Weakly precomplete computably enumerable equivalence relations

WEAKLY PRECOMPLETE COMPUTABLY ENUMERABLE EQUIVALENCE RELATIONS

SERIKZHAN BADAEV AND ANDREA SORBI

Abstract. Using computable reducibility $\leq$ on equivalence relations, we investigate weakly precomplete ceers (a "ceer" is a computably enumerable equivalence relation on the natural numbers), and we compare their class with the more restricted class of precomplete ceers. We show that there are infinitely many isomorphism types of universal (in fact uniformly finitely precomplete) weakly precomplete ceers , that are not precomplete; and there are infinitely many isomorphism types of non-universal weakly precomplete ceers. Whereas the Visser space of a precomplete ceer always contains an infinite effectively discrete subset, there exist weakly precomplete ceers whose Visser spaces do not contain infinite effectively discrete subsets. As a consequence, contrary to precomplete ceers which always yield partitions into effectively inseparable sets, we show that although weakly precomplete ceers always yield partitions into computably inseparable sets, nevertheless there are weakly precomplete ceers for which no equivalence class is creative. Finally, we show that the index set of the precomplete ceers is Σ^0_3-complete, whereas the index set of the weakly precomplete ceers is Π^0_3-complete. As a consequence of these results, we also show that the index sets of the uniformly precomplete ceers and of the e-complete ceers are Σ_3-complete.

1. Introduction

Precomplete equivalence relations, introduced by Mal'cev [13], and extensively studied by Ershov [9], play an important role in computability theory, and in the theory of numberings. Recall that an equivalence relation E on the set of natural numbers ω is *precomplete* if there exists a total computable function $f(e,x)$, called an *E-totalizer*, such that for all e, x,

$$\varphi_e(x)\downarrow \Rightarrow \varphi_e(x) \; E \; f(e,x),$$

or, equivalently, thanks to the Ershov Fixed Point Theorem [9], there exists a total computable function g, such that, for all e,

$$\varphi_e(g(e))\downarrow \Rightarrow \varphi_e(g(e)) \; E \; g(e).$$

A natural extension of precompleteness was proposed by Badaev [2], by weakening the above fixed point property:

Definition 1.1. [2] An equivalence relation E is *weakly precomplete* if there exists a partial computable function fix such that, for all e,

$$\varphi_e \text{ total} \Rightarrow [\mathrm{fix}(e)\downarrow \; \& \, \varphi_e(\mathrm{fix}(e)) \; E \; \mathrm{fix}(e)].$$

2010 *Mathematics Subject Classification.* 03D25.

Key words and phrases. Computably enumerable equivalence relation; weakly precomplete equivalence relation.

This work was partially supported by Grant 3952/GF4 of the Science Committee of the Republic of Kazakhstan. Part of the research contained in this paper was carried out while Badaev was INDAM-GNSAGA Visiting Professor at the Department of Mathematics and Computer Science "Roberto Magari" of the University of Siena, Italy, July 2011. Badaev wishes to thank INDAM-GNSAGA for supporting the visiting professorship. Sorbi is a member of INDAM-GNSAGA, and has been partially supported by PRIN 2102 *Logica, Modelli e Insiemi*.

This paper is almost entirely dedicated to the study of weakly precomplete computably enumerable equivalence relations. Recall that a computably enumerable equivalence relation is also called a *ceer*, or (in the Russian literature) a *positive* equivalence relation.

Precomplete ceers have been studied in several papers, of which, particularly relevant to our purposes are [1, 2, 3, 4, 5, 8, 11, 12, 14, 15, 18, 20]. For important examples of precomplete ceers that naturally arise in logic, see e.g. [20]. Precomplete ceers have been also investigated in relation to the following reducibility $\leq$ on equivalence relations, due to Ershov [9]: given equivalence relations R, S on ω, say that R is *reducible to* S, if there is a computable function f such that, for all x, y,

$$x \; R \; y \Leftrightarrow f(x) \; S \; f(y).$$

It was shown in [5] that every nontrivial (i.e., different from ω^2) precomplete ceer S is *universal* with respect to the class of ceers, i.e., for every ceer R, one has $R \leq S$. Lachlan [12] proved that any two nontrivial precomplete ceers R, S are *isomorphic* (notation: $R \simeq S$), i.e., there exists a computable permutation f of ω such that $R \leq S$ via f (and, consequently, $S \leq R$ via f^{-1}).

On the other hand, weakly precomplete ceers have not been extensively studied. A first useful observation is:

Lemma 1.2. *A ceer E is weakly precomplete if and only if*

$$(\forall e)[\varphi_e \text{ } total \ \Rightarrow (\exists n)[\varphi_e(n) \; E \; n]].$$

Proof. ($\Leftarrow$) Under the assumption, the partial function fix, defined by

$$\mathrm{fix}(e) = \text{first } n \, [\varphi_e(n)\downarrow \; E \; n],$$

is computable, and witnesses the fact that E is weakly precomplete.

($\Rightarrow$) On the other hand, if fix is a partial computable function witnessing that E is weakly precomplete, then, given e, if φ_e is total we have that fix(e) is the desired fixed point for φ_e, modulo E: this shows the claim. □

Definition 1.3. We say that a total function d is a *diagonal* function for an equivalence E if, for every x, $d(x) \not\!\!E \; x$.

Corollary 1.4. *A ceer E is weakly precomplete if and only if E has no computable diagonal function.*

Proof. Immediate. □

Contrary to the fact that all nontrivial precomplete ceers fall into a unique isomorphism type, it is known [2], that there are infinitely many distinct isomorphism types of weakly precomplete ceers.

Montagna [15] proposed the following definition, where, given a set X and an equivalence relation E, we define

$$[X]_E = \{x : (\exists y \in X)[x \; E \; y]\} :$$

Definition 1.5. [15] An equivalence relation E is *uniformly finitely precomplete* (abbreviated as *u.f.p.*) if there exists a total computable function $f(D, e, n)$ (where D is a finite set given by its canonical index) such that, for all D, e, n,

$$\varphi_e(n)\downarrow \in [D]_E \ \Rightarrow \varphi_e(n) \; E \; f(D, e, n).$$

Clearly, every precomplete equivalence relation is also u.f.p., and it can be shown, see [15], that nontrivial u.f.p. ceers are universal. Another important class of u.f.p. ceers is provided by the e-complete ceers, i.e., the u.f.p. ceers E which possess a computable diagonal function. Montagna [15], and independently Lachlan [12], proved that if R, S are e-complete ceers, then $R \simeq S$, hence, as for nontrivial precomplete ceers, e-complete ceers forms a single isomorphism type. By Corollary 1.4 it follows:

Corollary 1.6. [4] *Every u.f.p. ceer is either e-complete or weakly precomplete.*

Shavrukov [18] showed that there are u.f.p. ceers that are neither precomplete nor e-complete.

Remark 1.7. Unless otherwise specified, in the following, all ceers will be understood to be nontrivial.

We shall use the symbol Id to denote the equivalence relation $=$.

Following [1], we say that a sequence $\{R^s : s \in \omega\}$ of equivalence relations on ω is a *computable approximation to* a ceer R, if

(1) the set $\{\langle x, y, s\rangle : x\ R^s\ y\}$ is computable;
(2) $R^0 = \mathrm{Id}$;
(3) for all s, $R^s \subseteq R^{s+1}$; the equivalence classes of R^s are finite; there exists at most one pair $[x]_{R^s}, [y]_{R^s}$ of equivalence classes, such that $[x]_{R^s} \cap [y]_{R^s} = \emptyset$, but $[x]_{R^{s+1}} = [y]_{R^{s+1}}$ (we say in this case that the equivalence relation *R-collapses* x and y at stage $s + 1$);
(4) $R = \bigcup_t R^t$.

1.1. **A little excursus into numberings.** Recall that a *numbered set* is a pair $\langle \nu, S\rangle$, where S is a nonempty countable set, and $\nu : \omega \longrightarrow S$ is a *numbering*, i.e. an onto function. A *morphism* of two numbered sets $\langle \nu_1, S_1\rangle$ and $\langle \nu_2, S_2\rangle$ is a function $\mu : S_1 \longrightarrow S_2$, for which there exists a computable function f, such that $\nu_2 \circ f = \mu \circ \nu_1$ (we say that f *induces* μ). For more on the theory of numberings the reader is invited to look at [9], or [10].

Given an equivalence relation E on ω, let $\langle \nu_E, S_E\rangle$ be the numbered set in which $S_E = \{[x]_E : x \in \omega\}$ is the set of equivalence classes of E, and $\nu_E : \omega \longrightarrow S_E$ is given by $\nu_E(x) = [x]_E$. If R and E are equivalence relations and μ is a morphism of the numbered sets $\langle \nu_R, S_R\rangle$ and $\langle \nu_E, S_E\rangle$, we will also write $\mu : R \longrightarrow E$.

It is immediate to see that, for ceers R, E, one has that $R \leq E$ if and only if there exists a monomorphism, i.e. 1-1 morphism $\mu : R \longrightarrow E$. Similarly, if $R \simeq E$ then there exists a 1-1 and onto morphism $\mu : R \longrightarrow E$; moreover, the converse is true if the equivalence classes of R and E are infinite. (The converse need not be true in general: for instance, let 2ω and $2\omega + 1$ be the equivalence classes of R, and let $\{0\}$ and $\omega \smallsetminus \{0\}$ be the equivalence classes of E; then there there is 1-1 and onto morphism $\mu : R \longrightarrow E$, but $R \not\simeq S$, as is easily seen.)

2. Weakly precomplete ceers and universality

It is known, see [2], that there exist infinitely many weakly precomplete ceers which are pairwise not isomorphic. Badaev's construction in [2] was adapted by Shavrukov [18] to show that there exists a u.f.p. ceer which is neither precomplete nor e-complete, thus showing that there is a universal weakly precomplete ceer that is not precomplete.

In this section, we show (Corollary 2.4) that there exist infinitely many pairwise non-isomorphic weakly precomplete ceers, that are universal (in fact u.f.p.); and we show (Theorem 2.6) that there exist infinitely many pairwise non-isomorphic weakly precomplete ceers, that are not universal.

These results are visualized in Figure 1 (we recall that all u.f.p. ceers are universal).

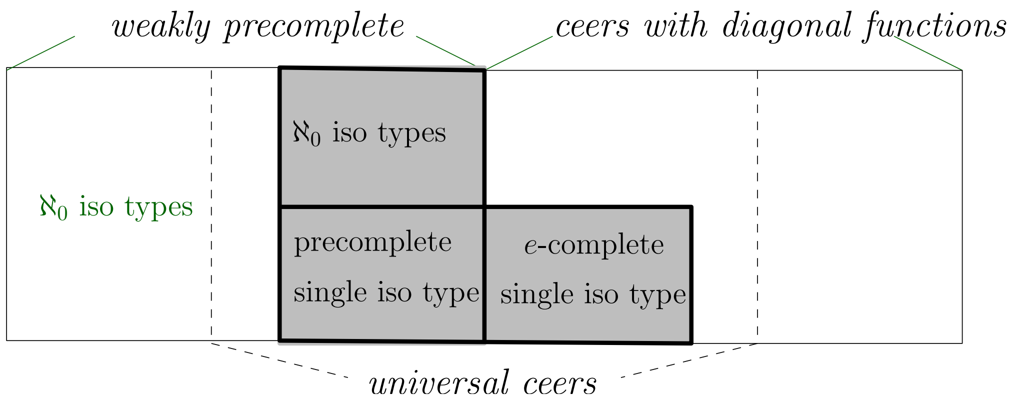

FIGURE 1. Weakly precomplete ceers vs. universal ceers (u.f.p. ceers are in gray)

2.1. **Universal weakly precomplete ceers.** We now prove a theorem, which implies (Corollary 2.4) that there exist infinitely many non isomorphic u.f.p. ceers, thus implying both Badaev's result and Shavrukov's result.

First, we recall, see [4]:

Definition 2.1. A *strong diagonal function* d for an equivalence relation E, is a total computable function satisfying, for every finite set D,

$$d(D) \notin [D]_E.$$

Thus, if d is a strong diagonal function for E then there is an effective procedure for finding, given any finite set D, a number which is not E-equivalent to any number in D. It is known, [4], that every e-complete ceer has a strong diagonal function.

Theorem 2.2. *If E is a ceer, such that E has a strong diagonal function, then there exist infinitely many ceers $\{E_i : i \in \omega\}$ such that, for every i, j, and*

$$E \subseteq E_i \,\&\, [i \neq j \Rightarrow E_i \not\cong E_j].$$

Proof. Let E be a given ceer, such that E has a strong diagonal function d.

The requirements. We want to construct a countable set $\{E_i : i \in \omega\}$ of ceers such that for every i, $E \subseteq E_i$, and satisfying the requirements, for every i, j, k, with $i \neq j$,

$$P_{i,j,k} : \varphi_k \text{ is total} \Rightarrow \varphi_k \text{ does not induce an isomorphism from } E_i \text{ onto } E_j.$$

Satisfaction of all requirements implies our claim, as for every isomorphism there is a total computable function inducing it. The *priority ordering* of requirements is given by

$$P_{i,j,k} < P_{i',j',k'} \Leftrightarrow \langle i, j, k\rangle < \langle i', j', k'\rangle.$$

Strategy to meet $P_{i,j,k}$. We describe the strategy to meet $P_{i,j,k}$ in isolation, which is of course implemented at certain stages s: hence E_i and E_j have to be understood as their approximations E_i^s and E_j^s, respectively, and in particular at each such stage, $[a_0]_{E_i}$ is a finite set:

(1) choose a witness b_0;
(2) wait for a number a_0 such that $\varphi_k(a_0){\downarrow}\, E_j\; b_0$;
(3) let $a_1 = d([a_0]_{E_i})$, and wait for $\varphi_k(a_1){\downarrow}$;
(4) if, say, $\varphi_k(a_1) = b_1$ then E_j-collapse b_0 and b_1, and restrain $a_0 \not\mathrel{E_i} a_1$.

Outcomes for the strategy to meet $P_{i,j,k}$. Here are the outcomes of the strategy:

(i) if we wait forever at (2), then we meet $P_{i,j,k}$ since φ_k, even if total, does not induce an onto morphism;
(ii) if we wait forever at (3), then we win $P_{i,j,k}$ since φ_k is not total;
(iii) if we act in (4), then we win $P_{i,j,k}$ since φ_k, even if total, does not induce a monomorphism.

Interactions between strategies. Let $P = P_{i,j,k}$ and $P' = P_{i',j',k'}$ be two requirements, and let us look at how P and P' interact: suppose that $P < P'$.

Each strategy may want to restrain two equivalence classes from collapsing with each other, in order to have the winning outcome (iii). So, suppose that P wants to restrain $a_0 \not\mathrel{E_i} a_1$: notice that P' may want to E_i-collapse a_0 and a_1 only if $j' = i$. When P makes the commitment of restraining $a_0 \not\mathrel{E_i} a_1$, it resets P', so that, using the strong diagonal function d, P' chooses a new witness b_0' which is not E-equivalent to any number in the current $[a_0]_{E_i} \cup [a_1]_{E_i}$. (This is done by taking $b_0' = d(X)$, where X is the set of all numbers so far mentioned in the construction.) Then the possible future collapsing action taken by P' through (4) may at most enlarge one, but not both, of the two classes $[a_0]_{E_i}$, $[a_1]_{E_i}$, by adjoining some $[b_1']_{E_i}$ to one of $[a_0]_{E_i}$, $[a_1]_{E_i}$, leaving however the two new classes distinct. On the other hand, no unwanted collpase is caused by E, by our choice of witnesses using the strong diagonal function.

The construction. The construction is by stages. At stage s we define, for every i, a ceer E_i^s: we begin with $E_i^0 = E$, and for $s > 0$, E_i^s extends E_i^{s-1}, by collapsing at most two classes; thus $\{E_i^s : s \in \omega\}$ is a uniformly c.e. tower of ceers, and $E_i = \bigcup_s E_i^s$ will be our desired ceer. Each requirements $P_{i,j,k}$ has four parameters $b_0(i,j,k,s), b_1(i,j,k,s), a_0(i,j,k,s), a_1(i,j,k,s)$. In the following, if a parameter, or E_i^s, is not explicitly defined at stage s, then it is understood that the value at s is the same as at the previous stage.

We *initialize* $P_{i,j,k}$ *at* s if we set each one of its parameters to be undefined.

We say that $P_{i,j,k}$ *requires attention at* s if one of the following in the given order happens:

(1) $b_0(i,j,k,s)$ is undefined; or
(2) there is some a such that $\varphi_{k,s}(a){\downarrow}\, E_j^s\; b_0(i,j,k,s)$, and $a_0(i,j,k,s)$ is undefined; or
(3) $\varphi_{k,s}(a_1(i,j,k,s)){\downarrow}$, and $b_1(i,j,k,s)$ is undefined.

Stage 0. Initialize all requirements, and let $E_i^0 = E$, for all i.

Stage $s+1$. Let $P = P_{i,j,k}$ be the least requirements to require attention: by initialization, there always is such a requirement. We *act* on P, according to the following:

(1) if P requires attention through item (1), then let X be the set of all numbers so far mentioned in the construction, and let $b_0(i,j,k,s+1) = d(X)$;
(2) if P requires attention through item (2), then choose such an a, let $a_0(i,j,k,s+1) = a$, and $a_1(i,j,k,s+1) = d(X)$, where X is the set of all numbers so far mentioned in the construction;
(3) if P requires attention through item (3), define $b_1(i,j,k,s+1) = \varphi_{k,s}(a_1(i,j,k,s+1))$, and E_j-collapse $b_0(i,j,k,s)$ and $b_1(i,j,k,s+1)$: in other words, let E_j^{s+1} be the equivalence relation obtained by adjoining the equivalence classes of these two numbers.

Initialize all requirements $R > P$, and go to Stage $s+2$. Finally, for every i, take $E_i = \bigcup_s E_i^s$.

Verification. Clearly $E \subseteq E_i$, for every i. The rest of the verification is based on the following lemma.

Lemma 2.3. *For every i,j,k, the requirement $P_{i,j,k}$ requires attention only finitely often, and $\lim_s b_0(i,j,k,s) = b_0(i,j,k)$ exists; moreover, if φ_k is total then $P_{i,j,k}$ is satisfied.*

Proof. By induction on $\langle i,j,k\rangle$ (with $i \neq j$). For the least triple $\langle i,j,k\rangle$ with $i \neq j$, we can assume that φ_k is the function with the empty graph: then $P_{i,j,k}$ requires attention at stage 1 only.

Now, suppose that the claim is true for every $\langle i',j',k'\rangle < \langle i,j,k\rangle$ (again, assume that $i \neq j$ and $i' \neq j'$, for all such i', j'). Let s_0 be the least stage at which no $P_{i',j',k'}$, with $\langle i',j',k'\rangle < \langle i,j,k\rangle$, requires attention. As the parameter $b_0(i,j,k,s_0-1)$ is undefined by initialization, $P_{i,j,k}$ requires attention at stage s_0 through item (1), hence, $b_0(i,j,k,s_0)$ equals the final value $b_0(i,j,k)$. If we never move to (2) at stages $s > s_0$ of the construction, then $P_{i,j,k}$ does no longer require attention, and φ_k does not induce an onto morphism, and thus $P_{i,j,k}$ is satisfied. Otherwise, let $s_1 > s_0$ be the least stage such that at stage s_1, $P_{i,j,k}$ requires attention through item (2): then we define at s_1 the final values $a_0(i,j,k)$, and $a_1(i,j,k)$, which are not E_i-equivalent. If $P_{i,j,k}$ will not require attention after s_1, then φ_k is not total, and thus $P_{i,j,k}$ is satisfied. Assume that φ_k is total: this implies that, at some stage $s_2 > s_1$, $P_{i,j,k}$ requires attention through item (3), and we act as in (3) of the construction; in particular we define the final value $b_1(i,j,k)$, and we E_j-collapse $b_0(i,j,k)$ and $b_1(i,j,k)$. After this, $P_{i,j,k}$ does not require attention any more; moreover $a_0(i,j,k)$ and $a_1(i,j,k)$ will never E_i-collapse, since lower-priority strategies choose, using the strong diagonal function d, witnesses that are not E_i-equivalent to either $a_0(i,j,k)$ or $a_1(i,j,k)$; hence φ_k does not induce a monomorphism, and thus $P_{i,j,k}$ is satisfied. □

This ends the proof of the theorem. □

Corollary 2.4. *There exist infinitely many non-isomorphic u.f.p. ceers.*

Proof. By a result by Bernardi and Montagna [4], the u.f.p. ceers coincide with the nontrivial quotients of any e-complete ceer. Moreover, again by [4], any e-complete ceer has a strong diagonal function. Thus, if in the previous theorem we start with an e-complete ceer E, then we have that the collection $\{E_i : i \in \omega\}$ consists of pairwise non-isomorphic u.f.p. ceers. □

Corollary 2.5. *There exist infinitely many weakly precomplete ceers that are u.f.p. (hence universal), not precomplete, and pairwise non-isomorphic.*

Proof. The result follows now from Corollary 1.6, Corollary 2.4 and the following facts: the precomplete ceers are all isomorphic ([12]); and the e-complete ceers are all isomorphic ([15, 12]). □

2.2. **Non-universal weakly precomplete ceers.** Although not isomorphic, the weakly precomplete ceers of the previous section are all u.f.p., and thus universal. In this section we prove that there exist infinitely many non-isomorphic weakly precomplete ceers that are not universal.

Theorem 2.6. *There exist infinitely many weakly precomplete ceers $\{E_i : i \in \omega\}$ such that each E_i is not universal, and if $i \neq j$ then $E_i \not\simeq E_j$.*

Proof. We want to build ceers $\{E_i : i \in \omega\}$ satisfying the following requirements (recall that Id denotes the identity relation):

Requirements. For every e, i, j, k with $i \neq j$ the requirements to be satisfied are:

$$\begin{aligned} N_{e,i}: &\quad \varphi_e \text{ total} \Rightarrow (\exists x_{e,i})[\varphi_e(x_{e,i})\ E_i\ x_{e,i}],\\ P_{i,j,k}: &\quad \varphi_k \text{ does not induce an isomorphism } E_i \simeq E_j,\\ Q_{i,k}: &\quad \varphi_k \text{ does not induce a reduction Id} \leq E_i. \end{aligned}$$

Strategy for $N_{e,i}$ in isolation. The strategy for $N_{e,i}$ is:

(1) appoint a new witness $x_{e,i}$;
(2) wait for $\varphi_e(x_{e,i})$ to converge;
(3) E_i-collapse $x_{e,i}$ and $\varphi_e(x_{e,i})$.

Outcomes. The outcomes of the strategy are clear.

Strategy for $P_{i,j,k}$ in isolation, with $i \neq j$. If $i \neq j$, the strategy for $P_{i,j,k}$ can be described as follows:

(1) pick new witnesses $b_0(i,j,k)$ and $b_1(i,j,k)$ (thus, not E_j-equivalent), and momentarily restrain $b_0(i,j,k)\ \cancel{E_j}\ b_1(i,j,k)$;
(2) wait for two numbers c_0, c_1 to appear such that $\varphi_k(c_r)\ E_j\ b_r(i,j,k)$, with $r = 0, 1$;
(3) once the first pair of such numbers c_0 and c_1 has appeared then
    (a) if $c_0\ \cancel{E_i}\ c_1$ then E_i-collapse $b_0(i,j,k)$ and $b_1(i,j,k)$, and restrain $c_0\ \cancel{E_i}\ c_1$;
    (b) if already $c_0\ E_i\ c_1$, then keep the restraint $b_0(i,j,k)\ \cancel{E_i}\ b_1(i,j,k)$.

Outcomes. The outcomes of the strategy are, once again, clear:

(i) if we wait at (2), then φ_k can not induce an onto morphism;
(ii) if (3) holds then either $c_0\ E_i\ c_1$ when they appear (case (3b)), and thus φ_k can not induce a morphism as we restrain $b_0(i,j,k)\ \cancel{E_j}\ b_1(i,j,k)$; or (case (3a)) $c_0\ \cancel{E_i}\ c_1$ and $b_0(i,j,k)\ E_i\ b_1(i,j,k)$, and thus φ_k can not induce a monomorphism.

Strategy for $Q_{i,k}$ in isolation. The strategy for $Q_{i,k}$ can be described as follows:

(1) appoint new witnesses $a_0(i,k)$, $a_1(i,k)$;
(2) wait for $\varphi_k(a_0(i,k))$ and $\varphi_k(a_1(i,k))$ to converge, say $\varphi_k(a_0(i,k)))\downarrow = d_0$ and $\varphi_k(a_1(i,k)))\downarrow = d_1$;
(3) E_i-collapse d_0 and d_1.

Outcomes. Here are the outcomes of the strategy:

(i) if we wait at (2), then φ_k is not total;
(ii) if (3) then φ_k can not induce a monomorphism;

Interactions between strategies. There are obvious conflicts between strategies that want to E_i-collapse pairs of numbers (for some i), and strategies that want to restrain non E_i-equivalent pairs of numbers, but all strategies are finitary, and can be easily combined by a standard finite injury argument. We suppose to fix a priority ordering $<$ on the requirements of order type ω: if $R < R'$, then we say that R has *higher-priority than* R', or, equivalently R' has *lower-priority than* R.

Notice that only P-requirements want to restrain equivalence classes from being collapsed. So, let $R = P_{i,j,k}$ be a P-requirement. How does R restrain two equivalence classes against the action of a lower-priority $R' > R$? If R' is a P-requirement, then the restraint is guaranteed simply by initialization when R acts, so that the lower-priority R' starts anew with new witnesses outside the two equivalence classes; therefore a possible future collapsing action performed by R' does not interfere at all with the two equivalence classes, and thus with the restraint imposed by R. Suppose next that R' is an N-requirement, say $R' = N_{e,i'}$, and i' is either i or j, according to whether R wants to restrain $c_0 \not\!\!E_i\, c_1$, or $b_0(i,j,k) \not\!\!E_j\, b_1(i,j,k)$: in this case, it is true that $x_{e,i'}$ is chosen outside of the two equivalence classes, which R wants to restrain, but $\varphi_e(x_{e,i'})$ may in the future converge to a number in one of the two equivalence classes, and R' may want to $E_{i'}$-collapse $x_{e,i'}$ and $\varphi_e(x_{e,i'})$; but this action merely enlarges one of the two equivalence classes, without $E_{i'}$-collapsing them. Finally, suppose $R' = Q_{i',k'}$, where again i' is either i or j: To make sure that R' will be respectful of the restraint imposed by R, let $n(R')$ be the number of requirements having higher priority than R': each one restrains at most 2 equivalence classes, so altogether, R' has to deal with at most $2n(R')$ restrained equivalence classes. For this, it is enough to slightly modify the strategy in isolation, and allow R' to work with $2n(R') + 2$ distinct witnesses $a_r(i', k')$. Either $\varphi_{k'}$ is not defined on all these witnesses, and thus we win the requirement; or for each $r < 2n(R')+2$ we find d_r such that $\varphi_{k'}(a_r(i', k'))\downarrow = d_r$. At this point, either, already $d_h\ E_{i'}\ d_k$, for some $h < k < 2n(R')+2$, and thus we win with no further action; or there exist $h < k < 2n(R') + 2$, such that neither of the equivalence classes of d_h and d_k is restrained, so that we can $E_{i'}$-collapse the two of them, without interfering with higher-priority requirements.

To facilitate recognition of restrained equivalence classes, whenever a requirement R wants to restrain two equivalence classes, we label these classes with a *marker* $[R]$, which may be subsequently cancelled, i.e., taken off the equivalence classes.

The construction. The construction is by stages. At stage s we define, for every i, j, k, an approximation E_i^s to E_i; we use parameters $x_{e,i}(s)$, $b_r(i,j,k,s)$ $(r < 2)$, $a_r(i,k,s)$ $(r < 2n(Q_{i,k})+2$, where $n(Q_{i,k})$ is the number of requirements having priority higher than $Q_{i,k})$; at any stage $s > 0$, unless otherwise specified, each parameter retains the same value from the previous stage.

Let us say that a number is *new* if it is bigger than all numbers already E_i-equivalent (all i)to numbers so far mentioned in the construction. We say that a requirement R is *initialized at stage* s if we set the relative parameters to be undefined, and we cancel the restraint imposed by it, by taking the marker $[R]$ off the equivalence classes labelled by this marker.

Requiring attention. At stage s, we say that $N_{e,i}$ *requires attention* if, in the order,

(1) $x_{e,i}(s)$ is undefined; or
(2) $\varphi_{e,s}(x_{e,i}(s))\downarrow$, but $\varphi_e(x_{e,i}(s))$ and $x_{e,i}(s)$ are not as yet E_i-equivalent.

We say that $P_{i,j,k}$ *requires attention* if,

(1) for every $r < 2$, $b_r(i,j,k,s)$ is undefined; or
(2) for every $r < 2$, $b_r(i,j,k,s)$ is defined, with $b_0(i,j,k,s) \not\mathrel{E_j^s} b_1(i,j,k,s)$, and a number c_r has appeared such that $\varphi_{k,s}(c_r)\ E_j^s\ b_r(i,j,k,s)$, and $P_{i,j,k}$ has not as yet acted since the last time the witnesses $b_r(i,j,k,s)$ were defined.

We say that $Q_{i,k}$ *requires attention* if, in the order, where $m = 2n(Q_{i,k}) + 2$,

(1) for every $r < m$, $a_r(i,k,s)$ is undefined; or
(2) for every $r < m$, $a_r(i,k,s)$ is defined and a number d_r has appeared for which $\varphi_{k,s}(a_r(i,k,s))\downarrow = d_r$, and $Q_{i,k}$ has not as yet acted since the last time the witnesses $a_r(i,k,s)$ were defined.

Stage 0. Let $E_i^0 = \mathrm{Id}$; initialize all strategies.

Stage $s+1$. Let R be the least requirement that requires attention at stage $s+1$: notice that by initialization, there always is such a requirement. We take action according to whether R is an N-requirement, or a P-requirement, or a Q-requirement:

(1) $R = N_{e,i}$:
  (a) If R requires attention through (1), then define $x_{e,i}(s+1)$ to be new (we may assume that φ_e is still undefined on it);
  (b) If R requires attention through (2), then E_i-collapse $\varphi_e(x_{e,i}(s))$ and $x_{e,i}(s)$;
(2) $R = P_{i,j,k}$:
  (a) If R requires attention through (1), then choose two distinct new numbers and use them for the values of parameters $b_r(i,j,k,s+1)$, $r < 2$; put the marker $[R]$ on the equivalence classes of $b_0(i,j,k,s+1)$, and $b_1(i,j,k,s+1)$;
  (b) If R requires attention through (2), and $c_0 \not\mathrel{E_i^s} c_1$, then E_j-collapse $b_0(i,j,k,s)$ and $b_1(i,j,k,s)$, take the marker $[R]$ off the equivalence classes of $b_0(i,j,k,s+1)$ and $b_1(i,j,k,s+1)$, and put it on the equivalence classes of c_0 and c_1. If already $c_0\ E_i^s\ c_1$, then do nothing, so leave $[R]$ on the equivalence classes of $b_0(i,j,k,s)$ and $b_1(i,j,k,s)$.
(3) $R = Q_{i,k}$:
  (a) If R requires attention through (1), then choose $m = 2n(R) + 2$ distinct new numbers (we may assume that φ_k is still undefined on all of them) and use them for the values of parameters $a_r(i,k,s+1)$, $r < m$;
  (b) If R requires attention through (2), and there are no $l, h < m$, $l \neq h$ such that already $d_l\ E_i\ d_h$, then choose and E_i-collapse d_l and d_h, where $l, h < m$, $l \neq h$, is the least pair such that the equivalence classes of d_l and d_h are not labelled by any R', with $R' < R$.

Initialize all $R' > R$ and go to stage $s+2$.

Finally, take $E_i = \bigcup_s E_i^s$, for every i.

Verification. The verification is based on the following lemma. We say that at stage s the current parameter $x_{e,i}(s)$ relative to a requirement $N_{e,i}$ is *active* if $\varphi_e(x_{e,i}(s))$ is still undefined, or $x_{e,i}(s)$ and $\varphi_e(x_{e,i}(s))$ are not as yet E_i-equivalent.

Lemma 2.7. *Each requirement requires attentions only finitely often and is eventually satisfied.*

Proof. Assume that the claim is true of all requirements $R' < R$, and let $s_0 > 0$ be the least stage such that for all $s \geq s_0$, no requirement $R' < R$ requires attention, no parameter relative to any $R' < R$ changes, and by s_0 all $R' < R$ have been satisfied. Notice that whenever a requirement acts, its initializes all lower-priority requirements: therefore at $s_0 - 1$, the requirement exactly preceding R has acted, and at s_0 the requirement R is the highest-priority requirement to require attention. So at stage s_0, we define the final values of the relevant parameters of R, which will not not change at any further stages $s > s_0$ due to requirements $R' < R$.

We now distinguish the three possible cases:

Case $R = N_{e,i}$. We define the final value of $x_{e,i}$ at stage s_0. If $\varphi_e(x_{e,i})$ does not converge then φ_e is not total, R is satisfied, and R will not require attention any stage $s > s_0$. Assume that $\varphi_e(x_{e,i})$ is defined at some stage $s_1 > s_0$. Then, at stage s_1, R requires attention through (2) again; we E_i-collapse $\varphi_e(x_{e,i})$ with $x_{e,i}$, so R is satisfied, and R will not require attention any more.

Case $R = P_{i,j,k}$, with $i \neq j$. At stage s_0, we define the final values of $b_r(i,j,k)$ for all $r < 2$. If $P_{i,j,k}$ does not require attention anymore after stage s_0 then at least one of the two numbers $b_r(i,j,k)$ is not in the E_j-closure of the range of φ_k, and, therefore, R is satisfied since φ_k does not induce an onto morphism.

Suppose now that $s_1 > s_0$ is the least stage at which c_0 and c_1 appear.

We first claim:

($\star$) for every s, with $s_0 \leq s < s_1$, the following hold: for every $R' > R$, if R' is a P-requirement and w is the current value of a parameter of R', or R' is an N-requirement and w is the current active parameter of R', then w does not lie in either of the E^s_j-equivalence classes of $b_0(i,j,k)$ and $b_1(i,j,k)$; and these classes are disjoint.

The proof of this claim is by induction on s. The claim is trivially true of s_0, by choice of $b_0(i,j,k)$ and $b_1(i,j,k)$, and initialization of all $R' > R$. If the claim is true of s, with $s_0 < s+1 < s_1$, then exactly one $R' > R$ acts at $s+1$. The claim clearly continues to be true, by initialization, if R' only acts by choosing its witnesses. Otherwise, if R' is a P-requirement, then the claim clearly continues to hold by the inductive assumption, and initialization of all $R'' > R'$. If R' is a Q-requirement, then the claim continues to hold by initialization of all $R'' > R'$, and the fact that if R' collapses two equivalence classes, then these classes are not marked by $[R]$. The only case which deserves some attention is when $R' = N_{e,j}$, and the witness $x_{e,j}$ ceases to become active, and R' collapses it to $\varphi_e(x_{e,j})$. But up to s, $x_{e,j}$ does not lie in the E^s_j-equivalence class of $b_0(i,j,k)$ nor in that of $b_1(i,j,k)$, and even if $\varphi_e(x_{e,j})$ lies in one of these equivalence classes, nonetheless the E^{s+1}_j-equivalence classes of $b_0(i,j,k)$ and $b_1(i,j,k)$, remain disjoint.

Having shown ($\star$), we can argue that at s_1, $b_0(i,j,k)$ and $b_1(i,j,k)$, are not E_j-equivalent, so R requires attention for the last time at s_1, R can act and achieve at s_1 that $c_0\ E_i\ c_1$ if and only if $b_0(i,j,k) \not{E}_j\, b_1(i,j,k)$, by restraining $c_0 \not{E}_i\, c_1$, or $b_0(i,j,k) \not{E}_j\, b_1(i,j,k)$, as needed, and putting the marker $[\not{R}]$ as needed. In order to conclude that R is satisfied, we must now show that the relevant restraint is preserved forever. We claim:

($\star\star$) Let w_0, w_1 be the witnesses whose equivalence classes are restrained by R, i.e. marked by $[R]$. For every $s \geq s_1$, the following hold: for every $R' > R$, if R' is a P-requirement and w is the current value of a parameter of R', or R' is an N-requirement and w is the current active parameter of R', then w does not lie in either of the E^s_j-equivalence classes of w_0 and w_1; and these classes are disjoint.

The verification of this claim is similar to that of ($\star$).

From claim ($\star\star$) it follows that R is satisfied.

Case $R = Q_{i,k}$. At s_0 we define the final values $a_r(i,k,s)$ for all $r < m = 2n(R)+2$. Thus, after s_0, either we wait forever for φ_k to converge on all $a_r(i,k,s)$, in which case the requirement no longer requires attention, and is satisfied since φ_k is not total; or, at some stage $s_1 > s_0$ we complete the list of the numbers d_r, $r < m$. At s_1, R requires attention for the last time. If there are $l, h < m$, $l \neq h$, such that already $d_l \; E_i \; d_h$, then R is satisfied. Otherwise, there is a least pair $l, h < m$, $l \neq h$ such that their equivalence classes are not labelled by any $[R']$, with $R' < R$: in this case we E_i-collapse them, and we win since a_h and a_k are not Id-equivalent. □

This completes the proof of the theorem. □

Corollary 2.8. *There exist infinitely many non-isomorphic weakly precomplete ceers that are not universal.*

Proof. The claim follows from the previous theorem, and the observation that all the ceers built by the theorem are weakly precomplete, with the exception of at most one ceer that may be trivial. □

3. Weak precompleteness, creativeness and the Visser topology

To any equivalence relation R on ω, one can associate a topological space. For this, recall from Section 1, that to every equivalence relation R one can associate a numbered set $\langle \nu_R, S_R \rangle$.

Definition 3.1. Given R, the *Visser space* V_R is the topological space for which

(1) the points are the elements of S_R, i.e. the R-equivalence classes;
(2) a basis is given by $\{S \subseteq S_R : \nu_R^{-1}[S] \in \Pi^0_1\}$.

It is clear that any morphism $\mu : R \longrightarrow S$ is in fact a continuous mapping from V_R to V_S. Since all precomplete ceers are isomorphic, the corresponding Visser spaces are all homeomorphic with each other. However, homeomorphism does not imply isomorphism, as is shown by the following lemma.

Lemma 3.2. *There exist ceers R, S such that the respective Visser spaces V_R and V_S are homeomorphic, but R and S are not isomorphic.*

Proof. We will show that in fact there exist two ceers whose Vissers spaces are homeomorphic, but neither of them can be embedded through a monomorphism in the other one.

Given a c.e. set X, let R_X be the ceer,

$$x \; R_X \; y \Leftrightarrow [x = y \vee \{x, y\} \subseteq X].$$

The Visser space V_{R_X} consists of the following set of points:

$$\{X, \{x\} : x \notin X\}.$$

Notice that for every number $x \notin X$, the singleton $\{\{x\}\}$ is open in V_{R_X}.

It is known, see e.g. [1, 6, 17, 9, 11], that for every pair X, Y of c.e. sets, where Y is infinite,

$$R_X \leq R_Y \Leftrightarrow X \leq_1 Y.$$

Let now X and Y be simple sets that are not computably isomorphic. (It is known that there are simple sets X, Y such that $X \not\leq_T Y$ and $Y \not\leq_T X$: see [7].) On the other hand, consider any bijection $F : V_{R_X} \longrightarrow V_{R_Y}$, such that $F(X) = Y$ (of course, X and Y are here points of the Visser spaces V_{R_X} and V_{R_X}, respectively). We claim that every such F is a homeomorphism. Let us show for instance that F is continuous. Let U be a basic open set in V_{R_Y}: then $\nu_{R_Y}^{-1}[U]$ is co-c.e., and its complement Z is c.e. There are two cases to consider:

(1) if $Y \notin U$, then U is union of points distinct from Y; but then in this case, $F^{-1}[U]$ consists of points distinct from X, and we know that the singleton of each such point is open; thus $F^{-1}[U]$ is open;
(2) If $Y \in U$, then by simplicity of Y, Z is finite, and thus $F^{-1}[U]$ is open, as $\nu_{R_X}^{-1}[F^{-1}[U]]$ is co-finite.

□

Definition 3.3. [20] Let R be an equivalence relation. An infinite subset X of V_R is *effectively discrete* if there is a computable function f such that

$$X = \{[f(i)]_R : i \in \omega\},$$

and

$$i \neq j \Rightarrow f(i) \not R f(j).$$

It is easy to see that if X is an effectively discrete subset of a Visser space V_R, then X is (topologically) discrete.

This notion can be characterized in terms of the reducibility $\leq$, as pointed out in the following lemma:

Lemma 3.4. *Let R be an equivalence relation. The Visser space V_R contains an effectively discrete subset if and only if* $\mathrm{Id} \leq R$.

Proof. Immediate. □

Corollary 3.5. *If X is simple, then V_{R_X} does not contain any infinite effectively discrete subset.*

Proof. Any infinite effectively discrete subset of V_{R_X} would consist of equivalence classes which are all, with at most one exception, singletons of elements contained in X^c: thus we would get a c.e. subset of X^c. □

We now give some sufficient condition for a Visser space relative to a ceer, to contain an infinite effectively discrete subset.

Lemma 3.6. *If R is a ceer, with an equivalence class $[a]_R$, which is creative, then V_R contains an infinite effectively discrete subset.*

Proof. Let $[a]_R$ be creative, wit a total productive function p. Define a computable function f by induction as follows:

Step 0. Let $W_{i_0} = \emptyset$, and let $f(0) = p(i_0)$.

Step $n+1$. Let $W_{i_n} = \bigcup_{i \leq n} [f(i)]_R$, and let $f(n+1) = p(i_n)$.

It is easy to see that the family $\{[f(i)]_R : i \in \omega\}$ is effectively discrete. □

Recall, see e.g. [16], that a pair of disjoint c.e. sets A, B is *effectively inseparable* if there is a partial computable function $p(u, v)$ (called, *productive* for the pair) such that, for all u, v,

$$A \subseteq W_u \,\&\, B \subseteq W_v \,\&\, W_u \cap W_v = \emptyset \Rightarrow p(u, v){\downarrow} \notin W_u \cup W_v,$$

A ceer E is *effectively inseparable* (see [1]) if every pair of disjoint equivalence classes is effectively inseparable. Recall also that if A is a set that is half of a pair of effectively inseparable sets, then A is creative.

As a consequence, we have the following corollary:

Corollary 3.7. *The Visser space of any effectively inseparable ceer contains an infinite effectivey discrete subset.*

Proof. Immediate. □

The corollary applies to all u.f.p. ceers (thus including precomplete ceers) which are known to be e.i., see [15].

Not all Visser spaces of weakly precomplete ceers contain infinite effectively discrete subsets. In fact:

Theorem 3.8. *There exist infinitely many non-isomorphic weakly precomplete ceers whose Visser spaces do not contain infinite effectively discrete subsets.*

Proof. By Theorem 2.6 and Lemma 3.4: recall that in the proof of Theorem 2.6, for every i, we have $\mathrm{Id} \not\leq E_i$. □

The following corollary remarks one more interesting difference between precompleteness and weak precompleteness.

Corollary 3.9. *There exist infinitely many non-isomorphic weakly precomplete ceers such that no equivalence class is creative.*

Proof. By the previous theorem and Lemma 3.6. □

Although not necessarily yielding creativity or effectively inseparability of equivalence classes, weakly precomplete ceers however do yield partitions into computably inseparable equivalence classes. We first recall the definition.

Definition 3.10. A pair of disjoint c.e. sets A, B is *computably inseparable* if there is no decidable set X such that $A \subseteq X$ and $B \subseteq X^c$.

Theorem 3.11. *If R is a weakly precomplete ceer, and a and b are not R-equivalent, then $[a]_R$ and $[b]_R$ are computably inseparable.*

Proof. Let a and b be non-R-equivalent, and suppose that X is a computable set that separates the pair $[a]_R, [b]_R$. Define

$$f(z) = \begin{cases} b, & \text{if } z \in X; \\ a, & \text{if } z \notin X. \end{cases}$$

Then f is total computable. By weakly precompleteness let n be a fixed point for f, i.e. $n \, R \, f(n)$. Then

$$n \in X \Rightarrow n \, R \, f(n) = b \Rightarrow n \in [b]_R \subseteq X^c$$
$$n \notin X \Rightarrow n \, R \, f(n) = a \Rightarrow n \in [a]_R \subseteq X,$$

a contradiction. □

Question 3.12. *Are the Visser spaces of weakly precomplete ceers all homeomorphic with each other?*

4. Index sets

With the intent of further separating the notion of a precomplete ceer from the notion of a weakly precomplete ceer, we also prove that the corresponding index sets are different. In the following, we refer to an acceptable indexing $\{R_x : x \in \omega\}$ of all ceers, as the ones used in [1] or in [11].

4.1. **The index set of the precomplete ceers.**

Theorem 4.1. *The index set* $\mathrm{P} = \{x : R_x \text{ precomplete}\}$ *is* Σ_3*-complete.*

Proof. First of all, a simple calculation shows that $\mathrm{P} \in \Sigma_3$. To show Σ_3-hardness, we use below that for every Σ_3-set S there exists a c.e. class $\{X_{\langle i,j\rangle} : i, j \in \omega\}$ (meaning that the set $\{\langle x, y\rangle : x \in X_y\}$ is c.e.) such that

$$i \in S \Leftrightarrow (\exists j)[X_{\langle i,j\rangle} \text{ is infinite}],$$

see [19, Corollary IV.3.7]. Uniformly in i, we construct a ceer $E = E_i$ such that, for every i,

$$i \in S \Leftrightarrow \; E \text{ is precomplete.}$$

Recall that, by definition, a precomplete ceer is nontrivial.

Requirements. We want to satisfy requirement C and all requirements N_e if $i \notin S$, and some P_j, if $i \in S$:

$$C: \quad 0 \not\mathrel{E} 1$$
$$N_e: \quad \varphi_e \text{ is not a reduction witnessing } \mathrm{Id} \leq E$$
$$P_j: \quad f_j(e, x) \text{ is an } E\text{-totalizer of } \varphi_e$$

where f_j is a computable function built by us. If all N-requirements are met, then E is not universal and thus not precomplete, as for every e, φ_e is not a reduction witnessing $\mathrm{Id} \leq E$; otherwise E is precomplete.

The *priority listing* of the requirements is given by

$$N_0 < P_0 < N_1 < P_1 < \cdots < N_e < P_e < \cdots .$$

The plan is the following: if $i \notin S$, i.e. all $X_{\langle i,j\rangle}$ are finite, then E meets all requirements N_e, thus E is not precomplete. Otherwise, if j is the least number such that $X_{\langle i,j\rangle}$ is infinite, then the requirement P_j is attacked infinitely often, and thus the construction builds a total computable function $f_j(e,x)$ which is a totalizer for E. In this case all N_e of lower-priority than P_j are initialized infinitely often.

During the construction, a number is said to be *new* if it is bigger than all numbers already equivalent to numbers so far used in the construction.

Strategy for N_e. Here is the strategy for N_e:

(1) appoint four new witnesses $a_{e,r}$, $r < 4$;
(2) wait for $\varphi_e(a_{e,r})$ to converge to, say, y_r, all $r < 4$;
(3) if the y_r are pairwise not E-equivalent, then at least two of then are not E-equivalent to either 0 or 1; choose such y_l, y_h and E-collapse them.

Outcomes. If we wait forever at (2), then φ_e is not total, and thus it can not be a reduction; if there are y_l, y_h which are already E-equivalent, then we win without any further action; otherwise, we E-collapse some y_l, y_h and we win.

Strategy for P_j. Here is the strategy for P_j:

(1) extend f_j to the least pair e, x on which f_j is undefined: for this, choose a new y, and let $f_j(e,x) = y$;
(2) for all pairs e, x on which $f_j(e,x)$ has been defined at a previous stage, if $\varphi_e(x)$ converges to, say, z, and $f_j(e,x)$ is not as yet E-equivalent to z, then E-collapse $f_j(e,x)\ E\ z$.

Outcomes. If $X_{<i,j>}$ is infinite then we build a total computable function $f_j(e,x)$ which is a totalizer for E, so E is precomplete. If $X_{<i,j>}$ is finite then the outcome is finitary, and the strategy only causes finite injury to lower priority strategies.

We use several parameters throughout the construction: $a_{e,r}(s)$ $(r < 4)$, $f_{j,s}$. We say that N_e *requires attention* at stage $s+1$, if either

(1) all $a_r(s)$ are undefined, or
(2) for each $r < 4$, $\varphi_{e,s}(a_{e,r}(s))$ converges ($\varphi_{e,s}(a_{e,r}(s))\downarrow = y_r$, say), and N_e has not as yet acted since when the $a_{e,r}(s)$ were last defined.

We say that P_j *requires attention* at stage $s+1$, if $s+1$ is an $\langle i,j\rangle$-*expansionary* stage, i.e.

$$X_{\langle i,j\rangle,s+1} - X_{\langle i,j\rangle,s} \neq \emptyset.$$

At stage s, we say that we *initialize* N_e, if we set $a_{e,r}$ to be undefined; and we *initialize* P_j if we cancel the current f_j, i.e. we set $f_{j,s} = \emptyset$.

The construction. The construction is by stages. At stage s we define an approximation E^s to E. In describing the construction, when referring to the various parameters, we omit for simplicity to mention the stage.

Stage 0. Initialize all requirements. Let $E^0 = \mathrm{Id}$.

Stage $s+1$. Let R be the highest-priority requirement that requires attention. (Notice that there always is a requirement that requires attention.)

Suppose that $R = N_e$, for some e. We distinguish the following cases:

(1) if N_e requires attention through (1), then pick four new numbers $a_{e,r}(s+1)$, $r < 4$, (we may assume that φ_e is still undefined on all these $a_{e,r}(s+1)$);
(2) if N_e requires attention through (2), and all y_r, $r < 4$, are pairwise non-E-equivalent, then choose y_l, y_h, $l \neq h$, such that neither of these numbers are currently E-equivalent to either 0 or 1, and E-collapse them.

Suppose that $R = P_j$, for some j.

(1) let e, x be the least pair of numbers on which f_j is undefined; choose a new y, and let $f_j(e,x) = y$;
(2) for all pairs e, x such that $f_j(e,x)$ has been defined at some previous stage, $\varphi_e(x)$ converges to, say, z, and $f_j(e,x)$ is not as yet E-equivalent to z, then E-collapse $f_j(e,x)\ E\ z$.

Let E^{s+1} be the ceer resulting from the E-collapses introduced at this stage.

Initialize all requirements $R' > R$, and go to stage $s+2$.

This ends the construction.

Verification. The verification is based on the following lemma.

Lemma 4.2. *The following hold:*

(1) if $i \notin S$, then each requirement requires attention, and is initialized, only finitely many times, and each N_e is met;
(2) if $i \in S$, then there are a least j and a stage s_0, such that after s_0, P_j is not initialized anymore, and the total computable function $f_j = \bigcup_{s \geq s_0} f_{j,s}$ is a totalizer of E;
(3) $0 \not\!\!E\ 1$.

Proof. Assume $i \notin S$. Assume that t_0 is the least stage such that, no requirement $R' < R$ requires attention or is initialized at any stage $s \geq t_0$. Then after this stage R is not initialized anymore. If $R = P_j$ for some j, then there is a least stage $t_1 \geq t_0$ such that no $s \geq t_1$ is $\langle i,j\rangle$-expansionary, thus R does not require attention at any $s \geq t_1$. If $R = N_e$ then R requires attention at t_0 when it defines the last values $a_{e,r}$ of its parameters. If it never stops waiting for $\varphi_e(a_{e,r})$ to converge, all $r < 4$, then N_e never requires attention after t_0 and is satisfied. Otherwise, there is a least $t_1 > t_0$ at which all these computations stop, and we complete the list of y_r, $r < 4$. If there are $l, h < 4$, $l \neq h$ such that already $y_l\ E\ y_h$, then R is satisfied as $a_{e,l}$ and $a_{e,h}$ are not Id-equivalent; otherwise we choose a pair y_l, y_h (with both numbers outside the current equivalence classes of 0 and 1) and we E-collapse these numbers, so that, again, R is satisfied. After t_1, R will not require attention any more.

Assume $i \in S$. Then there is a least j, such that there are infinitely many $\langle i,j\rangle$-expansionary stages. There exists a least stage t_0 after which no higher priority requirement R' ever requires attention or acts. So at every $\langle i,j\rangle$-expansionary stage following t_0, P_j acts, f_j is no longer initialized, and eventually f_j is a total computable function, which is a totalizer witnessing that E is precomplete.

Finally, we must show that $0 \not\mathrel{E} 1$.

This is an immediate consequence of the following claim, where we we say that a current witness $y = f_j(e, x)$ od an N_e-requirement is *active* if $\varphi_e(x)$ is still undefined, or $\varphi_e(x)$ converges to z, say, but y and z have not as yet made E-equivalent:

($\star$) for every s, the following hold: for every R, if R is a P-requirement and y is the current value of an active parameter of R, then y does not lie in either of the E^s-equivalence classes of 0 or 1; and these two classes are disjoint.

The proof of the claim is by induction on s. It holds for $s = 0$. Suppose it holds of s. If at $s+1$ a P-requirement acts, then the claim continues to be true; indeed, if an N-requirement acts, then the claim continues to be true, since this requirement may only E-collapse numbers that lie in neither of the equivalence classes of 0 and 1. □

This ends the proof of the theorem. □

Corollary 4.3. *The index set* $\mathrm{U} = \{x : R_x$ *is u.f.p.*$\}$ *is* Σ_3*-complete.*

Proof. It is an easy calculation to see that $\mathrm{U} \in \Sigma_3$. The rest of claim is an immediate consequence of the proof of the previous theorem, where we show that for every Σ_3 set S, and any i we can effectively find a ceer R_i such that if $i \in S$ then R_i is precomplete, and thus u.f.p.; and if $i \notin S$ then R_i is not universal, and thus not u.f.p.. □

4.2. **The index set of the weakly precomplete ceers.**

Theorem 4.4. *The index set* $\mathrm{WP} = \{x : R_x$ *weakly precomplete*$\}$ *is* Π_3*-complete.*

Proof. It follows from Lemma 1.2 that $\mathrm{WP} \in \Pi_3$. To show that WP is Π_3-hard, as in the proof of the previous theorem let us fix a c.e. class $\{X_{\langle i,j \rangle} : i, j \in \omega\}$, and let

$$S = \{i : (\forall j)[X_{i,j} \text{ is finite}]\} :$$

uniformly in i, we will construct a ceer $E = E_i$ such that

$$i \in S \Leftrightarrow \; E \text{ is weakly precomplete.}$$

Requirements. We want to satisfy requirement C and all requirements N_e if $i \in S$, and some P_j, if $i \notin S$,

$$\begin{aligned} C: &\quad 0 \not\mathrel{E} 1 \\ P_j: &\quad f_j \text{ is a (total) } E\text{-diagonal function} \\ N_e: &\quad \varphi_e \text{ total} \Rightarrow (\exists x)[\varphi_e(x) \mathrel{E} x] \end{aligned}$$

where f_j is a computable function built by us. If $i \notin S$, then f_j, for some j, is total, and has no fixed point.

The *priority listing* of the requirements is given by

$$N_0 < P_0 < N_1 < P_1 < \cdots < N_e < P_e < \cdots .$$

The plan is the following: if $i \in S$, i.e. all sets $X_{\langle i,j \rangle}$ are finite, then E meets all requirements N_e, thus E is weakly precomplete. Otherwise, if j is the least number such that $X_{\langle i,j \rangle}$ is infinite, then by attacking the requirement P_j infinitely often, the construction builds a total computable function f_j with no fixed point modulo E. In this case all N_e of lower-priority than P_j are initialized infinitely often.

During the construction, a number is said to be *new* if it is bigger than all numbers already E-equivalent to numbers so far used in the construction: in particular, a new number is out of the current approximation to the equivalence classes $[0]_E$, $[1]_E$.

Strategy for N_e. Here is the strategy for N_e:

(1) appoint a new witness x_e;
(2) wait for $\varphi_e(x_e)$ to converge;
(3) E-collapse x_e and $\varphi_e(x_e)$

Outcomes. if we wait forever at (2), then φ_e is not total; if we act through (3), then x_e is a fixed point modulo E of φ_e.

Strategy for P_j. The strategy for P_j can be described as follows:

(1) extend f_j to the least number x on which f_j is undefined: for this, choose a new y, let $f_j(x) = y$, and restrain $x \not\!\!E\, y$.

    Outcomes. If $X_{<i,j>}$ is infinite then P_j is attacked infinitely often, thus f_j is a total E-diagonal computable function, thus E is not weakly precomplete. If $X_{<i,j>}$ is finite then the outcome is finitary, and the strategy only causes finite injury to lower priority strategies.

There is also the overall goal of restraining $0 \not\!\!E\, 1$, to satisfy requirement C. The construction uses initialization to restrain equivalence classes. Whenever a requirement R acts, it initializes lower-priority requirements which, as a consequence of this initialization, choose new witnesses, whose possible future E-collapse with other numbers, does not interfere with the restraints imposed by R or by C.

We use parameters $x_e(s)$ and $f_{j,s}$ to approximate x_e and f_j as described above. We say that N_e *requires attention* at stage $s+1$, if either

(1) $x_e(s)$ is undefined, or
(2) $\varphi_{e,s}(x_e(s))$ converges, but we have not as yet E-collapsed $x_e(s)$ with $\varphi_{e,s}(x_e(s))$.

We say that P_j *requires attention* at stage $s+1$, if $s+1$ is an $\langle i,j\rangle$*-expansionary* stage, i.e.

$$X_{\langle i,j\rangle,s+1} - X_{\langle i,j\rangle,s} \neq \emptyset.$$

At stage s, we say that we *initialize* N_e, if we set $x_e(s)$ to be undefined; and we *initialize* P_j if we cancel the current f_j, i.e. we set $f_{j,s} = \emptyset$.

The construction. The construction is by stages. At stage $s+1$ we define an approximation E^s to E, which is obtained by E^s, plus possibly the collapse of two E^s-equivalence classes that is performed at stage $s+1$. In describing the construction, when referring to the various parameters, we omit for simplicity to mention the stage.

Stage 0. Initialize all requirements. Let $E^0 = \mathrm{Id}$.

Stage $s+1$. Let R be the highest-priority requirement that requires attention. (Notice that there always is a requirement that requires attention.)

Suppose that $R = N_e$, for some e. We distinguish the following cases:

(1) if N_e requires attention through (1), then pick x_e to be a new number;
(2) if N_e requires attention through (2), then E-collapse x_e and $\varphi_e(x_e)$.

Suppose that $R = P_j$, for some j: let x be the least number on which f_j is undefined; choose a new y, and let $f_j(x) = y$

Let E^{s+1} be the ceer resulting from the E-collapses introduced at this stage.

Initialize all requirements $R' > R$.

This ends the construction.

Verification. We just need to prove the following lemma.

Lemma 4.5. *The following hold:*

(1) if $i \in S$, then each requirement requires attention, and is initialized, only finitely many times, and each N_e is met;
(2) if $i \notin S$, then there are a least j and a stage s_0, such that after s_0, P_j is not initialized anymore, and the total computable function $f_j = \bigcup_{s \geq s_0} f_{j,s}$ is an E-diagonal function;
(3) $0 \not\mathrel{E} 1$.

Proof. Assume $i \in S$. Let R be a requirement, and assume by induction that t_0 is the least stage such that, no requirement $R' < R$ requires attention or is initialized at any stage $s \geq t_0$. Then after this stage R is not initialized anymore. If $R = P_j$ for some j, then there is a least stage $t_1 \geq t_0$ such that no $s \geq t_1$ is $\langle i, j\rangle$-expansionary, thus R does not require attention at any $s \geq t_1$. If $R = N_e$ then R can require attention at most once after t_0: if it stops waiting for $\varphi_e(x_e)$ to converge, then N_e is satisfied. Otherwise, we E-collapse x_e and $\varphi_e(x_{e)})$, hence φ_e has a fixed point and N_e is satisfied.

Assume $i \notin S$. Let R be a requirement, and assume by induction that the claim holds of all $R' < R$. Then there is a least j, such that there are infinitely many $\langle i, j\rangle$-expansionary stages, and there exists a least stage t_0 after which no higher-priority requirement R' ever requires attention or acts. So at every $\langle i, j\rangle$-expansionary stage following t_0, P_j acts, f_j is no longer initialized, and eventually f_j is a total computable function witnessing that E is not weakly precomplete. Notice that $f_j(x) \not\mathrel{E} x$ is restrained since $f_j(x)$ is chosen new, and, when we choose it, lower-priority requirements get initialized. Indeed, the fact that the restraints imposed by a P-requirement and by C are respected by lower-priority requirements is a consequence of the following claim, where we say that at stage s, a currently defined witness $x_{e,s}$ of a requirement N_e is *active* if it has not been as yet collapsed to $\varphi_e(x_{e,s})$:

($\star$) For every s, the following hold: for every R, if w is an active witness of an $R' > R$, then w does not lie in the equivalence classes restrained by R, nor does it lie in $[0]_{E^s} \cup [1]_{E^s}$; and the equivalence classes restrained by R are disjoint at s, and the equivalence classes of 0 and 1 are disjoint at s.

The claim is proved by induction on s, keeping in mind that the only requirements that E-collapse elements are the N-requirement. Assume that it is true at s. To show for instance that the action taken at $s+1$ by a requirement N_e does not collapse 0 and 1, notice that N_e can only E-collapse pairs of numbers of the form x_e and $\varphi_e(x_e)$ of which one of them, x_e, is chosen out of the current approximation of $[0]_E \cup [1]_E$, and thus, even if $\varphi_e(x_e) \in [0]_E \cup [1]_E$, we simply enlarge one of the two classes, without E-collapsing the two of them. □

This concludes the proof of the theorem. □

We now show how to modify the proof of the previous theorem, to show that the index set

$$\mathrm{E} = \{x : R_x \text{ is } e\text{-complete}\}$$

is Σ_3-complete.

Corollary 4.6. *The index set* E *is* Σ_3*-complete.*

Proof. It is routine to check that $\mathrm{E} \in \Sigma_3$. For Σ_3-hardness, we can modify that proof of the previous theorem to obtain that, whatever i one considers, the ceer E built uniformly in i, is u.f.p., and thus in case $i \notin S$, E is e-complete being u.f.p. and possessing a total E-diagonal computable function. The modified proof combines the proof of the previous theorem, and elements of the proof of Theorem 2.2. We outline a sketch of the proof.

Knowing that every ceer extending an e-complete ceer is u.f.p., see [4], we start up with a (non-trivial) e-complete ceer L, and we consider a total strong L-diagonal computable function d. Then, given i, we build E by stages, so that at each stage s, the current approximation E^s to E is a ceer extending E^{s-1} if $s > 0$, with $E^0 = L$. The only other difference is a suitably modified version of a "new" number: namely, where in the proof of the previous theorem a number is "new" if it is not already E-equivalent to any number so far mentioned in the construction, now a number w is *new* if $w = d(X)$ where X is the set of numbers (witnesses) so far used in the construction: this allows us to find numbers not already E-equivalent to any previous number, although now E is collapsing pairs of full L-equivalence classes, instead of just pairs of finite sets of numbers. □

References

[1] U. Andrews, S. Lempp, J. S. Miller, K. M. Ng, L. San Mauro, and A. Sorbi. Universal computably enumerable equivalence relations. *J. Symbolic Logic*, 79(1):60–88, March 2014.
[2] S. Badaev. On weakly precomplete positive equivalences. *Siberian Math. Journal*, 32:321–323, 1991.
[3] C. Bernardi. On the relation provable equivalence and on partitions in effectively inseparable sets. *Studia Logica*, 40:29–37, 1981.
[4] C. Bernardi and F. Montagna. Equivalence relations induced by extensional formulae: Classifications by means of a new fixed point property. *Fund. Math.*, 124:221–232, 1984.
[5] C. Bernardi and A. Sorbi. Classifying positive equivalence relations. *J. Symbolic Logic*, 48(3):529–538, 1983.
[6] S. Coskey, J. D. Hamkins, and R. Miller. The hierarchy of equivalence relations on the natural numbers. *Computability*, 1:15–38, 2012.
[7] J. C. E. Dekker. A theorem on hypersimple sets. *Proc. Amer. Math. Soc.*, 5:791–796, 1954.
[8] Yu. L. Ershov. Positive equivalences. *Algebra and Logic*, 10(6):378–394, 1973.
[9] Yu. L. Ershov. *Theory of Numberings*. Nauka, Moscow, 1977.
[10] Yu. L. Ershov. Theory of numberings. In E. G. Griffor, editor, *Handbook of Computability Theory*, volume 140 of *Studies Logic Found. Math.*, pages 473–503. North-Holland, 1999.
[11] S. Gao and P. Gerdes. Computably enumerable equivalence realations. *Studia Logica*, 67:27–59, 2001.
[12] A. H. Lachlan. A note on positive equivalence relations. *Z. Math. Logik Grundlag. Math.*, 33:43–46, 1987.
[13] A. I. Mal'tsev. Sets with complete numberings. *Algebra i Logika*, 2(2):4–29, 1963.

[14] A. I. Mal'tsev. Positive and negative enumerations. *Dokl. Akad. Nauk SSSR*, 160(2):278–280, 1965.

[15] F. Montagna. Relative precomplete numerations and arithmetic. *J. Philosphical Logic*, 11:419–430, 1982.

[16] H. Rogers, Jr. *Theory of Recursive Functions and Effective Computability.* McGraw-Hill, New York, 1967.

[17] L. San Mauro. Forma e complessità. Uno studio dei gradi delle relazioni di equivalenza ricorsivamente enumerabili. Master's thesis, University of Siena, July 2011. In Italian.

[18] V. Yu. Shavrukov. Remarks on uniformly finitely positive equivalences. *Math. Log. Quart.*, 42:67–82, 1996.

[19] R. I. Soare. *Recursively Enumerable Sets and Degrees.* Perspectives in Mathematical Logic, Omega Series. Springer-Verlag, Heidelberg, 1987.

[20] A. Visser. Numerations, λ-calculus & arithmetic. In J. P. Seldin and J. R. Hindley, editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 259–284. Academic Press, London, 1980.

Department of Mechanics and Mathematics, Al-Farabi Kazakh National University, Almaty, 050038, Kazakhstan

E-mail address: serikzhan.badaev@kaznu.kz

Dipartimento di Ingegneria Informatica e Scienze Matematiche, Università Degli Studi di Siena, I-53100 Siena, Italy

E-mail address: andrea.sorbi@unisi.it