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Stability of Memristor Neural Networks with Delays Operating in the Flux-Charge Domain

Mauro Di Marco, Mauro Forti, Luca Pancioni

Department of Information Engineering and Mathematics, University of Siena, 53100 Siena, Italy (e-mail: {dimarco, forti, pancioni}@dii.unisi.it)

Abstract

The paper considers a class of neural networks where flux-controlled dynamic memristors are used in the neurons and finite concentrated delays are accounted for in the interconnections. Goal of the paper is to thoroughly analyze the nonlinear dynamics both in the flux-charge domain and in the current-voltage domain. In particular, a condition that is expressed in the form of a linear matrix inequality, and involves the interconnection matrix, the delayed interconnection matrix, and the memristor nonlinearity, is given ensuring that in the flux-charge domain the networks possess a unique globally exponentially stable equilibrium point. The same condition is shown to ensure exponential convergence of each trajectory toward an equilibrium point in the voltage-current domain. Moreover, when a steady state is reached, all voltages, currents and power in the networks vanish, while the memristors act as nonvolatile memories keeping the result of computation, i.e., the asymptotic values of fluxes. Differences with existing results on stability of other classes of delayed memristor neural networks, and potential advantages over traditional neural networks operating in the typical voltage-current domain, are discussed.

1. Introduction

In 2008, the fundamental discovery at Hewlett-Packard laboratories of nanodevices displaying a memristive behavior [1], has boosted an unprecedented interest in the modeling, analysis and applications to signal processing tasks of memristors [2, 3, 4, 5, 6, 7, 8]. The memristor, a shorthand for memory-resistor, was theoretically predicted by Professor Leon Chua, on the basis of symmetry arguments, as the fourth basic passive circuit element in addition to the resistor, capacitor, and inductor, in a seminal paper published in 1971 [9]. However, up until 2008 it remained basically at the level of a theoretic device, mainly due to the difficulty of implementation. Actually, a memristor acts a resistor, with the key difference that the instantaneous value of the resistance is not fixed, but it depends on the past history of memristor voltage or current. When the current (or voltage) in a memristor turns off, the memristor is able to memorize in a nonvolatile way the last value assumed by its resistance (also called memristance) [10]. As such it can be used as a linear programmable resistor in analog processing circuits [11] or to implement programmable neural network interconnections [12]. It can also be used as a dynamic nonlinear time-dependent resistor for the implementation of oscillatory circuits or neuromorphic architectures for real-time signal processing [13, 14, 15].

One main bottleneck of the classical Von Neumann architecture is that processing and storing of information occur on *physically distinct locations*, such as in the CPU and in the random-access-memory. This limits the rate at which information can be transferred and processed. A possible and promising way to overcome this issue is to use a parallel computational approach, as that offered by a neuromorphic architecture, together with unconventional electric devices as the memristors, that are able to *process and store information on the same physical device* [16]. Along this line of reasoning, recent papers [17, 18, 19] have proposed a neural network architecture where the nonlinear resistors in the neurons are replaced by nonlinear memristors. Such memristors play a double role, i.e., they participate in the nonlinear dynamics that is used for processing signals and, at the same time, they are able to store in a nonvolatile way, in the final values of memristances, the result of processing. An intriguing feature of such neural architectures is that, due to the presence of memristors, the processing takes place in the flux-charge domain, rather than in the typical voltage-current domain. As a consequence, when the neural network reaches a steady state, all voltages, current and power in the network turn off, while the memristors act as nonvolatile memories handling the result of computation, i.e., the asymptotic values of fluxes. This is another potential advantage with respect to traditional neural networks

computing in the voltage-current domain, where power continues to be consumed in steady state and batteries are needed to memorize the result of computation.

In the modeling of neural network architectures it is needed and also desirable to account for the presence of delays in the neuron interconnections. Delays can be due for instance to the non-instantaneous transmission of signal stimuli between different neurons. It is also worth mentioning that delays can be deliberately introduced in order that a neural network is able to perform special signal processing tasks as motion detection [20]. It is known that delays can be the source of unwanted oscillations in an otherwise stable un-delayed neural network [21, 22], so that a relevant issue is to find conditions ensuring that, even in the presence of delays, a neural network is stable. One of the most investigated topics in neural network theory has been indeed to find conditions ensuring that a neural network with delays has a unique equilibrium point (EP) which is globally exponentially stable (GES). We refer the reader to the review article [23] for quite an exhaustive account of the relevant results and huge body of literature along this line of research. We stress that a GES neural network has relevant potential applications for solving global optimization problems in real time. Indeed, the GES property ensures that the network is able to compute the global optimal solution independently of the choice of initial conditions, i.e., GES rules out the risk that the network gets stuck at some local minimum of the cost function to be minimized, see, e.g., [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37], and references therein. Other closely related applications of GES neural networks are in field of synchronization [15, 35, 38, 39, 40].

Previous papers on neural networks operating in the flux-charge domain [17, 18, 19] have considered the ideal case where delays in the interconnections are negligible. Goal of this paper is to fill this gap by thoroughly investigating the dynamics of such neural networks and, in particular, GES of the unique EP, when finite concentrated delays are taken into account in the interconnections. We will refer to this class of neural networks as delayed memristor neural networks (DMNNs). The analysis will be conducted by using a recently introduced technique, named flux-charge analysis method (FCAM), for studying memristor circuits in the flux-charge domain [41, 42].

The structure of the paper is outlined as follows. First of all we extend FCAM in order to handle delay elements (Section 2). The considered DMNN model, and its foundation are discussed in Section 3. Then, the paper gives conditions on the interconnection matrix, delay interconnection matrix, and nonlinear memristor characteristic, ensuring that a DMNN has a unique GES EP, with a known convergence rate of solutions, for the dynamics in the flux-charge domain (Section 4). The convergent dynamics in the voltage current-domain is studied, starting from the dynamic results in the flux-charge domain, in Section 5, whereas simulations results verifying the theoretic findings are presented in Section 6. Finally, some pertinent remarks and conclusions are collected in Section 7.

Notation. If $x = (x_i) \in \mathbb{R}^N$, we denote by $\|x\|$ the Euclidean norm of x, while $\|x\|_\infty = \max_{i=1,2,...,N} |x_i|$ is the infinity norm of x. Given a square matrix $A = (a_{ij}) \in \mathbb{R}^{N \times N}$, $\|A\| = \sqrt{\rho(A^\top A)}$, where $\rho(\cdot)$ is the spectral radius and $\top$ is the transpose, is the induced Euclidean norm, and $\|A\|_\infty = \max_{i=1,2,...,N} \sum_{j=1}^N |a_{ij}|$ means the infinity norm of A. If A is symmetric, by $A > 0$ (resp., $A < 0$) we mean that A is positive (resp., negative) definite, whereas $A \geq 0$ (resp., $A \leq 0$) means that A is positive (resp., negative) semidefinite. Suppose that function $\gamma(\cdot) \in C([t_0 - \tau, t_0], \mathbb{R})$, the space of real-valued continuous functions defined on $[t_0 - \tau, t_0]$, where $t_0 \in \mathbb{R}$ and $\tau > 0$. Then, we let $\|\gamma\| = \max_{\sigma \in [t_0-\tau,t_0]} \|\gamma(\sigma)\|$.

2. Extension of FCAM to Delay Elements

The synthesis of the neurons in a DMNN, and the analysis in the flux-charge domain, are conducted via a new method, named FCAM, introduced in [41]. In this section, we briefly recall some facts about FCAM needed in the paper and, most importantly, we extend FCAM in order to handle two-ports delay elements as those considered in the delayed neural networks here studied.

FCAM can be applied to analyze directly in the flux-charge domain a large class of nonlinear dynamic circuits containing linear resistors, inductors, capacitors, current or voltage sources and nonlinear *ideal flux- or charge-controlled memristors*. In addition to the standard electrical variables of a two-terminal element, i.e., the voltage $v(t)$, current $i(t)$, flux $\varphi(t) = \int_{-\infty}^t v(z)dz$ and charge $q(t) = \int_{-\infty}^t i(z)dz$, let us also define the *incremental flux*

$$\varphi(t; t_0) = \int_{t_0}^t v(z)dz = \varphi(t) - \varphi(t_0)$$
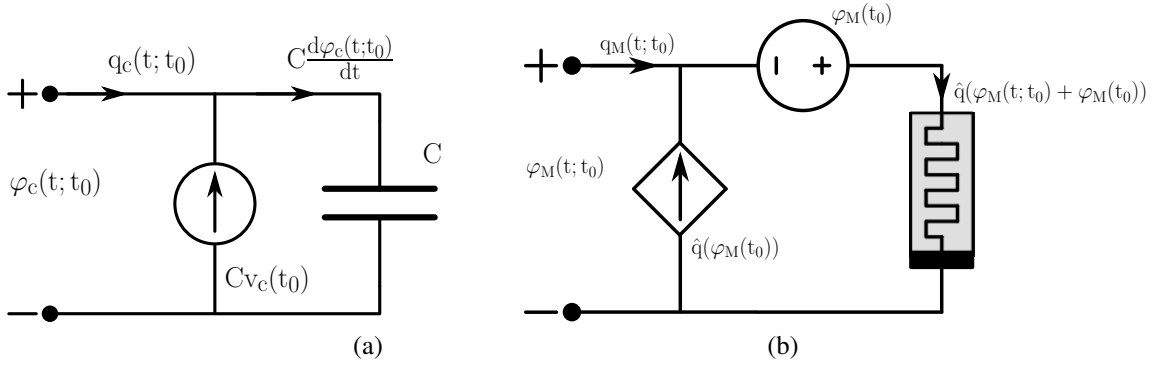
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Figure 1: Equivalent circuit in the flux-charge domain of an ideal capacitor and an ideal flux-controlled memristor.

and the *incremental charge*

$$q(t; t_0) = \int_{t_0}^{t} i(z)dz = q(t) - q(t_0)$$

for $t \geq t_0$, where t_0 is an assigned finite initial time instant.

Each two-terminal element can be represented by an equivalent circuit in the flux-charge domain and has a constitutive relation (CR) involving the incremental flux and charge at its terminals. The equivalent circuit of a linear capacitor is represented in Fig. 1(a), and the corresponding CR, which can be directly found also from the equivalent circuit, is

$$q_C(t; t_0) = -Cv_C(t_0) + C\frac{d}{dt}\varphi_C(t; t_0)$$

where $v_C(t_0)$ is the initial capacitor voltage. An ideal resistor has the CR $\varphi_R(t; t_0) = Rq_R(t; t_0)$.

Consider then an ideal flux-controlled memristor. According to the treatment in the seminal paper by Leon Chua [9], a flux-controlled memristor is defined by a nonlinear relation

$$q_M(t) = \hat{q}(\varphi_M(t)) \tag{1}$$

between flux and charge. The flux $\varphi_M(t)$ is the *state variable* of the memristor. It is shown in [41] that its CR in terms of incremental flux and charge is

$$q_M(t; t_0) = -\hat{q}(\varphi_M(t_0)) + \hat{q}(\varphi_M(t; t_0) + \varphi_M(t_0)) \tag{2}$$

whereas its equivalent circuit in the flux-charge domain is in Fig. 1(b).

Remark 1. *We remark that an ideal memristor is a* memoryless *element characterized by the static nonlinear relation (1), or (2), in the flux-charge domain. Instead, it is a* dynamic *element in the voltage-current domain, in fact, in that domain it satisfies the pair of equations $i_M(t) = \hat{q}'(\varphi_M(t))v_M(t)$, where $v_M(t) = d\varphi_M(t)/dt$, and the prime denotes the derivative with respect to the argument of $\hat{q}(\cdot)$. The quantity $q'(\varphi_M(t))$, with dimension of Ohm^{-1}, is called the memductance at $\varphi_M(t)$.*

Remark 2. *It is important to stress that the true model and CR of an ideal memristor are given by (1), or (2). An ideal memristor is known to display a number of peculiar fingerprints, as the fact that it gives rise to a pinched hysteresis loop in the voltage-current domain when a sinusoidal voltage or current is applied. However, as pointed out by Leon Chua in [43, p. 770], "the pinched loop itself is useless as a model since it cannot be used to predict the voltage response to arbitrarily applied current signals, and vice versa. The only way to predict the response of the device is to derive either the $\varphi - q$ constitutive relation, or the memristance vs. state map."*

Along similar lines we can obtain the equivalent circuits and CRs in the flux-charge domain of memoryless two-port networks as an operational amplifier (oa) or a voltage-controlled current-source.
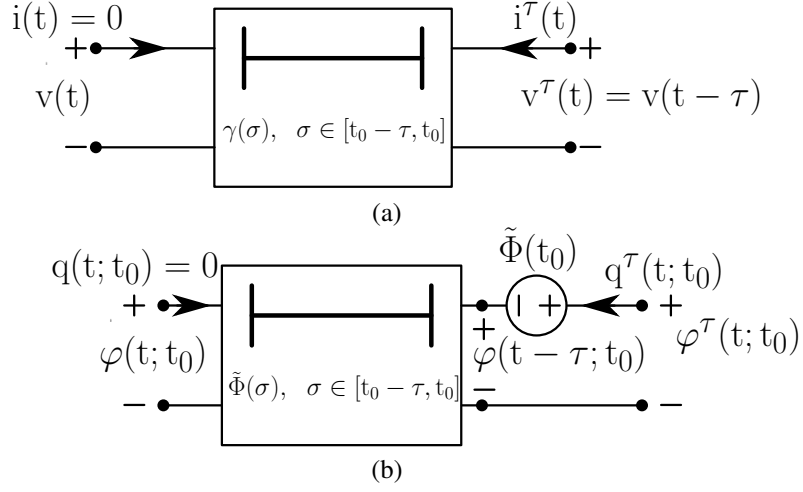
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Figure 2: Equivalent circuit of in the voltage-current domain (a) and in the flux-charge domain (b) of a delay element.

Let us now consider a delay element represented by a voltage-controlled voltage-source as in Fig. 2(a), where $v(t)$ is the input voltage, $i(t) = 0$ is the input current, $v^\tau(t) = v(t-\tau)$, $t \geq t_0$, is the output voltage and $0 < \tau < +\infty$ is a finite concentrated delay. We need to specify the initial conditions, which are given by

$$v(\sigma) = \gamma(\sigma), \quad \sigma \in [t_0 - \tau, t_0]$$

where $\gamma(\cdot) \in C([t_0 - \tau, t_0], \mathbb{R})$. Such a delay element has not been considered in [41], so in what follows we explicitly find its CRs in the flux-charge domain, i.e., the relationships between incremental charges and fluxes at its ports. For the input port we have

$$q(t; t_0) = 0 \tag{3}$$

for any $t \geq t_0$. For the output port

$$\varphi^\tau(t; t_0) = \int_{t_0}^t v^\tau(z) dz = \int_{t_0}^t v(z - \tau) dz = \int_{t_0 - \tau}^{t - \tau} v(z) dz = \int_{t_0}^{t - \tau} v(z) dz + \int_{t_0 - \tau}^{t_0} v(z) dz.$$

Let us consider the initial condition for the incremental flux

$$\varphi(\sigma; t_0) = \tilde{\Phi}(\sigma) = \int_{t_0 - \tau}^\sigma \gamma(z) dz, \quad \sigma \in [t_0 - \tau, t_0] \tag{4}$$

where $\tilde{\Phi} \in C([t_0 - \tau, t_0], \mathbb{R})$. It can be checked that

$$\varphi^\tau(t; t_0) = \varphi(t - \tau; t_0) + \tilde{\Phi}(t_0) = \varphi(t - \tau; t_0) + \int_{t_0 - \tau}^{t_0} \gamma(z) dz, \quad t \geq t_0. \tag{5}$$

We conclude that the CRs of the delay element in the flux-charge domain are given by (3) and (5), whereas the initial conditions are as in (4). The equivalent circuit representation in the flux-charge domain is given in Fig. 2(b).

Note that in the equivalent circuits in the flux-charge domain of the capacitor, memristor and delay element there are generators related to the initial conditions in the voltage-current domain.

On the basis of FCAM [41], electrical elements can be connected in the flux-charge domain via their terminals with incremental flux and charge as electrical variables. As a consequence, we can use Kirchhoff flux law (KφL) and Kirchhoff charge law (KqL) expressed in the usual way, i.e., the sum of incremental fluxes around any loop, and the sum of incremental charges in any cutset, are null. It is worth to stress that KφL and KqL hold when using incremental fluxes and charges, respectively, while they fail if we use the fluxes $\varphi(t)$ or the charges $q(t)$, see [41] for a detailed discussion.
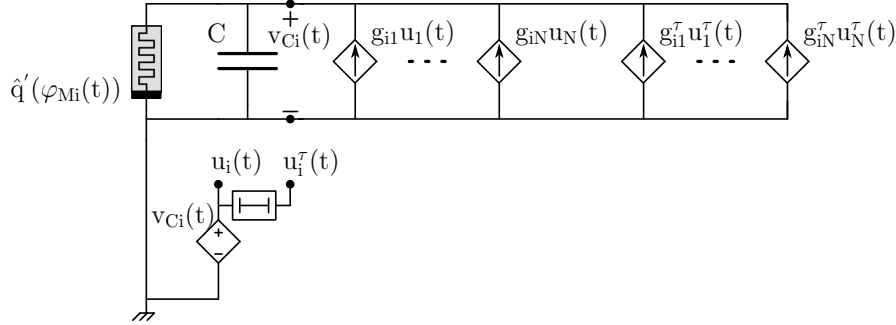
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Figure 3: Equivalent circuit of the *i*-th neuron in the voltage-current domain.

By using the CRs of circuit elements and KφLs, KqLs, it is possible to write a system of differential algebraic equations (DAEs) describing the memristor circuit in the flux-charge domain and, on this basis, to obtain also a state equation (SE) representation. The corresponding SEs in the voltage-current domain are obtainable via time differentiation of those in the flux-charge domain.

3. Delayed Memristor Neural Network Model

In this section we describe, first in the standard voltage-current domain, and then in the flux-charge domain via FCAM, the memristor neural network model with delays studied in the paper. Moreover, we give a mathematical foundation to the model under suitable assumptions.

We consider a 1D neural network array given by the interconnection of N neurons. Let t_0 be an assigned finite initial instant and $0 < \tau < +\infty$ be a finite concentrated delay. The equivalent circuit of the i-th neuron for $t \geq t_0$, which is represented in Fig. 3, contains a linear capacitor C and an ideal flux-controlled memristor M satisfying $q_{M_i}(t) = \hat{q}(\varphi_{M_i}(t))$, where $\hat{q} : \mathbb{R} \to \mathbb{R}$ is continuously differentiable in $\mathbb{R}$. The terminal $u_i(t) = v_{C_i}(t)$ is a buffered terminal for connections with other neurons. The main novelty with respect to previously considered models [17, 18, 19] is the presence of the terminal $u_i^\tau(t) = v_{C_i}(t - \tau)$, which provides a buffered *delayed voltage* for connections with other neurons. This can be obtained via a voltage-controlled voltage-source with a delay as described in Section 2. For simplicity, all delays $\tau > 0$ are assumed to be equal. The voltage-controlled current-sources $g_{ij}u_j(t)$ and $g_{ij}^\tau u_j^\tau(t)$, $j = 1, 2, \ldots, N$, represent the currents injected into the i-th neuron due to connections with other neurons. We have denoted by g_{ij} and g_{ij}^τ the corresponding interconnection conductances. For future reference we define the interconnection matrix $G = (g_{ij}) \in \mathbb{R}^{N \times N}$ and the delayed interconnection matrix $G^\tau = (g_{ij}^\tau) \in \mathbb{R}^{N \times N}$. We will refer henceforth to this network as a delayed memristor neural network (DMNN).

In order to study the dynamics of a DMNN for $t \geq t_0$, we need to provide for each neuron the initial conditions for the dynamic elements, i.e., the capacitor voltage $v_{C_i}(t_0)$, the memristor flux $\varphi_{M_i}(t_0)$, and the voltage $\gamma_i(\sigma)$, $\sigma \in [t_0 - \tau, t_0]$, for the delay element, where $\gamma_i \in C([t_0 - \tau, t_0], \mathbb{R})$. Note that we have $v_{C_i}(t_0) = \gamma_i(t_0)$.

The associated circuit of the i-th neuron in the flux-charge domain, obtained via FCAM, is represented in Fig. 4. In addition to the delay element, the capacitor C is the only dynamic element in the flux-charge domain. Indeed, as discussed in Section 2, the memristor is a memoryless element in the flux-charge domain with a nonlinear characteristic as given in (2).

Let us analyze the circuit in Fig. 4 by FCAM. The KqL yields

$$q_{M_i}(t; t_0) + q_{C_i}(t; t_0) = \sum_{j=1}^N g_{ij} \varphi_{u_j}(t; t_0) + \sum_{j=1}^N g_{ij}^\tau \varphi_{u_j}^\tau(t; t_0)$$

whereas by the KφL we have

$$\varphi_{M_i}(t; t_0) = \varphi_{C_i}(t; t_0) = \varphi_{u_i}(t; t_0)$$
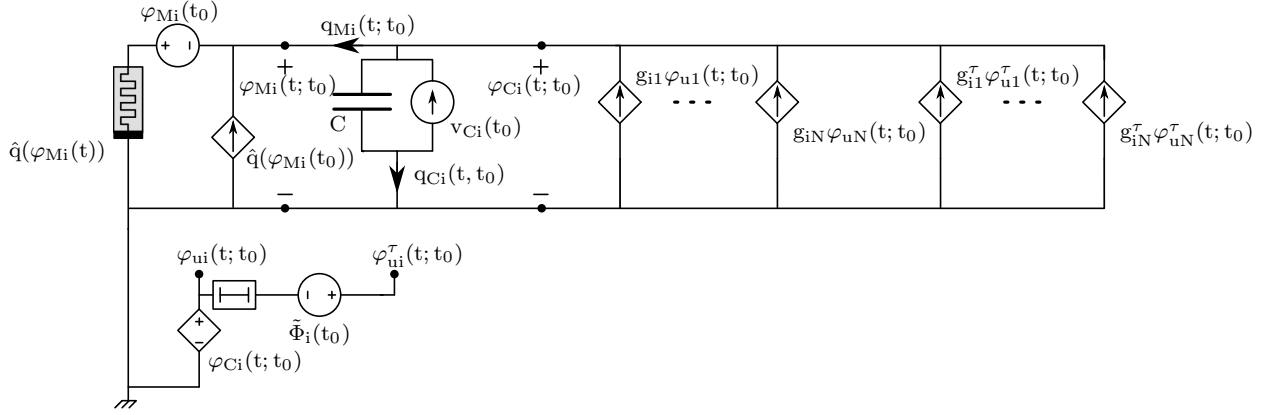
for any $t \geq t_0$.

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Figure 4: Associated circuit of the i-th neuron in the flux-charge domain as obtained via FCAM.

Recall that the CR of C is given by $q_{C_i}(t; t_0) = -Cv_{C_i}(t_0) + Cd\varphi_{C_i}(t; t_0)/dt$, and that of the memristor by $q_{M_i}(t; t_0) = -\hat{q}(\varphi_{M_i}(t_0)) + \hat{q}(\varphi_{M_i}(t; t_0) + \varphi_{M_i}(t_0))$. The CR of the delay element is

$$\varphi_{u_i}^{\tau}(t; t_0) = \varphi_{u_i}(t - \tau; t_0) + \int_{t_0-\tau}^{t_0} \gamma(z)dz, \quad t \geq t_0$$

with initial condition

$$\varphi_{u_i}(\sigma; t_0) = \tilde{\Phi}(\sigma) = \int_{t_0-\tau}^{\sigma} \gamma(z)dz, \quad \sigma \in [t_0 - \tau, t_0].$$

By substitution we obtain the system of N delay differential equations ($i = 1, 2, \ldots, N$)

$$C\frac{d}{dt}\varphi_{M_i}(t; t_0) = -\hat{q}(\varphi_{M_i}(t; t_0) + \varphi_{M_i}(t_0)) + \sum_{j=1}^{N} g_{ij}\varphi_{M_j}(t; t_0) + \sum_{j=1}^{N} g_{ij}^{\tau}\varphi_{M_j}(t - \tau; t_0)$$

$$+ Cv_{C_i}(t_0) + \hat{q}(\varphi_{M_i}(t_0)) + \sum_{j=1}^{N} g_{ij}^{\tau}\int_{t_0-\tau}^{t_0} \gamma_i(z)dz.$$

To simplify the notation, denote by $\phi_i(t) = \varphi_{M_i}(t)$, $i = 1, 2, \ldots, N$, the memristor fluxes and by $\phi(t) = (\varphi_{M_1}(t), \ldots, \varphi_{M_N}(t))^{\top} \in \mathbb{R}^N$ the vector of memristor fluxes. In matrix-vector notations, assuming $t_0 = 0$, we obtain that the DMNN satisfies in the flux-charge domain the following system of N nonlinear delay differential equations

$$C\frac{d}{dt}\phi(t) = -\hat{Q}(\phi(t)) + G\phi(t) + G^{\tau}\phi(t - \tau) + Q_0 \tag{6}$$

where $\hat{Q}(\cdot) = (\hat{q}(\cdot), \ldots, \hat{q}(\cdot))^{\top} : \mathbb{R}^N \to \mathbb{R}^N$ is a nonlinear diagonal mapping containing the memristor characteristic and where

$$Q_0 = Cv_0 + \hat{Q}(\phi_0) - (G + G^{\tau})\phi_0 + G^{\tau}\int_{-\tau}^{0} \gamma(z)dz$$

is a term depending in the initial conditions in the voltage-current domain. Here, we have let $v_0 = (v_{C_1}(0), \ldots, v_{C_N}(0))^{\top} \in \mathbb{R}^N$, $\phi_0 = (\varphi_{M_1}(0), \ldots, \varphi_{M_N}(0))^{\top} \in \mathbb{R}^N$ and $\gamma(\cdot) = (\gamma_1(\cdot), \ldots, \gamma_N(\cdot))^{\top} \in C([-\tau, 0], \mathbb{R}^N)$. The initial conditions are

$$\phi(\sigma) = \phi_0 + \int_{0}^{\sigma} \gamma(z)dz, \quad \sigma \in [-\tau, 0]. \tag{7}$$

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Remark 3. *In the DMNN (6) we use an ideal memristor model as that introduced by Leon Chua in the seminal paper [9]. We remark that, in the literature, other classes of delayed memristor NNs have been considered that use a different memristor model (cf. Remark 2) where the pinched hysteresis loop displayed by a memristor in response to a sinusoidal voltage or current is approximated by means of a device switching between two different memristance values, see, e.g., [38, 39, 44, 45, 46, 47, 48, 49, 40, 50, 51, 52], and references therein. We also note that, in model (6), each memristor is used as the only nonlinear element of a neuron, whereas in the quoted papers memristors are used as cell interconnections (see Remark 8 in Section 6 for further comparisons).*

Remark 4. *From an abstract mathematical viewpoint, model (6) is different also from a delayed standard cellular neural network (CNN) [53] and a Hopfield neural network (HNN). In fact, in (6) the interconnection terms $G\phi(t) + G^\tau\phi(t-\tau)$ are linear in $\phi(t)$ and $\phi(t-\tau)$, whereas the only nonlinearity is the diagonal map $\hat{Q}(\phi(t))$ with the memristor flux-charge characteristics. In a standard CNN or a Hopfield neural network there are instead nonlinear interconnection terms of the type $Gf(x(t)) + G^\tau f(x(t-\tau))$, where $x(\cdot)$ are the state variables and $f(\cdot)$ is a nonlinear sigmoidal map, moreover, the neuron self-inhibitions are linear. We mention that the interconnection structure of a DMNN is instead more similar to that of a full-range CNN [54, 55] or a CNN with resonant tunnelling diodes [56].*

3.1. Foundation of DMNN Model

Since, as discussed in Remarks 4, 3, model (6) differs from SCNNs, HNNs, and other previously considered delayed memristor neural network models, we first need to explicitly give a foundation to the model by studying existence, uniqueness and boundedness of solutions.

Let $T > 0$. We say that $\Phi(\cdot) : [-\tau, T] \to \mathbb{R}^N$ is a (local) solution of the initial value problem (IVP) given by (6) and (7) if $\Phi(\sigma) = \phi_0 + \int_0^\sigma \gamma(z)dz$, $\sigma \in [-\tau, 0]$, $\Phi(\cdot)$ is continuous on $[-\tau, T]$ and differentiable on $[0, T)$ and (6) is satisfied for $0 \le t < T$. We say that $\Phi(\cdot)$ is a (global) solution for $t \ge -\tau$ if it is a solution on $[-\tau, T]$ for any $T > 0$.

Consider the following assumption.

Assumption 1. *The memristor nonlinearity $\hat{q} \in C^1(\mathbb{R})$ satisfies*

$$\lim_{|\rho| \to \infty} \hat{q}'(\rho) = +\infty \tag{8}$$

where the prime denotes the derivative of $\hat{q}$ with respect to its argument, or otherwise it satisfies

$$\lim_{|\rho| \to \infty} \hat{q}'(\rho) = k_q > 0 \tag{9}$$

and we also have

$$\|G\|_\infty + \|G^\tau\|_\infty \doteq \max_{i \in \{1,2,...,N\}} \sum_{j=1}^N |g_{ij}| + \max_{i \in \{1,2,...,N\}} \sum_{j=1}^N |g_{ij}^\tau| < k_q. \tag{10}$$

Proposition 1. *If Assumption 1 is satisfied, then the following results hold.*

1. *For any $\gamma \in C([-\tau, 0], \mathbb{R}^N)$, there exists a unique solution $\Phi(t; 0, \gamma)$ of the IVP given by (6) and (7), which is bounded and hence defined for $t \ge -\tau$.*
2. *The solution $\Phi(t; 0, \gamma)$ is C^1 in $[-\tau, 0]$, and it is C^2 for $t \ge 0$.*

Proof. See Appendix A. ∎

Remark 5. *Let us briefly discuss the meaning of Assumption 1. First, we note that any charge-flux memristor characteristic of practical interest satisfies either (8) or (9). This is true for example of the frequently considered cubic characteristic $\hat{q}(\rho) = a\rho + b\rho^3$, where $b > 0$, which satisfies (8), and smooth approximations of the piecewise linear characteristic $\hat{q}(\rho) = a\rho + 0.5b(|\rho + 1| - |\rho - 1|)$, where $b > 0$, which instead satisfy (9) with $k_q = b$ [57, 10, 43].*

Under Assumption 1 there exists α such that

$$\hat{q}'(\rho) \ge \alpha > -\infty, \quad \rho \in \mathbb{R}. \tag{11}$$

If $\alpha \geq 0$, then $\hat{q}(\cdot)$ is a monotone nondecreasing function that models a passive memristor [9, Th. 1]. We stress that the treatment in the paper includes the case $\alpha < 0$, i.e., we allow for memristor characteristics that are not monotonically increasing (locally-active memristors), which are of special interest in several practical applications [57].

Consider now assumption (10). It is easy to see that, under (9), there may be unbounded solutions of (6) if the restriction (10) on the norm of G and G^τ fails. It is worth mentioning that, due to the practical values of k_q of memristors [10], (10) is not a restrictive assumptions on the interconnections, see also the simulation results in Section 6.

4. Global Exponential Stability of a DMNN

In this section we investigate, in the flux-charge domain, the fundamental dynamic property of GES of the unique EP of (6). By an EP we mean a constant solution $\phi(t) = \phi_e \in \mathbb{R}^N$, for $t \geq -\tau$, of (6). Note that ϕ_e is an EP of (6) if and only if ϕ_e satisfies the algebraic equation

$$0 = -\hat{Q}(\phi_e) + (G + G^\tau)\phi_e + Q_0. \tag{12}$$

Definition 1. *The EP $\phi_e \in \mathbb{R}^N$ of (6) is said to GES if there exist $h, k > 0$ such that, for any $\gamma \in C([-\tau, 0], \mathbb{R}^N)$, we have*

$$\|\Phi(t; 0, \gamma) - \phi_e\| \leq h \max_{-\tau \leq \theta \leq 0} \|\gamma(\theta) - \phi_e\| \exp(-kt), \quad t \geq 0.$$

Constant k is said to be the exponential convergence rate of solutions toward the EP.

In order to ensure GES of the EP we need to enforce suitable assumptions on G, G^τ, and the nonlinearities involved in model (6). In particular, we find it useful to state the assumptions in terms of an LMI as follows.

Assumption 2. *There exist a diagonal and positive definite matrix $P \in \mathbb{R}^{N \times N}$, and a symmetric and positive definite matrix $Q \in \mathbb{R}^{N \times N}$ such that*

$$\begin{bmatrix} -2\alpha\dfrac{P}{C} + 2\dfrac{[PG]_S}{C} + Q & \dfrac{PG^\tau}{C} \\ \left(\dfrac{PG^\tau}{C}\right)^\top & -Q \end{bmatrix} < 0. \tag{13}$$

We have denoted by $[\cdot]_S$ the symmetric part of an $N \times N$ matrix, i.e., $[PG]_S = \frac{1}{2}[PG + (PG)^\top]$. By applying Schur's complement, Assumption 2 is equivalent to requiring that matrix $\Sigma_0 \in \mathbb{R}^{N \times N}$ is such that

$$\Sigma_0 = -2\alpha\frac{P}{C} + Q + \frac{2}{C}[PG]_S + \frac{1}{C}PG^\tau Q^{-1}\left(\frac{1}{C}PG^\tau\right)^\top < 0. \tag{14}$$

We need to establish a preliminary result. Denote by $\mathcal{P}$ the class of $N \times N$ matrices such that all leading principal minors are positive [58]. Also denote by E_N the $N \times N$ identity matrix.

Lemma 1. *If Assumption 2 is satisfied, then matrix $\alpha E_N - (G + G^\tau) \in \mathcal{P}$.*

Proof. First of all we note that, given $Q = Q^\top > 0$, for any square matrix H we have

$$Q + HQ^{-1}H^\top - 2[H]_s \geq 0. \tag{15}$$

Indeed, since $(HQ^{-1/2} - Q^{1/2})(HQ^{-1/2} - Q^{1/2})^\top \geq 0$, we have $HQ^{-1}H^\top + Q - H - H^\top \geq 0$, which implies (15).
Since Assumption 2 holds true, from (14) we have

$$2\alpha\frac{P}{C} - \frac{2}{C}[PG]_S - \frac{2}{C}[PG^\tau]_S - \left(Q + \frac{1}{C}PG^\tau Q^{-1}\left(\frac{1}{C}PG^\tau\right)^\top - \frac{2}{C}[PG^\tau]_S\right) > 0.$$

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By letting $H = (PG^\tau)/C$ in (15) we have

$$Q + \frac{1}{C}PG^\tau Q^{-1}\left(\frac{1}{C}PG^\tau\right)^\top - \frac{2}{C}[PG^\tau]_S > 0$$

which implies

$$2\alpha\frac{P}{C} - \frac{2}{C}[PG]_S - \frac{2}{C}[PG^\tau]_S > 0$$

i.e., $[P(\alpha E_N - G - G^\tau)]_S > 0$. As a consequence, $\alpha E_N - (G + G^\tau) \in \mathcal{P}$ [59]. ∎

Theorem 1. *Suppose Assumptions 1 and 2 are satisfied. Then, for any $Q_0 \in \mathbb{R}^N$, (6) has a unique EP ϕ_e which is GES. Moreover, the convergence rate k of solutions toward the EP is given as*

$$k = \min\left\{\frac{1}{4p_M}|\lambda_M(\Sigma_0)|; \frac{1}{2\tau}\ln\left[1 + \frac{|\lambda_M(\Sigma_0)|}{2\lambda_M\left(\frac{1}{C}PG^\tau Q^{-1}\left(\frac{1}{C}PG^\tau\right)^\top\right)}\right]\right\} \tag{16}$$

where $p_M = \max_{i=1,\dots,N}(p_{ii})$, $p_m = \min_{i=1,\dots,N}(p_{ii})$ and $\lambda_M(\cdot)$ denotes the maximum eigenvalue of a symmetric matrix.

Proof. We first address existence and uniqueness of the EP, by showing that the algebraic equation (12) has a unique solution. Since Assumption 1 holds true, it is easy to verify that, for any $\varepsilon > 0$, $\bar{Q}(\phi_e) = \hat{Q}(\phi_e) - \alpha\phi_e + \varepsilon\phi_e$ is a diagonal mapping made of strictly increasing functions mapping $\mathbb{R}$ onto $\mathbb{R}$. We can rewrite (12) as

$$\bar{Q}(\phi_e) + (\alpha E_N - (G + G^\tau) - \varepsilon E_N)\phi_e = Q_0. \tag{17}$$

Lemma 1 yields $\alpha E_N - (G + G^\tau) \in \mathcal{P}$, which implies that there exists $\varepsilon > 0$ such that $\alpha E_N - (G + G^\tau) - \varepsilon E_N \in \mathcal{P}$ [59]. Then, we are under the hypotheses of [17, Th. 6, App. A], and consequently (17) has a unique solution.

If ϕ_e is an EP of (6), then $\psi(t) = \Phi(t; 0, \gamma) - \phi_e$ is ruled by

$$C\frac{d\psi(t)}{dt} = -\tilde{Q}(\psi(t)) + G\psi(t) + G^\tau\psi(t - \tau) \tag{18}$$

where $\tilde{Q}(\psi) = (\tilde{q}(\psi_1), \tilde{q}(\psi_2), \dots, \tilde{q}(\psi_N))^\top = \hat{Q}(\psi + \phi_e) - \hat{Q}(\phi_e)$ is such that $\tilde{q}'(\rho) \geq \alpha, \rho \in \mathbb{R}$.

Let us choose the following candidate Lyapunov function

$$\mathcal{V}(t) = e^{2kt}\psi^\top(t)P\psi(t) + \int_{t-\tau}^{t} e^{2kz}\psi^\top(z)Q\psi(z)dz \tag{19}$$

whose time derivative is

$$\frac{d\mathcal{V}(t)}{dt} = 2ke^{2kt}\psi^\top(t)P\psi(t) + 2e^{2kt}\psi^\top(t)P\frac{d\psi(t)}{dt} + e^{2kt}\psi^\top(t)Q\psi(t) - e^{2k(t-\tau)}\psi^\top(t-\tau)Q\psi(t-\tau)$$

$$= e^{2kt}\Big[2k\psi^\top(t)P\psi(t) + 2\psi^\top(t)P\frac{d\psi(t)}{dt} + \psi^\top(t)Q\psi(t) - e^{-2k\tau}\psi^\top(t-\tau)Q\psi(t-\tau)\Big].$$

By (18) we obtain

$$\frac{d\mathcal{V}(t)}{dt} = e^{2kt}[2k\psi^\top(t)P\psi(t) + 2\psi^\top(t)P\frac{-\tilde{Q}(\psi(t)) + G\psi(t) + G^\tau\psi(t-\tau)}{C} + \psi^\top(t)Q\psi(t) - e^{-2k\tau}\psi^\top(t-\tau)Q\psi(t-\tau)]$$

$$= e^{2kt}[2k\psi^\top(t)P\psi(t) - \frac{2\psi^\top(t)P\tilde{Q}(\psi(t))}{C} + \frac{2\psi^\top(t)PG\psi(t)}{C} + \psi^\top(t)Q\psi(t)$$

$$+ \frac{2\psi^\top(t)PG^\tau\psi(t-\tau)}{C} - e^{-2k\tau}\psi^\top(t-\tau)Q\psi(t-\tau)].$$

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Since $\tilde{q}(0) = 0$ and $\tilde{q}'(\rho) \geq \alpha$ (see (11)), we have $\rho\tilde{q}(\rho) \geq \alpha\rho^2$. In fact, $\rho > 0$ implies $\tilde{q}(\rho) \geq \alpha\rho$, while $\rho < 0$ implies $\tilde{q}(\rho) \leq \alpha\rho$. It follows that $\psi^\top(t)P\tilde{Q}(\psi(t)) = \sum_{i=1}^N \psi_i(t)p_i\tilde{q}(\psi_i(t)) \geq \alpha \sum_{i=1}^N p_i\psi_i^2(t) = \alpha\psi^\top(t)P\psi(t)$. Then, we obtain

$$\frac{d\mathcal{V}(t)}{dt} \leq e^{2kt}[2k\psi^\top(t)P\psi(t) - \frac{2\alpha\psi^\top(t)P(\psi(t))}{C}$$
$$+ \frac{2\psi^\top(t)PG\psi(t)}{C} + \psi^\top(t)Q\psi(t) + \frac{2\psi^\top(t)PG^\tau\psi(t-\tau)}{C} - e^{-2k\tau}\psi^\top(t-\tau)Q\psi(t-\tau)].$$

The previous equation can be rewritten in a matrix form as follows

$$\frac{d\mathcal{V}(t)}{dt} = e^{2kt}\begin{bmatrix} \psi^\top(t) & \psi^\top(t-\tau) \end{bmatrix} W \begin{bmatrix} \psi(t) \\ \psi(t-\tau) \end{bmatrix}$$

where

$$W = \begin{bmatrix} 2kP - 2\alpha\frac{P}{C} + \frac{2}{C}[PG]_S + Q & \frac{1}{C}PG^\tau \\ \left(\frac{1}{C}PG^\tau\right)^\top & -e^{-2k\tau}Q \end{bmatrix}.$$

We have $d\mathcal{V}(\cdot)/dt \leq 0$ if and only if $W \leq 0$, and such condition can be verified by using the Schur's complement for W. First, note that matrix $-e^{-2k\tau}Q$ is negative definite. Then, we have to check

$$\Sigma_k = 2kP - 2\alpha\frac{C}{P} + \frac{2\alpha}{C}[P]_S + Q + \frac{1}{C}PG^\tau e^{2k\tau}Q^{-1}\left(\frac{1}{C}PG^\tau\right)^\top \leq 0. \tag{20}$$

To this end, let us consider the expression Σ_0 in (14), and rewrite (20) as

$$\Sigma_k = 2kP + \Sigma_0 - \frac{1}{C}PG^\tau Q^{-1}\left(\frac{1}{C}PG^\tau\right)^\top + \frac{1}{C}PG^\tau e^{2k\tau}Q^{-1}\left(\frac{1}{C}PG^\tau\right)^\top$$

i.e.,

$$\Sigma_k = 2kP + \Sigma_0 - \frac{1}{C}PG^\tau Q^{-1}\left(\frac{1}{C}PG^\tau\right)^\top\left(e^{2k\tau} - 1\right). \tag{21}$$

We will now show that, under (14), there exists $k > 0$ such that $\Sigma_k < 0$.

If we pick $k > 0$ as in (16), it is seen that both of the following conditions are satisfied

$$2kp_M \leq \frac{1}{2}|\lambda_M(\Sigma_0)|$$

$$\left(e^{2k\tau} - 1\right)\lambda_M\left(\frac{1}{C}PG^\tau Q^{-1}\left(\frac{1}{C}PG^\tau\right)^\top\right) \leq \frac{1}{2}|\lambda_M(\Sigma_0)|. \tag{22}$$

Then, we have

$$\lambda_M(\Sigma_k) \leq \lambda_M(\Sigma_0) + 2kp_M + \left(e^{2k\tau} - 1\right)\lambda_M\left(\frac{1}{C}PG^\tau Q^{-1}\left(\frac{1}{C}PG^\tau\right)^\top\right) \leq 0.$$

This implies $W \leq 0$ and so $d\mathcal{V}(t)/dt \leq 0$, i.e., $\mathcal{V}(t)$ is a non increasing function. Then, $\mathcal{V}(0) \geq \mathcal{V}(t)$, i.e.,

$$\mathcal{V}(0) \geq e^{2kt}\psi(t)^\top P\psi(t) + \int_{t-\tau}^t e^{2kz}\psi(z)^\top Q\psi(z)dz \geq e^{2k\tau}\psi(t)^\top p_m\psi(t)$$

and

$$\mathcal{V}(0) = \psi(0)^\top P\psi(0) + \int_{-\tau}^0 e^{2kz}\psi(z)^\top Q\psi(z)dz \leq \psi(0)^\top p_M\psi(0) + \lambda_M(Q)\int_{-\tau}^0 \|\psi(z)\|^2 dz$$
$$\leq p_M\|\psi(0)\|^2 + \lambda_M(Q)\tau \max_{-\tau\leq\theta\leq 0}\|\psi(\theta)\|^2.$$

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The last two inequalities can be rewritten as

$$e^{2kt}\|\psi(t)\|^2 p_m \leq \mathcal{V}(0) \leq (p_M + \tau\lambda_M(Q)) \max_{-\tau \leq \theta \leq 0} \|\psi(\theta)\|^2$$

which leads to

$$\|\psi(t)\| \leq e^{-kt} \sqrt{\frac{p_M + \tau\lambda_M(Q)}{p_m}} \max_{-\tau \leq \theta \leq 0} \|\psi(\theta)\|. \tag{23}$$

This implies

$$\|\Phi(t; 0, \gamma) - \phi_e\| \leq e^{-kt} \sqrt{\frac{p_M + \tau\lambda_M(Q)}{p_m}} \max_{-\tau \leq \theta \leq 0} \|\Phi(\theta; 0, \gamma) - \phi_e\|. \tag{24}$$

∎

5. Voltage-current Domain

By FCAM [41], the SEs describing the dynamics of a DMNN in the voltage-current domain can be obtained by time differentiation of the SEs (6) in the flux-charge domain. We obtain

$$\begin{cases} C\dfrac{dv(t)}{dt} &= -M(\phi(t))v(t) + Gv(t) + G^\tau v(t - \tau) \\[2mm] \dfrac{d\phi(t)}{dt} &= v(t) \end{cases} \tag{25}$$

where $v(\cdot) = (v_{C_1}(t), v_{C_2}(t), \ldots, v_{C_N}(t))^\top$ is the vector of capacitor voltages and $M(\phi(t)) = \mathrm{diag}(\hat{q}'(\phi_1(t)), \hat{q}'(\phi_2(t)), \ldots, \hat{q}'(\phi_N(t)))^\top$, where $\hat{q}'(\phi_i(t))$ is the memristance of the i-th neuron [9]. The initial conditions are

$$v(\sigma) = \gamma(\sigma), \ \sigma \in [-\tau, 0]; \quad \phi(0) = \phi_0 \in \mathbb{R}^N \tag{26}$$

where $\gamma \in C([-\tau, 0], \mathbb{R}^N)$.

We remark that the system of nonlinear delay differential equations (25) describing a DMNN in the voltage-current domain has order $2N$, whereas the associated system in the flux-charge domain (6) is of order N. This order reduction is perfectly compatible with FCAM and is due to the fact that, as discussed in Section 2, each of the N memristors is a dynamic element in the voltage-current domain, whereas it is a memoryless element in the flux-charge domain (cf. Remark 1).

Let $T > 0$. We say that $[V(t; 0, (\gamma, \phi_0)), \Psi(t; 0, (\gamma, \phi_0))]$, with $V(\cdot) : [-\tau, T] \to \mathbb{R}^N$ and $\Psi(\cdot) : [0, T] \to \mathbb{R}^N$, is a solution of the IVP (25), (26) with initial conditions $\gamma \in C([-\tau, 0], \mathbb{R}^N)$ and $\phi_0 \in \mathbb{R}^N$, if we have the following: $V(\sigma; 0, (\gamma, \phi_0)) = \gamma(\sigma)$ for any $\sigma \in [-\tau, 0]$, $\Psi(0; 0, (\gamma, \phi_0)) = \phi_0$, $V(\cdot)$ is continuous on $[-\tau, T]$ and differentiable on $[0, T)$; $\Psi(\cdot)$ is differentiable on $[0, T)$ and (25) is satisfied for $0 \leq t < T$. We say that $[V(t; 0, (\gamma, \phi_0)), \Psi(t; 0, (\gamma, \phi_0))]$ is a solution for $t \geq -\tau$ if it is a solution on $[-\tau, T]$ for any $T > 0$.

Proposition 2. *If Assumption 1 is satisfied, then the following results hold.*

1. *For any $\gamma \in C([-\tau, 0], \mathbb{R}^N)$ and $\phi_0 \in \mathbb{R}^N$, there exists a unique solution $[V(t; 0, (\gamma, \phi_0)), \Psi(t; 0, (\gamma, \phi_0))]$ of the IVP (25) and (26), which is bounded and hence defined for $t \geq -\tau$.*
2. *The solution $[V(t; 0, (\gamma, \phi_0)), \Psi(t; 0, (\gamma, \phi_0))]$ is C^0 in $[-\tau, 0]$, and it is C^1 for $t \geq 0$.*

Proof. See Appendix B. ∎

In the paper we have assumed that $\hat{q}(\cdot)$ is C^1 in $\mathbb{R}^N$, so $M(\phi(\cdot))$ is in general only C^0 in $\mathbb{R}^N$. A remarkable fact proved in Proposition 1 is that, although the vector field at the right-hand side of (25) is only C^0 in $\mathbb{R}^{2N}$, the uniqueness of the solution for (25) is guaranteed. As a direct consequence of the proof of the same proposition it can be seen that the following relationship holds between the solution of the IPV (25), (26) for a DMNN in the voltage-current domain and that of the IVP (6), (7) in the flux-charge domain.

Property 1. *Suppose that Assumption 1 is satisfied. Fix any $\gamma \in C([-\tau, 0], \mathbb{R}^N)$ and $\phi(0) = \phi_0 \in \mathbb{R}^N$. Then the following results hold:*

1. *if $\Phi(t; 0, \gamma)$, $t \geq -\tau$, is the solution of the IVP (6), (7), then $[d\Phi(t; 0, \gamma)/dt, \Phi(t; 0.\gamma)]$, $t \geq -\tau$, is the solution of the IVP (25), (26).*

2. *Conversely, if $[V(t; 0, (\gamma, \phi_0)), \Psi(t; 0, (\gamma, \phi_0))]$, $t \geq -\tau$, is the solution of the IVP (25), (26), then*

$$\Phi(t; 0, \gamma) = \begin{cases} \phi_0 + \int_0^t \gamma(z)dz & \text{if} \quad t \in [-\tau, 0] \\ \Psi(t; 0, (\gamma, \phi_0)) & \text{if} \quad t \geq 0 \end{cases} \tag{27}$$

   is the solution of the IVP (6) and (7).

The EPs of (25) are obtained by letting $dv(t)/dt = 0$, $d\phi(t)/dt = 0$. The set of EPs is given by the N-dimensional manifold (linear subspace) in $\mathbb{R}^{2N}$

$$\mathcal{M}_{\text{EP}} = \{(v_e, \phi_e) \in \mathbb{R}^{2N} : v_e = 0, \phi_e \in \mathbb{R}^N\}$$

namely, there are infinitely many (a *continuum* of) nonisolated EPs. This is in agreement with the well-known fact that an ideal flux-controlled memristor can memorize any value of the flux as an equilibrium state in the voltage-current domain [43]. Note that the capacitor voltages vanish at any EP of (25).

Since there are infinitely many EPs, we cannot have GES for a DMNN in the voltage-current domain. However, we can prove the following convergence result.

Theorem 2. *Suppose Assumptions 1 and 2 are satisfied. Then, any solution of (25)-(26) exponentially converges toward an EP as $t \to +\infty$ and the convergence rate k is given in (16). Moreover, we have $v(t) \to 0$ as $t \to +\infty$, i.e., the capacitor voltages tend to 0 in steady state.*

Proof. Let $[V(t; 0, (\gamma, \phi_0)), \Psi(t; 0, (\gamma, \phi_0))]$ be a solution of (25)-(26). Point 2 of Property 1 implies that $\Phi(t; 0, \gamma)$ as in (27) is a solution of the IVP (6)-(7), while Theorem 1 guarantees that $\Phi(t; 0, \gamma)$ (and $\Psi(t; 0, (\gamma, \phi_0))$) converges exponentially to the unique EP ϕ_e and also $d\Phi(t; 0, \gamma)/dt \to 0$ as $t \to +\infty$. Since $V(t; 0, (\gamma, \phi_0)) = d\Psi(t; 0, (\gamma, \phi_0))/dt = d\Phi(t; 0, \gamma)/dt$, we have that $V(t; 0, (\gamma, \phi_0)) \to 0$ as $t \to +\infty$.

To prove exponential convergence to 0 of $V(t; 0, (\gamma, \phi_0))$ we observe that, from (18)

$$\begin{aligned} \|V(t; 0, (\gamma, \phi_0))\| &\leq \frac{1}{C} \left(\|\hat{Q}(\Phi(t; 0, \gamma) - \phi_e)\| + \|G(\Phi(t; 0, \gamma) - \phi_e)\| + \|G^\tau(\Phi(t - \tau; 0, \gamma) - \phi_e)\| \right) \\ &\leq \frac{1}{C} \left(\|\hat{Q}(\Phi(t; 0, \gamma) - \phi_e)\| + \|G\| \|(\Phi(t; 0, \gamma) - \phi_e)\| + \|G^\tau\| \|(\Phi(t - \tau; 0, \gamma) - \phi_e)\| \right) \end{aligned}$$

From (24) we have $\|(\Phi(t - \tau; 0, \gamma) - \phi_e)\| \leq e^{-kt} e^{k\tau} \sqrt{\frac{p_M + \tau \lambda_M(Q)}{p_m}} \max_{-\tau \leq \theta \leq 0} \|\Phi(\theta; 0, \gamma) - \phi_e\|$. Additionally, since $\hat{q}(\cdot) \in C^1(\mathbb{R})$, then $\hat{Q}(\cdot)$ is Lipschitz on any ball $B(\rho, \xi) = \{x \in \mathbb{R}^N : \|x - \xi\| \leq \rho\}$. Let us denote with $k_{\hat{Q}}$ the Lipschitz constant for $\xi = \phi_e$ and $\rho = \sqrt{(p_M + \tau \lambda_M(Q))/p_m} \max_{-\tau \leq \theta \leq 0} \|\Phi(\theta; 0, \gamma) - \phi_e\|$. Then we can write

$$\|V(t; 0, (\gamma, \phi_0))\| \leq \frac{1}{C} \left(k_{\hat{Q}} + \|G\| + e^{k\tau} \|G^\tau\| \right) e^{-kt} \sqrt{\frac{p_M + \tau \lambda_M(Q)}{p_m}} \max_{-\tau \leq \theta \leq 0} \|\Phi(\theta; 0, \gamma) - \phi_e\| \tag{28}$$

which proves the exponential convergence to 0 of $V(t; 0, (\gamma, \phi_0))$ with the same exponential rate of $\Psi(t; 0, (\gamma, \phi_0))$. ∎

Remark 6. *The dynamic analysis in the paper can be immediately extended to other classes of memristors, such as ideal extended memristors (memristor siblings), which can be brought back to an ideal memristor by suitable changes of variables [10]. Also the linear drift model of the HP memristor [1] can be brought back to an ideal memristor model. It is known that some practically implemented memristors, however, deviates from an ideal memristor and they need to be modeled by generic or extended memristors using state variables that are not necessarily the flux or the charge [60, 10]. The study of neural networks containing generic or extended memristors goes beyond the scope of the present work, but is considered a challenging issue for future works.*

6. Simulation Results

Under the assumptions of Theorem 1 the DMNN (6) has a unique EP $\phi_e \in \mathbb{R}^N$ for any input

$$Q_0 = Cv_0 + \hat{Q}(\phi_0) - (G + G^\tau)\phi_0 + G^\tau \int_{-\tau}^0 \gamma(z)dz \in \mathbb{R}^N.$$

From a mathematical viewpoint the DMNN implements a nonlinear mapping $\phi_e = \hat{\phi}_e(Q_0)$ between the space R^N of inputs to the space $\mathbb{R}^N$ of EPs toward which any solution of (6) exponentially converges. As discussed in Section 1, such a map is potentially useful for solving a number of signal processing tasks in real time. Among the many applications there is the solution of global optimization problems in real time, such as linear and programming problems or sorting problems. In such problems, the GES property prevents the solutions from getting stuck at some local minimum of the cost function to be minimized. It is important to stress that a convergent neural network that is multistable, i.e., it possesses multiple locally asymptotically stable EPs, would be instead unsuitable for addressing such classes of processing tasks. In fact, in the case of multistability, only convergence to a local minimum of the cost function may be guaranteed when solving optimization problems. Below we give some examples to verify and illustrate the property of GES of DMNNs in Theorem 1.

Example 1. Let us consider a third-order DMNN (6) such that

$$G = \begin{pmatrix} -4.0 & -0.2 & -1.0 \\ 0.3 & -3.5 & -0.1 \\ 0.1 & -1.0 & -4.5 \end{pmatrix}; \quad G^\tau = \begin{pmatrix} 0.1 & 0.3 & 0.2 \\ 1.0 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.1 \end{pmatrix}$$

and the memristors have a non-monotonic cubic nonlinearity $q = \hat{q}(\varphi) = -\varphi + \frac{1}{3}\varphi^3$ such that $\alpha = \min_{\mathbb{R}} \hat{q}'(\varphi) = \hat{q}'(0) = -1$. Note that we are dealing with locally-active memristors that can be implemented for instance by using passive memristors in combination with active conductances [57]. Moreover, let $C = 1$ F. Note that Assumption 1 is satisfied, since $\hat{q}'(\varphi) \to +\infty$ as $|\varphi| \to +\infty$. It can be verified that also Assumption 2 is satisfied by choosing $P = \text{diag}(1.856, 1.856, 1.856)$ and

$$Q = \begin{pmatrix} 5.568 & -0.062 & 0.555 \\ -0.062 & 4.954 & 0.678 \\ 0.555 & 0.678 & 6.1828 \end{pmatrix} > 0$$

so that, according to Theorem 1, the DMNN has a unique GES EP ϕ_e for any $Q_0 \in \mathbb{R}^3$.

Suppose $\tau = 2$ and $Q_0 = (1, -2, 2)^\top$. We conducted simulations of the DMNN by MATLAB routine dde23 when choosing $\varphi_M(0) = 0$ and three different sets of initial conditions, namely, $\gamma(\cdot) \in C([-2, 0], \mathbb{R}^3)$ is a constant, or a sinusoidal, or a polynomial function in $[-2, 0]$

$$\gamma(t) = (10.0, -17.857, -0.714)^\top, t \in [-2, 0]$$
$$\gamma(t) = (13.22 \cos(4t), 28.0 \sin(t), 8.621 \cos(2t))^\top, t \in [-2, 0]$$
$$\gamma(t) = (-0.566(t^2 + 3), -1.736(t^3 + 1), -2.686(t - 1))^\top, t \in [-2, 0].$$

It can be checked that in the three cases we have, as required, $Q_0 = (1, -2, 2)^\top$. Figure 5 depicts the results of simulations. It is seen that in all cases the DMNN converges to the same EP $\phi_e = (0.123, -0.640, 0.723)^\top$, in accordance with the result in Theorem 1. We simulated also the corresponding DMNN system (25) in the voltage-current domain for the same initial conditions $\varphi_M(0)$ and $\gamma(\cdot) \in C([-2, 0], \mathbb{R}^3)$. The results are reported in Fig. 5, where we can check that any solution converges to an EP $(v_e, \phi_e) = (0, 0, 0, 0.123, -0.640, 0.723)^\top$. Note that, according to Theorem 2, all capacitor voltages vanish in steady state.

Example 2. Let us now consider a third-order DMNN (6) such that

$$G = \begin{pmatrix} -2.0 & -0.1 & -1.0 \\ 0.3 & -3.5 & -0.1 \\ 0.1 & -1.0 & -1.5 \end{pmatrix}; \quad G^\tau = \begin{pmatrix} 0.1 & 0.3 & 0.2 \\ 0.1 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.1 \end{pmatrix}.$$

The memristors are passive and have a monotonic nonlinearity $q = \hat{q}(\varphi) = 6|\varphi|\arctan(\varphi)$. We have $\hat{q}(\cdot) \in C^1(\mathbb{R})$, moreover $0 \le \hat{q}'(\varphi) < 6$ for any $\varphi \in \mathbb{R}$, hence $\alpha = 0$. Also let $C = 1$ F. We can check that Assumption 1 is satisfied with $k_q = 6$. It can be verified that also Assumption 2 is satisfied by choosing $P = \text{diag}(2.16, 2.16, 2.16)$ and

$$Q = \begin{pmatrix} 4.795 & -0.144 & 0.650 \\ -0.144 & 6.941 & 0.789 \\ 0.650 & 0.789 & 4.079 \end{pmatrix} > 0$$

hence, according to Theorem 1, the DMNN has a unique GES EP ϕ_e for any $Q_0 \in \mathbb{R}^3$.

Let $\tau = 3$ and $Q_0 = (1, -2, 2)^\top$. We have conducted simulations of the DMNN when $\varphi_M(0) = 0$ and three different sets of initial conditions are chosen, namely, $\gamma(\cdot) \in C([-3, 0], \mathbb{R}^3)$ is given by

$$\gamma(t) = (-4.437, 4.451, 4.604)^\top, t \in [-3, 0]$$
$$\gamma(t) = (4.574\cos(4t), 5.668\sin(t), 4.627\cos(2t))^\top, t \in [-3, 0]$$
$$\gamma(t) = (1.672(t^2 + 3), -1.874(t^3 + 1), 11.149(t - 1))^\top, t \in [-3, 0].$$

It can be checked that in the three cases we have $Q_0 = (1, -2, 2)^\top$. The simulation results, shown in Figure 6, confirm that in all cases the DMNN converges to the same EP $\phi_e = (0.131, -0.382, 0.803)^\top$, in accordance with Theorem 1. For the corresponding DMNN system (25) in the voltage-current domain we have verified that, according to Theorem 2, all capacitor voltages tend to 0 as $t \to +\infty$.

Example 3. Consider a DMNN (6) with N neurons, having nearest-neighbor interconnections and periodic boundary conditions, namely G and G^τ are given by the circulant matrices

$$G = \text{circ}_N(a, s, 0, \ldots, 0, r) = \begin{pmatrix} a & s & 0 & 0 & \cdots & r \\ r & a & s & 0 & \cdots & 0 \\ 0 & r & a & s & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r & a & s \\ s & \cdots & 0 & 0 & r & a \end{pmatrix}$$

and $G^\tau = \text{circ}_N(\bar{a}, \bar{s}, 0, \ldots, 0, \bar{r})$ depending on the real parameters a, r, s and $\bar{a}, \bar{r}, \bar{s}$, respectively. Let us assume the memristors have a monotonic nonlinearity $q = \hat{q}(\varphi) = \beta\varphi + \arctan(\varphi)$ where $\beta > 0$. We have $\hat{q}(\cdot) \in C^1(\mathbb{R})$, moreover $\beta \le \hat{q}'(\varphi) \le \beta + 1$ for any $\varphi \in \mathbb{R}$, hence $\alpha = \beta$. Also let $C = 1$ F. We can check that Assumption 1 is satisfied with $k_q = \beta$, provided on the basis of (10) the constraint $|a| + |s| + |r| + |\bar{a}| + |\bar{s}| + |\bar{r}| < \beta$ holds.

This kind of circular neural networks are widely investigated in the literature as prototypical systems to study the potentials and limitations for information processing of neural network arrays [61, 62, 63]. We now want to find conditions on G and G^τ guaranteeing that Assumption 2 is satisfied by picking $P = E_N$ and $Q = E_N$. Following this choice, (14) simplifies to

$$-2\beta E_N + E_N + G + G^\top + G^\tau(G^\tau)^\top < 0. \tag{29}$$

Now, considering that $G + G^\top = \text{circ}_N(2a, r + s, 0, \ldots, 0, r + s)$, while $G^\tau(G^\tau)^\top = \text{circ}_N(\bar{a}^2 + \bar{s}^2 + \bar{r}^2, \bar{a}\bar{r} + \bar{a}\bar{s}, \bar{r}\bar{s}, 0, \ldots, 0, \bar{r}\bar{s}, \bar{a}\bar{r} + \bar{a}\bar{s})$, it turns out that (29) implies

$$\Sigma_0 = \text{circ}_N(2a + \bar{a}^2 + \bar{s}^2 + \bar{r}^2 + 1 - 2\beta, \bar{a}\bar{r} + \bar{a}\bar{s} + r + s, \bar{r}\bar{s}, 0, \ldots, 0, \bar{r}\bar{s}, \bar{a}\bar{r} + \bar{a}\bar{s} + r + s) < 0.$$

Since such a matrix is circulant and symmetric, there is an explicit expression for its eigenvalues [64]

$$\lambda_i(\Sigma_0) = 2a + \bar{a}^2 + \bar{s}^2 + \bar{r}^2 + 1 - 2\beta + 2(\bar{a}\bar{r} + \bar{a}\bar{s} + r + s)\cos\left(\frac{2\pi i}{N}\right) + 2(\bar{r}\bar{s})\cos\left(\frac{4\pi i}{N}\right); \quad i = 0, 1, \ldots, N - 1.$$

As a consequence, in order to have both Assumptions 1 and 2 verified, it suffices that the conditions $|a| + |s| + |r| + |\bar{a}| + |\bar{s}| + |\bar{r}| < \beta$ and $\max_{i=0,1,\ldots,N-1} \lambda_i(\Sigma_0) < 0$ are satisfied.

We have conducted simulations with MATLAB for a DMNN with $N = 10$ neurons in the case where G is defined by $a = 0.2$, $r = -1$ and $s = 0.1$, matrix G^τ by $\bar{a} = -1$, $\bar{r} = -0.8$ and $\bar{s} = 0.1$, moreover for the memristor
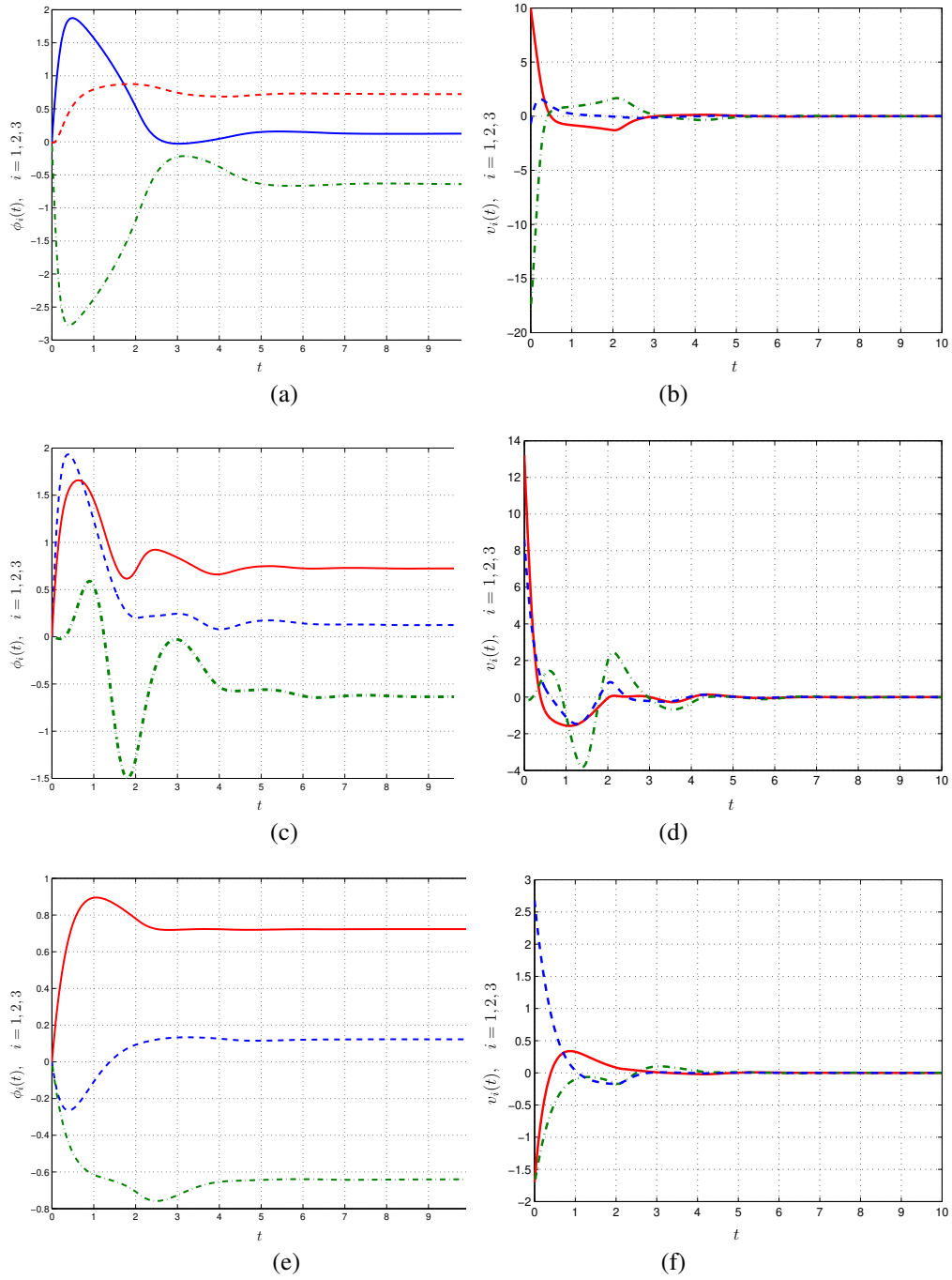
Figure 5: (a), (c), (e) Time-domain evolution of fluxes and (b), (d), (f) corresponding evolution of voltages for three different solutions of the DMNN in Example 1.
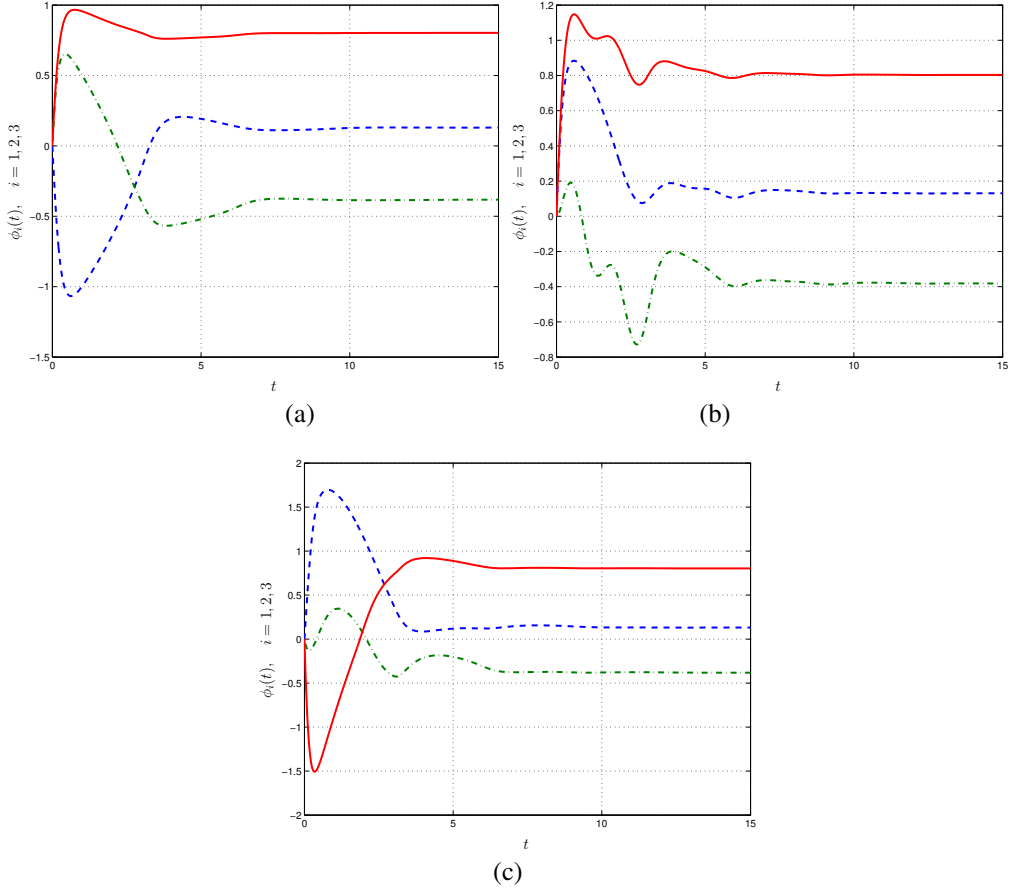
Figure 6: (a), (b), (c) Time-domain evolution of fluxes for three different solutions of the DMNN in Example 2.

nonlinearity we have $\beta = 4$ and $\tau = 3$. We have $|a| + |s| + |r| + |\bar{a}| + |\bar{s}| + |\bar{r}| = 3.2 < 4 = \beta$ so that Assumption 1 is satisfied. Moreover, $\max_{i=0,1,\ldots,N-1} \lambda_i(\Sigma_0) = -4.676 < 0$, hence also Assumption 2 is met. Figures 7(a), (b) depict the numerical results for two different sets of initial conditions $\gamma_i(\sigma)$, $\sigma \in [-3, 0]$, $i = 1, 2, \ldots, 10$, such that in both cases we have $Q_0 = (1, -2, 2, 1, 1, 2, -2, -1, -1, 2)^\top$. Again, it can be seen that both solutions converge to the same EP $\phi_e = (0.038, -0.343, 0.457, 0.036, 0.171, 0.278, -0.437, -0.042, -0.146, 0.395)^\top$, in accordance with Theorem 1.

Remark 7. *The examples demonstrate that the property of GES in Theorem 1 may be used for processing signals by means of a DMNN in the* flux-charge *domain. The input is Q_0, the processing takes place during the transient evolution of the fluxes, and the processing result is $\phi_e(Q_0)$. The processing result is memorized for further use by the memristors acting as nonvolatile devices in steady state. Note that the memristors are used both for the nonlinear processing of signals and to memorize the result of processing. In other words, computation and memorization is performed at the same physical location. Batteries are in principle not needed to memorize the processing results. Consider now the DMNN behavior in the voltage-current domain. As seen in Theorem 2, and as confirmed by simulations, all capacitor voltages, as well as all other voltages, current, and hence power in a DMNN, vanish when a steady state is reached. We stress that this is a potential advantage with respect to Hopfield [65], cellular [53], or Cohen-Grossberg [66] neural networks computing in the traditional* voltage-current *domain. Indeed, for those network, voltages, currents and power do not vanish in steady state, with an increase in power consumption. Moreover, for those networks the result of processing needs to be transferred to a memory location after the computation in a neuron ends, implying limitation in the computation speed, as discussed in Section 1. Batteries are also needed to memorize the result of processing.*
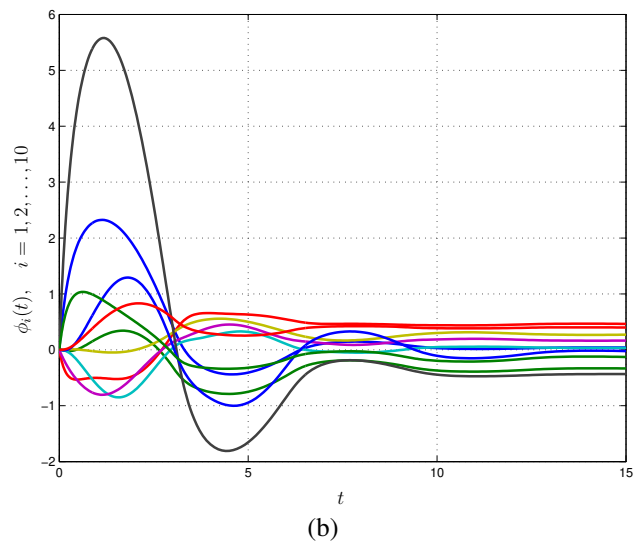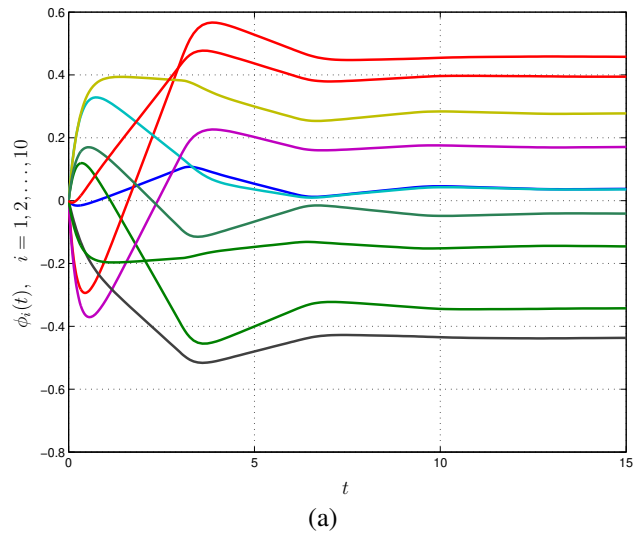
(a)

(b)

Figure 7: (a), (b) Time-domain evolution of fluxes for two different solutions of the DMNN in Example 3.

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Remark 8. *There are several relevant results in the literature concerning global stability of the EP, and related issues on global synchronization problems, for classes of delayed neural networks with memristors, see, e.g., [38, 39, 44, 45, 46, 47, 48, 49, 40, 49, 50, 51, 52], and references therein. As pointed out in Remark 3 in Section 3, the memristor model used in those papers is different from that considered here in the class of DMNNs (6). Indeed, in a DMNN we consider an ideal flux-controlled memristor defined by a nonlinear relationship $\hat{q}(\cdot)$ between flux and charge as that originally introduced by Leon Chua in the seminal paper [9]. This corresponds to the exact model and CR of an ideal memristor. Quite differently, the previously quoted papers use a memristor model that retains only one fingerprint of an ideal memristor, i.e., the fact that the memristance switches between two different values when a sinusoidal input signal is applied. We also note that the memristor neural networks in the quoted papers operate and compute in the traditional voltage-current domain, as confirmed by the fact that, differently from DMNNs, voltages and current do not tend to 0 when a steady state is reached. Potential advantages of DMNNs in terms of power consumption in steady state, with respect to other models of memristor neural networks, can be envisaged based on these considerations.*

7. Discussion and Conclusions

The paper has provided a thorough dynamic analysis of a class of memristor neural networks with concentrated delays, named DMNNs, operating in the flux-charge domain. A condition based on LMIs is given ensuring that a DMNN has a unique GES EP in the flux-charge domain and simulations are presented showing that this property is potentially useful for signal processing tasks. Under the same condition we have a convergent dynamics in the voltage-current domain and one main feature is that all voltages, currents and power drop off in steady steady, whereas the memristors are able to keep in memory in a nonvolatile way the result of processing. This is an advantage with respect to traditional neural network architectures operating in the voltage-current domain, where power does not drop off in steady state and batteries are needed to store the result of processing.

To the authors knowledge, the theoretic results in the paper are the first results on global stability for memristor neural networks operating in the flux-charge domain. The simulation results also show the potential usefulness of globally stable DMNNs for solving in real time signal processing tasks as global optimization problems. There is a huge body of literature devoted to global stability of delayed neural networks without memristors, where a rich variety of concentrated and distributed delays are considered, see [23], for a thorough review. Relevant recent contributions concerning global stability and dissipativity of delayed neural networks with impulses and interval time delays or time varying delays can be found in [67, 68]. It is arguable that a number of the techniques and results in those papers can be suitably adapted in order to apply them to analyze global stability in the flux-charge domain for the class of DMNNs in this paper or some of its extensions. We believe this might be an interesting topic for future research on memristor neural networks.

Appendix A. Proof of Proposition 1

We start by establishing a preliminary result. Let $A > 0$ and $\zeta(\cdot) \in C([-\tau, A], \mathbb{R}^N)$. We let $\zeta_t \in C([-\tau, 0], \mathbb{R}^N)$ be defined as $\zeta_t(\sigma) = \zeta(t + \sigma)$, $\sigma \in [-\tau, 0]$. Then, we can write (6) in vector form $d\zeta(t)/dt = F(\zeta_t)$ where $F : C([-\tau, 0], \mathbb{R}^N) \rightarrow \mathbb{R}^N$ is given by $F(\zeta_t) = -\hat{Q}(\zeta_t(0)) + G\zeta_t(0) + G^\tau \zeta_t(-\tau) + Q_0$.

Property 2. *Let $\hat{q}(\cdot) \in C^1(\mathbb{R})$, $M > \|\phi_0\|$, $\gamma \in C([-\tau, 0], \mathbb{R}^N)$ be such that $\|\gamma\| < (M - \|\phi_0\|)/\tau$. Then, there exist $A > 0$, depending only on M, such that there exists a unique solution $\Phi(t; 0, \gamma)$ of the IVP (6) and (7) defined for $t \in [-\tau, A]$. Moreover, the following estimate holds*

$$\max_{-\tau \leq \eta \leq A} \|\phi(\eta; \gamma)\| \leq \|\gamma\| e^{(N K_{q,M} + \|G\| + \|G^\tau\|)A} \tag{A.1}$$

where $k_{q,M}$ is the Lipschitz constant of $\hat{q}(\cdot)$ in $[-M, M]$.

Proof of Property 2. Since $\hat{q}(\cdot) \in C^1(\mathbb{R})$, $q(\cdot)$ is Lipschitz in $[-M, M]$, i.e., there exists $K_{q,M}$ such that $|\hat{q}(x) - \hat{q}(y)| \leq K_{q,M}|x - y|$ for any $x, y \in [-M, M]$. Consider now $\xi, \upsilon \in C([-\tau, 0], \mathbb{R}^N)$, such that $\|\xi\|, \|\upsilon\| < M$. We have

$$\|F(\xi) - F(\upsilon)\| = \| - \hat{Q}(\xi(0)) + \hat{Q}(\upsilon(0))\| + \|G(\xi(0) - \upsilon(0))\| + \|G^\tau(\xi(-\tau) - \upsilon(-\tau))\|$$
$$\leq NK_{q,M}\|\xi(0) - \upsilon(0)\| + \|G\|\,\|\xi(0) - \upsilon(0)\| + \|G^\tau\|\,\|\xi(-\tau) - \upsilon(-\tau)\|$$
$$\leq (NK_{q,M} + \|G\| + \|G^\tau\|)\|\xi - \upsilon\|$$

i.e., the vector field F satisfies the Lipschitz condition $\|F(\xi) - F(\upsilon)\| \leq K\|\xi - \upsilon\|$.

Let $\gamma \in C([-\tau, 0], \mathbb{R}^N)$ such that $\|\gamma\| < (M - \|\phi_0\|)/\tau$. We have that $\phi(\sigma)$ in (7) satisfies $\|\phi\| < M$. As a consequence, we are under the hypotheses of [69, Th. 3.7, p. 32]. Application of that theorem yields the result in the property. ∎

We are now in a position to complete the proof of Proposition 1.

1. Assume (8) holds true, then there exist finite $\widetilde{M}, \widehat{M} > 0$ such that

$$\hat{q}(\widetilde{M}) = (\|G\|_\infty + \|G^\tau\|_\infty)\widetilde{M} + \|Q_0\|_\infty, \quad \hat{q}(-\widehat{M}) = -(\|G\|_\infty + \|G^\tau\|_\infty)\widehat{M} - \|Q_0\|_\infty.$$

Let $\bar{M} = \max\{M, \widetilde{M}, \widehat{M}\}$. We want to show that $\sup_{-\tau \leq t \leq T} \|\Phi(t; 0, \gamma)\|_\infty = \sup_{-\tau \leq t \leq T}\{\max_{i=1,2,\ldots,N} |\phi_i(t; 0, \gamma)|\} \leq \bar{M}$, i.e., the solution of the of IVP (6) and (7) is bounded on $[-\tau, T]$. We observe that, since $\|\gamma\| < (M - \|\phi_0\|)/\tau$, then $\|\Phi(t; 0, \gamma)\|_\infty \leq M \leq \bar{M}$, $t \in [-\tau, 0]$. We wish to show that this implies $\|\Phi(t; 0, \gamma)\|_\infty \leq \bar{M}$, $t \in [0, \tau]$. Let $t_\mu \in [-\tau, 0]$ and let $\mu \in \{1, 2, \ldots, N\}$ be such that $|\phi_\mu(t_\mu; 0, \gamma)| = \max_{i \in \{1,2,\ldots,N\}} |\phi_i(t_\mu; 0, \gamma)| = \|\phi(t_\mu; 0, \gamma)\|_\infty$. We have

$$d\phi_\mu(t_\mu)/dt = -\hat{q}(\phi_\mu(t_\mu)) + \sum_{j=i}^{N} g_{\mu j}\phi_j(t_\mu) + \sum_{j=i}^{N} g_{\mu j}^\tau \phi_j(t - \tau) + Q_{0_\mu}.$$

If $\phi_\mu(t_\mu) > 0$, recalling that $\|\phi(t_\mu; 0, \gamma)\| \leq \bar{M}$, we obtain

$$d\phi_\mu(t_\mu)/dt \leq -\hat{q}(\phi_\mu(t_\mu)) + \|G\|_\infty\phi_\mu(t_\mu) + \|G^\tau\|_\infty\bar{M} + \|Q_0\|_\infty.$$

Il can be easily verified that, if $\phi_\mu(t_\mu) = \bar{M}$, then $d\phi_\mu(t_\mu)/dt \leq 0$. Similarly, if $\phi_\mu(t_\mu) < 0$, we have

$$d\phi_\mu(t_\mu)/dt \geq -\hat{q}(\phi_\mu(t_\mu)) - \|G\|_\infty\phi_\mu(t_\mu) - \|G^\tau\|_\infty\bar{M} + \|Q_0\|_\infty$$

which implies that, if $\phi_\mu(t_\mu) = -\bar{M}$, then $d\phi_\mu(t_\mu)/dt \geq 0$. As a consequence, we have that $\|\Phi(t; 0, \gamma)\|_\infty \leq \bar{M}$, $t \in [0, \tau]$.

Now, we proceed by induction. Assume that $\|\Phi(t; 0, \gamma)\|_\infty \leq \bar{M}$ when $t \in [(k-1)\tau, k\tau]$. Let $t_{\mu_k} \in [(k-1)\tau, k\tau]$ and let $\mu_k \in \{1, 2, \ldots, N\}$ be such that $|\phi_{\mu_k}(t_{\mu_k}; 0, \gamma)| = \|\phi(t_{\mu_k}; 0, \gamma)\|_\infty$. Similarly, we can prove that, when $\phi_{\mu_k}(t_{\mu_k}) > 0$

$$d\phi_{\mu_k}(t_{\mu_k})/dt \leq -\hat{q}(\phi_{\mu_k}(t_{\mu_k})) + \|G\|_\infty\phi_{\mu_k}(t_{\mu_k}) + \|G^\tau\|_\infty\bar{M} + \|Q\|_\infty.$$

which implies $d\phi_{\mu_k}(t_{\mu_k})/dt \leq 0$ if $\phi_{\mu_k}(t_{\mu_k}) = \bar{M}$. When $\phi_{\mu_k}(t_{\mu_k}) < 0$, we have instead

$$d\phi_{\mu_k}(t_{\mu_k})/dt \geq -\hat{q}(\phi_{\mu_k}(t_{\mu_k})) - \|G\|_\infty\phi_{\mu_k}(t_{\mu_k}) - \|G^\tau\|_\infty\bar{M} + \|Q_0\|_\infty$$

which implies $d\phi_{\mu_k}(t_{\mu_k})/dt \geq 0$ if $\phi_{\mu_k}(t_{\mu_k}) = -\bar{M}$. This shows that the solution of the of IVP (6) and (7) is bounded for $t \geq 0$ in the case (8).

Assume now (9), (10) hold true. Considering that $\lim_{|\rho| \to \infty} \hat{q}'(\rho) = k_q$ and $k_q > \|G\|_\infty + \|G^\tau\|_\infty$, there exists finite $\widetilde{M} > 0$ such that

$$\hat{q}(\widetilde{M}) = (\|G\|_\infty + \|G^\tau\|_\infty)\widetilde{M} + \|Q_0\|_\infty.$$

For the same reason, we can find a finite $\widehat{M} > 0$ such that

$$\hat{q}(-\widehat{M}) = -(\|G\|_\infty + \|G^\tau\|_\infty)\widehat{M} - \|Q_0\|_\infty.$$

We can define $\bar{M} = \max\{M, \widetilde{M}, \widehat{M}\}$, and use an analogous induction argument to prove that the solution of the of IVP (6) and (7) is bounded for $t \geq 0$ also in the case (9), (10).

Property 2 and boundedness of solution on $[-\tau, T]$ imply existence and uniqueness of a global solution $\Phi(t; 0, \gamma)$ of the IVP (6)-(7).

2. By definition of solution, we have $\Phi(\sigma; 0, \gamma) = \phi_0 + \int_0^\sigma \gamma(z)dz$, where $\gamma(\sigma)$ is a continuous function. Consequently, $\Phi(\cdot; 0, \gamma)$ is C^1 in $[-\tau, 0)$. Since the right hand side of (6) is a C^1 function ($\hat{q}(\cdot)$ is assumed to be C^1), then $\Phi(\cdot; 0, \gamma)$ is a C^2 function in $(0, \tau)$. Let us show that $\lim_{t \to 0^-} d\Phi(t; 0, \gamma)/dt = \lim_{t \to 0^+} d\Phi(t; 0, \gamma)/dt$. From (7) it turns out that

$$\lim_{t \to 0^-} \frac{d\Phi(t; 0, \gamma)}{dt} = \lim_{t \to 0^-} \gamma(t) = \gamma(0).$$

Evaluating the right hand size of (6) we have

$$\lim_{t \to 0^+} \frac{d\phi_i(t; 0, \gamma)}{dt} = (1/C)\Big(-\hat{q}(\phi_i(0; 0, \gamma)) + \sum_{j=1}^N g_{ij}\phi_j(0; 0, \gamma) + \sum_{j=1}^N g_{ij}^\tau \phi_j(-\tau; 0, \gamma) + C\gamma_i(0) + \hat{q}(\phi_i(0; 0, \gamma))$$

$$- \sum_{j=1}^N (g_{ij} + g_{ij}^\tau)\phi_j(0; 0, \gamma) + \sum_{j=1}^N g_{ij}^\tau \int_{-\tau}^0 \gamma_j(z)dz\Big).$$

Since $\phi_i(0; 0, \gamma) = \phi_{0_i}$ and $\phi_i(-\tau; 0, \gamma) = \phi_{0_i} - \int_{-\tau}^0 \gamma_i(z)dz$, we have

$$\lim_{t \to 0^+} \frac{d\Phi(t; 0, \gamma)}{dt} = \gamma(0) = \lim_{t \to 0^-} \frac{d\Phi(t; 0, \gamma)}{dt}$$

i.e., $\Phi(\cdot; 0, \gamma)$ is C^1 in $[-\tau, 0]$ and C^2 in $(0, \tau)$. A simple induction argument can be used to prove that $\Phi(\cdot; 0, \gamma)$ is C^2 in $(0, +\infty)$. ∎

Appendix B. Proof of Proposition 2

1. Let $\Phi(t; 0, \gamma)$, $t \geq -\tau$, be the solution of the IVP (6) and (7). We first show that, by means of $\Phi(\cdot)$ it is possible to construct a solution of the IVP (25)-(26). Let us consider $V(t; 0, (\gamma, \phi_0)) = d\Phi(t; 0, \gamma)/dt$, and $\Psi(t; 0, (\gamma, \phi_0)) = \Phi(t; 0, \gamma)$. By definition we have $\Psi(t; 0, (\gamma, \phi_0)) = \phi_0$. By differentiating both sides of (6), and recalling that $v_i(t; 0, (\gamma, \phi_0)) = d\psi_i(t; 0, (\gamma, \phi_0))/dt = d\phi_i(t; 0, \gamma)/dt$, $i \in \{1, 2, \ldots, N\}$, we have

$$C\frac{dv_i(t; 0, (\gamma, \phi_0))}{dt} = -\hat{q}'(\phi_i(t; 0, \gamma))\frac{d\phi_i(t; 0, \gamma)}{dt} + \sum_{j=1}^N g_{ij}\frac{d\phi_j(t; 0, \gamma)}{dt} + \sum_{j=1}^N g_{ij}^\tau \frac{d\phi_j(t - \tau; 0, \gamma)}{dt}$$

i.e.,

$$C\frac{dv_i(t; 0, (\gamma, \phi_0))}{dt} = -\hat{q}'(\psi_i(t; 0, (\gamma, \phi_0)))v_i(t; 0, (\gamma, \phi_0)) + \sum_{j=1}^N g_{ij}v_j(t; 0, (\gamma, \phi_0)) + \sum_{j=1}^N g_{ij}^\tau v_j(t - \tau; 0, (\gamma, \phi_0)).$$

From (7) we have $v_i(t; 0, (\gamma, \phi_0)) = \gamma_i(t) \in C([-\tau, 0], \mathbb{R}^N)$ and, from Property 1, we have that $V(t; 0, (\gamma, \phi_0)) = d\Phi(t; 0, \gamma)/dt$ is a continuous function on $[-\tau, 0]$ and it is differentiable for $t \geq 0$. As a consequence, we have that $[V(t; 0, (\gamma, \phi_0)), \Psi(t; 0, (\gamma, \phi_0))]$ is a solution of the IVP (25), (26) for $t \geq -\tau$.

Assume for contradiction that the IVP (25), (26) does not have a unique solution. Namely, suppose that, in addition to $[V(t; 0, (\gamma, \phi_0)), \Psi(t; 0, (\gamma, \phi_0))]$, there exists a different solution $[\widetilde{V}(t; 0, (\gamma, \phi_0)), \widetilde{\Psi}(t; 0, (\gamma, \phi_0))]$ of (25), (26). Integrating both sides of (25), and noting that $\int_0^t \hat{q}'(\tilde{\psi}_i(s; 0, (\gamma, \phi_0)))\tilde{v}_i(s; 0, (\gamma, \phi_0))ds = \int_{\tilde{\psi}_i(0; 0, (\gamma, \phi_0))}^{\tilde{\psi}_i(t; 0, (\gamma, \phi_0))} \hat{q}'(\tilde{\psi}_i)d\tilde{\psi}_i = \hat{q}(\tilde{\psi}_i(t; 0, (\gamma, \phi_0))) - \hat{q}(\tilde{\psi}_i(0; 0, (\gamma, \phi_0)))$, we have

$$C(\tilde{v}_i(t; 0, (\gamma, \phi_0)) - \tilde{v}_i(0; 0, (\gamma, \phi_0))) = -\hat{q}(\tilde{\psi}_i(t; 0, (\gamma, \phi_0))) + \hat{q}(\tilde{\psi}_i(0; 0, (\gamma, \phi_0))) + \sum_{j=1}^N g_{ij}\int_0^t \tilde{v}_j(z; 0, (\gamma, \phi_0))dz$$

$$+ \sum_{j=1}^N g_{ij}^\tau \int_0^t \tilde{v}_j(z - \tau; 0, (\gamma, \phi_0))dz$$

where $\tilde{v}_i(0; 0, (\gamma, \phi_0)) = \gamma_i(0)$. By letting $\sigma = z - \tau$ we have

$$C\frac{d\tilde{\psi}_i(t; 0, (\gamma, \phi_0))}{dt} - C\gamma_i(0) = -\hat{q}(\tilde{\psi}_i(t; 0, (\gamma, \phi_0))) + \hat{q}(\tilde{\psi}_i(0; 0, (\gamma, \phi_0))) + \sum_{j=1}^{N} g_{ij}(\tilde{\psi}_j(t; 0, (\gamma, \phi_0)) - \tilde{\psi}_j(0; 0, (\gamma, \phi_0)))$$

$$+ \sum_{j=1}^{N} g_{ij}^{\tau}\left(\int_0^{t-\tau} \tilde{v}_j(z; 0, (\gamma, \phi_0))dz + \int_{-\tau}^0 \tilde{v}_j(z; 0, (\gamma, \phi_0))dz\right).$$

From (25) and (7) we have $\int_0^{t-\tau} \tilde{v}_j(z; 0, (\gamma, \phi_0))dz = \tilde{\psi}_j(t - \tau; 0, (\gamma, \phi_0)) - \tilde{\psi}_j(0; 0, (\gamma, \phi_0))$ and consequently

$$C\frac{d\tilde{\psi}_i(t; 0, (\gamma, \phi_0))}{dt} = -\hat{q}(\tilde{\psi}_i(t; 0, (\gamma, \phi_0))) + \sum_{j=1}^{N} g_{ij}\tilde{\psi}_j(t; 0, (\gamma, \phi_0)) + \sum_{j=1}^{N} g_{ij}^{\tau}\tilde{\psi}_j(t - \tau; 0, (\gamma, \phi_0)) + C\gamma_i(0)$$

$$+ \hat{q}(\tilde{\psi}_i(0; 0, (\gamma, \phi_0))) - \sum_{j=1}^{N}(g_{ij} + g_{ij}^{\tau})\tilde{\psi}_j(0; 0, (\gamma, \phi_0)) + \sum_{j=1}^{N} g_{ij}^{\tau}\int_{-\tau}^0 \gamma_j(z)dz$$

for any $i \in \{1, 2, \ldots, N\}$. This shows that $\widetilde{\Phi}(t; 0, \gamma)$ defined as

$$\widetilde{\Phi}(t; 0, \gamma) = \begin{cases} \phi_0 + \int_0^t \gamma(z)dz & \text{if} \quad t \in [-\tau, 0] \\ \widetilde{\Psi}(t; 0, (\gamma, \phi_0)) & \text{if} \quad t \geq 0 \end{cases}$$

is solution of the IVP (6). Since the IVP (6) and (7) admits a unique solution, we necessarily have $\Psi(t; 0, (\gamma, \phi_0)) = \widetilde{\Psi}(t; 0, (\gamma, \phi_0))$ and then $V(t; 0, (\gamma, \phi_0)) = \widetilde{V}(t; 0, (\gamma, \phi_0))$. ∎

2. Derives from Proposition 1 and the link between the solutions of the two IVPs previously established. ∎

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