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RESEARCH ARTICLE

Analysis of a non-linear model of populations structured by size

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Abstract
The model we study deals with a population of marine invertebrates structured by size whose life stage is composed of adults and pelagic larvae such as barnacles contained in a local habitat. We prove existence and uniqueness of a continuous positive global mild solution and we give an estimate of it. We prove also that this solution is the strong solution of the problem.

Keywords Size-structured population dynamics · Semigroup theory · Affine semigroups · Mild solution · Strong solution

1 Introduction

A famous American zoologist of Swiss origin, Louis Agassiz, lived in the XIX century defined the barnacles like "little shrimps hanging from the rock with their heads, locked in a limestone house and kicked throwing food into their mouths".

They belong to a species of crustaceans, marine invertebrates whose life is composed of two stages, pelagic larvae and adult sessile.

Barnacles have two larval stages: the first (nauplius) spends its time as part of zooplankton floating wherever the wind, waves, currents, and tides may take it. In this period, which lasts of about two weeks, it can eat and moult; hence the second

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The authors dedicate this paper to the memory of Prof. Aldo Belleni Morante, who showed them the way of modelling and studying evolution problems by means of semigroup theory.

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stage is reached. At this point the nauplius metamorphoses into a non-feeding, more strongly swimming, cyprid larva.

When an appropriate place is found, the cyprid larva cements itself head first to the surface and then undergoes metamorphosis into a juvenile barnacle.

Typical barnacles develop six hard plates to surround and protect their bodies. For the rest of their lives they are cemented to the ground or even on shells of other animals, using their feathery legs, called cirrus, to capture plankton and gametes when spawning. They are usually found in the intertidal zone. Once metamorphosis is over and they have reached their adult form, barnacles will continue to grow, but not moult. Actually, they grow by adding new material to the ends of their heavily calcified plates.

The model we study, following He and Wang [7], consists of a non-linear system of two equations, the first one models the density of the adults, whereas the second one involves larval evolution. The equations are connected by means of the boundary conditions of the evolution of the adults, which takes into account the larval evolution.

The population dynamics of marine invertebrates such as barnacles, in which sessile adults and pelagic larvae are contained in a local area, are very much different from the population dynamics of vertebrates. Although the sessile adults can be viewed as living in limited area, their larvae can freely move from one area to another, because each area (patch) is connected by the pelagic pool containing the larvae. That is, such a population system in a local area is essentially "open", because newly settled larvae can be carried from outside the region, while the whole multipatch system can be "closed" if the larvae are produced by the sessile adults in each patch [10]. Moreover, it has been observed in sessile marine populations that the space to be settled by the larvae is a principal limiting resource, and the number of settlements is approximately proportional to the free space available to larvae.

In 2005, Kamioka [9], developed a rigorous mathematical approach to a closed age structured population model, which can be seen as a non-linear extension of Roughgarden–Iwasa–Baxter model [8, 13]. Originally, Roughgarden et al. recognized the absurd drawback of their model, whereby the population density may become negative for some initial conditions.

The main reason for this shortcoming is that the demographic parameters of the size growth rate and mortality are assumed to be independent of the population density. In reality, these parameters will depend on the density of the population or the available free space.

Also in [11] it is proposed an age-structured space-limited model, in which any references to the dimension is missing and it is proved that the solution of the model is a mild solution. Nothing is proved about the existence of a strong solution.

Field research during the last fifty years show that, for many populations, such as wild animals, plants and marine invertebrates, demographic and life history processes (e.g., growth, reproduction and death) depend on the size of individuals rather than age. Individuals' size is better than age to estimate the ecological and commercial values in ecology and resources economy. For these reasons, there have been many attempts to construct size-structured models (see, e.g., [1] and [14]). Although several discrete size-time models have been formulated for population in space-limited habitats, there are only few attempts, however, to analyze continuous time size-structured model with space-limited recruitment, since a rigorous mathematical analysis of continuous size-structured models usually involves some considerable complexities.

In this work, unlike [9], which studies an age-structured model, the equation for adults evolution takes into account size growth through a function that represents the individual growth rate.

Moreover, unlike studies already present in the literature that give the solution in an implicit way, we have defined, inspired by types of weaker solution studied for linear or semilinear problems, a new type of "mild" solution. This new type of solution is more explicit than that already present in the literature and give an immediate quantitative idea of what is being analyzed. We prove the existence of a mild solution global in time, by using affine semigroup techniques.

We get also estimates of the solution to prove that the local solution is global in time, and eventually, we also prove that this solution is a strong solution, that is it satisfies the equation of evolution in the classical sense.

The paper is organized as follows: we consider first that the boundary value of adults density is known and has some suitable properties. In this way, we find the mild solution of the evolution problem for the adults density by means of affine semigroup techniques and successive approximation procedures.

Then, by using another successive approximation procedure, we find that the function which represents the boundary value of adults density belongs to a suitable Banach space which ensures that it has the requested properties. At the end we prove that this solution is the global strong solution of the system.

2 The model

As we said in the introduction, we analyze the model studied by Zerong He and Haitao Wang [7]. The model under consideration reads as follows:

$$\begin{aligned}&\frac{\partial p(s,t)}{\partial t}+\frac{\partial}{\partial s}(g(s)p(s,t))=-\mu(s)p(s,t)-\delta(s,Q(t))p(s,t),\\&t>0,\ 0<s<m,\end{aligned} \tag{1}$$

$$g(0)p(0,t)=c(M-Q(t))L(t),\ t>0, \tag{2}$$

$$\frac{d}{dt}L(t)=-(v+c(M-Q(t)))L(t)+\int_0^m \beta(s)p(s,t)ds,\ t>0, \tag{3}$$

$$p(s,0)=p_0(s),\quad L(0)=L_0,\ 0\le s<m, \tag{4}$$

where $p(s,\ t)$ represents the size-density function of sessile adults living in a local habitat, depending on the size-variable $s\in(0,m)$ at time $t>0$, where $m>0$ is the maximum attainable size (finite).

Moreover, $g(s)$ is the growth rate, $\mu(s)$ is the natural death rate at size s and $\beta(s)$ is the fertility rate. The function $\delta(s, Q(t))$ represents the density-dependent death rate, where

$$Q(t) = \int_0^m \gamma(s)p(s,t)\,ds, \qquad t > 0, \tag{5}$$

is the size occupied space ($\gamma(s)$ is the area occupied by one individual of size s) with respect to the total available area M.

The unknown function $L(t)$ gives the abundance of larvae in the pelagic pool at time t, v is the natural death rate of the larvae and finally c is the settlement rate for unit free area.

The data $p_0(s)$ and L_0 give the initial conditions for larvae and adults respectively.

From a mathematical point of view, the following assumption are made:

1. the constants $c, v, M, m,$ are positive and L_0 is nonnegative;
2. the functions $\beta(s), p_0(s), \gamma(s)$ are positive and bounded functions on $(0, m)$. Moreover, $\int_0^m \beta(s)p_0(s)ds > 0$;
3. the growth rate $g(s)$ is positive and bounded for $s \in [0, m]$ (that is, $\exists\ \bar{g}$ such that $0 \le g \le \bar{g}$); moreover it is a differentiable function and its derivative is also bounded (that is, $\exists\ \bar{k}$ such that $|g'| \le \bar{k}$);
4. the death rate $\mu(s)$ is positive for all $s \in (0, m)$, locally integrable on $[0, m]$ and satisfies the condition

$$\int_0^m \frac{\mu(s)}{g(s)}ds = +\infty, \tag{6}$$

   which makes the maximum attainable size be finite;
5. the density-dependent death rate $\delta(s, Q)$ is nonnegative and uniformly bounded for $(s, Q) \in [0, m] \times [0, M]$, and it has a bounded, continuous and positive derivative with respect to Q. Therefore two positive constants $\bar{\delta}$ and δ' exist such that:

$$0 \le \delta(s, Q) \le \bar{\delta}, \qquad 0 \le \frac{\partial \delta(s, Q)}{\partial Q} \le \delta'. \tag{7}$$

3 Analysis of the model

Without loss of generality, we assume that the initial size of adult individuals is zero.

The survival rate, i.e., the proportion of newly settled larvae who can survive to size s, is given by

$$l(s) := e^{\{-\int_0^s \frac{\mu(r)}{g(r)}dr\}}. \tag{8}$$

Then, from assumption 4., we have $l(m) = 0$, $l'(s) = -l(s)\frac{\mu(s)}{g(s)}$.

In order to avoid mathematical troubles due to the singularity of the mortality rate μ, let us factor out the natural death rate $\mu(s)$ in the model (1–4).

Define a new function $q(s, t)$ by

$$p(s,t) := q(s,t)l(s). \tag{9}$$

Then is not difficult to see that the system (1–4) reduces as follows:

$$\frac{\partial q(s,t)}{\partial t} + \frac{\partial}{\partial s}(g(s)q(s,t)) = -\delta(s,Q(t))q(s,t),\ t > 0,\ 0 < s < m, \tag{10}$$

$$g(0)q(0,t) = c(M - Q(t))L(t),\ t > 0, \tag{11}$$

$$\frac{d}{dt}L(t) = -vL(t) - c(M - Q(t))L(t) + \int_0^m \phi(s)q(s,t)ds,\ t > 0, \tag{12}$$

$$q(s,0) = q_0(s), \quad L(0) = L_0,\ 0 \le s < m. \tag{13}$$

It is easy to see that the system (10–13) is equivalent to:

$$\frac{\partial q(s,t)}{\partial t} + \frac{\partial}{\partial s}(g(s)q(s,t)) = -\bar{\delta}q(s,t) + [\bar{\delta} - \delta(s,Q(t))]q(s,t), \\ t > 0,\ 0 < s < m \tag{14}$$

$$g(0)q(0,t) = \chi(t),\ t > 0, \tag{15}$$

$$\frac{d}{dt}L(t) = -vL(t) - \chi(t) + \int_0^m \phi(s)q(s,t)ds,\ t > 0, \tag{16}$$

$$q(s,0) = q_0(s) := \frac{p_0(s)}{l(s)}, \quad L(0) = L_0,\ 0 \le s < m. \tag{17}$$

where

$$Q(t) = \int_0^m \psi(s)q(s,t)ds,\ t > 0, \tag{18}$$

and $\phi(s) := \beta(s)l(s)$ is the new reproduction function of the adult population of size s and $\psi(s) := \gamma(s)l(s)$ is the expected space area occupied by an individual.

According to the assumptions on β and γ, also ϕ and ψ are bounded, i.e., there exist two constants $\bar{\phi}, \bar{\psi}$ such that

$$\phi(s) \le \bar{\phi}, \qquad \psi(s) \le \bar{\psi}, \qquad 0 < s < m. \tag{19}$$

Furthermore, we assume that

$$q_0(s) := \frac{p_0(s)}{l(s)} \in L^1(0,m), \tag{20}$$

is a nonnegative function and define the function

$$\chi(t) = c(M - Q(t))L(t), \qquad t > 0, \tag{21}$$

$$\chi(t) = 0, \qquad t \leq 0. \tag{22}$$

Remark 3.1 We introduce the system (14–17) because the term $\bar{\delta} - \delta(s, Q(t))$ is nonnegative and this fact is useful for the estimate of the solution.

Define the Banach space $X = L^1(0,m)$, with its usual norm $\|f\|_X = \int_0^m |f(s)|ds$, and denote by X^+ its positive cone $X^+ = \{f \in L^1(0,m) : f(s) \geq 0 \; a.e. \; in \; (0,m)\}$. By defining the operator:

$$Nf = -\frac{d}{ds}(gf), \quad D(N) = \left\{f \in X, \; \frac{d}{ds}(gf) \in X, (gf)(0) = 0\right\}, R(N) \subset X, \tag{23}$$

it is possible to prove the following lemma.

Lemma 3.1 *The operator N satisfies the following properties:*

1. *the operator* $(\lambda I - N)^{-1}$ *exists for all* $\lambda > 0$;
2. *the operator N is closed;*
3. *the domain* $D(N)$ of N *is dense in* X.

Proof The solution of the equation $(\lambda I - N)f = h$ reads as

$$f(s) = \frac{1}{g(s)} \int_0^s e^{-\lambda \int_{s'}^s \frac{1}{g(u)} du} h(s')ds'.$$

Hence if $h \in X^+$, then also $f \in X^+$.

Moreover $\|(\lambda I - N)^{-1}h\| \leq \frac{1}{\lambda}\|h\| \quad \forall \lambda > 0, \, h \in X^+$. Any $h \in X$ can be written as $h = h^+ - h^-$, with $h^+, h^- \in X^+$ ($h^+ = \frac{|h|+h}{2}, h^- = \frac{|h|-h}{2}$), and since $(\lambda I - N)^{-1}$ is a positive operator, it is possible to see that for $h \in X$, $\lambda > 0$,

$$\|(\lambda I - N)^{-1}h\| \leq \frac{1}{\lambda}\|h^+\| + \frac{1}{\lambda}\|h^-\| = \frac{1}{\lambda}\|h\|, \quad \forall \, h \in X. \tag{24}$$

Then, the operator $(\lambda I - N)^{-1} \in \mathscr{B}(X)$ and it is the resolvent operator $R(\lambda, N)$ of N. Hence Property 1. is proved.

Property 2. follows from the fact that $N = -\{(\lambda I - N)^{-1}\}^{-1} + \lambda I, \lambda > 0$ and $(\lambda I - N)^{-1} \in \mathscr{B}(X)$.

Property 3. is automatically proved because $C_0^\infty(0,m) \subset D(N)$ and since $\overline{C_0^\infty(0,m)} = X$, also $\overline{D(N)} = X$, hence $D(N)$ is dense in X. □

Thus, the operator $N \in \mathscr{G}(1,0;X)$ [4], and so it generates a strongly continuous semigroup of contractions. Now define the following operator:

$$A_t f = -\frac{d}{ds}(gf), D(A_t) = \left\{ f \in X, \frac{d}{ds}(gf) \in X, (gf)(0) = \chi(t) \right\}, R(A) \subset X. \quad (25)$$

Note that A_t depends on t because of its domain. For simplicity, from now on, we denote it simply by A.

The operator A is non-linear and it is affine to N [2]. In fact if $f_1, f_2 \in D(A)$, then $f_1 - f_2 \in D(N)$, and if $f \in D(A), g \in D(N)$, then $f + g \in D(A)$ and $A(f+g) = Af + Ag$.

Remark 3.2 We can write the operator N as the sum of two operators:

$$Nf = N_1 f + N_2 f, \quad (26)$$

where $N_1 = -g'f$, $N_2 = -gf'$ and $N_1 \in B(X)$ (i.e., $\|N_1\| \le \bar{k}$). Thus, $N_2 = N - N_1$ is a closed operator and $N_2 \in G(1,\bar{k};X)$.

Note that by adding and subtracting $-g'(s)q(s,t)$ to (10) we obtain the equivalent evolution equation

$$\frac{dq}{dt} = (A - N_1)q - g'q - \delta(Q)q, \quad (27)$$

where $\delta(Q) = \delta(\bullet, Q)$.

The operator $A - N_1$ is affine to $N - N_1$ and it is defined as follows:

$$(A - N_1)f = -g\frac{df}{ds}, \quad D(A - N_1) = \{f \in X, f' \in X, (gf)(0) = \chi(t)\},\ R(A - N_1) \subset X. \quad (28)$$

With these definitions the system (10–13) can be written as an evolution problem as follows:

$$\frac{dq(t)}{dt} = Aq(t) + F(q(t)), t > 0 \quad (29)$$

$$\frac{d}{dt}L(t) = -vL(t) - c(M - Q(t))L(t) + \int_0^m \phi(s)q(s,t)ds,\ t > 0, \quad (30)$$

$$q(0) = q_0, \quad L(0) = L_0, \quad (31)$$

where

$$Ff = (\bar{\delta} - \delta(s, Q(t)))f. \quad (32)$$

Note that $q(t) = q(\bullet, t) \in X$ and $L(t) \in \mathbb{R}$.

From the affine operator theory [2], the problem:

$$\frac{df}{dt} = (A - N_1)f, \qquad f(0) = f_0, \tag{33}$$

has the following solution

$$f(t) = e^{(A-N_1)t}f_0 = p(t) + e^{(N-N_1)t}[f_0 - p(0)],\ 0 \le t < t_0, \tag{34}$$

where $p(t)$ is a suitable function such that

$$p'(t) = (A - N_1)p(t), \quad p(t) \in D(A - N_1). \tag{35}$$

By choosing $p(t) = \frac{\chi(t - \int_0^s \frac{1}{g(u)}du)}{g(0)}$, we have $p(0) = 0$ by (22). Finally we obtain:

$$e^{(A-N_1)t}f_0 = \frac{\chi(t - \int_0^s \frac{1}{g(u)}du)}{g(0)} + e^{(N-N_1)t}f_0. \tag{36}$$

Now, we just have to understand how to write $e^{(N-N_1)t}$. Since $N \in \mathscr{G}(1, 0; X)$ and $N_1 \in \mathscr{B}(X)$, it follows that $N - N_1 \in \mathscr{G}(1, \bar{k}; X)$, and $e^{(N-N_1)t}$ can be find by means of the successive approximation procedures [4].

We assume that $\chi(t)$ is known. Obviously it has to be positive, continuous, differentiable and bounded. Of course $\chi(t)$ is bounded in any interval $[0, t_0]$, with $t_0 < \infty$, that is, if $t_0 > 0$ is fixed, a suitable $\bar{\chi} = \bar{\chi}(t_0) > 0$ exists such that $0 < \chi \le \bar{\chi}$, for any $t \in [0, t_0]$.

In the sequel, we shall prove that $\chi(t)$ really satisfies all these properties.

We want to formulate an integral equation, whose solution is an analogue of the "mild" solution of the linear problem.

In particular, from the system (10–13), we have the following equation for the adults:

$$q(t) = e^{(N-N_1)t}q_0 + \frac{\chi(t - \int_0^s \frac{1}{g(u)}du)}{g(0)} - \int_0^t e^{(N-N_1)(t-r)}[g' + \delta(s, Q(r))]q(r)dr. \tag{37}$$

Note that, if the solution of (37) is strongly differentiable with respect to t, it will be the strong solution of system (10–13). Obviously, this fact necessarily requires the differentiability of $\chi(t)$ with respect to t which will be proved in Sect. 5.

The system (33) is equivalent to the following integral equation [15]:

$$\begin{aligned} q(t) = {} & e^{(N-N_1)t}q_0 e^{-\bar{\delta}t}e^{-\bar{k}t} + \frac{\chi(t - \int_0^s \frac{1}{g(u)}du)}{g(0)} \\ & + \int_0^t e^{(N-N_1)(t-r)}e^{-(\bar{\delta}+\bar{k})(t-r)}(\bar{k} - g')q(r)dr \\ & + \int_0^t e^{(N-N_1)(t-r)}e^{-(\bar{\delta}+\bar{k})(t-r)}[\bar{\delta} - \delta(r, Q(r))]q(r)dr. \end{aligned} \tag{38}$$

We consider the form (38) because it is more useful to prove that its solution (a "mild" solution of our problem) belongs to X^+.

Define the Banach space $Y = C([0, \bar{t}\,], X)$, with the norm

$$\|f\|_Y = \sup\{\|f(t)\|_X,\ t \in [0, \bar{t}\,]\}, \tag{39}$$

where $\bar{t}$ will be chosen later, and consider the positive cone Y^+ of Y. Define also the operator K, $(D(K) = Y\, R(K) \subset Y)$ in the following way:

$$\begin{aligned} Kq = e^{(N-N_1)t} q_0 e^{-\bar{\delta}t} e^{-\bar{k}t} + \frac{\chi(t - \int_0^s \frac{1}{g(u)} du)}{g(0)} \\ + \int_0^t e^{(N-N_1)(t-r)} e^{-(\bar{\delta}+\bar{k})(t-r)} (\bar{k} - g') q(r) dr \\ + \int_0^t e^{(N-N_1)(t-r)} e^{-(\bar{\delta}+\bar{k})(t-r)} [\bar{\delta} - \delta(r, Q(r))] q(r) dr \end{aligned} \tag{40}$$

Hence, (38) can be written as:

$$q = Kq. \tag{41}$$

Lemma 3.2 *The operator K is locally Lipschitz on Y and:*

$$\|Kq_1 - Kq_2\|_Y \le \frac{(2\bar{\delta} + \bar{k} + \delta' \bar{\psi} \|q_2\|_Y)}{\bar{\delta} + \bar{k}} (1 - e^{-(\bar{\delta}+\bar{k})t}) e^{\bar{k}t} \|q_1 - q_2\|_Y. \tag{42}$$

Moreover, the operator K maps the positive cone Y^+ into itself.

Proof The inequality follows from (38) and the Lagrange Theorem. Finally, K maps Y^+ into itself by definition. □

Remark 3.3 Let $r > 0$ be suitably fixed, the sets $D_r = \{f \in X : \|f\| \le r\}$ and $C_r = \{f \in Y : f(t) \in D_r, \forall\, t \in [0, \bar{t}]\}$ are closed sets of X and Y respectively.

It is easy to prove that, for any $q \in C_r$,

$$\|Kq\|_Y \le \left[\frac{\|q_0\| e^{\bar{k}t}}{r} + \frac{\bar{\chi}}{g(0) r} + \frac{\bar{k}}{\bar{\delta}} (1 - e^{-\bar{\delta}t}) + \frac{\bar{\delta}}{\bar{k} + \bar{\delta}} e^{\bar{k}t} \right] r. \tag{43}$$

Moreover, for $q_1, q_2 \in C_r$ the inequality proved in Lemma 3.2 becomes

$$\|Kq_1 - Kq_2\| \le \frac{(2\bar{\delta} + \bar{k} + \delta' \bar{\psi} r)}{\bar{\delta} + \bar{k}} (1 - e^{-(\bar{k}+\bar{\delta})t}) e^{\bar{k}t} \|q_1 - q_2\|_Y,\ \forall\, q_1, q_2 \in C_r. \tag{44}$$

Hence, we can prove the following lemma.

Lemma 3.3 *The operator K is a contraction on C_r, if $\bar{t}$ is small enough.*

Proof As t goes to zero, the quantities $(1 - e^{-\bar{\delta}t}), (1 - e^{-(\bar{k}+\bar{\delta})t})$ vanish and $e^{\bar{k}t}$ tends to 1 . Moreover, a suitable r can be found, such that

$$\frac{\|q_0\|}{r} + \frac{\bar{\chi}}{g(0)r} + \frac{\bar{\delta}}{\bar{\delta}+\bar{k}} < 1. \tag{45}$$

Furthermore, we choose $\bar{t}$ suitably small, such that

$$c(\bar{t}) = \frac{(2\bar{\delta}+\bar{k}+\delta'\bar{\psi}r)}{\bar{\delta}}(1-e^{-\bar{\delta}\bar{t}})e^{\bar{k}\bar{t}} + \frac{\|q_0\|_Y e^{\bar{k}\bar{t}}}{r} + \frac{\bar{\chi}}{g(0)r} < 1.$$

Therefore $\|Kq\|_Y < c(\bar{t})r$, that is, K maps C_r into itself, and the proof follows directly from Lemma 3.2 and Remark 3.3. □

By using the fixed point theorem, the following theorem can be proved:

Theorem 3.1 *Equation* (41) *has a unique positive solution* $q = q(t)$ *defined for* $t \in [0, \bar{t}\,]$ *where* $\bar{t}$ *is small enough.*

Moreover

$$\|q(t)\| \le r,\ \forall\, t \in [0, \bar{t}\,] \text{ with a suitable } \bar{t} \text{ and } q(t) \in X^+,\ \forall\, t \in [0, \bar{t}\,]. \tag{46}$$

Remark 3.4 Note that $\bar{t}$ has to make the quantities $(1 - e^{-\bar{\delta}t}), (1 - e^{-(\bar{\delta}+\bar{k})t})$ suitably small and the constant r is only used to make a contraction in a bounded closed set of Y.

In fact, Theorem 3.1 gives a local solution of equation (37). To prove that the solution is defined in any interval $[0, t_0]$, $t_0 > 0$, we have to find some a priori estimates of the solution itself [4, 12].

Thus, let $t_0 > 0$ be fixed, from (40), by taking into account the nonnegativity of $q(t)$, and by using the Gronwall Lemma, we have:

$$\|q(t)\| \le \frac{\bar{\chi}}{g(0)} + \|q_0\|e^{\bar{k}t_0} + \frac{\bar{\chi}}{g(0)}\frac{\bar{k}+\bar{\delta}}{\bar{k}}e^{\bar{k}t_0}, \quad \forall\, t \in [0, t_0]. \tag{47}$$

Moreover

$$\|q(t)\|_\infty \le \frac{\bar{\chi}}{g(0)} + \|q_0\|_\infty e^{\bar{k}t_0} + \frac{\bar{\chi}}{g(0)}\frac{\bar{k}+\bar{\delta}}{\bar{k}}e^{\bar{k}t_0}, \tag{48}$$

where $\|\bullet\|_\infty$ is the norm in the space $L^\infty(0, m)$.

Thus, we have found a estimate of $\|q(t)\|$ which does not depend on t, but only on t_0, so the solution of (41) is defined in $[0, t_0]$ ([12], Chapter 6, Theorem 1.4).

Hence, we have just proved the following theorem.

Theorem 3.2 *The integral Eq.* (41) *has a unique continuous nonnegative solution* $q = q(t)$, *defined for* $t \in [0, t_0]$ *where* t_0 *is fixed a priori. Moreover,* $q(t)$ *belong to the closed cone* X^+, $\forall\, t \ge 0$ *and* $\|q(t)\| \le r$, $\forall\, t \ge 0$, *where* r *is fixed appropriately.*

Note that with similar techniques, it is possible to prove that (41) has a unique continuous nonnegative solution defined for any $t \in [0, t_0]$, belonging to the closed set of X:

$$\Sigma(\bar{r}) = \{f \in X \cap L^\infty(0, m),\ \|f\|_\infty \le \bar{r}\}, \tag{49}$$

where $\bar{r}$ is fixed. Moreover, from the estimate (48), we can conclude that the solution of (41) is a global solution also in $L^\infty(0, m)$.

As far as the larval equation of system (12) is concerned, we have that:

$$L(t) = L_0 e^{-\nu t} - \int_0^t \chi(r) e^{-\nu(t-r)} dr + \int_0^t e^{-\nu(t-r)} \int_0^m \phi(s) q(s, r) ds\, dr. \tag{50}$$

By substituting the expression of $\chi(t)$ given by (21) we can prove that

$$|L(t)| \le L_0 + \frac{\bar{\phi} M}{\bar{\psi} \nu} \tag{51}$$

with $M \ge Q(t)$ if $M \ge \bar{\psi}\left(\frac{\bar{\chi}}{g(0)} + \|q_0\| e^{\bar{k} t_0} + \frac{\bar{\chi}}{g(0)} \frac{\bar{\delta}+\bar{k}}{\bar{k}} e^{\bar{k} t_0}\right)$.

4 The function χ

In this section, we shall find the function $\chi(t)$, by applying again the Fixed Point Theorem.

Define the following Banach space [6]:

$$W = \{f \in C((0, \infty), X) : \sup\{e^{-kt} \|f(t)\|_X\} < \infty\}, \tag{52}$$

with the norm $\|f\|_W = \sup\{e^{-kt} \|f(t)\|_X\}$, where the constant k will be chosen later. Then define in W the operator $(\Lambda \chi)(t) = c(M - Q(t))L(t)$, with $D(\Lambda) = W$, $R(\Lambda) \subset W$.

Note that the function χ depends on both Q and L, which depend on χ themselves, and so

$$(\Lambda \chi)(t) = cML_\chi(t) - cQ_\chi(t) L_\chi(t). \tag{53}$$

Let $t_0 > 0$ be fixed, from the Theorem 3.2 we know that the solution $q(t)$ exists for $t \in [0, t_0]$, and we have:

$$|\Lambda \chi_1(t) - \Lambda \chi_2(t)| \le cM |L_{\chi_1}(t) - L_{\chi_2}(t)| + c|Q_{\chi_1}(t) L_{\chi_1}(t) - Q_{\chi_2}(t) L_{\chi_2}(t)|.$$

From (50) we have:

$$
\begin{aligned}
&|L_{\chi_1} - L_{\chi_2}| \\
&\quad \le \int_0^t e^{-vt} e^{vr} |\chi_1(r) - \chi_2(r)| dr + \int_0^t e^{-vt} e^{vr} \left(\int_0^m \phi(s) |q_1(s,r) - q_2(s,r)| ds \right) dr \\
&\quad \le \|\chi_1 - \chi_2\|_W e^{-vt} \int_0^t e^{(v+k)r} dr + \bar{\phi} \|q_1 - q_2\|_W e^{-vt} \int_0^t e^{(v+k)r} dr \\
&\quad \le \|\chi_1 - \chi_2\|_W \frac{e^{kt}}{v+k} + \bar{\phi} \|q_1 - q_2\|_W \frac{e^{kt}}{v+k}.
\end{aligned}
\tag{54}
$$

Multiplying each side by e^{-kt} and taking the supremum, we have:

$$
\|L_{\chi_1} - L_{\chi_2}\| \le \frac{\|\chi_1 - \chi_2\|_W}{v+k} + \frac{\bar{\phi}}{v+k} \|q_1 - q_2\|_W. \tag{55}
$$

In a similar way, we can find an estimate for $\|q_1 - q_2\|_W$.

By using the Lagrange Theorem and the Gronwall Lemma, we obtain:

$$
\|q_1 - q_2\|_W \le \frac{\bar{g}}{g(0)} \|\chi_1 - \chi_2\|_W \left(\frac{1}{k} + \frac{H}{k + \bar{\delta} - H} \right), \tag{56}
$$

where $H = 2\bar{\delta} + \delta' \bar{\psi} r + \bar{k}$ and $k > -\bar{\delta} + H$.

Finally, after many computations, by using the definition of Λ, we get the following estimate:

$$
\begin{aligned}
\|\Lambda\chi_1 - \Lambda\chi_2\|_W \le & \left[\frac{c(M + \bar{\psi} r)}{v+k} + \frac{c\bar{\phi}\bar{g}(M + \bar{\psi} r)}{(v+k) g(0)} \left(\frac{1}{k} + \frac{H}{k + \bar{\delta} - H)} \right) \right. \\
& \left. + \frac{c\bar{\psi}\bar{g}}{g(0)} \left(\frac{1}{k} + \frac{H}{k + \bar{\delta} - H)} \right) + \left(L_0 + \frac{\bar{\phi} M}{\bar{\psi} v} \right) \right] \|\chi_1 - \chi_2\|_W.
\end{aligned}
\tag{57}
$$

If $k > -\bar{\delta} + H$ is chosen large enough, the operator Λ is a contraction. Hence, it has a unique fixed point and thus a unique function $\chi = \chi(t) \in W$ exists as solution of $\chi = \Lambda\chi$.

From our calculations it turns out that $\chi(t)$ is a nonnegative function if $M \ge Q(t)$. From its definition $\chi(t)$ is bounded and

$$
\chi(t) = c(M - Q(t)) L(t) \le cML(t) \le cM \left(L_0 + \frac{\bar{\phi} M}{\bar{\psi} v} \right). \tag{58}
$$

5 Existence and uniqueness of the strong solution of the model

In the preceding section we proved that the system (29–31) has a unique nonnegative mild solution in $L^1(0, m)$, that is, we proved that (38) has a continuous solution and $L(t)$ is given by (50). Actually, we prove that the solution of (37) is strongly differentiable with respect to t. The differentiability of $L(t)$ follows immediately by standard theorems of ordinary differential equation theory. We observe that the solution of the system (10–13), found in [7], is actually a global integral solution, as we explain below. In fact, if we consider the following system in a Banach space Z:

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) + F(u(t)), & t > 0, \\ u(0) = u_0, \end{cases} \tag{59}$$

where the operator A has a non-dense domain in Z, it is said that $u(t)$ is an integral solution, $u \in C([0,T), Z)$ if

$$\int_0^t u(s)ds \in D(A), \ \forall\, t > 0 \tag{60}$$

and

$$u(t) = u_0 + A\int_0^t u(r)dr + \int_0^t F(u(r))dr. \tag{61}$$

In [5] it is proved that if a system has an integral solution it also has a mild solution. We have proved directly, in Sects. 3 and 4, that our system has a mild solution without using the concept of an "integral solution". Now, we shall prove that this solution is also a global strong solution of the system.

In [3], the following semilinear initial-value problem is studied

$$\begin{aligned} &\frac{du}{dt} = Bu(t) + G(u(t)), \ t > 0, \\ &u(0) = u_0, \end{aligned} \tag{62}$$

and the following theorem is proved.

Theorem 5.1 *Under the following assumptions:*

1. $B \in \mathscr{G}(M, \beta; X)$;
2. *G is a Lipschitz operator* $\|G(f_1) - G(f)\| \le \alpha(\|f_1\|, \|f\|)\|f_1 - f\|, \quad \forall f_1, f \in D(G) = X$, *where* $\alpha(\rho_1, \rho_2)$ *is a non-decreasing function of* ρ_1 *and of* ρ_2;
3. *G is Fréchet differentiable at any* $f \in D(G) = X$, *and its Fréchet derivative* G_f *is such that* $\|G_f g\| \le \alpha_1(\|f\|)\|g\|$, $\forall f, g \in X$, *where* $\alpha_1(\rho)$ *is a non-decreasing function of* ρ;
4. $\|G_{f_1} g - G_f g\| \to 0 \quad$ *as* $\quad \|f_1 - f\| \to 0 \quad \forall g, f_1, f \in X$;
5. $u_0 \in D(B)$;
6. *if a strong solution* $w = w(t)$ *of* (62) *exists over* $[0, t_1] \subset [0, t_0]$, *then* $\|w(t)\| \le \eta \ \forall t \in [0, t_1]$, *where* η *is a suitable constant that may depend on* u_0 *and* t_0;

the semilinear initial-value problem (62) *has a unique strong solution* $u(t)$ *defined over the whole* $[0, t_0]$ *where* t_0 *is a priori fixed.*

Remark 5.1 To prove Theorem 5.1, in [3] it is shown, first, that the integral equation

$$u(t) = exp(tB)u_0 + \int_0^t exp[(t-s)B]F(u(s))ds \tag{63}$$

has a continuous solution, then, in Lemma 5.4, page 200, it is proved that the solution of (63) is differentiable with respect to t. This is made by calculating directly the quantity $[u(t+h)-u(t)]h^{-1}$ and taking the limit as h goes to zero.

Lemma 5.1 *If F is defined by* (32) *the conditions* (2), (3), (4) *of Theorem* 5.1 *are satisfied.*

Proof Property 2. follows from the definition. In fact,

$$\begin{aligned}\|F(f)-F(h)\| =& \|(\bar{\delta}-\delta(s,Q(t)))f-(\bar{\delta}-\delta(s,Q(t)))h\| \\ =& \|(\bar{\delta}-\delta(s,Q(t)))(f-h) \le 2\bar{\delta}\|f-h\|, \quad \forall f,h \in D(F)=X.\end{aligned}$$

Hence F is a Lipschitz operator with Lipschitz constant $2\bar{\delta}$. It easy to prove that F is Fréchet differentiable for all $f \in D(F)=X$, with Frechét derivative $F_f(g) = \bar{\delta}g - \delta_{Qf}(s,Qg)(Qg)f - \delta(s,Qf)g$, where δ_{Qf} is the Fréchet derivative of δ. Moreover

$$\begin{aligned}\|F_f g\| =& \|\bar{\delta}g - \delta_{Qf}(s,Qg)(Qg)f - \delta(s,Qf)g\| \le \\ \le& \bar{\delta}\|g\| + \delta'\bar{\psi}\|g\|\|f\| + \bar{\delta}\|g\| = (2\bar{\delta}+\delta'\bar{\psi}\|f\|)\|g\|, \quad \forall f,g \in D(F)=X\end{aligned}$$

with $2\bar{\delta}+\delta'\bar{\psi}\|f\|$ is a non-decreasing function of $\|f\|$. Moreover, from the above inequality Property 4. of Theorem 5.1 also follows. □

Theorem 5.2 *Let $[0,t_0]$ be given, if $q_0 \in D(N)$ and $\chi(t)$ differentiable for $t \in [0,t_0]$, then the solution $q(t)$ of* (38) *is strongly differentiable for each $t \in [0,t_0]$.*

Proof If $\chi \equiv 0$ from Lemma 5.1 and Theorem 5.1 we have that $q(t)$ is differentiable with respect to t. If $\chi(t) \neq 0$ the function q will result differentiable as a sum of differentiable functions. □

Let us prove now that $\chi(t)$ is a differentiable function. If $t \in [0,t_0]$ we have

$$\begin{aligned}&\frac{\chi(t+h)-\chi(t)}{h} \\ &\quad = cM\left(\frac{L(t+h)-L(t)}{h}\right) - cL(t+h)\left(\frac{Q(t+h)-Q(t)}{h}\right) \\ &\qquad - cQ(t)\left(\frac{L(t+h)-L(t)}{h}\right).\end{aligned} \tag{64}$$

The first and the third term of the equality (64) have finite limit as h goes to zero. As regards the second term we have

$$\left|\frac{Q(t+h)-Q(t)}{h}\right| \leq \bar{\psi} \int_0^m \frac{|q(s,t+h)-q(s,t)|}{h} ds.$$

with

$$q(s,t+h)-q(s,t) = e^{(N-N_1)(t+h)} q_0 e^{-\bar{\delta}(t+h)} + \frac{\chi(t+h-\int_0^s \frac{1}{g(u)} du)}{g(0)} +$$
$$+ \int_0^{t+h} e^{(N-N_1)(t+h-r)} e^{-\bar{\delta}(t+h-r)} [-g' + \bar{\delta} - \delta(s,Q(r))] q(r) dr - e^{(N-N_1)t} q_0 e^{-\bar{\delta}t} +$$
$$- \frac{\chi(t-\int_0^s \frac{1}{g(u)} du)}{g(0)} - \int_0^t e^{(N-N_1)(t-r)} e^{-\bar{\delta}(t-r)} [-g' + \bar{\delta} - \delta(s,Q(r))] q(r) dr.$$

We know from Remark 5.1 that the quantity

$$e^{(N-N_1)(t+h)} q_0 e^{-\bar{\delta}(t+h)} + \int_0^{t+h} e^{(N-N_1)(t+h-r)} e^{-\bar{\delta}(t+h-r)} [-g' + \bar{\delta} - \delta(s,Q(r))] q(r) dr$$
$$- e^{(N-N_1)t} q_0 e^{-\bar{\delta}t} - \int_0^t e^{(N-N_1)(t-r)} e^{-\bar{\delta}(t-r)} [-g' + \bar{\delta} - \delta(s,Q(r))] q(r) dr,$$

divided by h has a finite limit as h goes to zero. Therefore, we have to check that the following quantity has a finite limit as h goes to zero:

$$\frac{\chi(t+h-\int_0^s \frac{1}{g(u)} du) - \chi(t-\int_0^s \frac{1}{g(u)} du)}{h}.$$

For a small enough h, if $t < \int_0^s \frac{1}{g(u)} du$ and $t+h < \int_0^s \frac{1}{g(u)} du$, we have that:

$$\frac{\chi(t+h-\int_0^s \frac{1}{g(u)} du) - \chi(t-\int_0^s \frac{1}{g(u)} du)}{h} = 0$$

for the definition of $\chi(t)$, (21), (22), hence, $\lim_{h\to 0} \frac{\chi(t+h)-\chi(t)}{h}$ exists and it is finite for $0 < t < \int_0^s \frac{1}{g(u)} du$.

On the other hand, if $t > \int_0^s \frac{1}{g(u)} du$, but $t < 2\int_0^s \frac{1}{g(u)} du$, that is $t - \int_0^s \frac{1}{g(u)} du < \int_0^s \frac{1}{g(u)} du$, since $\chi'(t)$ exists for $t < \int_0^s \frac{1}{g(u)} du$, we have that

$$\lim_{h\to 0} \frac{\chi(t+h-\int_0^s \frac{1}{g(u)} du) - \chi(t-\int_0^s \frac{1}{g(u)} du)}{h}$$

exists and it is finite, and so $\lim_{h\to 0} \frac{\chi(t+h)-\chi(t)}{h}$ exists and it is finite again for $0 < t < 2\int_0^s \frac{1}{g(u)} du$.

Therefore, we can repeat this process for $2\int_0^s \frac{1}{g(u)} du < t < 3\int_0^s \frac{1}{g(u)} du$ and so on, $\forall t \in (0, t_0)$.

Hence, we can conclude that $\chi(t)$ is differentiable $\forall t \in (0, t_0)$. Therefore the function q is strongly differentiable with respect to t.

The following Theorem is proved.

Theorem 5.3 *Let* $[0, t_0]$ *be fixed, if* $q_0 \in D(N)$ *then the function* $q(t)$ *solution of* (38) *is strongly differentiable and hence problem* (29–31) *has a unique global strong solution.*

Proof We have proved that $\chi(t)$ is a differentiable function, hence since $L(t)$ and $q(t)$ are strongly differentiable functions, the problem (29–31) has a unique global strong solution. □

6 Concluding remarks

In this paper, we have analysed a non-linear size-structured population model of a marine species.

We used the semigroup theory and the concept of affine semigroup.

We succeeded in finding a unique mild solution of the system, and we first proved that it is a local solution, that is it is defined for $t \in [0, \bar{t}]$ with $\bar{t}$ small enough and then we proved that is a global solution, that is it is defined in any interval $[0, t_0]$ with t_0 priori fixed. At the end, we also proved that the solution is strongly differentiable, hence it is a global strong solution of the system.

The model is a realistic extension of that in [7] and [11]. Paper [11] studied, also by means of the semigroup theory, a non-linear age-structured model but only the existence of a global mild solution is proved, whereas [7] studied a non-linear size-structured population model like ours but using the theory of non-densely defined operators.

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