On the scope of some formulas defining additive connectives in fuzzy logics

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Àngel García-Cerdaña
Institut d'Investigació en Intel.ligència Artificial (IIIA) CSIC
08193 Bellaterra
Catalonia, Spain

Carles Noguera
Institut d'Investigació en Intel.ligència Artificial (IIIA) CSIC
08193 Bellaterra
Catalonia, Spain

Francesc Esteva
Institut d'Investigació en Intel.ligència Artificial (IIIA) CSIC
08193 Bellaterra
Catalonia, Spain

Abstract

In [23] Wang, Wang and Pei defined a new fuzzy logic called NMG. They also introduced new formulas to define the additive connectives from multiplicative conjuction, residuated implication and bottom in NMG. However, they did not study the scope of these formulas in the general framework of fuzzy logics. This is the aim of this paper. Therefore, we add the definability formulas to known fuzzy logics as new axioms, following the method used in [7], and we obtain some families of logics presented in a simpler language. Finally, we discuss the standard completeness of these new logics.

KEYWORDS: Additive connectives, left-continuous t-norms, involutive left-continuous t-norms, IMTL-algebras, MTL-algebras, nilpotent minimum t-norm, NM-algebras, negation functions, non-classical logics, residuated lattices, residuated fuzzy many-valued logics.

1 Introduction

In [5] the authors defined the logics MTL (Monoidal T-norm based Logic) and IMTL (Involutive MTL) as fuzzy residuated multivalued systems, generalizations of BL logic, the Basic Fuzzy Logic defined by Hájek [14] and proved in [4] to be the logic of continuous t-norms and their residua. The primitive connectives of MTL are the multiplicative conjunction $*$, its residuated implication $\rightarrow$, the additive conjunction $\wedge$ and the constant 0. The claim of the authors was that MTL and IMTL are the logic of left-continuous t-norms and the logic of involutive left-continuous t-norms respectively. This claim was proved in [19] and [8]. Moreover, in [5] the authors studied a logic called NM (an axiomatic extension of IMTL) which is standard complete with respect to the nilpotent minimum t-norm, introduced by Fodor in

[9]. Finally, they also introduced the weak nilpotent minimum logic (WNM), a generalization of NM in which the corresponding left-continuous t-norms need not to be involutive.

In all these logics other connectives (the additive disjunction, negation, and the constant 1) are definable by means of the primitive ones. In addition, in BL the additive conjunction is also a definable connective thanks to the divisibility axiom.

Recently a new fuzzy logic NMG was defined in [23] as the axiomatic extension of WNM by the following single axiom: $(\neg\neg\varphi \to \varphi) \vee (\varphi \wedge \psi \to \varphi * \psi)$. In fact, this axiom is equivalent to the axiom (ŁG) introduced in [4]: $(((\varphi \to 0) \to 0) \to \varphi) \vee (\varphi \to \varphi * \varphi)$. Moreover, in [23] the authors prove that in NMG logic the additive disjunction and the additive conjunction are definable in terms of $*$, $\to$, and 0. Hence, NM logic can be presented in a simpler language without additive conjunction, since it is clearly an extension of NMG. The question that arises then is to characterize the fuzzy logics where these definitions are valid, in order to find all the fuzzy logics that admit this simplification of the language. To fulfill this task[1] we study the logics from the algebraic point of view (i.e. we focus on the algebraic counterparts of the logics, that are certain varieties of algebras). Therefore, we consider the translations of definability formulas into equations, $(D_\vee)$ for the additive disjunction and $(D_\wedge)$ for the additive conjunction, and the varieties of MTL-algebras defined by them, denoted by $\mathbf{MTL}[D_\vee]$ and $\mathbf{MTL}[D_\wedge]$. After some preliminaries, in section 3 we study the former, giving some of its standard algebras (i.e. algebras defined in the real unit interval). In section 4 we study $\mathbf{MTL}[D_\wedge]$ characterizing its chains as a special class of WNM-chains. We also study the equation in the more general framework of bounded residuated lattices (perhaps not prelinear) considering their relation with contraction and weak contraction properties. In section 5 we prove strong standard completeness for the logic associated to $\mathbf{MTL}[D_\wedge]$ and, finally, in section 6 we study the intersections of our new varieties with the known ones (i.e. the algebraic counterparts of the studied important fuzzy logics) obtaining a new hierarchy of algebras and logics.

2 Preliminaries

In this section we introduce all the basic notions that will be used throughout the paper.

Definition 2.1. *MTL (Monoidal T-norm based Logic) is the sentential logic in the language* $\mathcal{L} = \{*, \to, \wedge, 0\}$ *of type* $(2, 2, 2, 0)$ *defined by the Hilbert-style calculus with the rule of Modus Ponens and the following axioms (using implication as the least binding connective):*

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $\varphi * \psi \to \varphi$
(A3) $\varphi * \psi \to \psi * \varphi$
(A4) $\varphi \wedge \psi \to \varphi$
(A5) $\varphi \wedge \psi \to \psi \wedge \varphi$
(A6) $\varphi * (\varphi \to \psi) \to \varphi \wedge \psi$
(A7a) $(\varphi \to (\psi \to \chi)) \to (\varphi * \psi \to \chi)$
(A7b) $(\varphi * \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9) $0 \to \varphi$

[1] The new way for defining the additive disjunction was already presented in [22] and first studied in our note [6]. Moreover, our note [10] was devoted to all the definability formulas that appear in this paper.

Other usual connectives are defined by:
$\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$;
$\varphi \leftrightarrow \psi := (\varphi \to \psi) * (\psi \to \varphi)$; $\neg\varphi := \varphi \to 0$; $1 := \neg 0$.

Following the tradition of Linear Logics and Substructural Logics, we will usually refer to $*$, $\wedge$ and $\vee$ as the multiplicative conjunction, the additive conjunction and the additive disjunction respectively.

We denote by $Fm_{\mathcal{L}}$ the set of $\mathcal{L}$-formulas (built using a countable set of variables). If $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, we define: $\Gamma \vdash_{MTL} \varphi$ iff φ is derivable from Γ in the given calculus. We write $\vdash_{MTL} \varphi$ instead of $\emptyset \vdash_{MTL} \varphi$.

MTL can be seen as the axiomatic extension of Monoidal Logic, ML, (see [15], [12], [13]) with the axiom of prelinearity:

$$(\varphi \to \psi) \vee (\psi \to \varphi) \qquad \text{(Lin)}$$

Now let us recall some well-known extensions of ML and MTL:

- The *affine* Multiplicative Additive Linear Logic, aMALL (see [11]), is the axiomatic extension of ML with the axiom schema of involution:

$$((\varphi \to 0) \to 0) \to \varphi \qquad \text{(Inv)}$$

- The Involutive Monoidal T-norm based Logic, IMTL, is the axiomatic extension of MTL with (Inv) or equivalently the axiomatic extension of aMALL with (Lin).

- The Strict Monoidal T-norm based Logic, SMTL, is obtained from MTL by adding the pseudocomplementation schema:

$$(\varphi \wedge \neg\varphi) \to 0 \qquad \text{(Pseudo)}$$

- The Weak Nilpotent Minimum Logic, WNM, is defined by the axioms of MTL plus this schema:

$$(\varphi * \psi \to 0) \vee (\varphi \wedge \psi \to \varphi * \psi) \qquad \text{(WNM)}$$

- The Nilpotent Minimum Logic, NM, is defined by the axioms of IMTL plus (WNM).

- The Basic Fuzzy Logic, BL, can be obtained as the extension of MTL with the divisibility axiom:

$$\varphi \wedge \psi \to \varphi * (\varphi \to \psi) \qquad \text{(Div)}$$

We can obtain, as axiomatic extensions of BL, the Strict Basic Logic, SBL, and the well-known Łukasiewicz, Product and Gödel Logics by adding to BL the following axioms:

for SBL,
$$(\varphi \wedge \neg\varphi) \to 0$$

for Łukasiewicz,
$$((\varphi \to 0) \to 0) \to \varphi$$

for Product Logic, and
$$(\varphi \wedge \neg\varphi) \to 0,$$
$$\neg\neg\chi \to (((\varphi * \chi) \to (\psi * \chi)) \to (\varphi \to \psi)) \qquad (\Pi 1)$$

$$\varphi \to \varphi * \varphi \qquad \text{(Con)}$$

for Gödel Logic, which is also the extension of Intuitionistic Propositional Logic with the axiom (Lin) (see [14]). The lattice of these logics is depicted in Figure 1.

Figure 1: *Graph of main residuated many-valued logics with the shadowed part containing t-norm based logics.*

It is not difficult to see that the extension of ML by adding both (Pseudo) and (Inv) is the Classical Logic (see [20]). Thus in Figure 1 we can divide the axiomatic extensions of Monoidal Logic in three subclasses: logics satisfying pseudocomplementation (on the left-hand circle), logics with involutive negation (on the right-hand circle) and the rest.

Remark 2.2. *In BL the language needs only* $*, \to$ *and* 0 *as primitive connectives, since* $\wedge$ *becomes definable by taking* $\varphi \wedge \psi$ *as* $\varphi * (\varphi \to \psi)$*, i. e:*

$$\vdash_{BL} \varphi \wedge \psi \leftrightarrow \varphi * (\varphi \to \psi)$$

In IMTL the multiplicative conjunction is definable using another equivalence:

$$\vdash_{IMTL} \varphi * \psi \leftrightarrow \neg(\varphi \to \neg\psi)$$

and also in G we need only one conjunction since:

$$\vdash_{G} \varphi \wedge \psi \leftrightarrow \varphi * \psi$$

Now we turn to the algebraic semantics of these logics. For the necessary background in Universal Algebra see [2].

Definition 2.3. *A commutative integral bounded residuated lattice, or* RESIDUATED LATTICE *for short, is an algebra* $\mathcal{A} = \langle A, \wedge, \vee, *, \to, 0, 1\rangle$ *of type* $(2, 2, 2, 2, 0, 0)$ *satisfying:*

1. $\langle A, \wedge, \vee, 0, 1\rangle$ *is a bounded lattice.*
2. $\langle A, *, 1\rangle$ *is a commutative monoid with unit* 1*.*
3. *The operations* $*$ *and* $\to$ *form an adjoint pair:*

$$\forall a, b, c \in A,\ a * b \leq c \text{ iff } b \leq a \to c.$$

The class of all those algebras forms a variety, **RL**. Another operation, the negation, is defined in **RL** by: $\neg x := x \to 0$.

Let us recall the definition of MTL-algebra [5]:

Definition 2.4. *Let* $\mathcal{A} \in \mathbf{RL}$*. Then* $\mathcal{A}$ *is a MTL-algebra iff* $\mathcal{A} \vDash (x \to y) \vee (y \to x) \approx 1$*.* **MTL** *will be the variety of all MTL-algebras.*

In this algebras we can distinguish the sets of positive and negative elements:

Definition 2.5. *Let* $\mathcal{A} \in \mathbf{MTL}$*. An element* $a \in A$ *is said to be* POSITIVE *if* $a > \neg a$ *and is said to be* NEGATIVE *if* $a \leq \neg a$*. We will denote by* A_+ *and* A_- *the sets of positive and negative elements, respectively.*

Definition 2.6. *Given* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*, we define:* $\Gamma \vDash \varphi$ *iff for all* $\mathcal{A} \in \mathbf{MTL}$ *and for all evaluation* v *in* $\mathcal{A}$*, we have: If* $\forall \psi \in \Gamma$*,* $v(\psi) = 1$*, then* $v(\varphi) = 1$*.*

Then, one can prove this theorem of strong completeness for MTL logic:

Theorem 2.7. *If* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*, then* $\Gamma \vDash \varphi$ *iff* $\Gamma \vdash_{MTL} \varphi$*.*

But this result can be improved by means of the equational consequence:

Definition 2.8. *Let $Eq_{\mathcal{L}}$ be the set of $\mathcal{L}$-equations and let $\Pi \cup \{\varphi \approx \psi\} \subseteq Eq_{\mathcal{L}}$. We define the equational consequence by:*

$\Pi \models_{\mathbf{MTL}} \varphi \approx \psi$ *iff for all $\mathcal{A} \in$ **MTL** and for all evaluation v in $\mathcal{A}$, we have:*

$$\text{If } \forall\, \alpha \approx \beta \in \Pi,\ v(\alpha) = v(\beta), \text{ then } v(\varphi) = v(\psi).$$

Theorem 2.9. *The relation of derivability in the system MTL and the equational consequence in the variety* **MTL** *are mutually translatable:*

Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ and $\Pi \cup \{\varphi \approx \psi\} \subseteq Eq_{\mathcal{L}}$. The following conditions hold:

1. $\Gamma \vdash_{MTL} \varphi$ *iff* $\{\psi \approx 1 : \psi \in \Gamma\} \models_{\mathbf{MTL}} \varphi \approx 1$.

2. $\Pi \models_{\mathbf{MTL}} \varphi \approx \psi$ *iff* $\{\alpha \leftrightarrow \beta : \alpha \approx \beta \in \Pi\} \vdash_{MTL} \varphi \leftrightarrow \psi$.

In addition, each one of those translations is the inverse of the other, i.e.:

3. $\varphi \approx \psi \models_{\mathbf{MTL}} \varphi \leftrightarrow \psi \approx 1$ *and* $\varphi \leftrightarrow \psi \approx 1 \models_{\mathbf{MTL}} \varphi \approx \psi$.

4. $\varphi \vdash_{MTL} \varphi \leftrightarrow 1$ *and* $\varphi \leftrightarrow 1 \vdash_{MTL} \varphi$.

Therefore, MTL is an algebraizable logic in the sense of Blok and Pigozzi (see [1]) whose equivalent algebraic semantics is the variety **MTL**. So all the axiomatic extensions of MTL are also algebraizable in this sense and there is an order isomorphism between axiomatic extensions of MTL and subvarieties of **MTL**, using the translation of formulas into equations and viceversa. Let L be any logic of this family. We will refer to the algebras associated to L as L-algebras and **L** will be the variety of all L-algebras. If $\varphi \in Fm_{\mathcal{L}}$, L$[\varphi]$ will be the axiomatic extension of L by adding the schema φ and $\mathbf{L}[\varphi \approx 1]$ will be the equivalent algebraic semantics for this logic.

There are other kinds of completeness results that will be useful. The first is the completeness with respect to the totally ordered algebras (we will call them 'chains'):

Theorem 2.10 ([5])**.** *Each MTL-algebra is isomorphic to a subdirect product of MTL-chains.*

Corollary 2.11 ([5])**.** *Let $\varphi \in Fm_{\mathcal{L}}$. Then: $\vdash_{MTL} \varphi$ iff $\mathcal{A} \models \varphi \approx 1$ for every MTL-chain $\mathcal{A}$.*

The same kind of result is true for every axiomatic extension of MTL. Finally, we recall that it is also possible in some cases to restrict the semantics to the algebras defined in the real unit interval by a left-continuous t-norm and its residuum, obtaining the following standard completeness results:

Theorem 2.12. *Let $\varphi \in Fm_{\mathcal{L}}$.*

- $\vdash_{MTL} \varphi$ *iff $\varphi \approx 1$ is valid in every MTL-chain defined by a left-continuous t-norm ([19]).*

- $\vdash_{IMTL} \varphi$ *iff $\varphi \approx 1$ is valid in every IMTL-chain defined by an involutive left-continuous t-norm ([8]).*

- $\vdash_{BL} \varphi$ *iff $\varphi \approx 1$ is valid in every BL-chain defined by a continuous t-norm ([4]).*

- $\vdash_{WNM} \varphi$ *iff $\varphi \approx 1$ is valid in every WNM-chain defined by a weak nilpotent minimum t-norm ([5]).*

- $\vdash_{NM} \varphi$ *iff* $\varphi \approx 1$ *is valid in the NM-chain defined by the standard nilpotent minimum t-norm ([5]).*

Regarding negations, let us recall that negation functions corresponding to MTL-chains on $[0,1]$, defined as $n(x) = x \to 0$, are the so-called *(weak) negation functions* in the fuzzy literature (see [21]), i.e. functions $n\colon [0,1] \to [0,1]$ that are order reversing, satisfying $n(1) = 0$ and $x \leq n(n(x))$ for all $x \in [0,1]$. If a weak negation n is involutive, i.e. if it satisfies the equality

$$n(n(x)) = x \text{ for all } x \in [0,1],$$

then it is called a *strong* negation (for negations in MTL-chains see [5, 3]).

In [23] NMG is introduced as a new schematic extension of MTL by adding (WNM) and the following axiom:

$$(\neg\neg\varphi \to \varphi) \vee (\varphi \wedge \psi \to \varphi * \psi) \qquad \text{(NMG)}$$

It is easy to see that this axiom is equivalent to the axiom (ŁG) introduced in [4]:

$$(\neg\neg\varphi \to \varphi) \vee (\varphi \to \varphi * \varphi) \qquad \text{(ŁG)}$$

This new logic NMG also has standard completeness:

Theorem 2.13 ([23])**.** *Let* $\varphi \in Fm_{\mathcal{L}}$*.* $\vdash_{NMG} \varphi$ *iff* $\varphi \approx 1$ *is valid in the standard NMG-chain defined by the NMG-t-norm (the ordinal sum of nilpotent minimum in* $[0, \frac{1}{2}]$ *and minimum in* $[\frac{1}{2}, 1]$*).*

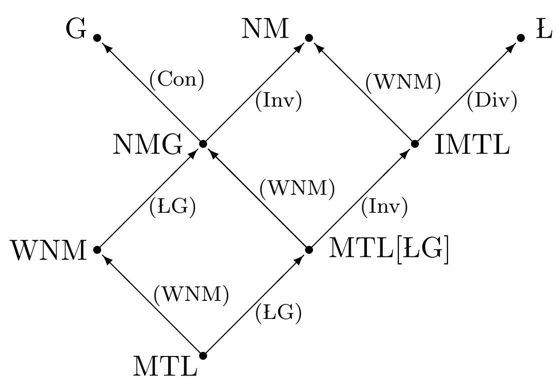

Figure 2: *Relationship between NMG and other well known fuzzy logics.*

In Figure 2 we consider the lattice of some of the logics presented so far. Note that the points in the diagram correspond to pairwise different logics. Indeed:

- WNM $\nvdash$ (ŁG) (see Example 14) hence MTL[ŁG] $\neq$ MTL and WNM $\neq$ NMG.
- MTL[ŁG] $\neq$ NMG, since any finite MV-chain with more than three elements obviously satisfies (ŁG) but does not satisfy (WNM).
- MTL[ŁG] $\neq$ IMTL, since the standard Gödel algebra satisfies (ŁG) and is not involutive.
- It is well known that the remaining logics are pairwise different.

Example 2.14. *Let* $\mathcal{A} = \langle \{0, a, b, 1\}, \vee, \wedge, *, \to, 0, 1 \rangle$ *be the chain formed by* $0 < a < b < 1$ *and where* $*$ *and* $\to$ *are the operations defined by the following tables:*

$*$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	b	b	1	1
1	0	a	b	1

Obviously $\mathcal{A}$ is a WNM-chain but does not satisfy the equation (ŁG):
$(\neg\neg a \rightarrow a) \vee (a \rightarrow a * a) = (\neg b \rightarrow a) \vee (a \rightarrow 0) = (b \rightarrow a) \vee b = b \vee b = b \neq 1.$

□

In [22, 23] the following equations are introduced to define the additive connectives $\vee$ and $\wedge$ from $\neg$ and $\rightarrow$ (where $\alpha(x, y) := (x \rightarrow y) \rightarrow y$):

$(D_\vee^\neg)$ $\quad x \vee y \approx \neg(\alpha(x, y) \rightarrow \neg(\alpha(x, y) \rightarrow \alpha(y, x)))$

$(D_\vee)$ $\quad x \vee y \approx \alpha(x, y) * (\alpha(x, y) \rightarrow \alpha(y, x))$

$(D_\wedge)$ $\quad x \wedge y \approx (x * y) \vee \neg(y \vee \neg y \vee \neg x)$

Moreover, they proved the following result:

Theorem 2.15 ([23])**.** *In every NMG-algebra $(D_\vee)$ and $(D_\wedge)$ are valid equations.*

Since involutive residuated lattices satisfy $x * y \approx \neg(x \rightarrow \neg y)$, in this variety $(D_\vee^\neg)$ and $(D_\vee)$ are equivalent. This is not true in non-involutive residuated lattices. Actually we have:

Proposition 2.16. *Let $\mathcal{A}$ be a residuated lattice. If $\mathcal{A}$ satisfies $(D_\vee^\neg)$, then $\mathcal{A}$ is involutive.*

Proof. Suppose it is not involutive. Then there must be an $a \in A$ such that $a < \neg\neg a$. One can check that $(D_\vee^\neg)$ fails for $x = y = a$. □

□

Therefore, for any variety of residuated lattices $\mathbf{K}$, we have

$$\mathbf{K}[D_\vee^\neg] = \mathbf{K}[D_\vee, \neg\neg x \approx x].$$

So, henceforth we need not consider $(D_\vee^\neg)$ and we will focus our attention on the remaining two equations.

3 Definability of additive disjunction

In this section we study the subvariety of **MTL** defined by the equation $(D_\vee)$ and we give some of its standard algebras. First let us consider two remarks about $(D_\vee)$:

Remark 3.1. *$(D_\vee)$ is true in every residuated lattice for $x \leq y$. So we only need to check the equation when $x > y$.*

Remark 3.2. *In* **MTL**, $(D_\vee)$ *is a kind of weak divisibility. Indeed, on the one hand MTL-algebras satisfy* $x \vee y \approx \alpha(x,y) \wedge \alpha(y,x)$ *and, on the other hand,* $(D_\vee)$ *is the equation* $x \vee y \approx \alpha(x,y) * (\alpha(x,y) \to \alpha(y,x))$*. Hence in MTL-algebras this definability is equivalent to:*

$$\alpha(x,y) \wedge \alpha(y,x) \approx \alpha(x,y) * (\alpha(x,y) \to \alpha(y,x))$$

that is, to the divisibility condition restricted to the pairs $\langle \alpha(x,y), \alpha(y,x) \rangle$*.*

Therefore, it is clear that $(D_\vee)$ *is valid in* **BL**, *since in this variety full divisibility holds. Furthermore,* $\mathbf{MTL}[D_\vee]$ *and* $\mathbf{IMTL}[D_\vee]$ *will be proved to be new intermediate varieties.*

In the following, even though we will not characterize those intermediate varieties, we will provide some partial results that give methods to obtain algebras satisfying the definability of additive disjunction. In particular, we will give some examples of left-continuous (non-continuous) t-norms satisfying $(D_\vee)$.

Definition 3.3 ([3])**.** *Let* $*$ *be a continuous t-norm and let* n *be a negation function in* $[0,1]$*. The binary operation* $*_n$ *on* $[0,1]^2$ *is defined as follows:*

$$x *_n y = \begin{cases} x * y & \text{if } x > n(y), \\ 0 & \text{otherwise.} \end{cases}$$

Let us recall that not all the operations defined in this way are t-norms. In [3] a set of necessary and sufficient conditions for $*_n$ being a t-norm is given. In this case we will say that n is *compatible* with $*$.

Definition 3.4 ([3])**.** *Given a negation* n *on* $[0,1]$*, a segment* $[a,b] \subseteq [0,1]$ *is said to be:*

1. positive *with respect to* n *if* $n(b) \leq a$*,*
2. semi-positive *with respect to* n *if* $b > n(b) > a$*, and*
3. negative *with respect to* n *if* $n(b) \geq b$*.*

Theorem 3.5 ([3], Theorem 3)**.** *Let* $*$ *be a continuous t-norm and* n *be a negation. Then* $*$ *is compatible with* n *iff the following conditions are satisfied:*

(i) If $[a,b]_*$ *is a positive Łukasiewicz component of* $*$ *and* $n(b) < a$*, then* n *is constant on* $[a,b]$*.*

(ii) If $[a,b]_*$ *is a positive Łukasiewicz component of* $*$ *and* $n(b) = a$*, then* n *is the corresponding Łukasiewicz negation on* $(a,b]$*, i.e.* $n(x) = x \to a$ *for all* $x \in (a,b]$*.*

(iii) If $[a,b]_*$ *is a positive Product component of* $*$ *and* $n(b) < a$*, then* n *is constant on* $(a,b]$*.*

(iv) If $[a,b]_*$ *is a positive Product component of* $*$ *and* $n(b) = a$*, then* n *is the corresponding Product negation on* $(a,b]$*, i.e.* $n(x) = a$ *for all* $x \in (a,b]$*.*

(v) If $[a,b]_*$ *is a semi-positive Łukasiewicz or Product component of* $*$ *then* n *coincides in* $(n(b),b]$ *with the Łukasiewicz negation* $n(x) = x \to n(b)$ *for all* $x \in (n(b),b]$*.*

(vi) n *can be arbitrary in every Gödel component and in every negative component of* $*$*.*

As a corollary when the negation n is involutive we have:

Corollary 3.6 ([16])**.** *If $*$ is a continuous t-norm and n is a compatible strong negation, then $*_n$ must be isomorphic to Łukasiewicz t-norm, to nilpotent minimum t-norm or to*

$$x \circ y = \begin{cases} 0 & \text{if } x \leq 1-y, \\ \frac{1}{3}+x+y-1 & \text{if } x, y \in [\frac{1}{3}, \frac{2}{3}] \text{ and } x > 1-y, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

From these results we can characterize the MTL-algebras over $[0,1]$ defined by a $*_n$ t-norm satisfying $(D_\vee)$.

Theorem 3.7. *Suppose that n is a negation compatible with a continuous t-norm $*$. Let $\mathcal{A}$ be the MTL-algebra defined by $*_n$. Then $\mathcal{A}$ satisfies $(D_\vee)$ if, and only if, $\forall a \in A_-$, $n(n(a)) = a$.*

Proof. First we prove the *only if* part. Suppose that there exists an element $a \in A_-$ such that $a < n(n(a))$. Then we would have $\alpha(a,0) *_n (\alpha(a,0) \to \alpha(0,a)) = n(n(a)) *_n (n(n(a)) \to a) = n(n(a)) *_n (n(a) \vee a) = n(n(a)) *_n n(a) = 0$ while $a \vee 0 = a$; hence $\mathcal{A}$ would not satisfy $(D_\vee)$.

Conversely, if $\mathcal{A}$ is a MTL-algebra defined over $[0,1]$ by a t-norm of type $*_n$ for a negation n such that $\forall a \in A_-$, $n(n(a)) = a$, then we will show that $(D_\vee)$ is satisfied. Taking into account Remark 3.1 and the characterization of $*_n$ in Theorem 3.5, we can suppose $a > b$ and we have to consider the following cases:

(i) If a and b belong to the same component, then the full divisibility is valid, and thus also $(D_\vee)$.

(ii) If a and b belong to a different component, then we have two cases:

(a) $a > n(b)$ and thus $a *_n b = min(a,b)$. We have $\alpha(a,b) = (a \to b) \to b = (n(a) \vee b) \to b = b \to b = 1$. So $(D_\vee)$ is satisfied.

(b) $a \leq n(b)$ and then $a *_n b = 0$. We have $\alpha(a,b) = (a \to b) \to b = (n(a) \vee b) \to b = n(a) \to b = n(n(a))$. Then $(D_\vee)$ is satisfied if $n(n(a)) *_n (n(n(a)) \to a) = a$. But this is true because, on the one hand, if $a \in A_-$, by supposition $n(n(a)) = a$ and the result is obvious and, on the other hand, if $a \in A_+$, a being idempotent, $n(n(a)) *_n (n(n(a)) \to a) = n(n(a)) *_n (n(a) \vee a) = n(n(a)) *_n a = a$. □

□

Notice that this proves $\mathbf{MTL}[D_\vee] \subsetneq \mathbf{MTL}$.

Corollary 3.8. *If n is an involutive negation compatible with a continuous t-norm $*$, then the IMTL-algebra defined by the left-continuous t-norm $*_n$ satisfies $(D_\vee)$.*

Corollary 3.9. *A weak nilpotent minimum t-norm satisfies $(D_\vee)$ if, and only if, for every negative element $a \in [0,1]$, $n(n(a)) = a$.*

Proof. Weak nilpotent minimum t-norms are of the form $*_n$ where n is an arbitrary weak negation and $*$ is the minimum t-norm. □

□

This result can be extended to all WNM-chains.

Theorem 3.10. *Let $\mathcal{A}$ be a WNM-chain. Then $\mathcal{A} \models (D_\vee)$ if, and only if, for every $a \in A_-$, $\neg\neg a = a$.*

Proof. First suppose that there is $a \in A_-$ such that $a < \neg\neg a$. Then $\mathcal{A} \not\models (D_\vee)$, since $\alpha(a,0) * (\alpha(a,0) \to \alpha(0,a)) = \neg\neg a * (\neg\neg a \to a) = \neg\neg a * (\neg a \vee a) = \neg\neg a * \neg a = 0$. Conversely, suppose that for every $a \in A_-$, $a = \neg\neg a$ and take $a, b \in A$ such that $a > b$ in order to check that $(D_\vee)$ holds for $\langle a, b\rangle$. Suppose that $a \in A_+$ and $b \in A_-$. If $a \le \neg b$, then $(a \to b) \to b = \neg a \vee b \to b = \neg a \to b = \neg\neg a \vee b = \neg\neg a$, so we obtain $\neg\neg a * (\neg\neg a \to a) = \neg\neg a * (\neg a \vee a) = \neg\neg a * a = \neg\neg a \wedge a = a$. If $a > \neg b$, then $(a \to b) \to b = \neg a \vee b \to b = b \to b = 1$, so we have $1 * (1 \to a) = a$. Suppose now that $a, b \in A_-$. Then $(a \to b) \to b = (\neg a \vee b) \to b = \neg a \to b = \neg\neg a \vee b = a \vee b = a$ (using that $\neg\neg a = a$); hence $a * (a \to a) = a * 1 = a = a \vee b$, as desired. The remaining case, $a, b \in A_+$, is easy to check. □

□

Next we consider a kind of IMTL-chains, namely those obtained by the so-called *rotation construction* from a left-continuous t-norm.

Definition 3.11 ([18])**.** *Let $*$ be a left-continuous t-norm. The rotation of $*$, noted by $\circ$, is defined by,*

$$x \circ y = \begin{cases} \frac{1}{2} + \frac{1}{2}((2x-1) * (2y-1)) & \text{if } x, y > \frac{1}{2}, \\ 0 & \text{if } x, y \le \frac{1}{2}, \\ \frac{1}{2} - \frac{1}{2}((2y-1) \to_* (1-2x)) & \text{if } y > \frac{1}{2}, x \le \frac{1}{2}, \\ \frac{1}{2} - \frac{1}{2}((2x-1) \to_* (1-2y)) & \text{if } y \le \frac{1}{2}, x > \frac{1}{2}. \end{cases}$$

Jenei proved in [18] that $\circ$ is a left-continuous t-norm with involutive negation if, and only if, either $*$ has no zero divisors or there is a $c \in (0,1]$ such that for every zero divisor x, $x \to_* 0 = c$.

Theorem 3.12. *Let $*$ be a continuous t-norm without zero divisors. Let $\circ$ be its rotation. Then, in the IMTL-algebra defined by $\circ$, $(D_\vee)$ is valid.*

Proof. It is easy to check the equation by cases whether x and y are in A_+ or in A_-. □

□

Proposition 3.13. *Let $*_n$ be a left-continuous t-norm obtained by a negation n and a compatible continuous t-norm $*$. Suppose that $*_n$ satisfies the second Jenei's condition, i.e. there is $c \in (0,1]$ such that for all zero divisor x, $n(x) = c$. Then $*_n$ does not satisfy $(D_\vee)$.*

Proof. Jenei's condition implies that n is defined by

$$n(x) = \begin{cases} 1 & \text{if } x = 0, \\ c & \text{if } x \in (0, c], \\ 0 & \text{if } x > c. \end{cases}$$

Thus, $n(n(\frac{c}{2})) = c \neq \frac{c}{2}$, so by Theorem 3.7 the result holds. □

□

Roughly speaking, from a continuous t-norm and by the rotation method we can obtain a left-continuous t-norm satisfying $(D_\vee)$ only if it has no zero divisors. Notice that continuous t-norms with zero divisors never satisfy Jenei's second condition. Moreover, any left-continuous t-norm $*_n$ obtained from a continuous t-norm and satisfying Jenei's second condition does not satisfy $(D_\vee)$ and therefore it also fails for its rotation. Nevertheless, the rotation method does not preserve the definability of the maximum as the following example proves.

Figure 3: *Example showing that rotation doesn't preserve $(D_\vee)$.*

Example 3.14. *Let $*$ be the t-norm defined in the square $[\frac{1}{2}, 1]^2$ in Figure 3. This t-norm is the ordinal sum of Min and a weak nilpotent minimum. Since the components of this sum satisfy $(D_\vee)$, by the next proposition $*$ also does. However its rotation does not satisfy $(D_\vee)$. Take $x = \frac{19}{24}$ and $y = \frac{3}{24}$ and check that it fails. This also shows that $\mathbf{IMTL}[D_\vee] \subsetneq \mathbf{IMTL}$.*

Finally, we prove that $(D_\vee)$ is preserved under ordinal sums of MTL-chains (for the definition of ordinal sum of chains see [4]).

Proposition 3.15. *Let I be a totally ordered set, let $\{\mathcal{A}_i : i \in I\}$ be a family of MTL-chains and let $\mathcal{A}$ be their ordinal sum. If $\mathcal{A}_i$ satisfies $(D_\vee)$ for every $i \in I$, then $\mathcal{A}$ also satisfies it.*

Proof. We only need to check this for $a \in A_i$ and $b \in A_j$ with $i \neq j$ and $a > b$. By the definition of ordinal sum, we have $a \to b = b$, so $(a \to b) \to b = 1$. Moreover, $(b \to a) \to a = a$; hence: $1 * (1 \to a) = 1 * a = a = a \vee b$. □

□

4 Definability of additive conjunction

It is well known that for every residuated lattice $\mathcal{A}$ the following are equivalent:

i) $\mathcal{A} \models x \approx x * x$ (Idempotency)

ii) $\mathcal{A} \models x \wedge y \approx x * y$ $(\wedge = *)$

iii) $\mathcal{A} \models x \to (x \to y) \approx x \to y$ (Contraction)

It is also known that the subvariety of **RL** satisfying the above equivalent conditions is termwise equivalent to the variety of Heyting algebras ($\mathbf{RL}[\text{Con}] = \mathbf{HA}$). In accordance with [20] we will say that a residuated lattice $\mathcal{A}$ has the *weak contraction property* (WCon) iff $\mathcal{A}$ satisfies the condition *iii)* in the case $y = 0$, that is, $x \to \neg x \approx \neg x$. Note that, by the property of residuation, this equation is obviously equivalent to the equation $\neg(x * x) \approx \neg x$. The weak contraction property in the setting of **RL** is also equivalent to the condition of *pseudocomplementation*, $\neg x \wedge x \approx 0$ (see [8] for a formal proof of this equivalence). It is obvious that contraction implies weak contraction, but the converse is not true in general. In fact, the equation $(D_\wedge)$ turns out to be the difference between weak contraction and contraction, as the following proposition shows:

Proposition 4.1. $\mathbf{RL}[D_\wedge, WCon] = \mathbf{RL}[Con] = \mathbf{HA}$.

Proof. Let $\mathcal{A}$ be a residuated lattice. We must prove that $\mathcal{A} \vDash (D_\wedge)$ and $\mathcal{A} \vDash \neg(x * x) \approx \neg x$ if, and only if, $\mathcal{A} \vDash x * x \approx x$. First suppose that $\mathcal{A}$ has the weak contraction property and satisfies the definability of the minimum. Then for all $a, b \in A$, $a \wedge b = (a * b) \vee \neg(b \vee \neg b \vee \neg a) = (a * b) \vee (\neg b \wedge \neg\neg b \wedge \neg\neg a) = (a * b) \vee (0 \wedge \neg\neg a) = a * b$. Conversely, suppose that $\mathcal{A}$ has the contraction property. Then $(a * b) \vee \neg(b \vee \neg b \vee \neg a) = (a * b) \vee (\neg b \wedge \neg\neg b \wedge \neg\neg a) = (a * b) \vee (0 \wedge \neg\neg a) = a * b = a \wedge b$. □

□

Corollary 4.2. *The equation $(D_\wedge)$ is valid in the variety of Heyting algebras.*

Recall Figure 1. If we add the equation $(D_\wedge)$ to any of the varieties corresponding to the logics on the left side of the picture not satisfying prelinearity we will obtain the variety of Heyting algebras. Similarly, if we add this equation to any variety contained in **SMTL** we will obtain the variety of G-algebras, in particular: $\mathbf{SMTL}[D_\wedge] = \mathbf{SBL}[D_\wedge] = \mathbf{G}$. Hence, it is clear that $(D_\wedge)$ is not valid in the variety $\mathbf{\Pi}$ of product algebras, so it also fails in any variety containing $\mathbf{\Pi}$. In fact, $\mathbf{\Pi}[D_\wedge] = \mathbf{BA}$, the variety of Boolean algebras. On the other side of the picture we realize that the equation $(D_\wedge)$ is not valid in the varieties of MV-algebras and IMTL-algebras, since in the MV-algebra $Ł_4$ the equation fails for $x = y = \frac{2}{3}$. In fact, we have:

Proposition 4.3. *If $\mathcal{A}$ is an IMTL-chain and there are $a, b \in A$ such that $a * b \neq 0$ and $a * b \neq a \wedge b$, then $\mathcal{A} \nvDash (D_\wedge)$.*

Proof. Suppose that $\mathcal{A}$ satisfies the hypothesis of the proposition for some pair such that $a \leq b$ (the other case is analogous). Assume also that $\mathcal{A} \vDash (D_\wedge)$. Since $\mathcal{A}$ is totally ordered and $a * b \neq a \wedge b$, we have: $a = a \wedge b = \neg(b \vee \neg b \vee \neg a) = \neg b \wedge \neg\neg b \wedge \neg\neg a = \neg b \wedge b \wedge a = \neg b \wedge a$, hence $a \leq \neg b$, but this is a contradiction since it implies $a * b = 0$. □

□

Therefore, we have characterized the variety of IMTL-algebras satisfying $(D_\wedge)$:

Corollary 4.4. *Let $\mathcal{A}$ be an IMTL-algebra. $\mathcal{A} \vDash (D_\wedge)$ if, and only if, $\mathcal{A}$ is an NM-algebra. Therefore,* $\mathbf{IMTL}[D_\wedge] = \mathbf{NM}$.

Proof. The *if* part follows from the fact that the variety of NM-algebras is included in the variety of NMG-algebras. The *only if* part is proved in the previous proposition. □

□

Corollary 4.5. *Let $\mathcal{A}$ be an arbitrary MV-chain such that $|A| \geq 4$. Then $\mathcal{A} \nvDash (D_\wedge)$.*

Proof. Since $|A| \geq 4$, $\exists a \in A$ such that $\neg a < a < 1$. Then $a * a \neq 0$ and $a * a \neq a$. □

□

However, it is easy to see that $Ł_3$ satisfies $(D_\wedge)$. So $Ł_3$ and the two-element Boolean algebra, $\mathcal{B}_2$, are the only non-trivial MV-chains satisfying the equation. This means that $\mathbf{MV}[D_\wedge] = \mathbb{V}(Ł_3)$, the variety generated by $Ł_3$.

In general, for the class of all MTL-algebras we have:

Theorem 4.6. *Let $\mathcal{A}$ be an MTL-algebra. If $\mathcal{A} \vDash (D_\wedge)$, then $\mathcal{A}$ is an WNM-algebra.*

Proof. Without loss of generality, suppose $\mathcal{A}$ totally ordered. Take $a, b \in A$, $a \leq b$. If $a \leq \neg b$, then $a * b = 0$. Suppose $a > \neg b$. We have: $a \wedge b = (a * b) \vee \neg(b \vee \neg b \vee \neg a)$, so $a = (a * b) \vee \neg(b \vee \neg a) = (a * b) \vee \neg b$. Hence, using $a > \neg b$, we obtain $a * b = a$. □

□

The converse is not true:

Proposition 4.7. **WNM** *does not satisfy the equation* $(D_\wedge)$.

Proof. The WNM-chain of Example 2.14 does not satisfy the equation $(D_\wedge)$. Indeed, take $x = y = a$ and check that it fails. □

□

Nevertheless, there is a huge family of WNM-chains enjoying the definability of the minimum, namely those which satisfy $(D_\vee)$:

Theorem 4.8. *Let $\mathcal{A}$ be a WNM-chain. Then, $\mathcal{A} \vDash (D_\vee)$ if, and only if, $\mathcal{A} \vDash (D_\wedge)$.*

Proof. Suppose that $\mathcal{A} \vDash (D_\vee)$ and take $a, b \in A$. First suppose $a \leq b$. If $a \leq \neg b$, then $(a*b)\vee\neg(b\vee\neg b\vee\neg a) = 0\vee\neg(\neg b\vee\neg a) = \neg\neg a$ (using that $b \leq \neg\neg b \leq \neg a$). Since $a \leq \neg b \leq \neg a$, using Theorem 3.10 we obtain $a = \neg\neg a$, so $(D_\wedge)$ holds. If $a > \neg b$, then $a * b = a \wedge b = a$, hence $(a*b)\vee\neg(b\vee\neg b\vee\neg a) = a\vee\neg(b\vee\neg a) = a\vee\neg b = a = a\wedge b$. The case $b < a$ is left to the reader. Conversely, suppose that $\mathcal{A} \nvDash (D_\vee)$. Then, by Theorem 3.10, there is $a \in A_-$ such that $a < \neg\neg a$. If $\mathcal{A} \vDash (D_\wedge)$, then $a = a\wedge\neg\neg a = (a*\neg\neg a)\vee\neg(\neg\neg a\vee\neg\neg\neg a\vee\neg a) = 0\vee\neg(\neg a) = \neg\neg a$, but this contradicts $a < \neg\neg a$. □

□

Corollary 4.9. *Let $\mathcal{A}$ be a MTL-algebra. If $\mathcal{A} \models (D_\wedge)$, then $\mathcal{A} \models (D_\vee)$ and for every $a \in A_-$, $\neg\neg a = a$.*

Proof. If $\mathcal{A} \models (D_\wedge)$, then $\mathcal{A}$ is a WNM-algebra so, by the last theorem it satisfies $(D_\vee)$. Moreover, by Theorem 3.10 this implies that for every $a \in A_-$, $\neg\neg a = a$. □

□

Therefore, we have proved $\mathbf{MTL}[D_\wedge] = \mathbf{WNM}[D_\wedge] = \mathbf{WNM}[D_\vee]$, and we have characterized the chains in this variety as those WNM-chains whose negative elements are involutive. Since the negative elements can be described as $A_- = \{a \wedge \neg a : a \in A\}$, we have: $\mathbf{MTL}[D_\wedge] = \mathbf{MTL}[WNM, \neg\neg(x \wedge \neg x) \approx x \wedge \neg x]$.

Notice that $\mathbf{MTL}[D_\wedge]$ is not closed under ordinal sums of chains. For instance Ł$_3\oplus$ Ł$_3$ does not satisfy the equation (is not even a WNM-chain).

Theorem 4.10. *The variety of BL-algebras satisfying $(D_\wedge)$ is $\mathbf{BL}[WNM]$ and is generated by G-chains, Ł$_3$ and ordinal sums of the form Ł$_3 \oplus \mathcal{G}$ (where $\mathcal{G}$ is a G-chain).*

Proof. First we argue that $\mathbf{BL}[D_\wedge] = \mathbf{BL}[WNM]$. Let $\mathcal{A}$ be a BL-chain. If $\mathcal{A}$ satisfies $(D_\wedge)$ then it is a WNM-chain (by Theorem 4.6). Conversely, if $\mathcal{A}$ is a WNM-chain, then it satisfies $(D_\vee)$ (since it is also a BL-chain), hence by Theorem 4.8 $\mathcal{A} \models (D_\wedge)$. Now let $\mathbf{K}$ be the variety of BL-algebras generated by G-chains, $Ł_3$ and the ordinal sums of the form $Ł_3 \oplus \mathcal{G}$. It is clear that all these generators are WNM-chains, so $\mathbf{K} \subseteq \mathbf{BL}[WNM]$. Now take a chain $\mathcal{A} \in \mathbf{BL}[WNM]$. Then $\mathcal{A}$ is embeddable in a saturated BL-chain (cf.[4]), say $\mathcal{C}$. Moreover, since $\mathcal{A}$ is a WNM-chain, $\mathcal{C}$ is an ordinal sum of WNM-chains. If there is some $Ł_3$ component, then it must be the first one (otherwise $\mathcal{C}$ would not be a WNM-chain). So $\mathcal{C}$ is one of the generators of $\mathbf{K}$, thus $\mathcal{A} \in \mathbf{K}$. This proves $\mathbf{BL}[WNM] \subseteq \mathbf{K}$. □

□

5 On the standard completeness of MTL$[D_\wedge]$

In order to obtain standard completeness for arbitrary theories for a logic L, one usually proves that every countable L-chain is embeddable in a standard L-chain, i. e., a L-chain over $[0,1]$, by means of Jenei and Montagna's method (see [19]). Unfortunately this method does not work when L is MTL$[D_\wedge]$, MTL$[D_\vee]$ or IMTL$[D_\vee]$ as the following examples show.

Example 5.1. *Let $A := \{0, \frac{1}{4}\} \cup ([\frac{1}{2}, 1] \cap \mathbb{Q})$ and define the BL-chain over A obtained as ordinal sum of $Ł_3$ (whose elements are $\{0, \frac{1}{4}, \frac{1}{2}\}$) and the standard rational G-chain (whose elements are $[\frac{1}{2}, 1] \cap \mathbb{Q}$). This algebra is a countable MTL-chain satisfying $(D_\wedge)$ and being a WNM-chain, as a consequence of Theorem 4.8, it also satisfies $(D_\vee)$. Nevertheless, applying Jenei and Montagna's method, we obtain the WNM-chain $\mathcal{B}$ over $[0,1]$ depicted in Fig.4. An easy computation shows that it does not satisfy either $(D_\wedge)$ or $(D_\vee)$. For instance, we have $\frac{1}{8} \in B_-$ and $\neg^{\mathcal{B}}\neg^{\mathcal{B}}\frac{1}{8} = \frac{1}{4}$ which proves that $\mathcal{B}$ does not satisfy $(D_\wedge)$ and by Theorem 4.8 does not satisfy $(D_\vee)$ either.*

Now consider the chain $Ł_5$ over the set $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, i.e. the five-element Łukasiewicz chain, and let $\mathcal{C}$ be its completion on $[0,1]$, depicted in Fig.5. $Ł_5$ is an IMTL-algebra that satisfies $(D_\vee)$ (since it is also a BL-algebra) but it is easy to check that $(D_\vee)$ fails in $\mathcal{C}$ for $x = \frac{5}{8}$ and $y = \frac{3}{8}$.

Figure 4: *Jenei and Montagna's completion of the ordinal sum of $Ł_3$ and the G-chain over $[\frac{1}{2}, 1] \cap \mathbb{Q}$.*

Figure 5: *Completion of $Ł_5$ where $(D_\vee)$ fails.*

The standard completeness of MTL$[D_\vee]$ and IMTL$[D_\vee]$ remain open problems but for the case of MTL$[D_\wedge]$ we obtain the corresponding completeness result in Theorem 5.3 by changing the construction of Jenei and Montagna in order to obtain a standard chain verifying $(D_\wedge)$.

Recall (see [3], Proposition 1) that weak negation functions in $[0,1]$ are left-continuous decreasing functions and are symmetric with respect to the identity mapping. So the points that make $(D_\wedge)$ false are the constant intervals of the negation in the set of negatives or, equivalently, the discontinuities in the set of non-negatives. Roughly speaking, our strategy to obtain an appropriate chain will be to add a real interval in every discontinuity point of the negation in the set of non-negatives, and define a bijective negation on those new intervals. Figure 6 illustrates the result of this method applied to the algebra of Example 5.1.

Figure 6: *Completion preserving the definability of the minimum. The dotted lines correspond to added intervals.*

Lemma 5.2. *Every countable MTL[$D_\wedge$]-chain is embeddable in a MTL[$D_\wedge$]-chain over* $[0,1]$.

Proof. Let $\mathcal{A}$ be a countable MTL[$D_\wedge$]-chain. By Corollary 4.9, for every $a \in A_-$, $\neg\neg a = a$. By Jenei and Montagna's method we obtain a WNM-chain $\mathcal{B}$ over $[0,1]$ and an embedding $\Phi : \mathcal{A} \to \mathcal{B}$. Possibly this standard chain does not satisfy ($D_\wedge$), i.e., there are some discontinuities of the negation in $\{x \in [0,1] \mid x \geq \neg^{\mathcal{B}} x\}$ (B_+ plus the negation fix point, if it exists). Our method consists in defining a new MTL[$D_\wedge$]-chain, $\mathcal{C}$, and an embedding $\Psi : \mathcal{A} \to \mathcal{C}$. Let $\Delta = \{x \in B_+ \text{ or } x \neg^{\mathcal{B}} x \mid x \text{ is a discontinuity point of } \neg^{\mathcal{B}}\}$. We will denote by $\neg^{\mathcal{B}}(x^+)$ and $\neg^{\mathcal{B}}(x^-)$ the upper and the lower limit of the negation function in x, respectively. Since $\neg^{\mathcal{B}}$ is decreasing, left-continuous and symmetric with respect to the diagonal, Δ is countable and for each $x \in \Delta$, $\neg^{\mathcal{B}}(x^+) < \neg^{\mathcal{B}}(x^-) = \neg^{\mathcal{B}}(x)$. Take $C\{\langle x, \alpha\rangle \mid x \in \Delta, \alpha \in [0,1]\} \bigcup \{\langle x, 0\rangle \mid x \notin \Delta\}$ with the lexicographic order.

Of course, C and $[0,1]$ are bijectable. Define $\neg^{\mathcal{C}}$ as the following left-continuous function:

$$\neg^{\mathcal{C}}(\langle x, \alpha\rangle) = \begin{cases} \langle \neg^{\mathcal{B}}(x), 0\rangle & \text{if } \alpha = 0 \text{ and } x \notin \Delta, \\ n_y(\langle x, \alpha\rangle) & \text{if } \alpha \in (0,1) \text{ and } y = x \in \Delta, \\ n_y^{-1}(\langle x, \alpha\rangle) & \text{if } \alpha \in (0,1),\ y \in \Delta \text{ and } x \in (\neg^{\mathcal{B}}(y^+), \neg^{\mathcal{B}}(y)). \end{cases}$$

where, for every $x \in \Delta$, n_x is a strictly decreasing bijection from $(\langle x, 0\rangle, \langle x, 1\rangle)$ into $(\langle \neg^{\mathcal{B}}(x^+), 0\rangle, \langle \neg^{\mathcal{B}}(x), 0\rangle)$.

Of course the WNM-algebra defined by $\neg^{\mathcal{C}}$ over the real unit interval satisfies ($D_\wedge$) since $\neg^{\mathcal{C}}$ is a continuous function on C_+. Taking into account that $A_- \subseteq \neg^{\mathcal{A}}[A]$, it is obvious that the mapping $\Psi : \mathcal{A} \to \mathcal{C}$ defined by $\Psi(x) := \langle \Phi(x), 0\rangle$ is an embedding. □

□

Theorem 5.3. *Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ be an arbitrary (possibly infinite) set of formulas. Then $\Gamma \vdash_{MTL[D_\wedge]} \varphi$ iff $\Gamma \models_{[0,1]} \varphi$, where $\models_{[0,1]}$ is the semantical consequence associated to the class of all left-continuous t-norm algebras satisfying ($D_\wedge$).*

Proof. Suppose $\Gamma \nvdash_{MTL[D_\wedge]} \varphi$. Then there exists a MTL[$D_\wedge$]-chain $\mathcal{A}$ and a homomorphism e from $Fm_{\mathcal{L}}$ into $\mathcal{A}$ such that $e(\gamma) = 1^{\mathcal{A}}$ for every $\gamma \in \Gamma$ and such that $e(\varphi) \neq 1^{\mathcal{A}}$. Since the number of variables is countable, $\Gamma \cup \{\varphi\}$ is also countable, hence we can suppose that $\mathcal{A}$ is a countable MTL[$D_\wedge$]-chain. Then by Theorem 5.2 there is a standard MTL[$D_\wedge$]-chain $\mathcal{B}$ and an embedding $f : \mathcal{A} \to \mathcal{B}$. Therefore, taking the evaluation $e' := f \circ e$ the theorem is proved. □

□

6 The hierarchy of these new logics

We have studied new ways of defining additive connectives in MTL and its extensions by adding the equations ($D_\vee$) and ($D_\wedge$) to the axiomatic of the varieties associated to these logics. The new hierarchy of logics obtained (or equivalently, new hierarchy of subvarieties) is drawn in Figure 7.

Proposition 6.1. *All inclusions shown in Figure 7 are proper.*

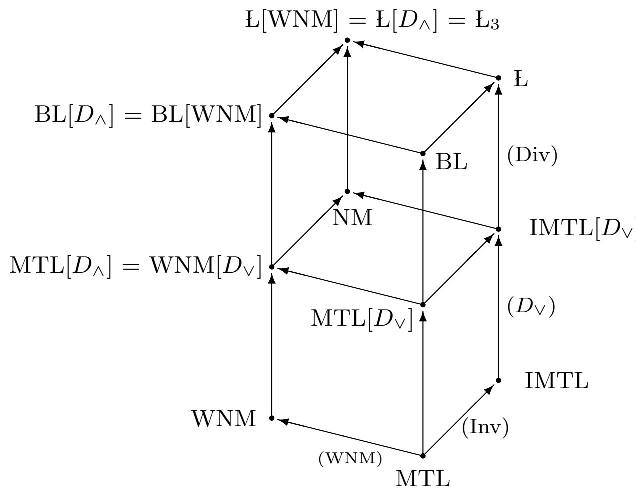

Figure 7: *The hierarchy of logics.*

Proof. The proof is given by cases and using the one-to-one correspondence between logics and their counterpart as algebraic varieties. From the fact that the variety generated by the MV-algebra of three elements is strictly contained in the variety of all the MV-algebras and in the variety **NM** we obtain:

- **BL**[WNM](= **BL**[$D_\wedge$]) $\subsetneq$ **BL**
- **NM** $\subsetneq$ **IMTL**[$D_\vee$]
- **WNM**[$D_\vee$] $\subsetneq$ **MTL**[$D_\vee$]
- **BL**[WNM] $\subsetneq$ **WNM**[$D_\vee$]
- **MV**$\subsetneq$ **IMTL**[$D_\vee$]
- **BL** $\subsetneq$ **MTL**[$D_\vee$]

From the fact that the variety of MV-algebras is strictly contained in **BL** we obtain:

- **IMTL**[$D_\vee$] $\subsetneq$ **MTL**[$D_\vee$]

As we have seen in Proposition 4.7, **WNM**[$D_\vee$] $\subsetneq$ **WNM**. As a consequence, we also have that **MTL**[$D_\vee$] $\subsetneq$ **MTL**.

Next we will show that **IMTL**[$D_\vee$] $\subsetneq$ **IMTL**. Let $\mathcal{A} = \langle\{0, a, b, c, 1\}, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ be the chain formed by $0 < a < b < c < 1$ and where $*$ and $\rightarrow$ are the operations defined by the following tables:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	a	b
c	0	0	a	a	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	c	c	1	1
1	0	a	b	c	1

An easy computation shows that $\mathcal{A}$ is a MTL-chain, obviously involutive. Nevertheless, $\mathcal{A}$ does not satisfy the equation $(D_\vee)$. Indeed, if we take $x = b$ and $y = a$ we have:

$$\alpha(b,a) * (\alpha(b,a) \to \alpha(a,b)) = c * (c \to b) = c * c = a \neq b = b \vee a.$$

Finally, note that $\mathbf{BL}[WNM] \subsetneq \mathbf{Ł}_3$. Indeed, the standard G-chain is not involutive. As a consequence, we also have that $\mathbf{NM} \subsetneq \mathbf{WNM}[D_\vee]$. This finishes the proof. □

□

7 Concluding remarks

$(D_\vee)$ can be seen as a kind of restricted divisibility, hence it gives intermediate logics between MTL and BL and between IMTL and Ł, namely MTL$[D_\vee]$ and IMTL$[D_\vee]$ respectively. In those logics the language can be simplified since we can delete $\wedge$. Indeed, $\vee$ is defined from $*$ and $\to$, and using that MTL proves[2] $\varphi \wedge \psi \leftrightarrow (\varphi * (\varphi \to \psi)) \vee (\psi * (\psi \to \varphi))$, $\wedge$ is also defined by means of $*$ and $\to$. Notice that in IMTL$[D_\vee]$ we can also choose De Morgan's law to define the additive conjunction from the additive disjunction.

But we have also studied $(D_\wedge)$ and proved it to be the difference between contraction and weak contraction. This equation implies $(D_\vee)$ and gives another way to define $\wedge$ and obtain a simpler language. Furthermore, we have seen that MTL$[D_\wedge]$ is a reasonable fuzzy logic since it also enjoys strong standard completeness. The problems that remain open are the standard completeness of MTL$[D_\vee]$ and IMTL$[D_\vee]$, since the usual method of Jenei and Montagna does not work for these logics.

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References

[1] W. J. Blok and D. Pigozzi. *Algebraizable logics.* Mem. Amer. Math. Soc. 396, vol 77, 1989.

[2] S. Burris and H. P. Sankappanavar. *A course in Universal Algebra.* Springer Verlag, New York, 1981.

[3] R. Cignoli, F. Esteva, L. Godo and F. Montagna. On a class of left-continuous t-norms. *Fuzzy Sets and Systems* 131 (2002) 283–296.

[4] R. Cignoli, F. Esteva, L. Godo and A. Torrens. Basic Fuzzy Logic is the logic of continuous t-norms and their residua. *Soft Computing* 4 (2000) 106–112.

[5] F. Esteva and L. Godo. Monoidal t-norm based Logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems* 124 (2001) 271–288.

[6] F. Esteva, A. García-Cerdaña and C. Noguera. On definability of maximum in left-continuous t-norms. *Proc. of EUSFLAT 2003* 609–613.

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[7] F. Esteva, L. Godo and A. García-Cerdaña. On the hierarchy of t-norm based residuated fuzzy logics. In *Beyond Two: Theory and Applications of Multiple-Valued Logic*, Ed. M.Fitting and E.Orlowska, Springer-Verlag, 2003, 251–272.

[8] F. Esteva, J.Gispert, L. Godo and F. Montagna. On the standard and Rational Completeness of some Axiomatic extensions of Monoidal t-norm Based Logic. *Studia Logica* 71 (2002) 199–226.

[9] J. Fodor. Nilpotent minimum and related connectives for fuzzy logic. *Proc. of FUZZ–IEEE'95*, 1995, pp. 2077–2082.

[10] A. García-Cerdaña, C. Noguera and F. Esteva. On definability of additive connectives in fuzzy logics. *Proc. of IPMU 2004*, 485 – 492.

[11] J. Y. Girard. Linear Logic. *Theoretical Computer Science*, 50 (1987) 1–102.

[12] S. Gottwald. *A Treatise on Many-valued Logics*, Studies in Logic and Computation 9, Research Studies Press Ltd., Baldock, UK, 2001.

[13] S. Gottwald, A. García-Cerdaña and F. Bou. Axiomatizing Monoidal Logic–A Correction. *Journal of Multiple-Valued Logic and Soft Computing* 9 (2003) 427–433.

[14] P. Hájek. *Metamathematics of Fuzzy Logic*, Trends in Logic, vol.4 Kluwer, 1998.

[15] U. Höhle. Commutative, residuated l-monoids. In Höhle, U. and Klement. E.P. eds., *Non-Classical Logics and Their Applications to Fuzzy Subsets*, Kluwer Acad. Publ., Dordrecht (1995) 55–106.

[16] S. Jenei. New family of triangular norms via contrapositive symmetrization of residuated implications. *Fuzzy Sets and Systems* 110 (1999) 157–174.

[17] S. Jenei. A note on the ordinal sum theorem and its consequence for the construction of triangular norms. *Fuzzy Sets and Systems* 126 (2002) 199–205.

[18] S. Jenei. A characterization theorem on the rotation construction for triangular norms. *Fuzzy Sets and Systems* 136 (2003) 283–289.

[19] S. Jenei and F. Montagna. A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica* 70 (2002) 183–192.

[20] H. Ono. Logic without contraction rule and residuated lattices. To appear in the book for the Festschrift of Prof. R. K. Meyer, edited by E. Mares.

[21] E. Trillas. Sobre funciones de negación en la teoría de conjuntos difusos. *Stochastica*, Vol. I, 1 (1979), 47–60. Also in: *Advances in Fuzzy Logic* (S. Barro et al. eds.), Public. Univ. S. de Compostela (1998) 31–45.

[22] S. M. Wang, B.S. Wang and G. J. Wang. A triangular-norm-based propositional fuzzy logic. *Fuzzy Sets and Systems* 136 (2003) 55–77.

[23] S. M. Wang, B. S. Wang and D. W. Pei. A fuzzy logic for an ordinal sum t-norm. *Fuzzy Sets and Systems* 149 (2005) 297–307.