Jumps of computably enumerable equivalence relations

JUMPS OF COMPUTABLY ENUMERABLE EQUIVALENCE RELATIONS

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Abstract. We study computably enumerable equivalence relations (or, ceers), under computable reducibility $\leqslant$, and the halting jump operation on ceers. We show that every jump is uniform join-irreducible, and thus join-irreducible. Therefore, the uniform join of two incomparable ceers is not equivalent to any jump. On the other hand there exist ceers that are not equivalent to jumps, but are uniform join-irreducible: in fact above any non-universal ceer there is a ceer which is not equivalent to a jump, and is uniform join-irreducible. We also study transfinite iterations of the jump operation. If a is an ordinal notation, and E is a ceer, then let $E^{(a)}$ denote the ceer obtained by transfinitely iterating the jump on E along the path of ordinal notations up to a. In contrast with what happens for the Turing jump and Turing reducibility, where if a set X is an upper bound for the A-arithmetical sets then $X^{(2)}$ computes $A^{(\omega)}$, we show that there is a ceer R such that $R \geqslant \mathrm{Id}^{(n)}$, for every finite ordinal n, but, for all k, $R^{(k)} \ngeqslant \mathrm{Id}^{(\omega)}$ (here Id is the identity equivalence relation). We show that if a, b are notations of the same ordinal less than ω^2, then $E^{(a)} \equiv E^{(b)}$, but there are notations a, b of ω^2 such that $\mathrm{Id}^{(a)}$ and $\mathrm{Id}^{(b)}$ are incomparable. Moreover, there is no non-universal ceer which is an upper bound for all the ceers of the form $\mathrm{Id}^{(a)}$ where a is a notation for ω^2.

1. Introduction

Recently, there has been a renewed interest in studying equivalence relations on the set ω of natural numbers under the reducibility $\leqslant$, where $R \leqslant S$ if there exists a computable function f such that, for all $x, y \in \omega$,

$$x \; R \; y \Leftrightarrow f(x) \; S \; f(y).$$

The first systematic investigation of this reducibility goes back to Ershov (see e.g., [9]). More recent papers, with applications to computable model theory and computable algebra, include [7, 10, 11]. A natural and interesting particular case is provided by restriction of $\leqslant$ to computably enumerable equivalence relations (which will be abbreviated as *ceers*). The earliest paper fully dedicated to ceers (therein called *positive* equivalence relations) is [8], followed by other papers motivated by applications to logic, in view of the numerous examples of ceers naturally arising in logic: see for instance [3, 4, 13, 14, 17, 19].

Let $\equiv$ be the equivalence relation given by $R \equiv S$ if $R \leqslant S$ and $S \leqslant R$. Then one can define, in the usual way, the degrees of equivalence relations (i.e., the $\equiv$-equivalence classes of equivalence relations), and in particular the poset $\mathcal{P}$ of degrees of ceers, which is bounded (see for instance [1]), i.e., with least element $\mathbf{0}$, consisting of just the trivial ceer, and greatest element $\mathbf{1}$: the ceers belonging to $\mathbf{1}$ are called *universal*. It is also worth recalling that this structure extends the

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structure of the 1-degrees of infinite c.e. sets: Given a set X define R_X to be the equivalence relation so that $x\ R_X\ y$ if and only if $x = y$ or $x, y \in X$. It is not difficult to show (see for instance [1]) that if X, Y are infinite c.e. sets then $X \leqslant_1 Y$ if and only if $R_X \leqslant R_Y$; moreover if Y is c.e. and $Z \leqslant R_Y$ then $Z \equiv R_X$ for some c.e. set X. It turns out in this way that the interval of degrees of ceers $[\deg(\mathrm{Id}), \deg(R_K)]$ (where Id denotes the identity realtion) is isomorphic to the interval of c.e. 1-degrees $[\mathbf{0}_1, \mathbf{0}_1']$, where $\mathbf{0}_1$ is the 1-degree of any infinite and coinfinite decidable set, and $\mathbf{0}_1'$ is the 1-degree of the halting set K. This fact has been exploited in [1] to show that the first order theory of the poset of degrees of ceers is undecidable.

Gao and Gerdes [12] define a useful notion of jump of a ceer.

Definition 1.1. Given a ceer R, define

$$x\ R'\ y \Leftrightarrow [x = y \text{ or } \varphi_x(x){\downarrow} R\ \varphi_y(y){\downarrow}]\,.$$

The ceer R' is called the *halting jump ceer of* R: in the following we simply call it the *jump* of R.

The main properties of the operation $R \mapsto R'$ are summarized in the following theorem:

Theorem 1.2. [12, 1] *The following properties hold:*

(1) $R \leqslant R'$*;*
(2) $R \leqslant S$ *if and only if* $R' \leqslant S'$*;*
(3) If R *is not universal then* R' *is not universal;*
(4) if R *is not universal, then* $R < R'$*.*

Proof. Item (1) is [12, Proposition 8.3(1)]; item (2) is [12, Theorem 4]; item (3) is [12, Corollary 8.5(2)]; item (4) is [1, Theorem 4.3]. □

In particular we have a well-defined jump operation on degrees, given by $(\deg(R))' = \deg(R')$: this jump operation is an order embedding, and takes every degree to a bigger degree, except when it can not become strictly bigger, i.e., on the greatest element of $\mathcal{P}$. A degree of a ceer is a *jump degree* if it it in the range of the jump operation on degrees.

Given equivalence relations R, S on ω, let $R \oplus S$ be defined by

$$x\ R \oplus S\ y \Leftrightarrow \begin{cases} u\ R\ v & \text{if } x = 2u \text{ and } y = 2v, \\ u\ S\ v & \text{if } x = 2u+1 \text{ and } y = 2v+1. \end{cases}$$

Notice that $\oplus$ induces a well defined binary operations on degrees: if $\mathbf{a}$ and $\mathbf{b}$ are the degrees of R and S, respectively, then $\mathbf{a} \oplus \mathbf{b}$ is the degree of $R \oplus S$. Moreover $R \oplus S$ satisfies $R, S \leqslant R \oplus S$.

Henceforth we will restrict our attention *only* to equivalence relations which are ceers, and consequently *only* to degrees of ceers, i.e., degrees in $\mathcal{P}$. Clearly, if R and S are ceers then $R \oplus S$ is a ceer, hence the degree of $R \oplus S$ is an upper bound of the degrees of R, S, but it need not be the least upper bound: in fact ([1]) in $\mathcal{P}$ there are degrees without least upper bound.

Definition 1.3. Define a degree $\mathbf{e}$ to be *uniform join-irreducible* if whenever $\mathbf{e} \leqslant \mathbf{a} \oplus \mathbf{b}$, then $\mathbf{e} \leqslant \mathbf{a}$, or $\mathbf{e} \leqslant \mathbf{b}$. A ceer E is *uniform join-irreducible* if its degree is uniform join-irreducible, i.e., whenever $E \leqslant R \oplus S$, then either $E \leqslant R$ or $E \leqslant S$.

As the degree of $\mathbf{a} \oplus \mathbf{b}$ is an upper bound of $\mathbf{a}$ and $\mathbf{b}$, if $\mathbf{e}$ is uniform join-irreducible, then its degree $\mathbf{e}$ is *join-irreducible*, i.e., for all degrees $\mathbf{a}, \mathbf{b}$ for which the join $\mathbf{a} \vee \mathbf{b}$ exists, if $\mathbf{e} \leqslant \mathbf{a} \vee \mathbf{b}$, then $\mathbf{e} \leqslant \mathbf{a}$, or $\mathbf{e} \leqslant \mathbf{b}$.

In Section 2 we investigate the interrelations between the property of being a jump degree, and uniform join-irreducibility. We show that every jump is uniform join-irreducible, and thus every jump degree is join-irreducible. Therefore, the uniform join of two incomparable degrees is not a jump. On the other hand, uniform join-irreducibility does not characterize the property of being a jump degree, as there exist degrees that are not jumps, but are uniform join-irreducible: in fact above any **a** strictly below **1** there is a degree which is not a jump, and is uniform join-irreducible. The notions of computable inseparability, and effective inseparability play an important role in this investigation. In Section 3 we study transfinite iterations of the jump operation. Given an ordinal notation a, and a ceer E, let $E^{(a)}$ denote the ceer obtained by transfinitely iterating the jump on E along the path of ordinal notations up to a. We show that if a and b are notations for the same ordinal less than ω^2, then $E^{(a)} \equiv E^{(b)}$ for every ceer E. On the other hand, the story is quite different at ω^2: there are notations a, b of ω^2 such that $\mathrm{Id}^{(a)}$ and $\mathrm{Id}^{(b)}$ are incomparable ceers (where we recall that Id is the identity equivalence relation). Furthermore, for any non-universal ceer Y, there is a notation $a \in O$ for ω^2 so that $\mathrm{Id}^{(a)} \nleqslant Y$. In contrast with what happens for the Turing jump and Turing reducibility, where if a set X is an upper bound for the A-arithmetical sets then X'' computes $A^{(\omega)}$, we show that there is a ceer R such that $R \geqslant \mathrm{Id}^{(n)}$, for every finite ordinal n, but, for all k, $R^{(k)} \ngeqslant \mathrm{Id}^{(\omega)}$.

Our main references for computability theory are the textbooks [6, 16, 18].

2. Jumps of ceers

In this section we prove that every degree in the range of the jump is not uniform join-reducible (Theorem 2.4), so there are degrees $\geqslant \mathbf{0}'$ that are not jump degrees, the degree of any uniform join of two incomparable ceers being so. However, uniform join-irreducibility does not characterize the property of being a jump, as (Theorem 2.11) above any degree of a non-universal ceer, there are uniform join-irreducible degrees that are not jumps.

The notions of computable inseparability, and effective inseparability will play a fundamental role in the rest of the paper. Recall that a pair of disjoint sets A, B is *computably inseparable* if there is no computable set X such that $A \subseteq X \subseteq \omega \smallsetminus B$. And a pair of disjoint sets A, B is *effectively inseparable* (shortly, *e.i.*) if there exists a partial computable function ψ (called a *productive function* for the pair) such that, for every pair of c.e. indices u, v,

$$A \subseteq W_u \,\&\, B \subseteq W_v \,\&\, W_u \cap W_v = \varnothing \Rightarrow \psi(u,v)\downarrow \,\&\, \psi(u,v) \notin W_u \cup W_v.$$

Clearly, effective inseparability implies computable inseparability. For every i, let

$$K_i = \{j : \varphi_j(j)\downarrow = i\} :$$

it is known, see e.g., [16, Theorem 7.XII(a)(c)], that K_i and K_j are effectively inseparable if $i \neq j$. Note that $K = \bigcup_i K_i$, where K is the halting set.

A ceer E is called *computably inseparable* (respectively, *effectively inseparable*, abbreviated as *e.i.*), if each pair of its distinct equivalence classes is computably inseparable (respectively, e.i.). Clearly, if a ceer is e.i. then it is also computably inseparable. It is known that there exist ceers that yield partitions of ω into e.i. sets, see [2], or [4]. If E is a computably inseparable ceer then E has no computable equivalence classes.

Recall:

Proposition 2.1. [12, Proposition 8.3(v)] *If E has no computable equivalence classes, and $E \leqslant R'$, then $E \leqslant R$.*

Proof. See [12]. $\square$

This yields the following corollary:

Corollary 2.2. *If E has no computable equivalence classes and is not universal, then E is not equivalent to a jump.*

Proof. Suppose $R' \leqslant E \leqslant R'$, and E has no computable classes. By the previous proposition, we have $E \leqslant R$, thus $R' \leqslant R$, which implies, by Theorem 1.2(4), that R is universal. Therefore E is universal. $\square$

The following lemma is motivated by the fact that in general, for given equivalence relations E, F, the ceer $E \oplus F$ need not be the least upper bound of E and F, or equivalently it is not true in general that $E \oplus E \leqslant E$.

Lemma 2.3. *If R and S are ceers then $R' \oplus S' \leqslant (R \oplus S)'$.*

Proof. By the s-m-n-Theorem, consider two 1-1 computable functions f, g so that, for all x, i, $\varphi_{f(i)}(x) = 2\varphi_i(i)$, $\varphi_{g(i)}(x) = 2\varphi_i(i) + 1$, and (using padding) f, g have disjoint ranges. We will show that the function h defined by $h(2x) = f(x)$ and $h(2x+1) = g(x)$ gives the needed reduction. For two even numbers $2x, 2y$,

$$\begin{aligned} 2x\ R' \oplus S'\ 2y &\Leftrightarrow x\ R'\ y \\ &\Leftrightarrow x = y \vee \varphi_x(x)\downarrow R\ \varphi_y(y)\downarrow \\ &\Leftrightarrow x = y \vee 2\varphi_x(x)\downarrow R \oplus S\ 2\varphi_y(y)\downarrow \\ &\Leftrightarrow f(x) = f(y) \vee \varphi_{f(x)}(f(x))\downarrow R \oplus S\ \varphi_{f(y)}(f(y))\downarrow \\ &\Leftrightarrow f(x)(R \oplus S)' f(y) \\ &\Leftrightarrow h(2x)(R \oplus S)' h(2y). \end{aligned}$$

Similarly, we can show that

$$2x + 1\ R' \oplus S'\ 2y + 1 \Leftrightarrow h(2x + 1)(R \oplus S)' h(2y + 1)$$

for two odd numbers. For $2x$ and $2y + 1$, these are never $R' \oplus S'$ equivalent: towards a contradiction, assume that $h(2x)(R \oplus S)' h(2y + 1)$, i.e., $f(x)(R \oplus S)' g(y)$. Since $f(x) \neq g(y)$, we have $\varphi_{f(x)}(f(x))\downarrow R \oplus S\ \varphi_{g(y)}(g(y))\downarrow$, but this can not be, since $\varphi_{f(x)}(f(x))$ is even and $\varphi_{g(y)}(g(y))$ is odd. $\square$

We are now ready to show that every jump is uniform join-irreducible, hence join-irreducible. Under the embedding of the 1-degrees of infinite c.e. sets into the ceers mentioned in the introduction, the halting set K is mapped to Id_1', the jump of the ceer with only one class. Thus, this theorem is a generalization of the well-known fact that the 1-degree (equivalently the m-degree) of K is join-irreducible.

Theorem 2.4. *If $E' \leqslant R \oplus S$, then $E' \leqslant R$ or $E' \leqslant S$.*

Proof. Suppose f witnesses that $E' \leqslant R \oplus S$. For $i, j \in K$, $f(i)$ and $f(j)$ have the same parity: otherwise, $f^{-1}(2\omega)$ would give a separation of $K_{\varphi_i(i)}$ and $K_{\varphi_j(j)}$, which are effectively inseparable. Suppose $f(i)$ is even for every $i \in K$ (a similar argument will apply if $f(i)$ is odd for every $i \in K$). Thus the set $Y = f^{-1}(2\omega + 1)$ is a decidable set in $\omega \smallsetminus K$. We will show that $E' \leqslant R$. For this, we first show that there exists an infinite decidable set X contained in $\omega \smallsetminus (K \cup Y)$. Use a productive function p of $\omega \smallsetminus K$ to enumerate an infinite c.e. set $X_0 = \{x_n : n \in \omega\}$ so that $X_0 \subseteq \omega \smallsetminus (K \cup Y)$: let $x_0 = p(y_0)$ where y_0 is a c.e. index of Y; having defined x_n, let $x_{n+1} = p(y_{n+1})$ where y_{n+1} is a uniformly found index of $Y \cup \{x_0, \ldots, x_n\}$. Since X_0 is an infinite c.e. set, it contains an infinite decidable set X which has the desired properties.

Now, we define the function g witnessing $E' \leqslant R$. Fix a computable injection $h : X \cup Y$ to X (notice that $f(h(z))$ is even for every $z \in X \cup Y$), and define

$$g(z) = \begin{cases} \frac{f(z)}{2}, & \text{if } z \in \omega \smallsetminus X \cup Y, \\ \frac{f(h(z))}{2}, & \text{if } z \in X \cup Y. \end{cases}$$

The claim follows from a case-by-case inspection:

- If $x, y \in X \cup Y$ then $x, y \notin K$; as also $h(x), h(y) \notin K$, we have $x \mathrel{E'} y$ if and only if $x = y$ if and only if $h(x) = h(y)$ if and only if $h(x) \mathrel{E'} h(y)$ if and only if $f(h(x)) \mathrel{R \oplus S} f(h(y))$ if and only if $\frac{f(h(x))}{2} \mathrel{R} \frac{f(h(y))}{2}$, which implies the claim;
- $x, y \notin X \cup Y$, then both $f(x), f(y)$ are even, and $g(x) = \frac{f(x)}{2}, g(y) = \frac{f(y)}{2}$: the claim follows trivially since f is a reduction from E' to $R \oplus S$;
- it remains to consider the case in which one of them is in $X \cup Y$ and the other is not: suppose for instance that $x \in X \cup Y$ and $y \notin X \cup Y$; notice that since at least one of x, y is not in K, and they are distinct, we have that $x \not\mathrel{E'} y$; on the other hand we have $h(x) \in X$, and thus $h(x) \notin K$, moreover $h(x) \neq y$, thus $h(x) \not\mathrel{E'} y$, from which the claim follows, since f is a reduction from E' to $R \oplus S$, and $f(h(x))$ and $f(y)$ are both even, the latter being even as $y \notin Y$.

□

Definition 2.5. If $\{S_i\}_{i \in \omega}$ is a family of equivalence relations, we define $\bigoplus_i S_i$ to be the equivalence relation

$$\langle u, x\rangle \bigoplus_i S_i \langle v, y\rangle \Leftrightarrow u = v \mathbin{\&} x \mathrel{S_u} y.$$

Theorem 2.6. *If $\{S_i\}$ is a uniformly c.e. family of ceers (i.e., the relation in x, y, i, which holds if $x \mathrel{S_i} y$, is c.e.), and $E' \leqslant \oplus_{i \in \omega} S_i$, then $E' \leqslant S_i$ for some i.*

Proof. Let f witness the reduction. As above, for some j, $f(i)$ is in the jth column for every $i \in K$. We complete the proof exactly as above using the equivalence: $\oplus_{i \in \omega} S_i \equiv S_j \oplus \oplus_{i \neq j} S_i$. □

Corollary 2.7. *If R and S are incomparable, then $R' \oplus S' < (R \oplus S)'$.*

Proof. Otherwise, by Lemma 2.3, we would have $(R \oplus S)' \equiv R' \oplus S'$. Then by Theorem 2.4, either $S' \leqslant (R \oplus S)' \leqslant R'$ or $R' \leqslant (R \oplus S)' \leqslant S'$. But then, by Theorem 1.2(2), $R \leqslant S$ or $S \leqslant R$, which is a contradiction. □

Corollary 2.8. *There are nonzero degrees which are not jumps.*

Proof. Take R and S to be incomparable ceers. Then $R \oplus S$ cannot be equivalent to any jump E', as otherwise by Theorem 2.4 E' would be below R or S, which would cause $R \leqslant S$ or vice versa. □

Let now E be a ceer such that there is no decidable set $X \neq \varnothing, \omega$ which is E-*closed* (i.e., satisfying that $X = [X]_E = \{y : (\exists x \in X)[x \mathrel{E} y]\}$), and let f be a computable function reducing $E \leqslant R \oplus S$. Then $E \leqslant R$ or $E \leqslant S$, otherwise the set $X = \{x : f(x) \text{ even}\}$ would be a nontrivial decidable set which is E-closed.

Theorem 2.4 extends the following previously known result to jumps.

Proposition 2.9. [8] *Let E be any ceer such that no E-closed set X is decidable, unless $X = \varnothing$, or $X = \omega$. Let $\{S_i\}_{i\in\omega}$ be a uniformly c.e. family of ceers. If $E \leqslant \bigoplus_i S_i$, then $E \leqslant S_i$ for some i.*

Proof. Suppose f gives a reduction $E \leqslant \bigoplus_i S_i$. If f has image in at least two different columns, then for some j, $f^{-1}(\omega^{[j]})$ is a decidable E-closed set, which is neither $\varnothing$ nor ω. □

Corollary 2.10. *If E is a computably inseparable ceer, then E is uniform-join irreducible, and hence join irreducible.*

Proof. Suppose E is either the join or the uniform join of R and S. Then $E \leqslant R \oplus S$, thus $E \leqslant R$ or $E \leqslant S$, because there is no nontrivial decidable set which is E-closed, as otherwise such a set would separate two distinct equivalence classes. □

One could hope that perhaps being a join or being a uniform join is the only obstruction to being a jump degree. The following theorem shows that this is false:

Theorem 2.11. *Given any non-universal ceer R, there exists a ceer $E \geqslant R$ so that E is not equivalent to a jump, and E is not uniform join-reducible and hence not join-reducible.*

In fact, this theorem follows by Corollary 2.2 and Corollary 2.10, together with the following theorem, which is of independent interest.

Theorem 2.12. *Let R be any non-universal ceer. Then there exists a non-universal e.i. ceer $E \geqslant R$.*

Proof. We construct E so that $2i \mathrel{E} 2j$ if and only if $i \mathrel{R} j$, thus $R \leqslant E$; we also construct an auxiliary ceer S such that $S \nleqslant E$, so that E is not universal; and we guarantee that E is e.i..

Requirements and their strategies. For $k \in \omega$ and $a < b \in \omega$, we have the requirements:

$$P_{a,b} : [a]_E \neq [b]_E \Rightarrow f_{a,b} \text{ is productive for } [a]_E \text{ and } [b]_E,$$
$$N_k : \varphi_k \text{ is not a reduction of } S \text{ to } E,$$

where $f_{a,b}$ is a computable function built by us, witnessing that the two equivalence classes $[a]_E$ and $[b]_E$ form an e.i. pair, if distinct.

We suppose in the following to have fixed a computable priority ordering $\{\mathcal{R}_i : i \in \omega\}$, of order type ω, of the requirements.

$P_{a,b}$-strategies. We construct the computable function $f_{a,b}$ by stages: when the strategy for $P_{a,b}$ acts, it extends $f_{a,b}$ to the least pair (u, v) on which $f_{a,b}$ is still undefined, putting $f_{a,b}(u, v) = m$, where m is new and odd; if $f_{a,b}(u', v') = m'$ has been already defined, and we see that $m' \in W_{u'}$ then we E-collapse m' and b; and if we see that $m' \in W_{v'}$ then we E-collapse m' and a.

We will give an inductive argument below to show that no two "active" odd numbers are ever equivalent. (As defined later, m is "active" if m is in the range of $f_{a,b}$ for a $P_{a,b}$ which may later cause it to collapse to either a or b.) From there, we will be able to see that no two even numbers become equivalent unless we collapse them for the sake of copying R.

N_k-strategies. The strategies for N_k requirements are slightly different. N_k starts by searching for four elements a_0, b_0, a_1, b_1 that are new in S in the k^{th} column so that φ_k of these elements are odd: if found then we implement the $N_k(a_0, b_0, a_1, b_1)$-module as described below. In the meantime, as φ_k on more and more elements in the k^{th} column converges to even numbers, we cause S on the k^{th} column to agree with a universal ceer, extending this agreement when S agrees with its φ_k image in E. We will argue that since R is not universal, this process must stop either with agreement failing, φ_k being partial, or φ_k giving four elements in the k^{th} column with odd images. The $N_k(a_0, b_0, a_1, b_1)$-module is described here:

The $N_k(a_0, b_0, a_1, b_1)$-module. The module aims at making S not reducible to E via φ_k; it is activated for the first time only if the computations $\varphi_k(a_0), \varphi_k(b_0), \varphi_k(a_1), \varphi_k(b_1)$ are all defined, give odd numbers, and a_0, b_0, a_1, b_1 are pairwise S-inequivalent.

(1) If already $\varphi_k(a_0)\ E\ \varphi_k(b_0)$ or already $\varphi_k(a_1)\ E\ \varphi_k(b_1)$; then do nothing.
(2) Otherwise, S-collapse a_0 and b_0, and initialize lower-priority strategies; and
(3) if later $\varphi_k(a_0)\ E\ \varphi_k(b_0)$, then go to stage (2) and proceed similarly with a_1, b_1 in place of a_0, b_0, respectively.
(4) After completing (3) for a_1, b_1, if already
$$\varphi_k(a_0)\ E\ \varphi_k(b_0)\ E\ \varphi_k(a_1)\ E\ \varphi_k(b_1),$$
then do nothing;
(5) otherwise, S-collapse a_0, b_0, a_1, b_1, and initialize lower-priority requirements.

Outcomes of $N_k(a_0, b_0, a_1, b_1)$. The module $N_k(a_0, b_0, a_1, b_1,)$ has the following outcomes:

(1) If (1) holds for the pair a_0, b_0 then $a_0 \not S\ b_0$ and $\varphi_k(a_0)\ E\ \varphi_k(b_0)$; similarly, if (1) holds for the pair a_1, b_1 then $a_1 \not S\ b_1$ and $\varphi_k(a_1)\ E\ \varphi_k(b_1)$.
(2) If we wait forever at (3) for the pair a_0, b_0 to see $\varphi_k(a_0)\ E\ \varphi_k(b_0)$ then $a_0\ S\ b_0$ and $\varphi_k(a_0) \not E\ \varphi_k(b_0)$; similarly, if we wait forever at (3) for the pair a_1, b_1 to see $\varphi_k(a_1)\ E\ \varphi_k(b_1)$ then $a_1\ S\ b_1$ and $\varphi_k(a_1) \not E\ \varphi_k(b_1)$.
(3) Otherwise, at some point, the strategy yields $a_i\ S\ b_i$ and $\varphi_k(a_i)\ E\ \varphi_k(b_i)$, for both $i = 0, 1$. When this happens, if already $\varphi_k(b_0)\ E\ \varphi_k(a_1)$, then we keep $b_0 \not S\ a_1$; if $\varphi_k(b_0) \not E\ \varphi_k(a_1)$, then our action in (5) makes $b_0\ S\ a_1$, and keeps $\varphi_k(b_0) \not E\ \varphi_k(a_1)$, unless next item holds.
(4) $\varphi_k(a_0)\ E\ \varphi_k(b_0)\ E\ \varphi_k(a_1)\ E\ \varphi_k(b_1)$: we will in fact show that eventually this outcome can not happen.

Note that outcomes (1) through (3) are all winning outcomes for N_k. On the other hand, we can rule out outcome (4), as we can exclude the possibility that we end up with
$$\varphi_k(a_0)\ E\ \varphi_k(b_0)\ E\ \varphi_k(a_1)\ E\ \varphi_k(b_1)$$

and we have already S-collapsed a_0, b_0, a_1, b_1. Note that when we defined $a_0 \mathrel{S} b_0$ we had $\varphi_k(a_0) \not\mathrel{E} \varphi_k(b_0)$. The E-collapse of $\varphi_k(a_0)$ and $\varphi_k(b_0)$ to, say, a number a (which is the least in its equivalence class) is due to the action of a higher-priority strategies of the form $P_{a,b}$; after this convergence of $\varphi_k(a_0)$ and $\varphi_k(b_0)$, the lower-priority P-requirements are initialized, and thus they cannot move $\varphi_k(a_0)$ or $\varphi_k(b_0)$ to new equivalence classes, since they can only move their markers, which by initialization are chosen to be different from all elements in the equivalence classes of $\varphi_k(a_0)$ and $\varphi_k(b_0)$. Similarly, when we defined $a_1 \mathrel{S} b_1$, we had $\varphi_k(a_1) \not\mathrel{E} \varphi_k(b_1)$. The E-collapse of $\varphi_k(a_1)$ and $\varphi_k(b_1)$ to, say, c (which is the least in its equivalence class) must be the effect of a later action of a higher-priority strategy of the form $P_{c,d}$. When we S-collapsed a_0, b_0, a_1, b_1, we had $\varphi_k(b_0) \not\mathrel{E} \varphi_k(a_1)$, hence $a \not\mathrel{E} c$. When later we E-collapse a and c, either a or c stops being the least representative in its equivalence class, and so either $P_{a,b}$ or $P_{c,d}$ becomes "obsolete", and initializes N_k. We will show, however, that eventually N_k is not initialized anymore, so there is a final choice of the witnesses which allows for N_k only winning outcomes.

Environments for the requirements. A $P_{a,b}$ requirement uses a parameter $f_{a,b,s}(\_)$ which is a finite function approximating the function $f_{a,b}$ as in the informal description given above. An N_k requirement uses parameters $c_k(s)$, $d_k(s)$ (which yield the interval $I_k(s) = \{\langle k, x\rangle : c_k(s) \leqslant x \leqslant d_k(s)\}$), and the four odd numbers $a_{k,0}(s)$, $a_{k,1}(s)$, $b_{k,0}(s)$, $b_{k,1}(s)$ which are used to implement the $N_k(a_{k,0}(s), b_{k,0}(s), a_{k,1}(s), b_{k,1}(s))$-module.

The construction. Let us fix a universal ceer T defined by $\langle x, i\rangle \mathrel{T} \langle y, j\rangle$ and and only if $i = j$ and $x \mathrel{A_i} y$ where A_i is the ceer generated by the set W_i, with computable approximation $\{T_s : s \in \omega\}$ as a c.e. set. Let us fix also a computable approximation $\{R_s : s \in \omega\}$ to R as a c.e. set.

For a requirement to be *initialized*, it means that its parameters are set to be undefined.

Stage 0. All requirements are initialized.

Stage $s+1$. Suppose we have already dealt with all requirements $\mathcal{R}_j$, with $j < i < s$, after skipping all P-requirements that have already been declared obsolete.

If $p(s)$ is a parameter, or a computation of a partial computable function, as evaluated at stage s, then for simplicity we will omit to mention the stage s, thus simply writing p, instead of $p(s)$. We distinguish the following three cases.

Case 1. If $\mathcal{R}_i = P_{a,b}$ and a, b are not the least in their respective E-equivalence classes, then declare $P_{a,b}$ *obsolete* (thus we never consider $P_{a,b}$ again), and end the stage (this will cause all lower priority requirements to be initialized);

Case 2. $\mathcal{R}_i = P_{a,b}$ and a, b are the least in their respective E-equivalence classes, then define $f_{a,b,s+1}$ to be the extension of $f_{a,b}$ to the least (by code) pair u, v on which $f_{a,b}$ is undefined, and define $f_{a,b,s+1}(u, v) = m$, where m is a new odd number.

If $f_{a,b}(u', v') = m'$ is already defined and $m' \in W_{u'}$ then E-collapse b and m'; if $m' \in W_{v'}$ then E-collapse a and m'.

Case 3. Suppose $\mathcal{R}_i = N_k$. Carry on the first action that applies below:

(1) if c_k is undefined, then choose $c_k(s+1)$ (call it c_k) to be new (so, all numbers $\langle k, j\rangle$ with $j \geqslant c_k$ are bigger than all numbers so far mentioned in the construction), let $d_k(s+1) = c_k+1$ (call it d_k), and
$$I_k(s+1) = \{\langle k, x\rangle : c_k \leqslant x \leqslant d_k\},$$
(call it I_k);

(2) if $\varphi_k(j)$ converges for all $j \leqslant \langle k, d_k\rangle$, and there are four S-inequivalent elements z in I_k so that $\varphi_k(z)$ is odd, then
   (a) if $a_{k,0}, a_{k,1}, b_{k,0}, b_{k,1}$ are undefined, then define $a_{k,0}(s+1), a_{k,1}(s+1), b_{k,0}(s+1), b_{k,1}(s+1)$ to be the least four S-inequivalent numbers $j \in I_k$ such that $\varphi_k(j)$ is odd (call them $(a_{k,0}, a_{k,1}, b_{k,0}, b_{k,1})$;
   (b) if $a_{k,0}, a_{k,1}, b_{k,0}, b_{k,1}$ are defined, then perform the $N_k(a_{k,0}, a_{k,1}, b_{k,0}, b_{k,1})$-module;

(3) if $\varphi_k(j)$ converges for all $j \leqslant \langle k, d_k\rangle$, and there are not four S-inequivalent elements in I_k with odd φ_k-images, then code T in S by S-collapsing all pairs $\langle k, i\rangle, \langle k, j\rangle \in I_k$ such that their φ_k-images are even and so that $i \; T_s \; j$.

(4) if after (3), we have that for all $\langle k, i\rangle, \langle k, j\rangle \in I_k$ such that their φ_k-images are even,
$$\langle k, i\rangle \; S \; \langle k, j\rangle \Leftrightarrow \varphi_k(\langle k, i\rangle) \; E \; \varphi_k(\langle k, j\rangle)$$
then let $d_k(s+1) = d_k+1$, and consequently $I_k(s+1) = \{\langle k, x\rangle : c_k(s+1) \leqslant x \leqslant d_k(s+1)\rangle\}$.

If N_k has acted, or $i+1 = s$, then end the stage; otherwise go to the next $\mathcal{R}_{i+1}$.

Define E_{s+1} to be the equivalence relation generated by E_s plus the pairs (i, j) that have been E-collapsed at $s+1$, plus the pairs $(2i, 2j)$ such that $i \; R_s \; j$.

When we end the stage, if $\mathcal{R}$ is the last requirement that has acted, then we initialize all requirements of lower priority than $\mathcal{R}$.

Verification. The verification is based on the following lemmata.

We call an odd number n *active* at stage s if $n = f_{a,b,s}(u, v)$ and $n \not\mathrel{E_s} a$ and $n \not\mathrel{E_s} b$.

Lemma 2.13. *At every stage s:*

(1) $2i \; E_s \; 2j \Rightarrow i \; R_s \; j$
(2) If n is active, then $n \not\mathrel{E_s} 2i$ for every i.
(3) If $n \neq m$ are active, then $n \not\mathrel{E_s} m$.

Proof. All three conditions clearly hold at stage 0. Supposing all three conditions hold at stage s, we verify the conditions at stage $s+1$. If $2i \; E_{s+1} \; 2j$ but $2i \not\mathrel{E_s} 2j$, then this collapse must have either been caused by collapse in Case 2 or by the final instruction that we E_{s+1}-collapse $2i$ and $2j$ if $i \; R_s \; j$. In the latter case, we have $i \; R_s \; j$. In the former case, we must have the odd element $f_{a,b}(u', v')$ being E_s-equivalent to either $2i$ or $2j$, which would contradict (2) at the previous stage.

Suppose n is active and $n \; E_{s+1} \; 2i$ for some i. Then this collapse is either caused by collapse in Case 2 or by the final instruction that we E_{s+1}-collapse $2i$ and $2j$ if $i \; R_s \; j$. In the latter case, we would have $n \; E_s \; 2j$, which contradicts (2) at the previous stage. In the former case, we would have $n \; E_{s+1} \; 2i$ is caused by the collapse of an odd element m with some other element. Thus either $n \; E_s \; m$ or $2i \; E_s \; m$. If $n \; E_s \; m$, then we would have two active odd elements being equivalent at stage s contradicting (3). If $2i \; E_s \; m$, then we have an active element being equivalent to an even element, contradicting (2) at stage s.

Suppose $n \neq m$ are active at stage $s+1$ and $n\ E_{s+1}\ m$. Then again, this collapse is either caused by collapse in Case 2 or by the final instruction that we E_{s+1}-collapse $2i$ and $2j$ if $i\ R_s\ j$. As neither can be E_s-equivalent to an even number by (2) at the previous stage, we rule out the latter case. Then there is some m' active which collapses with another element and either $n\ E_s\ m'$ or $m\ E_s\ m'$. Either way, this contradicts (3) at stage s. □

Lemma 2.14. *For every i, j, we have $i\ R\ j \Leftrightarrow 2i\ E\ 2j$. Hence $R \leqslant E$.*

Proof. By Lemma 2.13, we have the right-to-left direction. By the final step of every stage, we ensure that if $i\ R\ j$ then $2i\ E\ 2j$. □

Lemma 2.15. *Each requirement $\mathcal{R}$ initializes lower-priority strategies only finitetely often, and if $\mathcal{R} = N_k$, for some k, then $\mathcal{R}$ requires attention finitely often: in particular $\lim_s d_k(s)$ exists.*

Proof. We prove this by induction. Assume the claim true of every requirement $\mathcal{R}'$, with $\mathcal{R}' < \mathcal{R}$. If $\mathcal{R} = N_k$, after all $\mathcal{R}'$ with $\mathcal{R}' < N_k$ stop initializing lower priority strategies, we have that N_k cannot be further initialized, and when this happens we appoint the last value c_k of $c_k(s)$. To see that N_k requires attention only finitely often, since the N-module itself is finitary, it is enough to show that $\lim_s d_k(s)$ exists. Suppose that this is not the case, then $d_k(s)$, after the last initialization of N_k monotonically increases to ∞. Then for all but finitely many T-classes $[x]_T$, we have that every $\varphi_k(\langle k, i\rangle)$ is even for all $i \in [x]_T$ with $i > c_k$. By padding and the definition of T, there is a reduction of T to itself given by $x \mapsto \langle l, x\rangle$ which misses these finitely many classes, where $l \in \omega$. Thus,

$$\begin{aligned} i\ T\ j &\Leftrightarrow \langle l, i\rangle\ T\ \langle l, j\rangle \\ &\Leftrightarrow \langle k, \langle l, i\rangle\rangle\ S\ \langle k, \langle l, j\rangle\rangle \\ &\Leftrightarrow \varphi_k(\langle k, \langle l, i\rangle\rangle)\ E\ \varphi_k(\langle k, \langle l, j\rangle\rangle) \\ &\Leftrightarrow \frac{\varphi_k(\langle k, \langle l, i\rangle\rangle)}{2}\ R\ \frac{\varphi_k(\langle k, \langle l, j\rangle\rangle)}{2}, \end{aligned}$$

This gives a computable reduction of T to R, but this is impossible since R is not universal. This contradicts the assumption that $\lim_s d_k(s)$ does not exist.

Thus all outcomes of N_k are finitary, and thus N_k also initializes lower priority requirements only finitely often. Similarly, if $\mathcal{R} = P_{a,b}$: after its last initialization, $P_{a,b}$ may initialize lower-priority strategies at most once, namely, when it becomes obsolete. □

Lemma 2.16. *Each requirement is satisfied, or eventually obsolete.*

Proof. Let $P_{a,b}$ be given, and $P_{a,b}$ not eventually obsolete. By Lemma 2.15, there is a least stage after which $P_{a,b}$ is not initialized any more. Then after this stage, we construct $f_{a,b}$ witnessing that $([a]_E, [b]_E)$ is an e.i. pair.

Let us now consider the case of an N_k-requirement, and let s_0 be a stage after which N_k is never again initialized, so no higher-priority N-requirement requires attention after s_0, nor does any higher-priority P-requirement become obsolete after s_0. Let c_k denote the limit value of $c_k(s)$, and by Lemma 2.15, let d_k be the limit vale of $d_k(s)$. So we have that $I_k = \{\langle k, x\rangle : c_k \leqslant x \leqslant d_k\rangle\}$ is the limit value of $I_k(s)$.

If φ_k is not total on $[0, \langle k, d_k\rangle]$ or for some x, y, we have

$$\neg[x\ T\ y \Leftrightarrow \varphi_k(\langle k, x\rangle)\ E\ \varphi_k(\langle k, y\rangle)],$$

then φ_k is not a reduction function and thus N_k is satisfied. Otherwise, we eventually appoint four permanent witnesses $a_{k,0}$, $b_{k,0}$, $a_{k,1}$, $b_{k,1}$. For simplicity, for $i = 0, 1$, write $a_i = a_{k,i}$ and $b_i = b_{k,i}$. We may suppose that action taken by N_k makes $a_0 \; S \; b_0$ and $a_1 \; S \; b_1$; otherwise, again N_k is satisfied. We must exclude the possibility that the numbers $\varphi_k(a_0), \varphi_k(b_0), \varphi_k(a_1), \varphi_k(b_1)$ all E-collapse, and the numbers a_0, b_0, a_1, b_1 all S-collapse. But, as explained in the informal description of the outcomes of the $N_k(a_0, b_0, a_1, b_1)$-module, this possibility would require some $P < N_k$ to become obsolete at some stage after s_0, thus providing one more initialization of N_k, which is impossible by the choice of s_0. □

This concludes the proof of the theorem. □

3. Transfinite iterations of the jump operation

For a finer analysis of the properties of the jump operation on ceers, we use computable ordinals, and Kleene's system $\mathcal{O}$ of ordinal notations: for all unexplained notions in this regard (including the function $|a|_{\mathcal{O}}$ expressing the ordinal with notation a; the sum $+_{\mathcal{O}}$ on ordinal notations, and Kleene's strict partial order $<_{\mathcal{O}}$ on notations) the reader is referred to Rogers' textbook [16, §11.7-8].

Definition 3.1. We define, for each ceer E and $a \in \omega$, a ceer $E^{(a)}$ by recursion as follows:

If $a = 1$, then $E^{(a)} = E$.

If $a = 2^b$, then $E^{(a)} = (E^{(b)})'$.

If $a = 3 \cdot 5^e$, then $E^{(a)} = \bigoplus_{n \in \omega} E^{(\varphi_e(n))}$.

If a is not of the form 2^b or $3 \cdot 5^e$, then $E^{(a)} = \mathrm{Id}$.

Lemma 3.2. *For every $a \in \mathcal{O}$, $E \leqslant E^{(a)}$. If $a, b \in \mathcal{O}$ and $a <_{\mathcal{O}} b$, then $E^{(a)} \leqslant E^{(b)}$, and if E is non-universal then $E^{(a)} < E^{(b)}$. Moreover, the reduction witnessing $E^{(a)} \leqslant E^{(b)}$ is uniform in a, b and does not depend on E.*

Proof. The proofs follow (using Theorem 1.2) by standard inductive arguments on notations. □

Theorem 3.3. *If E is any computably inseparable ceer, R is any ceer, and $a \in \mathcal{O}$ is any notation, then $E \leqslant R^{(a)}$ if and only if $E \leqslant R$.*

Proof. The right to left direction is clear since $R \leqslant R^{(a)}$. We now prove the left to right direction by transfinite induction on notations for a. If $a = 2^b$, and $E \leqslant R^{(a)} = (R^{(b)})'$, then by Observation 2.2, $E \leqslant R^{(b)}$, thus $E \leqslant R$ by inductive hypothesis. If $a = 3 \cdot 5^e$, and $E \leqslant R^{(a)}$, then by Proposition 2.9, $E \leqslant R^{(\varphi_e(n))}$ for some n, thus $E \leqslant R$ by inductive hypothesis.

□

Corollary 3.4. *Above any non-universal ceer R there is a non-universal ceer E such that $E \nleqslant R^{(a)}$ for any $a \in \mathcal{O}$.*

Proof. Given a non-universal ceer R, take E to be computably inseparable and $\geqslant R'$, as constructed in Theorem 2.11 (R' is not universal by Theorem 1.2(4)). If $E \leqslant R^{(a)}$ for any $a \in O$, then $E \leqslant R$ by the previous theorem, but then $R' \leqslant E \leqslant R$, contradicting the assumption that R is not universal. □

Theorem 3.5. *If $a, b \in O$ and $|a|_{\mathcal{O}} < |b|_{\mathcal{O}}$, then for any non-universal ceer X, $X^{(b)} \nleqslant X^{(a)}$.*

Proof. Suppose towards a contradiction that there are pairs of notations $a, b \in O$ so that $|a|_{\mathcal{O}} < |b|_{\mathcal{O}}$ and $X^{(b)} \leqslant X^{(a)}$. Choose such a pair which minimizes $|a|_{\mathcal{O}}$. Since X is not universal, and $X' \leqslant X^{(b)} \leqslant X^{(a)}$, we see that $X^{(a)} \nleqslant X$, so $|a|_{\mathcal{O}}$ is not 0. Let $c <_O b$ be so $|c|_{\mathcal{O}} = |a|_{\mathcal{O}}$.

First, suppose that $|a|_{\mathcal{O}}$ is a successor ordinal, so $a = 2^d$. Then, since $(X^{(c)})' \leqslant X^{(b)} \leqslant X^{(a)} = (X^{(d)})'$, we have that $X^{(c)} \leqslant X^{(d)}$ contradicting the minimality of a.

Next, suppose that $|a|_{\mathcal{O}}$ is a limit ordinal, so $a = 3 \cdot 5^e$. Then $(X^{(c)})' \leqslant X^{(b)} \leqslant X^{(a)}$. By Theorem 2.6, there is some n so that $(X^{(c)})' \leqslant X^{(\varphi_e(n))}$. But then $\varphi_e(n) <_O a$ contradicts the minimality of $|a|_{\mathcal{O}}$. □

Throughout what follows, we also use the notation

$$\exp^0(b) = b$$
$$\exp^{x+1}(b) = 2^{\exp^x(b)}.$$

Theorem 3.6. *If $a, b \in \mathcal{O}$ are notations for $\alpha < \omega^2$, then for any E, $E^{(a)} \equiv E^{(b)}$.*

Proof. Suppose that the theorem is false, and let $\alpha < \omega^2$ be least so that the theorem fails. First, we show that $\alpha \neq \omega$. Let a, b be any notations for ω. This means $a = 3 \cdot 5^{e_a}$ and $b = 3 \cdot 5^{e_b}$ where each $\varphi_{e_a}(n)$ is of the form $\exp^k(1)$ for some m, and similarly for $\varphi_{e_b}(n)$. Then to see $E^{(a)} \leqslant E^{(b)}$, we give a reduction: For each n, find some new m so that $\varphi_{e_b}(m) > \varphi_{e_a}(n)$. Then we can uniformly find a reduction from $E^{(\varphi_{e_a}(n))}$ to $E^{(\varphi_{e_b}(m))}$ since these are just finite jumps. Putting these reductions together yields a reduction witnessing $E^{(a)} \leqslant E^{(b)}$. Clearly the least α where the result fails cannot be a successor ordinal since $X \leqslant Y$ if and only if $X' \leqslant Y'$. For any other limit ordinal $\omega \cdot (k+1) < \omega^2$, with notations $a = 3 \cdot 5^{e_a}$ and $b = 3 \cdot 5^{e_b}$, we simply fix some n and m so that $\varphi_{e_a}(n)$ is a notation for an ordinal $> \omega \cdot k$ and $\varphi_{e_b}(m)$ is a notation for an ordinal $> |\varphi_{e_a}(n)|_{\mathcal{O}}$. Then by induction, we can get a reduction of the first n columns of $E^{(a)}$ into the first $m + n$ columns of $E^{(b)}$, and above these the rest is exactly as in the case of $\alpha = \omega$. □

In the rest of this section we will often use the jumps of the identity equivalence relation Id (where $x \text{ Id } y$ if $x = y$). If κ is the partial computable function $\kappa(x) = \varphi_x(x)$, then (as observed in [12]), for every $x, y, n \in \omega$,

$$x \text{ Id}^{(n)} y \Leftrightarrow (\exists i \leqslant n)[\kappa^i(x)\downarrow = \kappa^i(y)\downarrow],$$

where k^i is the i-th iterate of κ, starting with κ^0, the identity function. More generally, if X is an equivalence relation, then

$$x\ X^{(n)}\ y \Leftrightarrow (\exists i < n)[\kappa^i(x)\downarrow = \kappa^i(y)\downarrow \vee \kappa^n(x)\downarrow X\ \kappa^n(y)\downarrow].$$

In particular, we have the following lemma.

Lemma 3.7. *For every n, $\text{Id}^{(n)}$ is properly contained in $\text{Id}^{(n+1)}$.*

Proof. The proof is trivial. Properness of the inclusion follows from Theorem 1.2. □

The following theorem stands in contrast to the situation for the Turing degrees. In Turing reducibility, if a set X is $\geqslant_T$ all A-arithmetical sets, then the double Turing jump X'' of X computes $A^{(\omega)}$.

In the proof of next theorem and some of the following ones, we make use of infinite computable lists of fixed points as given by the recursion theorem, which is formally justified as follows. We fix a single index j so that we control φ_j by the Recursion Theorem, and we then take a computable list $(j_i)_{i\in\omega}$ of indices for the columns $\varphi_{j_i}(k) = \varphi_e(\langle i, k\rangle)$. We can then control the φ_{j_i} in any order we wish, as we are simply controlling the single function φ_j. Alternatively, since a computable sequence of indices can be viewed as the range of a computable function f, a formal justification to the argument is also provided by the Case Functional Recursion Theorem [5]: see also [15] for useful comments about this theorem.

Theorem 3.8. *There is a ceer X so that for all n, $X \geqslant \mathrm{Id}^{(n)}$ and for all k, $X^{(k)} \ngeqslant \mathrm{Id}^{(\omega)}$*

Proof. We present the ideas of the proof and omit some details. We build X with the property that only numbers in the same column can ever collapse, and we ensure that X satisfies two types of requirements. Firstly, we code each $\mathrm{Id}^{(n)}$ as a column of X. Secondly, we ensure φ_j is not a reduction of $\mathrm{Id}^{(\omega)}$ to $X^{(k)}$. Obviously, any finite number of requirements of the first kind commit to entire columns of X, but these together only force X to code some $\mathrm{Id}^{(n)}$ for a finite n. Similarly, we will make sure that actions for the requirements of the second type only effect X on finitely many columns, allowing infinitely many remaining columns for later requirements of the first type. Action for requirements of the first type are clear: You choose an unused column and code.

We now discuss action for requirements of the second type: We want to ensure that φ_j is not a reduction of $\mathrm{Id}^{(\omega)}$ to $X^{(k)}$. Let n be largest so that some higher priority requirement has some column of X committed to coding $\mathrm{Id}^{(n)}$, and let $m = n + k + 1$. The idea is that no higher priority requirement gives a way to decode $\mathrm{Id}^{(m)}$ from $X^{(k)}$. We will use this for our diagonalization.

We fix an infinite collection F of indices which we control via the recursion theorem, and let x_0, x_1 be from F. The m^{th} column of $\mathrm{Id}^{(\omega)}$ is $\mathrm{Id}^{(m)}$, so we consider the reduction $\varphi_{j'}(x) = \varphi_j(\langle m, x\rangle)$ from $\mathrm{Id}^{(m)}$ to $X^{(k)}$. We now diagonalize against $\varphi_{j'}$ being a reduction from $\mathrm{Id}^{(m)}$ to $X^{(k)}$. This allows us to forget about $\mathrm{Id}^{(\omega)}$ and work with the more managable $\mathrm{Id}^{(m)}$.

The goal is to force the enemy to cause $\kappa^k(\varphi_{j'}(x_0)$ and $\kappa^k(\varphi_{j'}(x_1))$ to converge while we have not yet caused $\kappa^m(x_0)$ or $\kappa^m(x_1)$ to have converged. Once this happens, we will have enough power to diagonalize as needed. We now describe how we entice $\kappa^k(\varphi_{j'}(x_0))$ to converge. Note that while we control φ_{x_0} via the recursion theorem, we have no power over the identity (κ^0) of $\varphi_{j'}(x_0)$, and thus have no direct control over convergences of κ on $\varphi_{j'}(x_0)$. Our only control is via the supposed reduction $\varphi_{j'}$ from $\mathrm{Id}^{(m)}$ to $X^{(k)}$. Using an auxiliary element, $x_2 \in F$, we first wait for $\varphi_{j'}(x_2)$ to converge, and then make $\kappa(x_0)\downarrow = \kappa(x_2) = x_3$ for an $x_3 \in F$. This implies that $x_0 \,\mathrm{Id}^{(m)}\, x_2$. In order to cause $\varphi_{j'}(x_0)\ X^{(k)}\ \varphi_{j'}(x_2)$, it must be that $\kappa(\varphi_{j'}(x_0))$ converges. Using similar strategies and many auxiliary elements, we can cause $\kappa^k(\varphi_{j'}(x_0))$ and $\kappa^k(\varphi_{j'}(x_1))$ to converge. In the meantime, we have caused $\kappa^k(x_0)$ and $\kappa^k(x_1)$ to have converged.

Now, $\varphi_{j'}(x_0)\ X^{(k)}\ \varphi_{j'}(x_1)$ if and only if $\kappa^k(\varphi_{j'}(x_0))$ is X-equivalent to $\kappa^k(\varphi_{j'}(x_1))$. Since we control X, we are in a position to diagonalize. That is true unless $\kappa^k(\varphi_{j'}(x_0)) = \langle l, a\rangle$ and $\kappa^k(\varphi_{j'}(x_1)) = \langle l, b\rangle$ where the l^{th} column is already a coding column for a higher priority requirement. In this case, that column is coding precisely $\mathrm{Id}^{(s)}$ for some $s \leqslant n$. Thus we have $\varphi_{j'}(x_0)\ X^{(k)}\ \varphi_{j'}(x_1)$ if and only if $a \,\mathrm{Id}^{(s)}\, b$.

We once more employ the strategy above to force $\kappa^s(a)$ and $\kappa^s(b)$ to converge. This causes us to cause $\kappa^{k+s}(x_0)$ and $\kappa^{k+s}(x_1)$ to converge to new members of F. Critically, $k+s \leqslant k+n < m$. That is, we have not yet caused $\kappa^m(x_0)$ or $\kappa^m(x_1)$ to converge. At this point, it has been determined whether or not $\kappa^s(a) = \kappa^s(b)$, and thus whether or not $\varphi_{j'}(x_0)\ X^{(k)}\ \varphi_{j'}(x_1)$, and we can still determine whether or not $x_0 \operatorname{Id}^{(m)} x_1$. This allows direct diagonalization. □

We now turn our attention to the case of understanding jumps of a ceer based on notations for ω^2. Unlike in the case of notations for ordinals $< \omega^2$, we will see that the degree of the jump depends on the notation. We will see that for any degree of a ceer X, there is a least degree of a ceer of the form $X^{(a)}$ where a is a notation for ω^2. We will see that there are incomparable degrees of this form, and that the only upper bound to the set of degrees of this form is the universal degree.

Theorem 3.9. *There is a notation $c \in \mathcal{O}$ for the ordinal ω^2 so that for any ceer X and notation $b \in \mathcal{O}$ for an ordinal $\geqslant \omega^2$, then $X^{(c)} \leqslant X^{(b)}$. Moreover, the reduction witnessing $X^{(c)} \leqslant X^{(b)}$ can be found uniformly in b and does not depend on X.*

Proof. We fix c to be a notation generated as follows: the notation c_0 for ω is given by the function $\varphi_e(n) = \exp^n(1)$. Let c_k be the chosen notation for $\omega \cdot (k+1)$. The notation c_{k+1} for $\omega \cdot (k+2)$ is given by the computable function $\varphi(n) = \exp^n(c_k)$. Finally, the notation for c is given by the computable function $\psi(n) = c_n$. This particular choice of c_0 is special for the purpose of the following lemma. The choice of c_{k+1} is so that $X^{(c_{k+1})} = (X^{(c_k)})^{(c_0)}$.

Lemma 3.10. *Given a reduction of the ceer X to $Y^{(m)}$, $m \in \mathcal{O}$, and given any notation $n = 3 \cdot 5^e \in \mathcal{O}$ so that $m <_{\mathcal{O}} n$, we can uniformly find a reduction of $X^{(c_0)}$ to $Y^{(n)}$.*

Proof. We have that $m <_{\mathcal{O}} \varphi_e(l) <_{\mathcal{O}} \varphi_e(l+1)$ for some l. Thus $2^{\varphi_e(l)} \leqslant_{\mathcal{O}} \varphi_e(l+1)$. Thus, since $X \leqslant Y^{(m)}$, we have that $X \leqslant Y^{(m)} \leqslant Y^{(\varphi_e(l))}$ and thus $X' \leqslant Y^{(2^{\varphi_e(l)})}$ and thus $X' \leqslant Y^{(\varphi_e(l+1))}$ uniformly. Repeating this process, we see that $X'' \leqslant Y^{(\varphi_e(l+2))}$ uniformly, and more generally $X^{(\exp^k(1))} \leqslant Y^{(\varphi_e(l+k))}$ uniformly, which gives a reduction of $X^{(c_0)}$ to $Y^{(n)}$ □

Let $b = 3 \cdot 5^e$ be any notation for an ordinal $\geqslant \omega^2$. It suffices to find uniformly an increasing sequence $n_0 < n_1 < \ldots$ and reductions of $X^{(c_i)}$ to $X^{(\varphi_e(n_i))}$. To find n_0, we wait until we see some $3 \cdot 5^{e_0} <_{\mathcal{O}} \varphi_e(y)$ for some e_0 and y. We then declare $n_0 = y$. Using the lemma above, we can uniformly find a reduction of $X^{(c_0)}$ to $X^{(3 \cdot 5^{e_0})}$, and thus to $X^{(\varphi_e(n_0))}$. Next, we search for some $n_1 > n_0$ and an e_1 so that $\varphi_e(n_0) <_{\mathcal{O}} 3 \cdot 5^{e_1} <_{\mathcal{O}} \varphi_e(n_1)$. Since $X^{(c_0)}$ reduces to $\varphi_e(n_0)$, the lemma allows us to uniformly reduce $X^{(c_1)} = (X^{(c_0)})^{(c_0)}$ to $X^{(3 \cdot 5^{e_1})}$, and thus to $X^{(n_1)}$. Repeating as such gives the desired reduction. □

Now we show that there can be incomparable degrees of the form $X^{(a)}$ where a is a notation for ω^2. In fact, this happens even with $X = \operatorname{Id}$.

Theorem 3.11. *There are two notations $a, b \in \mathcal{O}$ for the ordinal ω^2 so that $\operatorname{Id}^{(a)}$ and $\operatorname{Id}^{(b)}$ are incomparable ceers.*

Proof. Let $\{e_n : n \in \omega\}$ and $\{i_n : n \in \omega\}$ be recursive sets of indices that we control by the recursion theorem. Let e be so $\varphi_e(n) = 2^{3 \cdot 5^{e_n}}$, and $a = 3 \cdot 5^e$. Similarly, let i be so $\varphi_i(n) = 2^{3 \cdot 5^{i_n}}$ and let $b = 3 \cdot 5^i$.

Requirements. We satisfy requirements:

$$P_n : \varphi_n \text{ is not a reduction of } \mathrm{Id}^{(a)} \text{ to } \mathrm{Id}^{(b)}$$
$$R_n : \varphi_n \text{ is not a reduction of } \mathrm{Id}^{(b)} \text{ to } \mathrm{Id}^{(a)}$$

Fix a priority ordering of order type ω.

Throughout the course of the construction, we will be choosing values for $\varphi_{e_n}(m)$ and $\varphi_{i_n}(m)$ for each $m \in \omega$. At stage s, we determine the values of $\varphi_{e_n}(m)$ and $\varphi_{i_n}(m)$ when $n, m \leqslant s$. We choose $\varphi_{e_0}(0) = \varphi_{i_0}(0) = 1$. We always choose $\varphi_{e_{n+1}}(0) = e_n$ to ensure that $e_n <_{\mathcal{O}} e_{n+1}$ as needed to make $e \in \mathcal{O}$. Similarly, we choose $\varphi_{i_{n+1}}(0) = i_n$. Unless some instruction is given otherwise, we will always choose $\varphi_{e_n}(m+1) = 2^{\varphi_{e_n}(m)}$. At stage s, for any $n \in \omega$, we let N_n^s be 1 plus the number of x so that we have determined $\varphi_{e_n}(x)$ to be an odd number (i.e. a notation for a limit ordinal). Similarly, we let M_n^s be 1 plus the number of x so that we have determined $\varphi_{i_n}(x)$ to be an odd number. In other words, if we give no special instructions, we will continue the construction to make e_n be a notation for the ordinal $\omega \cdot (\Sigma_{k \leqslant n} N_k^s)$ and i_n be a notation for the ordinal $\omega \cdot (\Sigma_{k \leqslant n} M_k^s)$.

Actions. For the sake of requirement P_n, we act as follows: Choose l fresh, and let $k = \langle l, x \rangle \in \{l\} \times K$. We wait for $\varphi_n(k) \downarrow = \langle j, g \rangle$. Once we see this convergence, we mention $\varphi_n(k)$ so that future requirements will choose fresh numbers larger than this. Then, we let m be largest so that $\varphi_{e_l}(m)$ is defined. We then make $\varphi_{e_l}(m+1) = \varphi_{e_l}(m) +_{\mathcal{O}} c_{\Sigma_{k \leqslant j} M_k}$. Recall from above that c_k is a canonical notation for the ordinal $\omega \cdot k$.

The action for the sake of the requirement R_n is symmetric to the action for requirement P_n.

At each stage, we say that a requirement P_n or R_n *requires attention* if it has not yet acted since being initialized or if $\varphi_n(k)$ has converged (thus ending its waiting phase). At each stage, we let the highest priority requirement requiring attention act.

Whenever we act for the sake of requirement P_n (or R_n) by choosing l or by mentioning $\varphi_n(k)$ and choosing $\varphi_{e_l}(m+1)$ (or $\varphi_{i_l}(m+1)$), we initialize all lower priority requirements.

It is straightforward to see that each requirement is reinitialized only finitely often. Thus, each $3 \cdot 5^{e_{j+1}}$ is a notation for some ordinal of the form $3 \cdot 5^{e_j} +_{\mathcal{O}} c$ where c is a notation for an ordinal $< \omega^2$. It follows that $3 \cdot 5^e$ is a notation for ω^2. Similarly, $3 \cdot 5^i$ is a notation for ω^2.

Lemma 3.12. *Each requirement is satisfied.*

Proof. Let s be a stage at which P_n is last initialized. Then the value of l and k chosen is final. If $\varphi_n(k)$ does not converge, then P_n is satisfied. So, suppose it does converge to $\langle j, g \rangle$. Since every lower priority requirement is initialized, thus will choose a fresh l and only ever gives instructions for convergence on its own φ_{e_l} or φ_{i_l}, no lower priority requirement will ever give special instructions for φ_{e_d} or φ_{i_d} for any $d \leqslant j$. Similarly, no higher priority requirement will ever give any instructions as that would reinitialize P_n after stage s, contrary to the choice of s. This shows that $3 \cdot 5^{i_j}$ will be a notation for $\omega \cdot (\Sigma_{k \leqslant n} M_k^s)$. The instruction given for φ_{e_l} ensures that $3 \cdot 5^{e_l}$ is a notation for some ordinal greater than this. It follows from Theorem 3.5 that $\mathrm{Id}^{2^{3 \cdot 5^{e_l}}} \nleqslant \mathrm{Id}^{2^{3 \cdot 5^{i_j}}}$. Suppose now towards a contradiction that φ_n is a reduction of $\mathrm{Id}^{(a)}$ to $\mathrm{Id}^{(b)}$. By the proof of Theorem 2.6, since $\mathrm{Id}^{2^{3 \cdot 5^{e_l}}}$ reduces to $\mathrm{Id}^{(b)}$ with an element of K being sent to the jth column, it follows that $\mathrm{Id}^{2^{3 \cdot 5^{e_l}}}$ reduces to the jth column of $\mathrm{Id}^{(b)}$. But this just means that $\mathrm{Id}^{2^{3 \cdot 5^{e_l}}} \leqslant \mathrm{Id}^{2^{3 \cdot 5^{i_j}}}$, which is a contradiction.

□

The proof of the theorem is now complete. □

We now see that there is no non-universal ceer which is an upper bound for all the ceers of the form $Y^{(a)}$ where a is a notation for ω^2. Since for every ceer Y, $Y' > \mathrm{Id}$, it suffices to show the result for $Y = \mathrm{Id}$.

Theorem 3.13. *For any non-universal ceer X, there is a notation $a \in \mathcal{O}$ for the ordinal ω^2 so that* $\mathrm{Id}^{(a)} \nleqslant X$.

Proof. Let $\{e_k^n : n, k \in \omega\}$ and $I = \{i_{jk}^n : n, j, k \in \omega\}$ be recursive sets of indices that we control by the recursion theorem. We let $a = 3 \cdot 5^e$ where $\varphi_e(n) = 3 \cdot 5^{e_0^n}$ for every $n \in \omega$. As we progress, we define $\varphi_{e_0^n}(m)$ for each n and m. We make $\varphi_{e_0^0}(0) = 1$ and $\varphi_{e_0^{n+1}}(0) = 2^{3 \cdot 5^{e_0^n}}$ for each $n \in \omega$.

At various stages in the construction, we may choose $\varphi_{e_k^n}(m+1)$ to be $3 \cdot 5^{e_{k+1}^n}$. When we do this, we then define $\varphi_{e_{k+1}^n}(0) = 2^{\varphi_{e_k^n}(m)}$. The idea of the construction is as follows: We fix a universal ceer E. In order to diagonalize against φ_n being a reduction from $\mathrm{Id}^{(a)}$ into X, we attempt to code E into $\mathrm{Id}^{(a)}$. In order to do this, we create a sequence of notations $\{3 \cdot 5^{e_k^n}\}_{k \in \omega}$ so that for some m_k, we have $\varphi_{e_k^n}(m_k + 1) = 3 \cdot 5^{e_{k+1}^n}$. Of course, if this proceeded infinitely often, then $3 \cdot 5^{e_0^n}$ and thus a would not be in O, and certainly not a notation for ω^2, as needed. On the other hand, if this proceeded infinitely often, we would have $E \leqslant X$ contradicting the assumption that X is non-universal. Thus, this process will stop at some finite stage, and thus $3 \cdot 5^{e_0^n}$ will be a notation for some ordinal of the form $\omega \cdot k$ for some $k \in \omega$. This is the strategy to satisfy a single requirement diagonalizing against φ_n giving a reduction from $\mathrm{Id}^{(a)}$ to X. We run one such strategy for each φ_n.

Requirements. Here are the requrements to be satisfied, for each φ_n:

$$P_n : \varphi_n \text{ is not a reduction of } \mathrm{Id}^{(a)} \text{ to } X$$

The P_n strategy. If $n = 0$, we make $\varphi_{e_0^n}(0) = 1$. Otherwise, we make $\varphi_{e_0^n}(0) = 2^{3 \cdot 5^{e_0^{n-1}}}$, let $\varphi_{e_0^n}(1) = 3 \cdot 5^{e_1^n}$, $\varphi_{e_0^n}(x+1) = 2^{\varphi_{e_0^n}(x)}$ for all $x > 1$. Lastly, we define $\varphi_{e_1^n}(0) = 2^{\varphi_{e_0^n}(0)}$. Let $m_0 = 0$. We will attempt to ensure that if φ_n is a reduction of $\mathrm{Id}^{(a)}$ into X, then $j \mapsto \langle 2, i_{0j}^n \rangle$ is a reduction of E into $\mathrm{Id}^{(3 \cdot 5^{e_0^n})}$. This would contradict X being non-universal. Note that when first considered, $\varphi_{i_{0j}^n}(x)$ is undefined for all x, and we declare every $j \in \omega$ to be *active*.

We define a counter $c = 1$. We allow this requirement to act at stage $s > n$ if for all $j, j' \leqslant c$, $j \; \mathrm{Id}^{(a)} \; j'$ if and only if $\varphi_n(j) \; X \; \varphi_n(j')$.

If the P_n strategy acts at stage s, let $j < j'$ be the first pair so that $j \; E_s \; j'$ and we have not yet acted on behalf of this pair. We will now act on behalf of this pair. If j or j' is not active, then we do nothing. Otherwise, let k be largest so that $\varphi_{e_k^n}(0)$ is defined, and let m_k be largest so that $\varphi_{e_k^n}(m_k)$ is defined. Let $\varphi_{e_k^n}(m_k + 1) = 3 \cdot 5^{e_{k+1}^n}$ and $\varphi_{e_k^n}(x+1) = 2^{\varphi_{e_k^n}(x)}$ for all $x \geqslant m_k + 1$. We define $\varphi_{e_{k+1}^n}(0) = 2^{\varphi_{e_k^n}(m_k)}$. We make $\varphi_{i_{(k-1)j}^n}(x) = \varphi_{i_{(k-1)j'}^n}(x) = \langle m_k + 2, i_{kj}^n \rangle$ for all x and $\varphi_{i_{(k-1)g}^n}(x) = \langle m_k + 2, i_{kg}^n \rangle$ for all x and $g \neq j, j'$. We now declare j' to be *inactive* (we do not need to collapse $i^n_{0j'}$ with other elements because we will already cause $i^n_{0j'}$ $\mathrm{Id}^{(3\cdot 5^{e^n_0})}$ i^n_{0j} and $j < j'$). We now increase the counter c by 1.

Note that if j and j' are active, then for all $d < k-1$, we have made $\varphi_{i^n_{dj}}(x) = \langle m_d + 2, i^n_{(d+1)j}\rangle$ and similarly for j'. Thus, we have ensured for each $d < k-1$ that

$$\begin{aligned}\langle m_d + 2, i^n_{dj}\rangle \,\mathrm{Id}^{(3\cdot 5^{e^n_d})}\, \langle m_d + 2, i^n_{dj'}\rangle &\Leftrightarrow i^n_{dj} \,\mathrm{Id}^{(2^{3\cdot 5^{e^n_{d+1}}})}\, i^n_{dj'} \\ &\Leftrightarrow \langle m_{d+1} + 2, i^n_{(d+1)j}\rangle \,\mathrm{Id}^{(3\cdot 5^{e^n_{d+1}})}\, \langle m_{d+1} + 2, i^n_{(d+1)j'}\rangle.\end{aligned}$$

In particular, by making $\varphi_{i^n_{(k-1)j}}(x) = \varphi_{i^n_{(k-1)j'}}(x)$, we make $i^n_{(k-1)j}$ $\mathrm{Id}^{(2^{3\cdot 5^{e^n_{d+1}}})}$ $i^n_{(k-1)j'}$, and thus $\langle 2, i^n_{0j}\rangle \,\mathrm{Id}^{(3\cdot 5^{e^n_0})}\, \langle 2, i^n_{0j'}\rangle$.

At each stage, for each function $\varphi_{e^l_k}$ so that $\varphi_{e^l_k}(0)$ is defined, we let m be largest so that $\varphi_{e^l_k}(m)$ is already defined. In the absence of any instruction from any acting requirement as to what value to give $\varphi_{e^l_k}(m+1)$, we define $\varphi_{e^l_k}(m+1) = 2^{\varphi_{e^l_k}(m)}$.

Lemma 3.14. *Every strategy acts only finitely often and thus succeeds.*

Proof. Suppose towards a contradiction that some requirement acts infinitely often. Let n be so that P_n acts infinitely often. Thus we have φ_n is a reduction of $\mathrm{Id}^{(e)}$ to X. We now will show that $E \leqslant \mathrm{Id}^{(3\cdot 5^{e^n_0})}$ via the map $j \mapsto \langle 2, i^l_{0j}\rangle$. By construction, we have ensured that if $j \mathrel{E} j'$, then $\langle 2, i^l_{0j}\rangle \,\mathrm{Id}^{(e^n_0)}\, \langle 2, i^l_{0j'}\rangle$. Suppose j and j' are least in their E-equivalence classes. Then at every stage in the construction, the analysis above shows that $\langle 2, i^n_{0j}\rangle \,\mathrm{Id}^{(3\cdot 5^{e^n_0})}\, \langle 2, i^n_{0j'}\rangle$ is equivalent to $i^n_{(k-1)j}$ $\mathrm{Id}^{(2^{3\cdot 5^{e^n_k}})}$ $i^n_{(k-1)j'}$ where k is greatest so that $\varphi_{e^n_k}(0)$ is defined. But now this is impossible, since $\mathrm{Id}^{(2^{3\cdot 5^{e^n_k}})}$ is a jump, but $\varphi_{i^n_{(k-1)j}}(x)$ and $\varphi_{i^n_{(k-1)j'}}(x)$ have not yet converged. Thus at no stage can it be that $\langle 2, i^n_{0j}\rangle \,\mathrm{Id}^{(3\cdot 5^{e^n_0})}\, \langle 2, i^n_{0j'}\rangle$. Thus $j \mapsto \langle 2, i^n_{0j}\rangle$ gives a reduction of E to $\mathrm{Id}^{(3\cdot 5^{e^n_0})}$, but this is the nth column of $\mathrm{Id}^{(a)}$, thus we have E reduces to $\mathrm{Id}^{(a)}$. Thus since φ_n gives a reduction of $\mathrm{Id}^{(a)}$ to X, we have a reduction of E to X contradicting the non-universality of X. $\square$

Thus, each $e^0_i <_{\mathcal{O}} e^0_{i+1}$ and there are only finitely many notations n for limit ordinals so that $e^0_i <_{\mathcal{O}} n <_{\mathcal{O}} e^0_{i+1}$. Thus a is a notation for ω^2. $\square$

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