Order Fuzzy Logics</b> Petr Hájek, Franco Montagna, and Carles Noguera	853

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# CONTENTS

Preface to Volume 3	v
Contents of Volumes 1 and 2	vii

## *Volume 3*

The list of authors of Volume 3	ix
---------------------------------	----

### **XII Algebraic Semantics: The Structure of Residuated Chains**

Thomas Vetterlein	929
1 Introduction	929
2 Residuated totally ordered monoids	930
3 The partial-algebra method for the representation of totally ordered monoids	932
3.1 The idea	932
3.2 D.p.r. tomonoids and R-chains	933
3.3 Ordinal sum decomposition of R-chains	936
3.4 Naturally ordered R-chains	937
3.5 The representation of d.p.r. tomonoids	941
4 Coextensions of totally ordered monoids	942
4.1 The idea	942
4.2 Q.n.c. tomonoids and their quotients	943
4.3 The chain of quotients of a q.n.c. tomonoid	946
4.4 The Cayley tomonoid	948
4.5 Real Archimedean coextensions	955
5 Historical remarks and further reading	964
5.1 Partial algebras	964
5.2 Coextensions of tomonoids	965

**XIII Semantic Games for Fuzzy Logics**

Christian G. Fermüller

969

1	Introduction .....	969
2	Hintikka's game — from classical to many-valued logics .....	970
2.1	Hintikka's classical semantic game .....	970
2.2	Hintikka's game in a fuzzy logic setting .....	971
2.3	Limits of Hintikka-style games .....	973
3	An explicit evaluation game for Łukasiewicz logic .....	975
4	Giles's game for Łukasiewicz logic .....	978
5	Generalizing Giles's game .....	984
5.1	General payoff principles .....	984
5.2	Dialogue principles for logical connectives .....	986
5.3	Extracting truth functions .....	989
5.4	Revisiting the game for Łukasiewicz logic .....	992
5.5	Finitely-valued Łukasiewicz logics .....	993
5.6	Cancellative hoop logic .....	993
5.7	Abelian logic .....	994
5.8	Alternative aggregation functions .....	994
6	Giles's game and hypersequents — the case of Abelian logic .....	995
6.1	Abelian logic revisited .....	996
6.2	A Giles-style game for Abelian logic .....	996
6.3	Disjunctive states .....	998
6.4	A hypersequent system for Abelian logic .....	999
7	Backtracking games .....	1001
7.1	A backtracking game for Łukasiewicz logic .....	1001
7.2	A backtracking game for Gödel logic .....	1003
7.3	An implicit backtracking game for Gödel logic .....	1005
7.4	An implicit backtracking game for Product logic .....	1006
8	Propositional random choice games .....	1007
9	Random choice rules for semi-fuzzy quantifiers .....	1011
9.1	A note on binary quantifiers .....	1013
9.2	Blind choice quantifiers .....	1014
9.3	Deliberate choice quantifiers .....	1020
10	A brief synopsis of semantic games .....	1023
11	Historical remarks and further reading .....	1024

**XIV Ulam–Rényi Game Based Semantics for Fuzzy Logics**

Ferdinando Cicalese and Franco Montagna

1029

1	Introduction .....	1029
2	Preliminaries .....	1030



3	Ulam–Rényi game	1033
3.1	Ulam–Rényi game without lies	1033
3.2	Ulam–Rényi game with a fixed maximum number of lies	1036
3.3	Ulam–Rényi games and many-valued logics	1036
4	The Combinatorics of Ulam–Rényi games	1039
4.1	The fundamental bound on the strategy size	1039
4.2	The basic case — a two batch “optimal” strategy	1040
4.3	Half lies and asymmetric channels	1043
4.4	The combinatorics of the half-lie Ulam–Rényi game	1043
5	Multichannel Ulam–Rényi games and Hájek Basic Logic: preliminaries	1045
6	A game semantics for Gödel logic	1048
6.1	Gödel games	1048
6.2	Some strategies	1049
6.3	Truth functions in Gödel games and completeness	1050
7	A game semantics for product logic	1051
7.1	The Product game	1051
7.2	Some strategies	1053
7.3	Truth functions in Product game and completeness	1054
8	Game semantics for BL and for SBL	1055
8.1	BL games	1055
8.2	Truth functions and completeness for BL-games	1056
8.3	Completeness theorem	1058
9	Historical remarks and further reading	1058

## **XV Fuzzy Logic with Evaluated Syntax**

	Vilém Novák	1063
1	Introduction	1063
2	Language and truth values of $Ev_{\mathcal{L}}$	1064
2.1	The algebra of truth values	1064
2.2	Language and formulas	1066
3	Traditional and evaluated syntax	1067
3.1	Classical syntactic consequence	1068
3.2	Evaluated syntax and graded syntactic consequence	1069
3.3	Semantics	1072
3.4	Formal fuzzy theories	1073
3.5	Soundness and completeness in evaluated syntax	1075
3.6	Restrictions when choosing the algebra of truth values	1075
4	Syntax of $Ev_{\mathcal{L}}$	1078
4.1	Logical axioms	1078
4.2	Inference rules and validity	1079

4.3	Special provable tautologies and properties of the provability degree . . .	1080
4.4	Consistency of fuzzy theories . . . . .	1081
4.5	Extension of fuzzy theories and deduction theorem . . . . .	1083
4.6	Complete fuzzy theories . . . . .	1087
4.7	Henkin fuzzy theories . . . . .	1091
5	Completeness theorem . . . . .	1092
6	Model theory . . . . .	1094
7	Some additional properties . . . . .	1097
7.1	Enriching the structure of truth values . . . . .	1097
7.2	Sorites fuzzy theories . . . . .	1098
7.3	$Ev_L$ and Łukasiewicz logic . . . . .	1099
8	Historical remarks and further reading . . . . .	1100

## **XVI Fuzzy Description Logics**

Fernando Bobillo, Marco Cerami, Francesc Esteva,

Àngel García-Cerdaña, Rafael Peñaloza, Umberto Straccia 1105

1	Introduction . . . . .	1105
2	Algebras of truth values defined by triangular norms . . . . .	1107
2.1	Triangular norms and conorms . . . . .	1107
2.2	Divisible finite t-norms and t-conorms . . . . .	1110
2.3	Expanding the standard chain with an involutive negation . . . . .	1111
2.4	Expanding the standard chain with other operators . . . . .	1112
3	Introduction to Fuzzy Description Logics . . . . .	1112
3.1	Basic concept constructors in FDLs . . . . .	1114
3.2	Semantics for basic concept constructors . . . . .	1115
3.3	The hierarchy of fuzzy attributive languages . . . . .	1116
3.4	Knowledge bases . . . . .	1121
3.5	Reasoning tasks . . . . .	1122
4	Fuzzy Description Logics from the point of view of MFL . . . . .	1123
4.1	A logical framework for FDLs: The propositional logics . . . . .	1123
4.2	A logical framework for FDLs: The predicate logics . . . . .	1124
4.3	Basic fuzzy description logics as fragments of fuzzy first-order logics . . . . .	1126
5	Fuzzy Description Logic languages with higher expressivity . . . . .	1128
5.1	Fuzzy concept constructors . . . . .	1128
5.2	Fuzzy role constructors . . . . .	1131
5.3	Fuzzy datatypes . . . . .	1132
5.4	New axioms . . . . .	1133
5.5	Some important languages . . . . .	1135

6 Reasoning algorithms . . . . .	1135
6.1 Equivalences between reasoning tasks in finitely-valued FDLs . . . . .	1135
6.2 Types of algorithms . . . . .	1137
7 Decidability and complexity issues . . . . .	1152
7.1 Undecidability of consistency in $\mathfrak{JALCE}$ . . . . .	1153
7.2 Further undecidability results . . . . .	1159
7.3 Complexity bounds . . . . .	1161
7.4 Reasoning with finite chains . . . . .	1163
8 Some historical remarks and further reading . . . . .	1164
8.1 Classical Description Logics in short . . . . .	1164
8.2 Fuzzy Description Logics . . . . .	1165
8.3 Computational issues in a historical perspective . . . . .	1169

## **XVII States of MV-algebras**

Tommaso Flaminio and Tomáš Kroupa 1183

1 Introduction . . . . .	1183
2 Basic notions . . . . .	1184
2.1 MV-algebras . . . . .	1184
2.2 Łukasiewicz logic . . . . .	1187
2.3 Probability measures . . . . .	1188
2.4 Compact convex sets . . . . .	1189
3 States . . . . .	1191
3.1 Basic properties . . . . .	1192
3.2 Examples of states . . . . .	1195
3.3 States of finitely presented MV-algebras . . . . .	1196
3.4 States of $\ell$ -groups . . . . .	1200
4 Integral representation . . . . .	1202
4.1 Characterization of state space . . . . .	1205
4.2 Existence of invariant states and faithful states . . . . .	1206
5 De Finetti's coherence criterion for many-valued events . . . . .	1208
5.1 Betting on formulas of Łukasiewicz logic . . . . .	1212
5.2 Complexity . . . . .	1216
6 MV-algebras with internal states . . . . .	1218
6.1 Subdirectly irreducible SMV-algebras . . . . .	1220
6.2 States of MV-algebras and internal states . . . . .	1221
6.3 Dealing with coherent books in SMV-algebraic theory . . . . .	1224
7 Conditional probability and Dutch Book argument . . . . .	1225
7.1 Bookmaking on many-valued conditional events . . . . .	1227
8 Historical remarks and further reading . . . . .	1232

**XVIII Fuzzy Logics in Theories of Vagueness**

Nicholas J.J. Smith

1237

1 Introduction .....	1237
1.1 Characteristics of vague predicates .....	1237
1.2 Other phenomena .....	1239
2 Non-fuzzy theories of vagueness .....	1240
2.1 Theory of vagueness .....	1240
2.2 Epistemicism .....	1241
2.3 Gaps and third values .....	1244
2.4 Plurivaluationism .....	1247
2.5 Contextualism .....	1249
3 Fuzzy theories of vagueness .....	1250
3.1 Degrees of truth .....	1250
3.2 Truth rules .....	1253
3.3 The intended model .....	1256
4 Arguments for .....	1256
4.1 The nature of vagueness .....	1256
4.2 Solving the sorites .....	1261
5 Arguments against .....	1266
5.1 Artificial precision .....	1267
5.2 Truth functionality .....	1272
6 Historical remarks and further reading .....	1276

INDEX

1283