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(Article begins on next page)

Varieties of Bounded K-lattices*

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Abstract

In this paper we continue to study varieties of K-lattices, focusing on their bounded versions. These (bounded) commutative residuated lattices arise from a specific kind of construction: the *twist-product* of a lattice. Twist-products were first considered by Kalman in 1958 to deal with order involutions on plain lattices, but the extension of this concept to residuated lattices has attracted some attention lately. We describe the lower part of the lattice of subvarieties of bounded K-lattices, showing that there is only one atom and describing up to a certain extent the covers of the atom. We also consider some special subvarieties of bounded K-lattices, and study the lattice of subvarieties for those cases.

Introduction

This paper is a natural continuation of the investigation in [3] on varieties of K-lattices. These are commutative residuated lattices coming from a specific kind of construction: the *twist-product*.

The idea of considering the twist-product construction goes back to Kalman's paper [18], where only pure lattices were considered. The extension of this

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concept to residuated lattices is due to Tsinakis and Wille [23]; they considered the twist-product of a residuated lattice \mathbf{L} having a greatest element \top such that the element $(\top, 1)$ (1 the monoid identity) is the dualizing element relative to the natural involution. In other words for all $(x, y) \in L \times L^\partial$, $\sim(x, y) = (x, y) \rightarrow (\top, 1)$ and so $((x, y) \rightarrow (\top, 1)) \rightarrow (\top, 1) = (x, y)$.

K-lattices were introduced by [10] and they are residuated lattices that are subalgebras of the algebras obtained by applying the Tsinakis-Wille construction to *integral* commutative residuated lattices; in this case $\top = 1$ and the dualizing pair is $(1, 1)$.

The lattice of subvarieties of K-lattices has been investigated in [3] and this paper is mostly based on the results therein, but here we choose a different starting point, as we consider twist-products coming from *bounded* commutative residuated lattices.

We will not try to develop a theory of bounded K-lattices, paralleling the one in [3] and the reason is that many parts would be just a repetition of our previous work. In this paper we will focus on the differences in the various parts of the theory, quoting freely [3] for the parts that are clearly identical. Moreover we will use the same notation for residuated lattices that are lower bounded and their *bounded* version, letting the context clear any possible ambiguity.

The paper is organized as follows. In Section 1, we enumerate all general results needed to tackle the problem at hand, as well as those results from K-lattices that will be needed. In Section 2 we describe the lower part of the lattice of subvarieties of bounded K-lattices, specifically we show that there is only one atom, describe (up to a certain extent) all finitely generated covers of the atom, and mention some infinite covers. Finally in Section 3 we consider some special subvarieties of bounded K-lattices, and study the lattice of subvarieties for those cases.

1 Preliminaries

As in [3], we first mention the classical result by B. Jónsson (see [17, 9] for details). We use the standard notation of $\mathbf{S}, \mathbf{H}, \mathbf{P}, \mathbf{P}_u, \mathbf{V}$ for the operators subalgebras, homomorphic images, products, ultraproducts and varieties, acting on classes of algebras.

Lemma 1.1. (*Jónsson's Lemma*) *Let \mathbf{K} be a class of algebras such that $\mathbf{V}(\mathbf{K})$ is congruence distributive; then*

1. *if \mathbf{A} is a finitely subdirectly irreducible algebra in $\mathbf{V}(\mathbf{K})$, then $\mathbf{A} \in \mathbf{HSP}_u(\mathbf{K})$;*
2. *if \mathbf{A}, \mathbf{B} are finite subdirectly irreducible algebras in $\mathbf{V}(\mathbf{K})$ then $\mathbf{V}(\mathbf{A}) = \mathbf{V}(\mathbf{B})$ if and only if $\mathbf{A} \cong \mathbf{B}$.*

In particular if \mathbf{K} is a finite class of finite algebras and $\mathbf{V}(\mathbf{K})$ is congruence distributive, then all the finitely subdirectly irreducible algebras in $\mathbf{V}(\mathbf{K})$ are in $\mathbf{HS}(\mathbf{K})$.

A **commutative residuated lattice** is a structure $(A, \vee, \wedge, \cdot, \rightarrow, 1)$ such that

1. (A, \vee, \wedge) is a lattice;
2. $(A, \cdot, 1)$ is a commutative monoid;
3. (\cdot, \rightarrow) form a residuated pair w.r.t. the ordering, i.e. for all $a, b, c \in A$

$$ab \leq c \quad \text{if and only if} \quad a \leq b \rightarrow c.$$

We denote this variety by CRL; if 1 is the largest element in the ordering the lattice is said to be **integral**. Commutative and integral residuated lattices form a variety which we call CIRL. Note that algebras in CRL are congruence distributive, since they have a lattice reduct.

If we enlarge the type of CIRL with a constant 0 and we add the axiom $0 \leq x$ then we get the variety of **bounded** commutative residuated lattices, denoted by BCRL¹. Any finite algebra in CIRL is naturally bounded and most subvarieties of CIRL have their bounded version. In BCRL we have a natural negation given by $\neg x = x \rightarrow 0$.

There are two equations that result in interesting subvarieties of CIRL and BCRL

$$(x \rightarrow y) \vee (y \rightarrow x) \approx 1; \tag{P}$$

$$x(x \rightarrow y) \approx y(y \rightarrow x). \tag{D}$$

The subvariety of BCRL given by equation (P) is the variety of MTL-algebras, and the one given by equations (P) and (D) is the variety of BL-algebras.

Algebras in CIRL satisfying (D) are called hoops. For hoops usually there is a convention when dealing with their bounded counterparts: we use the name *algebra* to identify the bounded version; so we talk about Wajsberg algebras (or MV-algebras), product algebras, Gödel algebras and of course the bounded version of generalized Boolean algebras are just Boolean algebras. While the term *basic algebra* is used sometimes in the literature the most common name for the bounded version of basic hoops is **BL-algebras** and there is no standard name for the bounded version of hoops. The theory of the bounded version of any variety is almost equal to the theory of the unbounded version. There is one main difference though: since all the algebras are bounded they cannot be cancellative, so there is no variety corresponding to cancellative hoops. Of course cancellation comes back through the window as soon as we start talking about ordinal sums (see the definition below); in fact it is evident that any ordinal sum in which the first component is an algebra in BCRL and all the others are algebras in CIRL is an algebra in BCRL. Another general fact is the following: adding a constant changes the subalgebras of an algebra, in the sense that one has less subalgebras to consider. This usually simplifies the

¹We follow the common usage that bounded implies integral, see [14].

lattice of subvarieties: it is well known (and an easy exercise) that the lattice of subvarieties of product algebras is the three element chain (and the atom is the variety of Boolean algebras). The reader can compare this with the lattice of subvarieties of product hoops described in [1].

The connection however is strong; it can be proved that if \mathbf{V} is a subvariety of BCRL then the class $\mathbf{S}^0(\mathbf{V})$ of its zero-free subreducts (that is, subalgebras as CIRL of the zero-free reducts of algebras in \mathbf{V}) is a subvariety of CIRL (see [1], where the result is stated and proved for hoops, but the divisibility is never used in the proof). But there are also more subtle differences; it is tempting to conjecture that if a set of zero-free equations axiomatizes a subvariety \mathbf{V} of BCRL then the variety $\mathbf{S}^0(\mathbf{V})$ of its zero-free subreducts is axiomatized by the same set of equations. Though this happens very often it does not always happen, as shown in [4] p. 372.

A powerful tool for constructing is the ordinal sum. If $\mathbf{A}_0 \in \text{BCRL}$, $\mathbf{A}_1 \in \text{CIRL}$ we put a structure on the set $A_0 \setminus \{1\} \cup A_1 \setminus \{1\} \cup \{1\}$. The ordering is given by

$$a \leq b \quad \text{if and only if} \quad \begin{cases} b = 1, \text{ or} \\ a \in A_0 \setminus \{1\} \text{ and } b \in A_1 \setminus \{1\} \text{ or} \\ a, b \in A_i \setminus \{1\} \text{ and } a \leq_{A_i} b, i = 0, 1. \end{cases}$$

and we define

$$a \cdot b = \begin{cases} a, & \text{if } a \in A_0 \setminus \{1\} \text{ and } b \in A_1, \\ b, & \text{if } a \in A_1 \text{ and } b \in A_0 \setminus \{1\}, \\ a \cdot_{A_i} b, & \text{if } a, b \in A_i, i = 0, 1, \end{cases}$$

$$a \rightarrow b = \begin{cases} 1, & \text{if } a \in A_0 \setminus \{1\} \text{ and } b \in A_1, \\ b, & \text{if } a \in A_1 \text{ and } b \in A_0 \setminus \{1\}, \\ a \rightarrow_{A_i} b, & \text{if } a, b \in A_i, i = 0, 1. \end{cases}$$

If we call $\mathbf{A}_0 \oplus \mathbf{A}_1$ the resulting structure, then it is easily checked that $\mathbf{A}_0 \oplus \mathbf{A}_1$ is a semilattice ordered integral and commutative residuated monoid (and so the ordinal sum of two hoops in the sense of [8] always exists). It might not be a residuated lattice though and the reason is that if 1_{A_0} is not join irreducible and \mathbf{A}_1 is not bounded we run into trouble. In fact if $a, b \in A_0 \setminus \{1\}$ and $a \vee_{A_0} b = 1_{A_0}$ then the upper bounds of $\{a, b\}$ all lie in A_1 ; and since \mathbf{A}_1 is not bounded there can be no least upper bound of $\{a, b\}$ in $\mathbf{A}_0 \oplus \mathbf{A}_1$ and the ordering cannot be a lattice ordering. However this is the only case we have to worry about; if 1_{A_0} is join irreducible, then the problem disappears, and if 1_{A_0} is not join irreducible but \mathbf{A}_1 is bounded, say by u , then we can define

$$a \vee b = \begin{cases} a, & a \in A_1 \text{ and } b \in A_0; \\ b, & a \in A_0 \text{ and } b \in A_1; \\ a \vee_{A_1} b, & \text{if } a, b \in A_1; \\ a \vee_{A_0} b, & \text{if } a, b \in A_0 \text{ and } a \vee_{A_0} b < 1; \\ u, & \text{if } a, b \in A_0 \text{ and } a \vee_{A_0} b = 1; \end{cases}$$

We will call $\mathbf{A}_0 \oplus \mathbf{A}_1$ the **ordinal sum** and we will say that the ordinal sum **exists** if $\mathbf{A}_0 \oplus \mathbf{A}_1 \in \text{BCRL}$. We will now list some examples of varieties in which ordinal sums play a special role.

It is easy to see that the 3-element Gödel chain \mathbf{G}_3 is isomorphic to $\mathbf{2} \oplus \mathbf{2}$. If we consider \mathbf{GA} the variety of Gödel algebras, it can be shown that \mathbf{GA} is a locally finite variety and it is generated by all the finite Gödel chains $\mathbf{G}_n = \mathbf{2} \oplus \dots \oplus \mathbf{2}$ ($n - 1$ summands).

More generally, algebras of the form $\mathbf{2} \oplus \mathbf{D}$ for some $\mathbf{D} \in \text{CIRL}$ are exactly the directly indecomposable Stonean algebras (see Section 3.4).

On the other hand, subdirectly irreducible BL-algebras are of the form $\mathbf{A} \oplus \mathbf{D}$, with \mathbf{A} a (bounded) Wajsberg chain and \mathbf{D} a basic hoop.

A large class of algebras in BCRL can be obtained as follows. For any $\mathbf{A} \in \text{CIRL}$ we construct a new algebra, called the **connected rotation** of \mathbf{A} in the following way; if $\mathbf{L}_2 = \{0, \frac{1}{2}, 1\}$ then the universe is

$$A^{\delta_2} = (\{0\} \times A) \cup \left\{ \left(\frac{1}{2}, 1 \right) \right\} \cup (\{1\} \times A)$$

and the operations are defined as

$$\begin{aligned} (x, a) \vee (y, b) &= \begin{cases} (1, a \vee b) & \text{if } x = y = 1; \\ (0, a \wedge b), & \text{if } x = y = 0; \\ (y, b), & \text{if } x < y. \end{cases} \\ (x, a) \wedge (y, b) &= \begin{cases} (1, a \wedge b) & \text{if } x = y = 1; \\ (0, a \vee b), & \text{if } x = y = 0; \\ (y, b), & \text{if } x < y. \end{cases} \\ (x, a)(y, b) &= \begin{cases} (1, ab), & \text{if } x = y = 1; \\ (xy, 1), & \text{if } x, y \neq 1; \\ (y, b), & \text{if } x = 1 \text{ and } y = \frac{1}{2}; \\ (y, a \rightarrow b), & \text{if } x = 1 \text{ and } y = 0. \end{cases} \\ (x, a) \rightarrow (y, b) &= \begin{cases} (1, a \rightarrow b), & \text{if } x = y = 1; \\ (1, b \rightarrow a), & \text{if } x = y = 0; \\ (0, ab), & \text{if } x = 1 \text{ and } y = 0; \\ (1, 1), & \text{if } x < y; \\ (x \rightarrow y, 1), & \text{otherwise.} \end{cases} \end{aligned}$$

It is not hard to check that $\mathbf{A}^{\delta_2} \in \text{BCRL}$ and it is also involutive; moreover the set $A^{\delta_2} \setminus \{(\frac{1}{2}, 1)\}$ is a subalgebra of \mathbf{A}^{δ_2} that we will call the **disconnected rotation** of \mathbf{A} and denote by \mathbf{A}^{δ_1} . Connected and disconnected rotations of algebras in CIRL have been studied in [12], [7] and [5].

On one hand, $\mathbf{2}^{\delta_1}$, the disconnected rotation of the 2-element Boolean algebra, is the 4-element nilpotent minimum chain \mathbf{N}_4 . In general, the variety of nilpotent minimum algebras is generated by connected and disconnected rotations of finite Gödel chains (see Section 3.2).

On the other hand, the Chang algebra \mathbf{L}_1^ω (see Section 3.1.1) is isomorphic to the disconnected rotation of the cancellative hoop \mathbf{C}_ω .

Let $\mathbf{A} \in \text{CIRL}$; the **K-expansion** $K(\mathbf{A})$ of \mathbf{A} is a structure whose universe is $A \times A$ and the operations are defined as:

$$\begin{aligned}(a, b) \vee (c, d) &:= (a \vee c, b \wedge d) \\ (a, b) \wedge (c, d) &:= (a \wedge c, b \vee d) \\ (a, b)(c, d) &:= (ac, (a \rightarrow d) \wedge (c \rightarrow b)) \\ (a, b) \rightarrow (c, d) &:= ((a \rightarrow c) \wedge (d \rightarrow b), ad).\end{aligned}$$

Theorem 1.2. [10] *For every $\mathbf{A} \in \text{CIRL}$, $K(\mathbf{A})$ is a commutative residuated lattice that is also*

1. 1-involutive;
2. **1-distributive**, i.e. it satisfies both distributive laws for lattices whenever at least one of the elements is equal to 1.
3. If we set $\sim x = x \rightarrow 1$ then it satisfies the equations

$$xy \wedge 1 \approx (x \wedge 1)(y \wedge 1) \tag{K1}$$

$$((x \wedge 1) \rightarrow y) \wedge ((\sim y \wedge 1) \rightarrow \sim x) \approx x \rightarrow y. \tag{K2}$$

A **Kalman lattice** or just **K-lattice** is a commutative residuated lattice that is 1-involutive, 1-distributive and satisfies (K1) and (K2); the variety of K-lattices is denoted by **KL**.

If we start with an algebra $\mathbf{A} \in \text{BCRL}$, then $K(\mathbf{A})$ will have $(0, 1)$ as a lower bound, and $(1, 0) = \sim (0, 1)$ as an upper bound. Thus we define **bounded K-lattices** as the variety of K-lattices with an extra constant 0 and the axiom $0 \leq x$ (observe that they will also be upper bounded by $\top = \sim 0$). We will denote the variety of bounded K-lattices by **BKL**.

Most of the basic definitions about K-lattices in [3] can be formulated in the bounded case as well. In particular if $\mathbf{B} \in \text{BKL}$ we denote by \mathbf{B}^- the algebra whose universe is $\{b \in B : b \leq 1\}$ endowed with same lattice operations, constants 0, 1 and multiplication as \mathbf{B} and a new implication $a \rightarrow^- b = (a \rightarrow b) \wedge 1$. It is clear that $\mathbf{B}^- \in \text{BCRL}$ and moreover $\text{Con}(\mathbf{B})$ and $\text{Con}(\mathbf{B}^-)$ are isomorphic via the mappings

$$\alpha \mapsto \alpha^- = \alpha \cap (B^- \times B^-) \quad \theta \mapsto \text{Cg}_{\mathbf{B}}(\theta).$$

In particular for each $\alpha \in \text{Con}(\mathbf{B})$ there is an $\alpha^- \in \text{Con}(\mathbf{B}^-)$ such that $(\mathbf{B}/\alpha)^- \cong \mathbf{B}^-/\alpha^-$ (see [16] for details). Moreover we have two lemmas whose proofs are identical to the analogous one for the unbounded case (the first appears in [3] and the second in [10]).

Lemma 1.3. *Let $\mathbf{A}, \mathbf{B} \in \text{BCRL}$ and let $(\mathbf{A}_i)_{i \in I} \subseteq \text{BCRL}$. Then*

1. if $\mathbf{A} \leq \mathbf{B}$, then $\mathbf{A}^- \leq \mathbf{B}^-$;
2. $(\prod_{i \in I} \mathbf{A}_i)^- \cong \prod_{i \in I} \mathbf{A}_i^-$.

Lemma 1.4. [10] If $\mathbf{L} \in \text{BCRL}$ then $\mathbf{L} \cong K(\mathbf{L})^-$. If $\mathbf{A} \in \text{BKL}$, then $f : a \mapsto (a \wedge 1, \sim a \wedge 1)$ is an embedding of \mathbf{A} in $K(\mathbf{A}^-)$.

By the same token, all the properties of the operator K transfer to the bounded case without any change. If \mathbf{K} is a class of algebras in BCRL, then $K(\mathbf{K}) = \{\mathbf{A} \in \text{BKL} : \mathbf{A} \leq K(\mathbf{L}) \text{ for some } \mathbf{L} \in \mathbf{K}\}$; if \mathbf{W} is a class of algebras in BKL we define $\mathbf{W}^- = \mathbf{V}(\{\mathbf{A}^- : \mathbf{A} \in \mathbf{W}\})$. As in [3], a subvariety \mathbf{W} of BKL is a **Kalman variety** if $\mathbf{W} = K(\mathbf{W}^-)$.

Lemma 1.5. Let \mathbf{V} be any subvariety of BCRL, \mathbf{W} any subvariety of BKL and \mathbf{K} any subclass of BCRL:

1. $K(\mathbf{V})$ is a subvariety of BKL and $K(\mathbf{V}) = \{\mathbf{A} \in \text{BKL} : \mathbf{A}^- \in \mathbf{V}\}$;
2. $K(\mathbf{V}(\mathbf{K})) = \mathbf{V}(K(\mathbf{K}))$;
3. $\mathbf{HSP}_u(K(\mathbf{K})) \subseteq \mathbf{SK}(\mathbf{HSP}_u(\mathbf{K}))$;
4. $K(\mathbf{V})^- = \mathbf{V}$ and $\mathbf{W} \subseteq K(\mathbf{W}^-)$;
5. $K(\mathbf{W}^-)$ is the smallest Kalman variety containing \mathbf{W} ;
6. $K : \Lambda(\text{BCRL}) \mapsto \Lambda(\text{BKL})$ is a lattice homomorphism;
7. $\mathbf{W}^- \subseteq \mathbf{V}$ if and only if $\mathbf{W} \subseteq K(\mathbf{V})$, hence²

$$\mathbf{W}^- = \bigwedge \{\mathbf{U} : \mathbf{W} \subseteq K(\mathbf{U})\}.$$

8. K is also injective, i.e. it is an embedding.

If $\mathbf{A} \in \text{BKL}$ and \mathbf{B} is a subalgebra of \mathbf{A} , we say that \mathbf{B} is an **admissible subalgebra** if $\mathbf{B}^- = \mathbf{A}^-$. Since it is evident that the intersection of any family of admissible subalgebras is still admissible, admissible subalgebras form a complete meet-semilattice (indeed algebraic) under inclusion.

If $\mathbf{A} \in \text{BCRL}$, we define $K_0(\mathbf{A})$ as the minimal admissible subalgebra of $K(\mathbf{A})$. This notation will come in handy when we look at some of the examples.

In some special cases admissible subalgebras can be totally characterized.

An algebra $\mathbf{A} \in \text{BCRL}$ is **involutive** if for all $a \in A$, $\neg\neg a = a$; in this case we may define $a \oplus b = (a \rightarrow 0) \rightarrow b$.

Theorem 1.6. Let $\mathbf{A} \in \text{BCRL}$ be involutive; then there is a one to one and onto correspondence between the lattice filters of \mathbf{A} and the admissible subalgebras of $K(\mathbf{A})$. More precisely, if F is a lattice filter of \mathbf{A} , then

$$K(\mathbf{B}, F) = \{(a, b) \in K(\mathbf{B}) : a \oplus b \in F\} \leq K(\mathbf{B})$$

²i.e. the operators $K, ^-$ form a left adjoint pair between $\Lambda(\text{BCRL})$ and $\Lambda(\text{BKL})$.

is admissible. Conversely if $\mathbf{A} \in \mathbf{BKL}$ is such that \mathbf{A}^- is involutive, then

$$F = \{((a \wedge 1) \rightarrow^- 0) \rightarrow^- \sim a : a \in A\}$$

is a lattice filter of \mathbf{A}^- and $K(\mathbf{A}^-, F)$ is an admissible subalgebra of $K(\mathbf{A}^-)$.

Wajsberg algebras and rotations (both connected and disconnected) are examples of involutive BCRL.

Let now \mathbf{H} be a Heyting algebra; an element $a \in H$ is **dense** if $\neg a = 0$. A filter F of \mathbf{H} is **regular** if it contains all the dense elements; it is easy to see that F is regular if and only if \mathbf{H}/F is a Boolean algebra.

Theorem 1.7. *Let \mathbf{H} be a Heyting algebra; then there is a one to one correspondence between the regular filters of \mathbf{H} and the algebras in \mathbf{KL} whose negative cone is isomorphic with \mathbf{H} . More precisely, if F is a regular filter of \mathbf{H} , then*

$$K(\mathbf{H}, F) = \{(a, b) \in K(\mathbf{H}) : a \vee b \in F\} \leq K(\mathbf{H})$$

and $K(\mathbf{H}, F)^- \cong \mathbf{H}$. Conversely if $\mathbf{A} \in \mathbf{BKL}$ is such that \mathbf{A}^- is a Heyting algebra, then $F = \{(a \vee \sim a) \wedge 1 : a \in A\}$ is a regular filter of \mathbf{A}^- such that $K(\mathbf{A}^-, F) \cong \mathbf{A}$.

For a proof of the two previous results we quote [10].

We also have the following result, concerning ordinal sums.

Theorem 1.8. ([3], Theorem 3.4) *The admissible subalgebras of $K(\mathbf{A} \oplus \mathbf{B})$ are in one to one correspondence with the admissible subalgebras of $K(\mathbf{A})$. Moreover, if \mathbf{S} is an admissible subalgebra of $K(\mathbf{A})$, then $T_S^{\mathbf{B}} = S \cup (A \times B) \cup (B \times A) \cup (B \times B)$ is the universe of an admissible subalgebra of $K(\mathbf{A} \oplus \mathbf{B})$. And if \mathbf{T} is an admissible subalgebra of $K(\mathbf{A} \oplus \mathbf{B})$, then $S_T = T \cap A \times A$ is the universe of an admissible subalgebra of $K(\mathbf{A})$ that satisfies $T_{S_T}^{\mathbf{B}} = T$.*

As a particular case, we can completely describe the admissible subalgebras of $K(\mathbf{A} \oplus \mathbf{B})$ if $\mathbf{A} \in \mathbf{BCRL}$ is either involutive or Heyting, using Theorems 1.6 and 1.7.

2 Atoms and covers

If \mathbf{V} is any variety we denote by $\Lambda(\mathbf{V})$ its lattice of subvarieties; a variety $\mathbf{W} \in \Lambda(\mathbf{V})$ that is a cover of an atom is called **almost minimal**. The lattice of subvarieties of (non necessarily bounded) K-lattices has been investigated at length in [3]. If $\mathbf{2}$ is the two element Boolean algebra then $K(\mathbf{2})$ has four elements and it is the only four element bounded K-lattice, so it makes sense to denote it by \mathbf{K}_4 . Moreover if $2 = \{0, 1\}$, then $\{(0, 1), (1, 1), (1, 0)\}$ is the universe of admissible subalgebra \mathbf{K}_3 of $K(\mathbf{2})$. As in the unbounded case [10] \mathbf{K}_3 is the only totally ordered algebra in \mathbf{BKL} and therefore $\mathbf{V}(\mathbf{K}_3)$ is the only representable subvariety of \mathbf{BKL} . $K(\mathbf{V}(\mathbf{2}))$ is of course the variety generated by

the *kalmanization* of the variety \mathbf{BA} of Boolean algebras; and in the same way that [3] we can check that $\mathbf{V}(\mathbf{K}_3)$ is not a Kalman variety. Here is the first result that is different for bounded K-lattices.

Theorem 2.1. $\mathbf{V}(\mathbf{K}_3)$ is the only atom in $\Lambda(\mathbf{BKL})$.

There is another subtler difference of behavior in bounded K-lattices. Here, the element $(0,0)$ always exists in $K(\mathbf{A})$ for $\mathbf{A} \in \mathbf{BCRL}$, and can be described equationally.

Lemma 2.2. If $\mathbf{A} \in \mathbf{BKL}$, then there exists at most one element $\mathbf{o} \in A$ such that

- $\sim \mathbf{o} = \mathbf{o}$,
- $\mathbf{o} \wedge 1 = 0$.

Proof. If $z \in A$ satisfies both $\sim z = z$ and $z \wedge 1 = 0$, then clearly $z \wedge 1 = \mathbf{o} \wedge 1$ and $\sim z \wedge 1 = \sim \mathbf{o} \wedge 1$, and in Kalman lattices these two equations imply that $z = \mathbf{o}$ (see [10]). \square

Lemma 2.3. If $\mathbf{A} \in \mathbf{BKL}$, the following are equivalent

1. $\mathbf{A} \cong K(\mathbf{A}^-)$,
2. $\mathbf{K}_4 \leq \mathbf{A}$,
3. there exists $\mathbf{o} \in A$ such that $\sim \mathbf{o} = \mathbf{o}$ and $\mathbf{o} \wedge 1 = 0$.

Proof. 1. \Rightarrow 2 and 2. \Rightarrow 3. are immediate, as $(0,0)$ satisfies the equations for \mathbf{o} . To show that 3. \Rightarrow 1., observe that we always have $\mathbf{A} \leq K(\mathbf{A}^-)$ as $x \mapsto (x \wedge 1, \sim x \wedge 1)$ is an embedding, by Lemma 1.4. But if 3. holds, it is also onto: if $a, b \in A^-$, then $z = a \vee (\sim b \wedge \mathbf{o})$ satisfies $z \wedge 1 = a$ and $\sim z \wedge 1 = b$. \square

Lemma 2.3 implies that Kalman subvarieties of \mathbf{BKL} will always be above $\mathbf{V}(\mathbf{K}_4)$. More precisely:

Theorem 2.4. $\mathbf{V}(\mathbf{K}_4) = K(\mathbf{BA})$ is almost minimal in $\Lambda(\mathbf{BKL})$; moreover it is the only almost minimal Kalman variety.

2.1 Finitely generated almost minimal varieties

In Section 5 of [3], we introduced a subclass of \mathbf{CIRL} from which we could construct finitely generated almost minimal varieties in $\Lambda(\mathbf{KL})$. An algebra $\mathbf{A} \in \mathbf{CIRL}$ is **tight** if

- $|A| > 2$;
- \mathbf{A} is bounded by 0 and any element different from 0, 1 generates \mathbf{A} .

It turns out that finite tight algebras describe almost all finitely generated almost minimal varieties in \mathbf{CIRL} :

Theorem 2.5. [2] *If $\mathbf{A} \in \text{CIRL}$ is a finite subdirectly irreducible algebra generating an almost minimal variety, then either \mathbf{A} is tight, or it is isomorphic with the 0-free reduct of either \mathbf{G}_3 or \mathbf{N}_4 .*

The definition of tight algebras would make sense also for algebras in BCRL but there is a relevant difference: in CIRL every filter is a subalgebra, so the second condition above implies that any tight algebra is simple. Since in BCRL a filter is not in general a subalgebra, simplicity may not hold. Therefore we give a different and more general definition.

We say an algebra $\mathbf{A} \in \text{BCRL}$ is **rigid** if:

- $|A| > 2$;
- \mathbf{A} is subdirectly irreducible;
- \mathbf{A} has no proper subalgebras different from $\{0, 1\}$;
- for all proper nontrivial $\theta \in \text{Con}(\mathbf{A})$, $\mathbf{A}/\theta \cong \mathbf{2}$.

It is clear from the definition that the algebras \mathbf{G}_3 and \mathbf{N}_4 are rigid. Observe that any stiff algebra in the sense of [14, 19] is rigid; the following results will show the importance of finite rigid algebras.

Lemma 2.6. *Let $\mathbf{A} \in \text{BCRL}$ be rigid; then \mathbf{A} has at most one proper nontrivial congruence, i.e. $\text{Con}(\mathbf{A})$ is either the two or three-element chain.*

Proof. If μ is the monolith and it is a proper congruence (otherwise we are done), then $\mathbf{A}/\mu \cong \mathbf{2}$. If $\theta \supsetneq \mu$, it must be the congruence $\theta = A \times A$, as it must have a pair (a, b) such that $(a, 0), (b, 1) \in \mu$. \square

Theorem 2.7. *The following results hold in BCRL.*

1. *If $\mathbf{A} \in \text{BCRL}$ is a finite subdirectly irreducible algebra, then $\mathbf{V}(\mathbf{A})$ is an almost minimal variety if and only if \mathbf{A} is rigid.*
2. *If $\mathbf{A} \in \text{BCRL}$ satisfies that the 0-free reduct of \mathbf{A} is a tight algebra in CIRL, then it is rigid.*

Proof. The first part is immediate, as the definition of rigid algebras and Lemma 2.6 characterize the finitely generated covers of $\mathbf{V}(\mathbf{2}) = \text{BA}$.

For the second part, it is clear that if the 0-free reduct of \mathbf{A} is tight, then any $a \neq 0, 1$ generates the whole \mathbf{A} as an algebra in CIRL, but as it contains 0 it implies that \mathbf{A} has no subalgebras other than \mathbf{A} and $\{0, 1\}$ as an algebra in BCRL. Moreover, as filters are subalgebras in CIRL, we have that \mathbf{A} is subdirectly irreducible and does not have any non-trivial proper congruences. Therefore \mathbf{A} is rigid. \square

The converse of the second part of Theorem 2.7 is false, however, as there are new covers of the atom $\mathbf{V}(\mathbf{2}) = \text{BA}$. For instance, the reader can verify that the algebra described in Figure 1 is rigid, but it is clearly not tight as an algebra in CIRL (and it is neither \mathbf{G}_3 nor \mathbf{N}_4 , so by Theorem 2.5 it is a distinct cover).

We will now proceed to show the importance of rigid algebras for BKL.

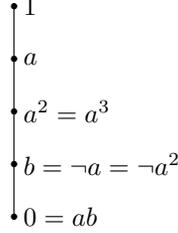


Figure 1: An algebra that generates a cover of $\mathbf{V}(2)$ in BCRL but not in CIRL.

Lemma 2.8. *Let $\mathbf{A} \in \text{BCRL}$ be rigid having a non-trivial proper congruence; then there exists a proper admissible subalgebra of $K(\mathbf{A})$ such that its only proper nontrivial quotient is isomorphic to \mathbf{K}_3 .*

Proof. By Lemma 2.6, let μ be the proper monolith of \mathbf{A} and let $\theta = \text{Cg}_{K(\mathbf{A})}(\mu)$, so that $\theta^- = \mu$. Clearly $K(\mathbf{A})/\theta \cong \mathbf{K}_4$, and consider the set

$$B = \{(a, b) \in K(\mathbf{A}) : (a, b)/\theta \in \mathbf{K}_3\} = \{(a, b) \in K(\mathbf{A}) : ((a, b), (0, 0)) \notin \theta\}.$$

Then B is nonempty $((0, 1), (1, 0), (1, 1) \in B)$; clearly it is the universe of an admissible subalgebra \mathbf{B} of $K(\mathbf{A})$ and moreover $\mathbf{B}/(\theta \cap B \times B) \cong \mathbf{K}_3$. As $\theta \cap B \times B$ is the only proper nontrivial congruence of \mathbf{B} , the Lemma holds. \square

Theorem 2.9. *Let $\mathbf{A} \in \text{BKL}$ be a finite subdirectly irreducible algebra that generates an almost minimal variety different from $\mathbf{V}(\mathbf{K}_4) = K(\text{BA})$. Then \mathbf{A}^- is rigid.*

On the other hand, let \mathbf{A} be a finite rigid algebra in BCRL such that 0 is meet irreducible. Then $K(\mathbf{A})$ has a subalgebra that generates an almost minimal variety different from $K(\text{BA})$.

Proof. Suppose that \mathbf{A} is a finite subdirectly irreducible algebra generating an almost minimal variety different from $K(\text{BA})$. Clearly $|A^-| > 2$, and any proper subalgebra of \mathbf{A} must be isomorphic with \mathbf{K}_3 . This implies that \mathbf{A}^- cannot have proper subalgebras different from $\{0, 1\}$. Consider now a proper nontrivial $\theta \in \text{Con}(\mathbf{A}^-)$ and let $\alpha = \text{Cg}_{\mathbf{A}}(\theta)$. Then α is proper nontrivial in $\text{Con}(\mathbf{A})$ and hence we must have $\mathbf{A}/\alpha \cong \mathbf{K}_3$. This implies that $(\mathbf{A}/\alpha)^- = \mathbf{A}^-/\theta = \mathbf{2}$ so \mathbf{A}^- is rigid.

Conversely, let \mathbf{A} be a finite rigid algebra such that 0 is meet irreducible. Since \mathbf{A} is finite and subdirectly irreducible, so is $K(\mathbf{A})$. As \mathbf{A} is rigid, it does not have proper subalgebras different from $\{0, 1\}$, so the subalgebras of $K(\mathbf{A})$ are only \mathbf{K}_3 , \mathbf{K}_4 and the admissible subalgebras. As 0 is meet irreducible in \mathbf{A} , $K(\mathbf{A}) \setminus \{(0, 0)\}$ is the universe of an admissible subalgebra \mathbf{B} of $K(\mathbf{A})$ (see Lemma 4.8 in [3]) that does not contain $(0, 0)$, so \mathbf{K}_4 is not a subalgebra of \mathbf{B} (see Lemma 2.3).

By Lemma 2.8 there is an admissible subalgebra $\mathbf{C} \leq K(\mathbf{A})$ with the property that its only nontrivial quotient is \mathbf{K}_3 (if it has one).

Therefore the minimal admissible subalgebra $K_0(\mathbf{A})$ satisfies that its only subalgebras are \mathbf{K}_3 and $K_0(\mathbf{A})$, and its only non-trivial quotient (if any) is \mathbf{K}_3 . By Jónsson's Lemma we conclude that $K_0(\mathbf{A})$ generates a cover of the atom in $\Lambda(\text{BKL})$. □

2.2 Other almost minimal varieties

We will now consider some examples of non-finitely generated covers of $\mathbf{V}(\mathbf{K}_3)$.

Define a unary term $\tau(x) = (x \wedge 1) \vee (\sim x \wedge 1)$ in the language of BKL, and if $p(x_1, \dots, x_n)$ is any term in the language of BCRL we define $\widehat{p}(x_1, \dots, x_n) = p(\tau(x_1), \dots, \tau(x_n))$, where we replace every instance of \rightarrow in p with \rightarrow^- .

Lemma 2.10. *Let $\mathbf{A} \in \text{BCRL}$ be subdirectly irreducible such that there is a term p with every element $x_1, \dots, x_n \neq 0$ satisfying $p(x_1, \dots, x_n) = 1$, and assume also that $K_p(\mathbf{A}) = K(\mathbf{A}) \setminus \{(0, 0)\}$ is the universe of an admissible subalgebra of $K(\mathbf{A})$. Then $K_p(\mathbf{A})$ satisfies $\widehat{p} \approx 1$ and $K(\mathbf{A})$ does not satisfy it.*

Proof. The first part is clear by definition, since $\tau(a, b) = a \vee b > 0$ and therefore $\widehat{p}((a_1, b_1), \dots, (a_n, b_n)) = 1$, and for the second part observe that $\widehat{p}((0, 0), \dots, (0, 0)) \neq 1$. □

We will use this Lemma to show that $K_0(\mathbf{2} \oplus \mathbf{C}_\omega)$ generates a non-finitely generated cover of $\mathbf{V}(\mathbf{K}_3)$.

Observe that by Theorem 1.8, we have that the universe of $K_0(\mathbf{2} \oplus \mathbf{C}_\omega)$ is $K(\mathbf{2} \oplus \mathbf{C}_\omega) \setminus \{(0, 0)\}$.

Now, in \mathbf{C}_ω cancellativity holds, therefore by Lemma 2.10 $K_0(\mathbf{2} \oplus \mathbf{C}_\omega)$ satisfies

$$(\tau(x) \rightarrow^- \tau(x)\tau(y)) \rightarrow^- \tau(y) \approx 1,$$

and neither $K(\mathbf{2} \oplus \mathbf{C}_\omega)$ nor \mathbf{K}_4 satisfy it. Thus $\mathbf{V}(K_0(\mathbf{2} \oplus \mathbf{C}_\omega))$ is different from $\mathbf{V}(K(\mathbf{2} \oplus \mathbf{C}_\omega))$ and $\mathbf{V}(\mathbf{K}_4) = K(\text{BA})$.

To show that the variety is minimal, if $\mathbf{B} \in \mathbf{HSP}_u(K_0(\mathbf{2} \oplus \mathbf{C}_\omega))$, by using Lemma 1.3 and the well-known properties of cancellative hoops we can deduce that either $\mathbf{B}^- \cong \mathbf{2}$ or $\mathbf{B}^- \cong \mathbf{2} \oplus \mathbf{A}$ for some totally ordered cancellative hoop \mathbf{A} . In case $\mathbf{B}^- \cong \mathbf{2}$, then $\mathbf{B} \cong \mathbf{K}_3$ (as $\mathbf{K}_4 \notin \mathbf{HSP}_u(K_0(\mathbf{2} \oplus \mathbf{C}_\omega))$). Otherwise $\mathbf{B} \leq K(\mathbf{2} \oplus \mathbf{A})$ and $(0, 0) \notin \mathbf{B}$; but then $\mathbf{B} \cong K_0(\mathbf{2} \oplus \mathbf{C})$ for some subalgebra $\mathbf{C} \leq \mathbf{B}$ and $\mathbf{2} \oplus \mathbf{C}_\omega \leq \mathbf{2} \oplus \mathbf{C}$. It follows that $K_0(\mathbf{2} \oplus \mathbf{C}_\omega) \leq K_0(\mathbf{2} \oplus \mathbf{C}) \cong \mathbf{B}$. This is enough to prove that $K_0(\mathbf{2} \oplus \mathbf{C}_\omega)$ generates an almost minimal variety different from $K(\text{BA})$.

We will now use a similar argument to show that $K_0(\mathbf{L}_1^\omega)$ generates an almost minimal variety above $\mathbf{V}(\mathbf{K}_3)$. Observe that the Chang algebra satisfies $\mathbf{L}_1^\omega \cong (\mathbf{C}_\omega)^{\delta_1}$, that is the disconnected rotation of the cancellative hoop \mathbf{C}_ω , and we can identify \mathbf{C}_ω with the radical (and only filter) of \mathbf{L}_1^ω . By Theorem 1.6

we have that the universe of $K_0(\mathbf{L}_1^\omega)$ is $\{(a, b) \in K(\mathbf{L}_1^\omega) : a \oplus b = 1\} = \{(a, b) \in K(\mathbf{L}_1^\omega) : \neg a \rightarrow b = 1\}$. Therefore $K_0(\mathbf{L}_1^\omega)$ satisfies the equation

$$((x \wedge 1) \rightarrow^- 0) \rightarrow^- (\sim x \wedge 1) \approx 1,$$

and it is clearly the only admissible subalgebra of $K(\mathbf{L}_1^\omega)$ that satisfies it, moreover \mathbf{K}_4 does not satisfy it.

To show that the variety is minimal, if $\mathbf{B} \in \mathbf{HSP}_u(K_0(\mathbf{L}_1^\omega))$, we can deduce that either $\mathbf{B}^- \cong \mathbf{2}$ or $\mathbf{B}^- \cong \mathbf{A}^{\delta_1}$ for some totally ordered cancellative hoop \mathbf{A} . In case $\mathbf{B}^- \cong \mathbf{2}$, then $\mathbf{B} \cong \mathbf{K}_3$ (as $\mathbf{K}_4 \notin \mathbf{HSP}_u(K_0(\mathbf{L}_1^\omega))$). Otherwise by Theorem 1.6 we have that $\mathbf{B} \cong K(\mathbf{A}^{\delta_1}, F)$ for some lattice filter F of \mathbf{A}^{δ_1} , and therefore $K_0(\mathbf{L}_1^\omega) \leq K_0(\mathbf{A}^{\delta_1}) \leq \mathbf{B}$. This is enough to prove that $K_0(\mathbf{L}_1^\omega)$ generates an almost minimal variety different from $K(\mathbf{BA})$.

3 Lattices of subvarieties

In this section we will consider the lattices of subvarieties of $K(\mathbf{V})$ for some specific $\mathbf{V} \subseteq \mathbf{BCRL}$. Some results are very similar to the ones obtained for the unbounded cases in [3], some are substantially different and some are totally new in the sense that deal with varieties that have no corresponding known unbounded counterpart. We will omit all the proofs that are simply rewritings of the proofs in [3] and we will illustrate in more details the *new* cases.

3.1 Bounded Basic K-Lattices

In this Section we will deal with different subvarieties of $K(\mathbf{BL})$ but first we will mention some basic results.

The finitely generated almost minimal varieties are $K(\mathbf{BA})$, $\mathbf{V}(K_0(\mathbf{G}_3))$ and all $\mathbf{V}(K_0(\mathbf{L}_p))$ for p prime. The non-finitely generated ones are $\mathbf{V}(K_0(\mathbf{2} \oplus \mathbf{C}_\omega))$ and $\mathbf{V}(K_0(\mathbf{L}_1^\omega))$.

Observe also that if $\mathbf{A} \in \mathbf{BL}$ is subdirectly irreducible, then $\mathbf{A} \cong \mathbf{B} \oplus \mathbf{D}$, where \mathbf{B} is a Wajsberg algebra (actually a chain) and \mathbf{D} is a basic hoop. Then all the admissible subalgebras of $K(\mathbf{A})$ are in one to one correspondence with the admissible subalgebras of $K(\mathbf{B})$, which in turn are in one to one correspondence with the lattice filters of \mathbf{B} by Theorems 1.8 and 1.6.

3.1.1 Bounded Wajsberg K-lattices

Let \mathbf{WA} be the variety of Wajsberg algebras; a bounded K-lattice \mathbf{A} is a **bounded Wajsberg K-lattice** if $\mathbf{A}^- \in \mathbf{WA}$. The lattice $\Lambda(\mathbf{WA})$ is well known [20] and it is simpler than the lattice of subvarieties of Wajsberg hoops [4]; this simplification is reflected also in the structure of $\Lambda(K(\mathbf{WH}))$. Since Wajsberg algebras are involutive, if \mathbf{L}_n is the $n + 1$ -element Wajsberg chain, then its admissible subalgebras can be described exactly as in [3], Section 5: if a is the only coatom of \mathbf{L}_n then

$$K_{m,n} = \{(u, v) : u \oplus v \geq a^m\}$$

is the universe of an admissible subalgebra $\mathbf{K}_{m,n} = K(\mathbf{L}_n, F)$ of $K(\mathbf{L}_n)$, for $F = \langle a^m \rangle$ the filter generated by a^m . Now similarly to [3] we can show:

Theorem 3.1. *The only almost minimal varieties in $\Lambda(K(\mathbf{WA}))$ are $K(\mathbf{BA})$, $\mathbf{V}(\mathbf{K}_{0,p})$ for p prime, and $\mathbf{V}(K_0(\mathbf{L}_1^\omega))$.*

We define

$$\mathbf{L}_n^\omega = \Gamma(\mathbb{Z} \times^l \mathbb{Z}, (n, 0)),$$

where \times^l is the lexicographic product and Γ is the Mundici's functor [21].

Each proper subvariety of \mathbf{WA} has only finitely many subvarieties (we say sometimes that has *finite height*): if \mathbf{V} is proper then there is a finite subset X of $\{\mathbf{L}_n : n \in \mathbb{N}\}$ and a finite subset $Y \subseteq \{\mathbf{L}_m^\omega : m \in \mathbb{N}\}$ with $\mathbf{V} = \mathbf{V}(X \cup Y)$ [4].

Recalling Theorem 1.6, we have that

- $K(\mathbf{L}_n, F) \leq K(\mathbf{L}_m, G)$ if and only if $n|m$ and $\langle F \rangle_m \subseteq G$, where $\langle F \rangle_m$ is the lattice filter generated by $F \subseteq \mathbf{L}_n$, viewed as a subset of \mathbf{L}_m . This can be simplified as each lattice filter in \mathbf{L}_n is principal, and therefore if $F = \langle x^r \rangle_n$ and $G = \langle x^s \rangle_m$, then $\mathbf{K}_{r,n} = K(\mathbf{L}_n, F) \leq K(\mathbf{L}_m, G) = \mathbf{K}_{s,n}$ if and only if there is a $k \in \mathbb{N}$ with $nk = m$ and $s \geq rk$.
- $K(\mathbf{L}_n, F) \leq K(\mathbf{L}_m^\omega, G)$ if and only if $n|m$ and $\langle F \rangle_m \subseteq G$.
- $K(\mathbf{L}_n^\omega, F) \leq K(\mathbf{L}_m^\omega, G)$ if and only if $n|m$ and $\langle F \rangle_m \subseteq G$.

We stress that we can obtain all the results in Section 6.3 for finitely generated varieties of Wajsberg \mathbf{K} -lattices with an even simpler presentation. This is a straightforward, albeit lengthy, exercise and we leave it to the interested reader. As a small example, in Figure 2 we include the lattice of subvarieties of $\mathbf{V}(K(\mathbf{L}_4))$.

For the case of non-finitely generated varieties, observe that in $K(\mathbf{L}_n^\omega, F)$, if F is a lattice filter properly containing the radical, and if θ is the congruence in $K(\mathbf{L}_n^\omega, F)$ generated by the radical of \mathbf{L}_n^ω , then $K(\mathbf{L}_n^\omega, F)/\theta \cong \mathbf{K}_4$. Therefore in this case the structure of the lattice of subvarieties becomes much more complex.

3.1.2 Bounded Gödel and Product \mathbf{K} -lattices

Let's consider the variety \mathbf{GA} of Gödel algebras; it is obvious that its lattice of subvarieties is identical to the one of Gödel hoops. If we look at $K(\mathbf{GA})$ things are different though. Observe first that the following Theorem from [3] still holds.

Theorem 3.2. *The nontrivial subalgebras (up to isomorphism) of \mathbf{K}_{n^2} are the algebras \mathbf{K}_{m^2} and \mathbf{K}_{m^2-1} for $m = 2, \dots, n$.*

However, it is no longer true (as in the unbounded case) that \mathbf{K}_{n^2-1} has a subalgebra isomorphic with \mathbf{K}_{m^2} for $m < n$, neither it is true that \mathbf{K}_{n^2-1} has \mathbf{K}_{m^2} as a homomorphic image for $m < n$ (this is due to the fact that a homomorphism $f : \mathbf{G}_n \rightarrow \mathbf{G}_m$ satisfies $f(x) = 0$ only for $x = 0$). In particular

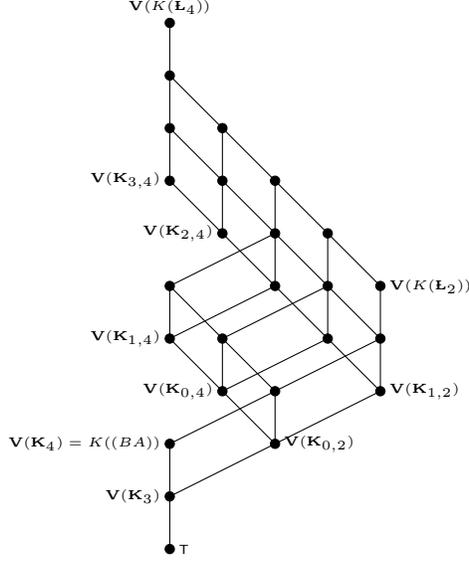


Figure 2: $\Lambda(\mathbf{V}(K(\mathbf{L}_4)))$

$\Lambda(K(\mathbf{GA}))$ is no longer a chain, and it is more complicated (in fact unlike $\Lambda(\mathbf{GA})$ there are proper subvarieties with infinitely many subvarieties).

Define $\mathbf{K}_\infty = K([0, 1]_G)$, where $[0, 1]_G$ is the unit interval with the standard Gödel operations. By Theorem 1.7, its only proper admissible subalgebra has universe $K([0, 1]_G) \setminus \{(0, 0)\}$, and we call it $\mathbf{K}_{0,\infty}$. As every element $x \in (0, 1]$ satisfies $\neg\neg x = 1$, by Lemma 2.10 we have that $\mathbf{K}_{0,\infty}$ (as well as any \mathbf{K}_{n^2-1}) satisfies

$$(\tau(x) \rightarrow^- 0) \rightarrow^- 0 \approx 1,$$

but neither \mathbf{K}_{n^2} nor \mathbf{K}_∞ satisfy it.

With the previous results one can show that it is the lattice in Figure 3.

A BL-algebra is a **product algebra** if it satisfies the equation

$$\neg y \vee ((x \rightarrow xy) \rightarrow y) \approx 1.$$

The variety PA of product algebras has been studied in [13]; joining the results therein with the results about product K-lattices in [3] we get:

Lemma 3.3. *The subdirectly irreducible algebras in $K(\mathbf{PA})$ are exactly \mathbf{K}_3 , \mathbf{K}_4 , $K_0(\mathbf{2} \oplus \mathbf{C})$ and $K(\mathbf{2} \oplus \mathbf{C})$ for any totally ordered cancellative hoop \mathbf{C} .*

Then it is easily checked that $\Lambda(K(\mathbf{PA}))$ is the lattice in Figure 4.

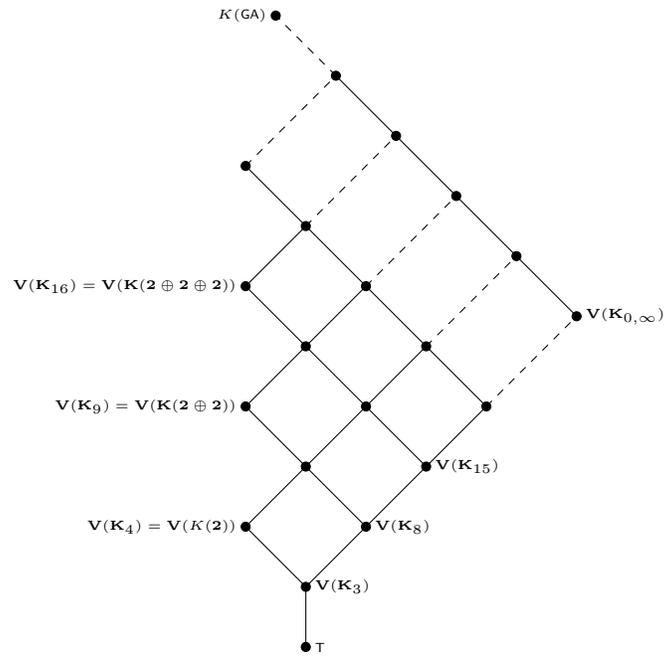


Figure 3: $\Lambda(K(GA))$

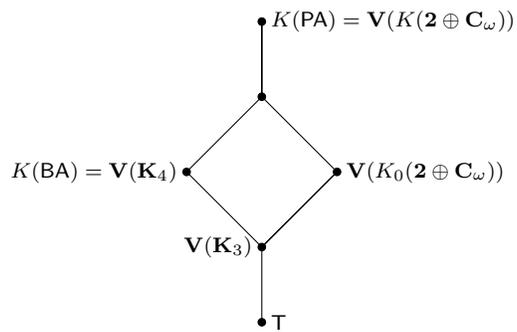


Figure 4: $\Lambda(K(PA))$

3.2 Nilpotent minimum K-lattices

Let's consider the variety generated by connected and disconnected rotations of Gödel chains, that is the variety **NM** of **nilpotent minimum algebras**. This variety has been completely described in [15]; these are the highlights:

- the variety **NM** is the subvariety of **BCRL** axiomatized by (P), i.e. they are **MTL** algebras, $\neg\neg x \approx x$ (which gives involution) and

$$(xy \rightarrow 0) \vee ((x \wedge y) \rightarrow xy) \approx 1;$$

- the finite **NM**-chains are exactly connected or disconnected rotations of Gödel chains; more precisely if \mathbf{N}_k denotes the k -element nilpotent minimum chain, then we have that \mathbf{N}_{2n} is the disconnected rotation of the Gödel chain \mathbf{G}_n and \mathbf{N}_{2n+1} is the connected rotation of \mathbf{G}_n (of course \mathbf{G}_0 is the trivial algebra). Clearly $\mathbf{N}_2 = \mathbf{2}$ and $\mathbf{N}_3 = \mathbf{L}_2$;
- we define \mathbf{N}_ω to be the nilpotent minimum algebra whose universe is the real interval $[0, 1]$ the order is the natural order and the operations are:

$$xy = \begin{cases} 0, & \text{if } y \leq 1 - x; \\ \min(x, y), & \text{otherwise.} \end{cases} \quad x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y; \\ \max(1 - x, y), & \text{otherwise.} \end{cases}$$

It is clear that $\neg x = 1 - x$ and hence $\neg(1/2) = 1/2$. It follows that $[0, 1] \setminus \{1/2\}$ is the universe of a subalgebra of \mathbf{N}_ω that we denote by \mathbf{N}_ω^* ;

- $\mathbf{N}_{2k}, \mathbf{N}_{2k+1} \leq \mathbf{N}_{2n+1}$ for $k \leq n$; $\mathbf{N}_{2k} \leq \mathbf{N}_{2n}$ for $k \leq n$; $\mathbf{N}_{2n} \leq \mathbf{N}_\omega^*$ for all n ; $\mathbf{N}_{2n+1} \leq \mathbf{N}_\omega$ for all n ;
- $\mathbf{H}(\mathbf{N}_{2n+1}) = \{\mathbf{N}_k : k \leq 2n + 1\}$, $\mathbf{H}(\mathbf{N}_{2n}) = \{\mathbf{N}_{2k} : k \leq n\}$;
- the variety **NM** is generated by all the chains $\{\mathbf{N}_{2n+1} : n \in \mathbb{N}\}$ or by any infinite chain having an element a such that $\neg a = a$; hence $\mathbf{NM} = \mathbf{V}(\mathbf{N}_\omega)$;
- every subvariety of **NM** is generated by its finite algebras, so it has the finite model property and each splitting algebra is finite;
- $\mathbf{NM}^* = \mathbf{V}(\mathbf{N}_\omega^*) = \mathbf{V}(\{\mathbf{N}_{2k} : k \in \mathbb{N}\})$ is a proper subvariety of **NM**.

Now putting together all we know, we observe that:

Lemma 3.4. *For every $n \in \mathbb{N}$, $\mathbf{HSP}_u(K(\mathbf{N}_{2n+1})) \subseteq \mathbf{SK}(\{\mathbf{N}_k : k \leq 2n + 1\})$ and $\mathbf{HSP}_u(K(\mathbf{N}_{2n})) \subseteq \mathbf{SK}(\{\mathbf{N}_{2k} : k \leq n\})$.*

Proof. We compute:

$$\begin{aligned} \mathbf{HSP}_u(K(\mathbf{N}_{2n+1})) &\subseteq \mathbf{SK}(\mathbf{HSP}_u(\mathbf{N}_{2n+1})) = \mathbf{SK}(\mathbf{HS}(\mathbf{N}_{2n+1})) \\ &= \mathbf{SK}(\mathbf{HS}(\mathbf{G}_n^{\delta_2})) = \mathbf{SK}(\mathbf{HS}(\mathbf{G}_n)^{\delta_2}) \\ &= \mathbf{SK}(\mathbf{HS}(\{\mathbf{G}_k : k \leq 2n + 1\}^{\delta_2})) \\ &= \mathbf{SK}(\{\mathbf{N}_k : k \leq 2n + 1\}); \end{aligned}$$

the second point follows by a similar argument. \square

It is easy to see that the non-admissible subalgebras of $K(\mathbf{N}_{2n})$ are exactly the admissible subalgebras of $K(\mathbf{N}_{2k})$ with $k \leq n$ and the non admissible subalgebras of $K(\mathbf{N}_{2n+1})$ are exactly the admissible subalgebras of $K(\mathbf{N}_{2k})$ and $K(\mathbf{N}_{2k+1})$ with $k \leq n$. Since NM is involutive the admissible subalgebras can be computed using Theorem 1.6; it follows that that $K(\mathbf{N}_k)$ has k admissible subalgebras and they can be found by computing $x \oplus y = (x \rightarrow 0) \rightarrow y$ on each \mathbf{N}_k . As an example in Figure 5 we draw the proper admissible subalgebras of $K(\mathbf{N}_5)$, where $N_5 = \{0 < \neg a < b = \neg b < a < 1\}$.

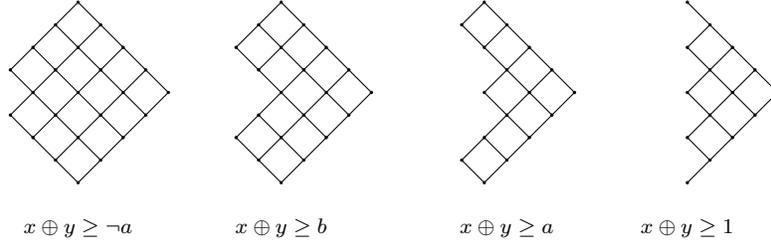


Figure 5: The admissible subalgebras of $K(\mathbf{N}_5)$

In order to describe the lattice of subvarieties of $K(\mathbf{NM})$, we first observe that \mathbf{L}_2 and \mathbf{N}_4 are the only rigid algebras in NM, therefore each one generates a cover of $\mathbf{V}(\mathbf{K}_3)$ different from $K(\mathbf{BA})$.

Then we observe that there exist retractions $\gamma_{2n} : \mathbf{N}_{2n} \rightarrow \mathbf{2}$ for each k . Thus if we consider $K(\mathbf{N}_{2n}, F)$ for some lattice filter F containing x such that $\gamma_{2n}(x) = 0$ (that is lattice filters strictly greater than the radical), then we will have $\mathbf{K}_4 \in \mathbf{H}(K(\mathbf{N}_{2n}, F))$ by considering the morphism $f(a, b) = (\gamma_{2n}(a), \gamma_{2n}(b))$. It is also easy to see that if the filter F is contained in the radical (i.e. $\gamma_{2n}(x) = 1$ for each $x \in F$), then $\mathbf{K}_4 \notin \mathbf{H}(K(\mathbf{N}_{2n}, F))$. Moreover, if we consider a morphism $\gamma_{2n,2k} : \mathbf{N}_{2n} \rightarrow \mathbf{N}_{2k}$, then $K(\mathbf{N}_{2k}) \in \mathbf{H}(K(\mathbf{N}_{2n}, F))$ if and only if the lattice filter F contains an element x such that $\gamma_{2n,2k}(x) = 0$.

The case of \mathbf{N}_{2n+1} is similar, if we consider a morphism $\gamma_{2n+1,2k+1} : \mathbf{N}_{2n+1} \rightarrow \mathbf{N}_{2k+1}$, then $K(\mathbf{N}_{2k+1}) \in \mathbf{H}(K(\mathbf{N}_{2n+1}, F))$ if and only if the lattice filter F contains an element x such that $\gamma_{2n+1,2k+1}(x) = 0$. In Figure 6 we describe the lattice of subvarieties of $\mathbf{V}(K(\mathbf{N}_5))$.

To simplify notation, we name $K_0(\mathbf{L}_2) \leq K_1(\mathbf{L}_2) \leq K(\mathbf{L}_2)$, $K_0(\mathbf{N}_4) \leq K_1(\mathbf{N}_4) \leq K_2(\mathbf{N}_4) \leq K(\mathbf{N}_4)$ and $K_0(\mathbf{N}_5) \leq K_1(\mathbf{N}_5) \leq K_2(\mathbf{N}_5) \leq K_3(\mathbf{N}_5) \leq K(\mathbf{N}_5)$ the admissible subalgebras of \mathbf{L}_2 , \mathbf{N}_4 and \mathbf{N}_5 , respectively. From the previous observations, we recall that

- $\mathbf{K}_4 \leq K(\mathbf{L}_2)$ and $K(\mathbf{N}_4) \leq K(\mathbf{N}_5)$,
- $K_0(\mathbf{L}_2) \leq K_0(\mathbf{N}_5)$ and $K_0(\mathbf{N}_4) \leq K_0(\mathbf{N}_5)$,
- $K_1(\mathbf{L}_2) \leq K_2(\mathbf{N}_5)$,
- $K_1(\mathbf{N}_4) \leq K_1(\mathbf{N}_5)$,

- $K_2(\mathbf{N}_4) \leq K_3(\mathbf{N}_5)$,
- $\mathbf{K}_4 \in \mathbf{H}(K_2(\mathbf{N}_4))$,
- $K(\mathbf{L}_2) \in \mathbf{H}(K_3(\mathbf{N}_5))$.

With this information, the reader can verify that the lattice of subvarieties of $\mathbf{V}(K(\mathbf{N}_5))$ is effectively the one drawn in Figure 6.

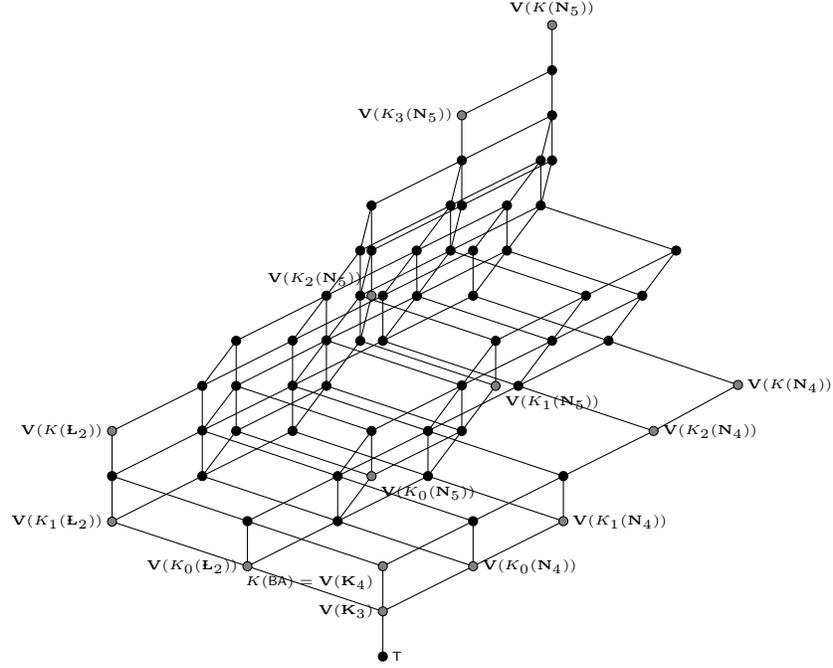


Figure 6: $\Lambda(\mathbf{V}(K(\mathbf{N}_5)))$

3.3 Drastic product \mathbf{K} -lattices

A **drastic product chain**, briefly a DP-chain, is chain in BCRL where the monoid operation (which determines the entire structure) is defined as

$$ab = \begin{cases} 0, & \text{if } a, b \neq 1; \\ a \wedge b, & \text{otherwise.} \end{cases}$$

We denote by \mathbf{DP}_n the DP-chain with n -elements; note that each DP-chain is simple and \mathbf{DP}_n has subalgebras isomorphic with \mathbf{DP}_k for all $k \leq n$. In particular $\mathbf{DP}_3 \cong \mathbf{L}_2$. DP-chains have been considered (under a different name) by C. Noguera in his PhD Thesis [22] and investigated in [6]. In details:

- the variety DP generated by all DP-chains is axiomatized with respect to MTL by $x \vee \neg(x^2) \approx 1$;
- DP is generated by any infinite chain or by any infinite set of finite chains;
- hence any proper subvariety of DP is generated by a single finite chain and thus $\Lambda(\text{DP})$ is a chain of order $\omega + 1$.

We remark that any DP-chain has a coatom a_1 ; in fact if \mathbf{D} is an infinite DP-chain and $u, v \in D \setminus \{0, 1\}$ then $\neg u = \neg v$ and we can take a to be that common value; moreover in a DP-chain with coatom a_1

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y; \\ a_1, & \text{if } 1 > x > y; \\ y, & \text{if } x = 1. \end{cases}$$

The main difference between this example and the previous ones is that in general the admissible subalgebras of $K(\mathbf{DP}_n)$ do not form a chain; as a matter of fact they form a chain if and only if $n \leq 4$ and this will be clear after our description of the lattice of admissible subalgebras of $K(\mathbf{DP}_n)$.

To investigate the admissible subalgebras of the Kalman product of each DP-chain \mathbf{DP}_n , we assume $n \geq 4$ (as $\mathbf{DP}_2 \cong \mathbf{2}$ and $\mathbf{DP}_3 \cong \mathbf{L}_2$, those cases have already been considered), and let

$$DP_n = \{0 = a_{n-1} \prec a_{n-2} \prec \cdots \prec a_1 \prec 1\}.$$

Define

$$K_n^\emptyset = \{(x, 1), (1, x), (x, a_1), (a_1, x) : x \in DP_n\}.$$

Lemma 3.5. *For $n \geq 4$, K_n^\emptyset is the smallest admissible subalgebra of $K(\mathbf{DP}_n)$. Moreover, every sublattice of $K(\mathbf{DP}_n)$ closed under \sim and containing K_n^\emptyset is the universe of an admissible subalgebra of $K(\mathbf{DP}_n)$.*

Proof. To show that each admissible subalgebra of \mathbf{DP}_n contains K_n^\emptyset , observe that, for $i = 1, \dots, n-2$,

$$(a_i, 1) \cdot (1, 0) = (a_i, \neg a_i) = (a_i, a_1),$$

and then if $i = 2, \dots, n-2$,

$$(a_1, a_i) \cdot (a_1, a_1) = (0, a_1) = (a_{n-1}, a_1),$$

so the elements $(a_i, a_1), (a_1, a_i)$ belong to each admissible subalgebra.

As K_n^\emptyset is clearly the universe of a sublattice closed under \sim , we only need to show that any sublattice of $K(\mathbf{DP}_n)$ closed under \sim and containing K_n^\emptyset is closed under the product. To show this, observe that for $i, j, k, l = 1, \dots, n-1$,

$$\begin{aligned} (a_i, a_j) \cdot (a_k, a_l) &= (0, (a_i \rightarrow a_l) \wedge (a_k \rightarrow a_j)), \\ (a_i, a_j) \cdot (a_k, 1) &= (0, a_k \rightarrow a_j), \\ (a_i, a_j) \cdot (1, a_k) &= (a_i, a_j), \end{aligned}$$

and each implication is either a_1 or 1. □

To describe all the subalgebras, consider the lattice with universe

$$X_n = \{(a_i, a_j) : i \geq j > 1\}$$

where the ordering is given coordinate-wise.

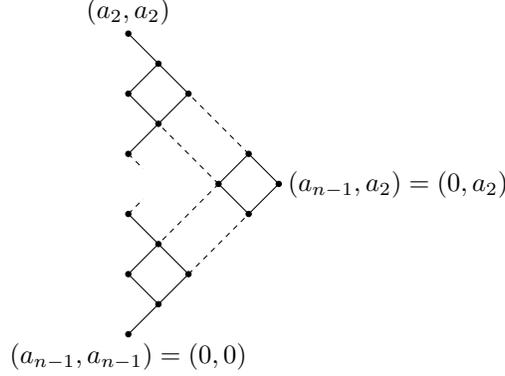


Figure 7: The lattice X_n .

Now, for each up-set U of X_n consider

$$K_n^U = K_n^\emptyset \cup \{(x, y) : (x, y) \in U \text{ or } (y, x) \in U\}.$$

From this definition and the previous results, the following Theorem is immediate.

Theorem 3.6. *For $n \geq 4$, the admissible subalgebras of $K(\mathbf{DP}_n)$ are \mathbf{K}_n^U , for each U up-set of X_n . Moreover, $\mathbf{K}_n^U \subset \mathbf{K}_n^W$ if and only if $U \subset W$.*

We also note:

Corollary 3.7. *The lattice of admissible subalgebras of $K(\mathbf{DP}_n)$ is a chain if and only if $n \leq 4$.*

Figures 8 and 9 show all the admissible subalgebras of $K(\mathbf{DP}_4)$ and $K(\mathbf{DP}_5)$.

Since the non admissible subalgebras of $K(\mathbf{DP}_n)$ are exactly the admissible subalgebras of $K(\mathbf{DP}_m)$ with $m < n$, in principle we can draw the entire lattice of subalgebras. For that we use the following result, in which we use that in the inclusion $\mathbf{DP}_m \leq \mathbf{DP}_n$, the coatom $a_1^{(m)}$ goes to the coatom $a_1^{(n)}$, each $a_k^{(m)}$ for $k = 2, \dots, m-2$ goes to $a_k^{(n)}$, and $a_{m-1}^{(m)} = 0$ goes to $a_{n-1}^{(n)} = 0$.

Theorem 3.8. 1. *For each $4 \leq n$, $\mathbf{K}_{0,2} \leq \mathbf{K}_{1,2} \leq \mathbf{K}_n^\emptyset$;*

2. *for each $4 \leq m < n$, $\mathbf{K}_m^U \leq \mathbf{K}_n^{\tilde{U}}$, where \tilde{U} is the up-set of X_n generated by each $(a_i, a_j) \in U$ with $i, j < m-1$, each (a_{n-1}, a_j) if $(a_{m-1}, a_j) \in U$ and $j < m-1$, and (a_{n-1}, a_{n-1}) if $(a_{m-1}, a_{m-1}) \in U$.*

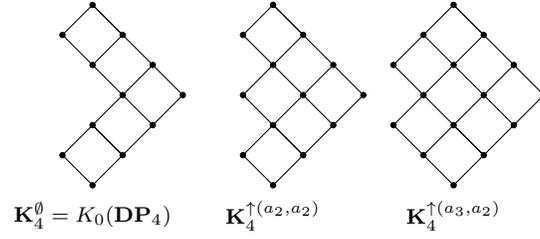


Figure 8: The proper admissible subalgebras of $K(\mathbf{DP}_4)$

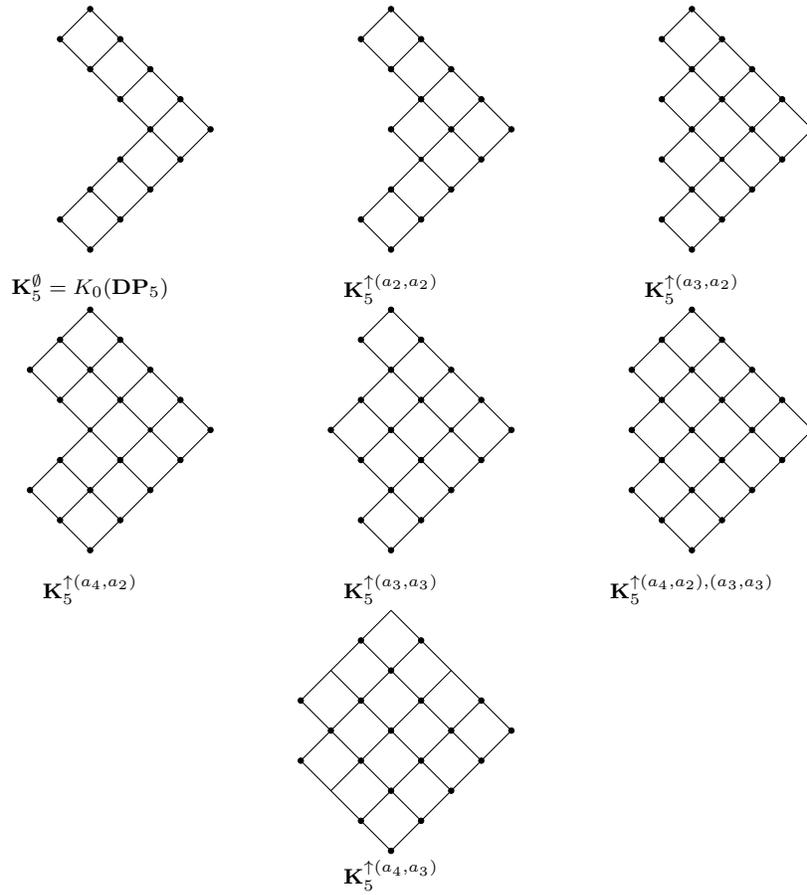


Figure 9: The proper admissible subalgebras of $K(\mathbf{DP}_5)$

From here, and using the fact that none of \mathbf{K}_n^U has proper non-trivial congruences, using the same techniques that we have employed throughout this paper, we can describe $\Lambda(K(\mathbf{DP}_n))$.

The lattices $\Lambda(K(\mathbf{DP}_4))$ and $\Lambda(K(\mathbf{DP}_5))$ are in Figures 10 and 11, respectively; we have used the same names before for the admissible subalgebras of $K(\mathbf{DP}_3) = K(\mathbf{L}_2)$. Unfortunately $\Lambda(K(\mathbf{DP}_n))$ becomes very intricate as soon as $n > 4$, and it is borderline impossible to draw it by hands.

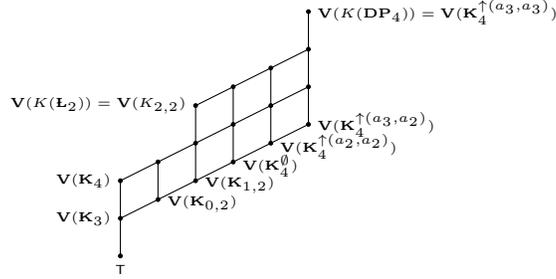


Figure 10: The lattice $\Lambda(K(\mathbf{DP}_4))$

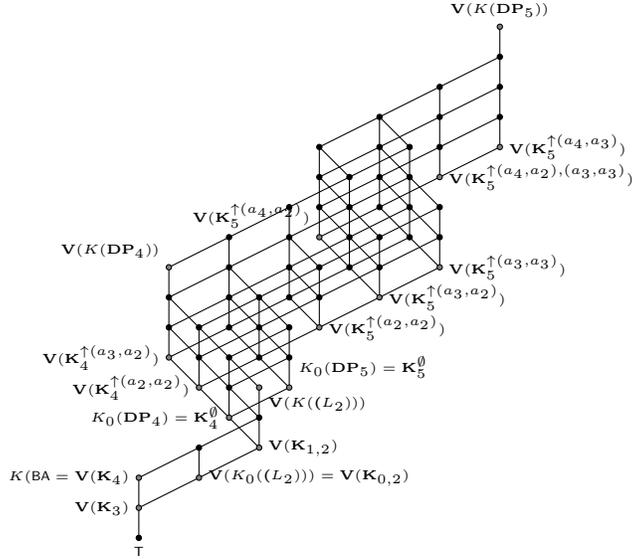


Figure 11: The lattice $\Lambda(K(\mathbf{DP}_5))$

3.4 Stonean K-lattices

Stonean residuated lattices are bounded residuated lattices satisfying

$$\neg x \vee \neg\neg x \approx 1.$$

We call the variety of Stonean residuated lattices **SRL**. Here we list some properties of Stonean residuated lattices that will be useful in what follows; they can be found in [11].

- Stonean residuated lattices are pseudocomplemented, i.e. they satisfy $x \wedge \neg x = 0$.
- They satisfy $\neg(x \wedge y) = \neg x \vee \neg y$. Moreover all elements of the form $\neg x$ are Boolean.
- They satisfy $x = \neg\neg x(\neg\neg x \rightarrow x)$, and as $\neg\neg x$ is Boolean this is equivalent to $x = \neg\neg x \wedge (\neg x \vee x)$.
- All directly indecomposable Stonean residuated lattices are of the form $\mathbf{2} \oplus \mathbf{D}$ for some $\mathbf{D} \in \mathbf{CIRL}$ and vice versa.
- As subdirectly irreducible algebras are directly indecomposable, and filters in $\mathbf{2} \oplus \mathbf{D} \in \mathbf{SRL}$ are either filters of \mathbf{D} or the whole algebra $\mathbf{2} \oplus \mathbf{D}$, subdirectly irreducible algebras in **SRL** are of the form $\mathbf{2} \oplus \mathbf{D}$ for some subdirectly irreducible $\mathbf{D} \in \mathbf{CIRL}$.

In [10], as pseudocomplemented lattices satisfy the Glivenko equation

$$\neg\neg(\neg\neg x \rightarrow x) = 1,$$

the authors present (Theorem 6.17) a bijective correspondence between regular lattice filters of a pseudocomplemented distributive residuated lattice \mathbf{A} and regular admissible subalgebras of $K(\mathbf{A})$. This result can be adapted and improved for Stonean residuated lattices.

Let $\mathbf{A} \in \mathbf{SRL}$; a lattice filter F in \mathbf{A} is called *regular* if $\neg\neg x \in F$ implies $x \in F$.

Lemma 3.9. *Regular lattice filters in $\mathbf{A} \in \mathbf{SRL}$ are exactly the lattice filters containing all dense elements $D = \{x \in A : \neg x = 0\}$.*

Proof. If F is a regular lattice filter and $\neg x = 0$, then clearly $\neg\neg x = 1 \in F$, so $x \in F$ and the filter contains all dense elements. Reciprocally, if F is a lattice filter containing all dense elements and $\neg\neg x \in F$, recalling that $x = \neg\neg x \wedge (\neg x \vee x)$ and that $\neg x \vee x$ is dense, $\neg\neg x, \neg x \vee x \in F$ so $x \in F$. \square

We say that an admissible subalgebra \mathbf{S} of $K(\mathbf{A})$ is *regular* provided $(\neg\neg a, \neg\neg b) \in S$ implies $(a, b) \in S$.

Lemma 3.10. *Let $\mathbf{A} \in \mathbf{SRL}$. All admissible subalgebras of $K(\mathbf{A})$ are regular.*

Proof. Let \mathbf{S} be an admissible subalgebra of $K(\mathbf{A})$ and suppose $(\neg\neg a, \neg\neg b) \in S$. As $(1, b), (a, \neg a), (\neg b, b), (\neg\neg b, \neg\neg a) \in S$ and recalling that the elements $\neg a, \neg b$ are Boolean, the following element is in S :

$$\begin{aligned} (((\neg b, b) \wedge (\neg\neg b, \neg\neg a)) \vee (a, \neg a)) \wedge (1, b) &= ((0, b \vee \neg\neg a) \vee (a, \neg a)) \wedge (1, b) \\ &= (a, (b \vee \neg\neg a) \wedge \neg a) \wedge (1, b) \\ &= (a, (b \wedge \neg a) \vee (\neg\neg a \wedge \neg a)) \wedge (1, b) \\ &= (a, b \wedge \neg a) \wedge (1, b) \\ &= (a, b). \end{aligned}$$

□

From Theorem 1.8 it is clear that the only admissible subalgebras of $K(\mathbf{2} \oplus \mathbf{D})$ for $\mathbf{D} \in \text{CIRL}$ are $K(\mathbf{2} \oplus \mathbf{D})$ and the subalgebra with universe $K(\mathbf{2} \oplus \mathbf{D}) \setminus \{(0, 0)\}$. However, from the previous results and Theorem 6.10 in [10] we have a better description of admissible subalgebras for $K(\mathbf{A})$ where \mathbf{A} is any Stonean residuated lattice.

Theorem 3.11. *For each $\mathbf{A} \in \text{SRL}$, the correspondence*

$$F \mapsto \mathbf{S}_F$$

defines a bijection from the set of lattice filters of \mathbf{A} containing all dense elements onto the set of all admissible subalgebras of $K(\mathbf{A})$, where \mathbf{S}_F is the subalgebra of $K(\mathbf{A})$ with universe

$$\{(a, b) \in K(\mathbf{A}) : \neg a \rightarrow \neg\neg b \in F\}.$$

Observe that in particular if $\mathbf{A} \in \text{SRL}$ is directly indecomposable, then there is only one proper filter containing all dense elements, so the only proper admissible subalgebra of $K(\mathbf{A})$ will have universe $K(\mathbf{A}) \setminus \{(0, 0)\}$.

With these results in consideration, we can investigate the lattice $\Lambda(K(\text{SRL}))$. Not surprisingly it is highly complex, but we will get some information about the bottom part of the lattice.

Clearly $\mathbf{V}(\mathbf{K}_3)$ is the only atom, and the next result is also immediate.

Theorem 3.12. *The only finitely generated almost minimal varieties of $K(\text{SRL})$ are $\mathbf{V}(\mathbf{K}_4) = K(\text{BA})$ and $\mathbf{V}(\mathbf{K}_8) = \mathbf{V}(K_0(\mathbf{2} \oplus \mathbf{2}))$.*

Proof. It is clear that $\mathbf{V}(\mathbf{K}_4)$ is an almost minimal variety. Moreover, the only (up to isomorphism) rigid finite Stonean residuated lattice is $\mathbf{2} \oplus \mathbf{2} = \mathbf{G}_3$, and the conclusion follows. □

From the properties of Stonean residuated lattices, we can say something more going upwards in the lattice of subvarieties of $K(\text{SRL})$.

Lemma 3.13. *If $\mathbf{A} \in \text{SRL}$ and $f : \mathbf{A} \rightarrow \mathbf{2}$ is a morphism, then $f(x) = 0$ implies $x = 0$.*

Proof. It is enough to show this for subdirectly irreducible algebras. In this case we have that $\mathbf{A} \cong \mathbf{2} \oplus \mathbf{D}$ for some $\mathbf{D} \in \text{CIRL}$, and if $x \in D$ $\neg x = 0$, so $f(\neg x) = 0$ and $f(x) = 1$. \square

Corollary 3.14. *If $\mathbf{A} \in \text{SRL}$ is subdirectly irreducible, then $\mathbf{K}_4 \notin \mathbf{V}(K_0(\mathbf{A}))$.*

Proof. Recall that “ $\mathbf{K}_4 \not\leq \mathbf{A}$ ” is first-order definable, so as it holds in $K_0(\mathbf{A})$ it will be true for any ultrapower. From the previous result we have that $\mathbf{K}_4 \notin \text{HSP}_u(K_0(\mathbf{A}))$. \square

Lemma 3.15. *Let $\mathbf{D} \in \text{CIRL}$ be finite and subdirectly irreducible. Then $\mathbf{V}(\mathbf{2} \oplus \mathbf{D})$ covers $\mathbf{V}(\mathbf{2} \oplus \mathbf{2})$ in $\Lambda(\text{SRL})$ if and only if $\mathbf{V}(\mathbf{D})$ covers $\mathbf{V}(\mathbf{2}) = \text{GBA}$ in $\Lambda(\text{CIRL})$.*

From these results, we obtain the following.

Theorem 3.16. *Let $\mathbf{A} \in K(\text{SRL})$ be finite and subdirectly irreducible. Then $\mathbf{V}(\mathbf{A})$ is a cover of $\mathbf{V}(\mathbf{K}_8)$ but not of $\mathbf{V}(\mathbf{K}_4) = K(\text{BA})$ if and only if $\mathbf{A} \cong K_0(\mathbf{2} \oplus \mathbf{D})$ for some $\mathbf{D} \in \text{CIRL}$ finite and subdirectly irreducible that generates a cover of $\mathbf{V}(\mathbf{2}) = \text{GBA}$ in $\Lambda(\text{CIRL})$.*

Therefore if $\mathbf{D} \in \text{CIRL}$ is finite, subdirectly irreducible and generates a cover of $\mathbf{V}(\mathbf{2}) = \text{GBA}$ in $\Lambda(\text{CIRL})$, the lattice of subvarieties of $\mathbf{V}(K(\mathbf{2} \oplus \mathbf{D}))$ will be as in Figure 12.

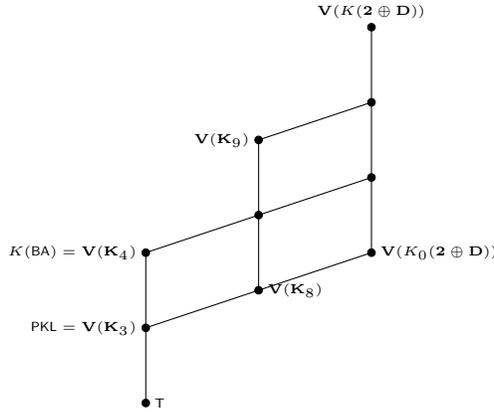


Figure 12: $\Lambda(\mathbf{V}(K(\mathbf{2} \oplus \mathbf{D})))$

For the case of non-finitely generated almost minimal varieties, an example will be $\mathbf{V}(K_0(\mathbf{2} \oplus \mathbf{C}_\omega))$. If $\mathbf{A} \in \text{CIRL}$ is infinite and generates an almost minimal variety in $\Lambda(\text{CIRL})$ different from cancellative hoops CH, then $\mathbf{V}(K_0(\mathbf{2} \oplus \mathbf{A}))$ will be another example of the cover. In Figure 13 we consider the lattice of subvarieties of $\mathbf{V}(K(\mathbf{2} \oplus \mathbf{C}_\omega), K(\mathbf{2} \oplus \mathbf{L}_2))$.

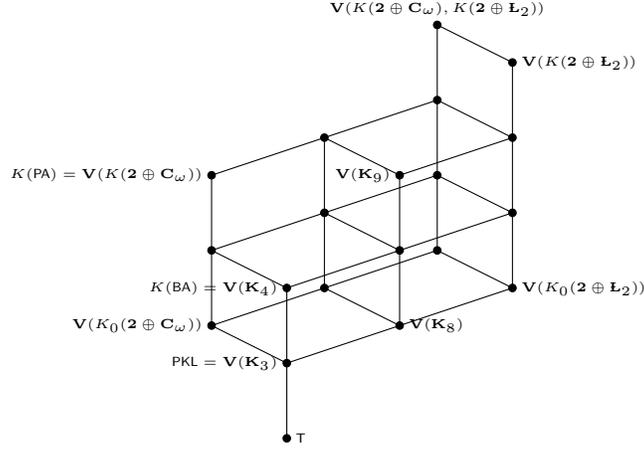


Figure 13: $\Lambda(\mathbf{V}(K(\mathbf{2} \oplus \mathbf{C}_\omega), K(\mathbf{2} \oplus \mathbf{L}_2)))$

Conclusions and future work

One of the most interesting features of the Kalman construction (at least from our point of view) is that it allows us to explore previously unexplored parts of the lattice of subvarieties of CRL, by *lifting* properties of *integral* commutative residuated lattices that we already know. In fact Section 3 of this paper and Section 6 of [3] are completely devoted to the purpose and we believe that we have shown that such an enterprise is at least useful. In particular the combination of the Kalman construction with other well known construction, such as the ordinal sum or the disconnected rotation, seems to be a very powerful tool and deserves to be investigated more.

Another possibility is to concentrate on a specific variety, e.g. $K(\text{BL})$ and investigate its algebraic properties. What are its splitting algebras? What are its projective members? Is there a canonical representation for subdirectly irreducible algebras? We believe that these are all very interesting questions and we propose to investigate them further.

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