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Carles Noguera
Institut d'Investigació en Intel·ligència Artificial - CSIC
Catalonia, Spain

Francesc Esteva
Institut d'Investigació en Intel·ligència Artificial - CSIC
Catalonia, Spain

Joan Gispert
Universitat de Barcelona
Catalonia, Spain

Abstract

The study of perfect, local and bipartite IMTL-algebras presented in [29] is generalized in this paper to the general non-involutive case, i. e. to MTL-algebras. To this end we describe the radical of MTL-algebras and characterize perfect MTL-algebras as those for which the quotient by the radical is isomorphic to the two-element Boolean algebra, and a special class of bipartite MTL-algebras, $\mathbb{BP}_0$, as those for which the quotient by the radical is a Boolean algebra. We prove that $\mathbb{BP}_0$ is the variety generated by all perfect MTL-algebras and give some equational bases for it. We also introduce a new way to build MTL-algebras by adding a negation fixpoint to a perfect algebra and also by adding some set of points whose negation is the fixpoint. Finally, we consider the varieties generated by those algebras, giving equational bases for them, and we study which of them define a fuzzy logic with standard completeness theorem.

Keywords: Algebraic logic, bipartite algebras, filters, fuzzy logics, Glivenko-style theorems, IMTL-algebras, local algebras, radical, MTL-algebras, perfect algebras, standard algebras, standard completeness.

1 Introduction

In [21] Petr Hájek presented a new logic, called *BL*, intended to be the basic fuzzy logic. He gave an algebraic semantics for BL logic introducing the variety of BL-algebras. Indeed, BL is weaker than all the systems of fuzzy logic based on continuous t-norms and their residua known at that time: Łukasiewicz logic, Product logic and Gödel logic. Afterwards Cignoli, Esteva, Godo and Torrens proved in [7] that BL is the logic of all continuous t-norms and their residua. However, the sufficient and necessary condition for a t-norm to have a residuated implication is the left-continuity; hence it makes sense to consider fuzzy logics based not on continuous t-norms but on left-continuous t-norms. To this goal, Esteva and Godo proposed in [14] a new logic, called *MTL*, as the basic fuzzy logic in this more general sense. The proposal was successful when Jenei and Montagna proved in [26] that MTL is indeed the logic of all left-continuous t-norms and their residua.

Furthermore, in [14] a new class of algebras is defined, the variety of MTL-algebras, that we denote by $\mathbb{MTL}$. This variety contains all BL-algebras and it is an equivalent algebraic semantics for MTL, so MTL is an algebraizable logic in the sense of Blok and Pigozzi [3] and there is an order-reversing isomorphism between the set of axiomatic extensions of MTL and the set of subvarieties of $\mathbb{MTL}$. Therefore, the algebraic research on varieties of MTL-algebras is equivalent to the task of finding axiomatic extensions of MTL and, among them, new t-norm based fuzzy logics. Some parts of the lattice of varieties of BL-algebras are well-known [10], but in the general framework of MTL very few algebraic investigations have been carried out till now. We only had the description of all varieties of NM-algebras given in [19], the description of all varieties of semisimple 4-contractive IMTL-algebras in [20], the structure of standard ΠMTL-chains [23, 24], some varieties of MTL-algebras where the lattice operations are definable [18] and, finally, some methods to construct IMTL-algebras [25]. In order to characterize one of those methods, the one called by Jenei *disconnected rotation*, the authors have generalized in [29] some notions that had been used in the study of MV-algebras (see [11, 1, 2]). Indeed, in that paper the notions of perfect, local and bipartite algebras are studied in the wider variety of IMTL-algebras and perfect IMTL-algebras are proved to be exactly those IMTL-algebras obtained by disconnected rotation.

Our purpose in this paper is to generalize the study of perfect and bipartite algebras to $\mathbb{MTL}$. This is actually a double generalization since the study of these kinds of algebras also has been done for BL-algebras in [12]. The content of the paper is as follows. After some algebraic preliminaries, first we will study the radical of MTL-algebras generalizing the description that had been given for MV-algebras, BL-algebras and IMTL-algebras. Then we will define perfect, local and bipartite MTL-algebras and also the class $\mathbb{BP}_0$ of algebras bipartite by all maximal filters and we will study the varieties generated by those algebras. Finally, we will define new kinds of MTL-algebras by adding new points to perfect algebras. These constructions will give a characterization of connected rotations as perfect IMTL-algebras plus an added negation fixpoint, and some new varieties of MTL-algebras, hence new fuzzy logics; we will axiomatize them and we will decide which of them have standard completeness theorem, i. e. are t-norm based.

2 Preliminaries

MTL logic is introduced by Esteva and Godo in [14]. It is presented by means of a Hilbert style calculus in the language $\mathcal{L} = \{*, \rightarrow, \wedge, 0\}$ of type $(2, 2, 2, 0)$. The only inference rule is *Modus Ponens* and the axiom schemata are the following (taking $\rightarrow$ as the least binding connective):

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi * \psi \to \varphi$
(A3)	$\varphi * \psi \to \psi * \varphi$
(A4)	$\varphi \wedge \psi \to \varphi$
(A5)	$\varphi \wedge \psi \to \psi \wedge \varphi$
(A6)	$\varphi * (\varphi \to \psi) \to \varphi \wedge \psi$
(A7a)	$(\varphi \to (\psi \to \chi)) \to (\varphi * \psi \to \chi)$
(A7b)	$(\varphi * \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9)	$0 \to \varphi$

The usual defined connectives are introduced as follows:

$\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$;

$\varphi \leftrightarrow \psi := (\varphi \to \psi) * (\psi \to \varphi)$;

$\neg\varphi := \varphi \to 0$;

$1 := \neg 0$.

We denote the set of $\mathcal{L}$-formulas built over a countable set of variables by $Fm_{\mathcal{L}}$. Given $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, we write $\Gamma \vdash_{MTL} \varphi$ if, and only if, φ is provable from Γ in the system MTL.

In the same paper some important extensions of MTL are also defined:

IMTL is the axiomatic extension of MTL obtained by adding the law of involution:

$\neg\neg\varphi \to \varphi$,

WNM logic (resp. NM logic) is obtained by adding to MTL (resp. to IMTL) the following axiom:

$(\varphi * \psi \to 0) \vee (\varphi \wedge \psi \to \varphi * \psi)$.

Recall also that BL is the axiomatic extension of MTL obtained by adding the divisibility axiom:

$\varphi \wedge \psi \to \varphi * (\varphi \to \psi)$,

Łukasiewicz logic can be obtained by adding to IMTL the divisibility axiom or by adding the law of involution to BL,[1]

Gödel logic can be obtained by adding to MTL (or to BL) the contraction axiom:

$\varphi \to \varphi * \varphi$,

and the classical propositional calculus can be obtained by adding to any of these logics the excluded middle axiom:

[1]Łukasiewicz logic can be also obtained from MTL by just adding the axiom $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$.

$\varphi \vee \neg\varphi$.

Definition 2.1 ([14]). *Let $\mathcal{A} = \langle A, *, \rightarrow, \wedge, \vee, 0, 1\rangle$ be an algebra of type $(2, 2, 2, 2, 0, 0)$. $\mathcal{A}$ is an MTL-algebra iff it is a commutative integral bounded residuated lattice satisfying the prelinearity equation:*

$$(x \rightarrow y) \vee (y \rightarrow x) \approx 1$$

We define a unary operation by $\neg a := a \rightarrow 0$.
$\mathcal{A}$ is an IMTL-algebra iff in addition it satisfies the equation of involution:

$$\neg\neg x \approx x$$

If the lattice order is total we will say that $\mathcal{A}$ is an MTL-chain (resp. IMTL-chain). The MTL-chains defined over the real unit interval $[0, 1]$ are those where $*$ is a left-continuous t-norm and are called standard chains. If $\circ$ is a left-continuous t-norm, $[0, 1]_\circ$ will denote the standard chain given by $\circ$.

$\mathbb{MTL}$ (resp. $\mathbb{IMTL}$) will denote the class of all MTL-algebras (resp. IMTL-algebras). It is well known that those classes are definable by equations; hence they are varieties.[2]

Now we can define the equational consequence in $\mathbb{MTL}$:

Definition 2.2. *Let $Eq_\mathcal{L}$ be the set of $\mathcal{L}$-equations, i.e. the set of expressions of the form $\varphi \approx \psi$ where $\varphi, \psi \in Fm_\mathcal{L}$. Let $\Sigma \cup \{\varphi \approx \psi\} \subseteq Eq_\mathcal{L}$. We define:*
$\Sigma \models_{\mathbb{MTL}} \varphi \approx \psi$ iff for every $\mathcal{A} \in \mathbb{MTL}$ and every evaluation v in $\mathcal{A}$, it holds:
If for every $\alpha \approx \beta \in \Sigma$, $v(\alpha) = v(\beta)$, then $v(\varphi) = v(\psi)$.

Then the next theorem is easy to prove:

Theorem 2.3. *The relation of derivability in MTL and the equational consequence given by the variety $\mathbb{MTL}$ are mutually translatable. Indeed, given $\Gamma \cup \{\gamma\} \subseteq Fm_\mathcal{L}$ and $\Sigma \cup \{\varphi \approx \psi\} \subseteq Eq_\mathcal{L}$, we have:*

1. *$\Gamma \vdash_{MTL} \gamma$ iff $\{\psi \approx 1 : \psi \in \Gamma\} \models_{\mathbb{MTL}} \gamma \approx 1$.*
2. *$\Sigma \models_{\mathbb{MTL}} \varphi \approx \psi$ iff $\{\alpha \leftrightarrow \beta : \alpha \approx \beta \in \Sigma\} \vdash_{MTL} \varphi \leftrightarrow \psi$.*

Moreover, each one of these translations is the inverse of the other one in the following sense:

3. *$\varphi \approx \psi \models_{\mathbb{MTL}} \varphi \leftrightarrow \psi \approx 1$ and $\varphi \leftrightarrow \psi \approx 1 \models_{\mathbb{MTL}} \varphi \approx \psi$.*
4. *$\gamma \vdash_{MTL} \gamma \leftrightarrow 1$ and $\gamma \leftrightarrow 1 \vdash_{MTL} \gamma$.*

Therefore, using the theoretical apparatus introduced by Blok and Pigozzi in [3], we can say that MTL is an algebraizable logic and $\mathbb{MTL}$ is its equivalent algebraic semantics. Furthermore, we obtain that all axiomatic extensions of MTL are also algebraizable and their equivalent algebraic semantics are the subvarieties of $\mathbb{MTL}$ defined by the translations of the axioms into equations. In particular:

[2]For any unexplained notion on Universal Algebra see [5].

- The equivalent algebraic semantics for IMTL is the variety of IMTL-algebras
- The equivalent algebraic semantics for NM is the variety of NM-algebras, i.e. those IMTL-algebras satisfying $(x * y \to 0) \vee (x \wedge y \to x * y) \approx 1$.
- The equivalent algebraic semantics for WNM is the variety of WNM-algebras, i.e. those MTL-algebras satisfying $(x * y \to 0) \vee (x \wedge y \to x * y) \approx 1$.
- The equivalent algebraic semantics for BL is the variety of BL-algebras, i.e. those MTL-algebras satisfying $x \wedge y \approx x * (x \to y)$.
- The equivalent algebraic semantics for Gödel logic is the variety of G-algebras, i.e. those MTL-algebras satisfying $x * x \approx x$.
- The equivalent algebraic semantics for Łukasiewicz logic is the variety of MV-algebras, i.e. those IMTL-algebras satisfying $x \wedge y \approx x * (x \to y)$.
- The equivalent algebraic semantics for the classical propositional calculus is the variety of Boolean algebras (denoted as $\mathbb{BA}$) i.e. those MTL-algebras satisfying $x \vee \neg x \approx 1$.

It will be useful later on to recall now the definition of some examples of these algebras:

- $\mathcal{B}_2$ and $\mathcal{B}_4$ will be the Boolean algebras of two elements and four elements respectively, with the usual definitions.
- For every $n \geq 3$, Ł_n is the MV-algebra defined over the set $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ and $[0,1]_L$ is the MV-algebra defined over the real unit interval. Recall that the operations of strong conjunction and negation in all these algebras have the following expressions: $a * b = \max\{a + b - 1, 0\}$ and $\neg a = 1 - a$. The remaining operations are defined from the former in the following way: $a \to b := \neg(a * \neg b)$, $a \wedge b := a * (a \to b)$ and $a \vee b := (a \to b) \to b$.
- For every $n \geq 3$, G_n is the G-algebra defined over the set $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ and $[0,1]_G$ is the G-algebra defined over the real unit interval. Recall that the operations in all these algebras have the following expressions: $a*b = a \wedge b = min\{a, b\}$, $a \vee b = max\{a, b\}$ and
$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$
- There is, up to isomorphism, only one NM-algebra defined over the real unit interval $[0,1]$. We will denote it as $[0,1]_{NM}$. It is given by the nilpotent minimum t-norm (introduced by Fodor in [16]):
$$a * b = \begin{cases} \min\{a, b\} & \text{if } a > 1 - b, \\ 0 & \text{otherwise,} \end{cases}$$
its residuated implication:
$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ \max\{1 - a, b\} & \text{otherwise,} \end{cases}$$
$a \wedge b = min\{a, b\}$ and $a \vee b = max\{a, b\}$.

The completeness of MTL with respect to the variety $\mathbb{MTL}$ can be improved using the following important theorem:

Theorem 2.4 ([14])**.** *Each MTL-algebra is representable as a subdirect product of MTL-chains.*

Corollary 2.5. *Given* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$*,*
$\Gamma \vdash_{MTL} \varphi$ *if, and only if,* $\{\psi \approx 1 : \psi \in \Gamma\} \vDash_{\{MTL-chains\}} \varphi \approx 1$.

The same results are true for every axiomatic extension of MTL (i.e. for every subvariety of $\mathbb{MTL}$).

The completeness with respect to chains was further improved in [26] and in [13] proving the so-called strong standard completeness theorems for MTL and IMTL, i.e. completeness with respect to chains defined over $[0, 1]$. For the further discussion of the paper it will be convenient to give now a sketch of their proofs.[3]

Theorem 2.6 ([26])**.** *Let* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ *be any set of formulas (maybe infinite). Then:*
$\Gamma \vdash_{MTL} \varphi$ *iff* $\{\psi \approx 1 : \psi \in \Gamma\} \vDash_{[0,1]_*} \varphi \approx 1$ *for every left-continuous t-norm* $*$*.*

Proof. The *only if* part is consequence of the previous corollary. The other implication is proved by contraposition. Suppose $\Gamma \nvdash_{MTL} \varphi$. Then there is an MTL-chain $\mathcal{A}$ and an evaluation v on $\mathcal{A}$ such that $v[\Gamma] \subseteq \{1^{\mathcal{A}}\}$ and $v(\varphi) \neq 1^{\mathcal{A}}$. Since $Fm_{\mathcal{L}}$ is countable, we can suppose that $\mathcal{A}$ is also countable. The proof now consists in constructing a standard MTL-chain $[0, 1]_*$ and an embedding $h : \mathcal{A} \hookrightarrow [0, 1]_*$. Then, taking the evaluation $h \circ v$, we will have the desired counterexample. The construction has the following steps:

- Consider the set $B := \{\langle 0^{\mathcal{A}}, 1\rangle\} \cup \{\langle a, q\rangle : a \in A \setminus \{0^{\mathcal{A}}\}, q \in Q \cap (0, 1]\}$.
- Consider the lexicographical order $\preceq$ on B.
- Define the following monoidal operation on B:

$$\langle a, q\rangle \circ \langle b, r\rangle := \begin{cases} min\{\langle a, q\rangle, \langle b, r\rangle\} & \text{if } a * b = min\{a, b\} \\ \langle a * b, 1\rangle & \text{otherwise.} \end{cases}$$

- A is embeddable in B by mapping every $a \in A$ to $\langle a, 1\rangle$.
- $\mathcal{B} = \langle B, \circ, \preceq\rangle$ is a densely ordered countable monoid with maximum and minimum, so it is isomorphic to a monoid $\mathcal{B}'$ over $Q \cap [0, 1]$. Obviously, A is also embeddable in this monoid. Let h the such embedding.
- $\mathcal{B}'$ is completed to $[0, 1]$ by defining:

$$\forall \alpha, \beta \in [0, 1] \quad \alpha \otimes \beta := sup\{h(x * y) : h(x) \leq \alpha, h(y) \leq \beta\}.$$

- $\otimes$ is a left-continuous t-norm, so it defines a standard MTL-algebra $[0, 1]_\otimes$, and h is the desired embedding.

[3] The rest of the logics mentioned before enjoy also some version of standard completeness theorem, but we will not discuss it here. We refer the interested reader to [21, 14, 7].

□

Theorem 2.7 ([13]). *Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ be any set of formulas (maybe infinite). Then:*
$\Gamma \vdash_{IMTL} \varphi$ *iff* $\{\psi \approx 1 : \psi \in \Gamma\} \models_{[0,1]_*} \varphi \approx 1$ *for every left-continuous t-norm* $*$ *with an involutive negation.*

Proof. The proof is a modification of the previous one. Suppose that $\mathcal{A}$ is a countable IMTL-chain and v is an evaluation on $\mathcal{A}$ such that $v[\Gamma] \subseteq \{1^{\mathcal{A}}\}$ and $v(\varphi) \neq 1^{\mathcal{A}}$. Now the construction goes like this:

- For every $a \in A$, $suc(a)$ is defined the successor of a in the order of $\mathcal{A}$ if it exists, or $suc(a) = a$ otherwise.
- $B := \{\langle a, 1\rangle : a \in A\} \cup \{\langle a, q\rangle : \exists a' \in A$ such that $a \neq a'$ and $suc(a') = a$, $q \in Q \cap (0, 1)\}$.
- Consider the lexicographical order $\preceq$ on B.
- As before, we define the following monoidal operation on B:

$$\langle a, q\rangle \circ \langle b, r\rangle := \begin{cases} min\{\langle a, q\rangle, \langle b, r\rangle\} & \text{if } a * b = min\{a, b\} \\ \langle a * b, 1\rangle & \text{otherwise.} \end{cases}$$

- The operation is modified in the following way:

$$\langle a, q\rangle \times \langle b, r\rangle := \begin{cases} \langle 0^{\mathcal{A}}, 1\rangle & \text{if } a = suc(\neg b), q + r \leq 1 \\ \langle a, q\rangle \circ \langle b, r\rangle & \text{otherwise} \end{cases}$$

- A is embeddable in B by mapping every $a \in A$ to $\langle a, 1\rangle$.
- $\mathcal{B} = \langle B, \times, \preceq\rangle$ is a densely ordered countable monoid with maximum and minimum, so it is isomorphic to a monoid $\mathcal{B}'$ over $Q \cap [0, 1]$. Obviously, A is also embeddable in this monoid. Let h be such embedding.
- $\mathcal{B}'$ is completed to $[0, 1]$ by defining:

$$\forall \alpha, \beta \in [0, 1] \quad \alpha \otimes \beta := sup\{h(x * y) : h(x) \leq \alpha, h(y) \leq \beta\}.$$

- $\otimes$ is a left-continuous t-norm with an involutive negation, so it defines a standard MTL-algebra $[0, 1]_{\otimes}$, and h is the desired embedding.

□

Given an MTL-algebra $\mathcal{A}$ and an element $a \in A$, we say that a is the *fixpoint* of $\mathcal{A}$ if, and only if, $a = \neg a$. In [22] is proved that there exists at most one fixpoint.[4]

Definition 2.8. *Given an MTL-algebra, $\mathcal{A}$, the sets of positive and negative elements are respectively defined as:*
$A_+ := \{a \in A : a > \neg a\}$
$A_- := \{a \in A : a \leq \neg a\}$

[4]Actually, Höhle states it for the involutive algebras, but the same proof gives the result for the general non-involutive case.

Consider the terms $p(x) := x \vee \neg x$ and $n(x) := x \wedge \neg x$. The next proposition is an easy but useful result describing these sets:

Proposition 2.9. *Let $\mathcal{A}$ be an MTL-algebra. Then:*

- $A_+ = \{p(a) : a \in A, \neg a \neq \neg\neg a\}$.
- $A_- = \{n(a) : a \in A\}$.

Notice that if $\neg a = \neg\neg a$, then $a \vee \neg a$ is the fixpoint.

Recall that a filter in an MTL-algebra is any set F such that:

- $1 \in F$
- If $a \in F$ and $a \leq b$, then $b \in F$
- If $a, b \in F$, then $a * b \in F$.

F is proper iff $0 \notin F$. F is a prime filter iff F is proper and $\forall a, b \in A$ if $a \vee b \in F$, then $a \in F$ or $b \in F$.

Using Zorn's Lemma one can prove that for each proper filter F, there is a maximal proper filter G containing F. Moreover, every maximal filter is prime. $Max(\mathcal{A})$ will denote the set of all maximal filters. The radical of $\mathcal{A}$ is defined as $Rad(\mathcal{A}) = \bigcap Max(\mathcal{A})$. Note that in a chain the set of filters is totally ordered, hence the radical is the maximum proper filter and $Rad(\mathcal{A}) \subseteq A_+$.

Recall this known property of maximal filters:

Proposition 2.10. *Let $\mathcal{A}$ be an MTL-algebra and $M \subseteq A$ a maximal filter. Then for every $a \in A$ ($a \notin M \Leftrightarrow \exists n\ \neg a^n \in M$).*

We state also, for the reader's convenience, the known correspondence between filters and congruences in MTL-algebras.

Proposition 2.11. *Let $\mathcal{A}$ be an MTL-algebra. For every filter $F \subseteq A$ we define $\Theta(F) := \{\langle a, b\rangle \in A^2 : a \leftrightarrow b \in F\}$, and for every congruence θ of $\mathcal{A}$ we define $Fi(\theta) := \{a \in A : \langle a, 1\rangle \in \theta\}$. Then, Θ is an order isomorphism from the set of filters onto the set of congruences and Fi is its inverse.*

We will need two special ways to construct IMTL-algebras, the so-called rotation methods, that were introduced by Jenei (see [25]). To this end we need first the notion of prelinear semihoop.

Definition 2.12 ([15])**.** *An algebra $\mathcal{A} = \langle A, *, \rightarrow, \wedge, 1\rangle$ of type $(2, 2, 2, 0)$ is semihoop iff:*

- *$\mathcal{A} = \langle A, \wedge, 1\rangle$ is an inf-semilattice with upper bound.*
- *$\langle A, *, 1\rangle$ is a commutative monoid isotonic w.r.t. the inf-semilattice order.*
- *For every $a, b \in A$ ($a \leq b$ iff $a \rightarrow b = 1$).*
- *For every $a, b, c \in A$ $a * b \rightarrow c = a \rightarrow (b \rightarrow c)$.*

If in addition satisfies the prelinearity equation, it is a prelinear semihoop.

Proposition 2.13. *Filters of MTL-algebras and prelinear semihoops are equivalent notions in the following sense:*

(1) Let $\mathcal{A}$ be an MTL-algebra and $F \subseteq A$ a filter. Then restricting the operations of $\mathcal{A}$ to F, we obtain a prelinear semihoop.

(2) Let $\mathcal{F}$ be a prelinear semihoop. Then there is an MTL-algebra $\mathcal{A}$ such that $F \subseteq A$, F is a proper filter of $\mathcal{A}$ and the operations of $\mathcal{A}$ extend the operations of $\mathcal{F}$.

Proof. (1) Just notice that F is closed under $*$, $\rightarrow$ and $\wedge$.

(2) It is enough to take the algebra $\mathcal{A} := \mathcal{B}_2 \oplus \mathcal{F}$ of [[15], Lemma 3.13], i.e. the addition of a bottom element to $\mathcal{F}$.

$\square$

Definition 2.14. *Let $\mathcal{A}$ be a prelinear semihoop. We introduce the disconnected rotation of $\mathcal{A}$ as an algebra denoted $\mathcal{A}^*$ and defined as follows. Let $A \times \{0\}$ be a disjoint copy of A. For every $a \in A$ we write a' instead of $\langle a, 0\rangle$. Consider $\langle A' = \{a' : a \in A\}, \leq\rangle$ with the inverse order and let $A^* := A \cup A'$. We extend these orderings to an order in A^* by putting $a' < b$ for every $a, b \in A$. Finally, we take the following operations in $\mathcal{A}^*$:*

$1^{\mathcal{A}^*} := 1^{\mathcal{A}}$, $0^{\mathcal{A}^*} := (1^{\mathcal{A}})'$, $\wedge^{\mathcal{A}^*}$ *the minimum w.r.t. the ordering,* $\vee^{\mathcal{A}^*}$ *the maximum w.r.t. the ordering,*

$$\neg^{\mathcal{A}^*} a := \begin{cases} a' & \text{if} \quad a \in A \\ b & \text{if} \quad a = b' \in A' \end{cases}$$

$$a *^{\mathcal{A}^*} b := \begin{cases} a *^{\mathcal{A}} b & \text{if} \quad a, b \in A \\ \neg^{\mathcal{A}^*}(a \rightarrow^{\mathcal{A}} \neg^{\mathcal{A}^*} b) & \text{if} \quad a \in A, b \in A' \\ \neg^{\mathcal{A}^*}(b \rightarrow^{\mathcal{A}} \neg^{\mathcal{A}^*} a) & \text{if} \quad a \in A', b \in A \\ 0^{\mathcal{A}^*} & \text{if} \quad a, b \in A' \end{cases}$$

$$a \rightarrow^{\mathcal{A}^*} b := \begin{cases} a \rightarrow^{\mathcal{A}} b & \text{if} \quad a, b \in A \\ \neg^{\mathcal{A}^*}(a *^{\mathcal{A}} \neg^{\mathcal{A}^*} b) & \text{if} \quad a \in A, b \in A' \\ 1^{\mathcal{A}^*} & \text{if} \quad a \in A', b \in A \\ \neg^{\mathcal{A}^*} b \rightarrow^{\mathcal{A}} \neg^{\mathcal{A}^*} a & \text{if} \quad a, b \in A' \end{cases}$$

Definition 2.15. *Let $\mathcal{A}$ be an MTL-algebra satisfying one of the following conditions:*

- *$\mathcal{A}$ does not have zero divisors.*
- *$\exists c \in A$ such that $\forall a \in A$ zero divisor, $\neg a = c$.*

Then, the connected rotation of $\mathcal{A}$ is denoted $\mathcal{A}^\star$ and defined as follows.
Take $\langle A' = \{a' : a \in A, a \neq 0^{\mathcal{A}}\}, \leq\rangle$, a disjoint copy of $A \setminus \{0^{\mathcal{A}}\}$ with the inverse order, and define $\neg^{\mathcal{A}^\star} 0^{\mathcal{A}} := 0^{\mathcal{A}}$ and all the operations as in the disconnected rotation.

Proposition 2.16 ([25])**.** *Disconnected rotations are IMTL-algebras without fixpoint and connected rotations are IMTL-algebras with fixpoint.*

Finally, we need to recall another algebraic property, the congruence extension property (CEP, for short). Given a class of algebras $\mathbb{K}$ of the same type, we say that $\mathbb{K}$ has the CEP if, and only if, for every algebra $\mathcal{A} \in \mathbb{K}$, every subalgebra $\mathcal{B} \subseteq \mathcal{A}$ and every congruence $\theta \in \mathcal{C}o(\mathcal{B})$, there exists a congruence $\theta' \in \mathcal{C}o(\mathcal{A})$ such that $\theta = \theta' \cap B^2$. This property holds for $\mathbb{MTL}$ as a consequence of the following bridge theorem of Abstract Algebraic Logic connecting the CEP with the local deduction-detachment theorem:

Theorem 2.17 ([4])**.** *If L is an algebraizable logic and $\mathbb{K}$ is its equivalent algebraic semantics, then:*

L has the local deduction-detachment theorem if, and only if, $\mathbb{K}$ has the CEP.

Since MTL has local deduction-detachment theorem [14], we obtain:

Corollary 2.18. $\mathbb{MTL}$ *has the CEP.*

3 Main results

3.1 The radical of MTL-algebras

The radical has been a useful notion in the study of MV-algebras and BL-algebras. In [17] the following characterization of the radical is given for MV-algebras (in the equivalent form of Wajsberg algebras):

If $\mathcal{A}$ is an MV-algebra, then $Rad(\mathcal{A}) = \{a \in A : a^n > \neg a \quad \forall n \geq 1\}$.

We have obtained the same characterization in the general involutive case, i.e. for IMTL-algebras, in [29]. Moreover, the radical of BL-algebras has been studied by Sessa and Turunen in [30], obtaining this description:

If $\mathcal{A}$ is a BL-algebra, then $Rad(\mathcal{A}) = \{a \in A : \neg\neg a^n > \neg a \quad \forall n \geq 1\}$.

Afterwards this result has been improved by Cignoli and Torrens in [8], obtaining:

If $\mathcal{A}$ is a BL-algebra, then $Rad(\mathcal{A}) = \{a \in A : a^n > \neg a \quad \forall n \geq 1\}$,

i.e. the same expression as in the involutive case. However, the property of divisibility was used in the proofs of both characterizations for the radical of BL-algebras. So it was not obvious how to generalize this result to MTL-algebras. Here we present a new proof for the whole class of MTL-algebras.

First we do it for chains:

Lemma 3.1. *Let $\mathcal{A}$ be an MTL-chain. Then,*

$Rad(\mathcal{A}) = \{a \in A : a^n > \neg a \quad \forall n \geq 1\}$.

Proof. If $a \in Rad(\mathcal{A})$, then for every $n \geq 1$, $a^n \in Rad(\mathcal{A}) \subseteq A_+$. Since $a^n \leq a$, we obtain $\neg a \leq \neg a^n < a^n$. Conversely, take $a \in A$ such that for every $n \geq 1$, $a^n > \neg a$. Then, in particular, for every n, $a^n \neq 0$, so the filter generated by a, $\mathbb{F}i(a)$, is proper. Thus, $a \in \mathbb{F}i(a) \subseteq Rad(\mathcal{A})$, since the set of filters of $\mathcal{A}$ is totally ordered. □

In order to extend the characterization to all MTL-algebras we will need some previous lemmas.

Lemma 3.2. *Let $\mathcal{A}$ be an MTL-algebra and F a maximal filter of $\mathcal{A}$. Then for any subalgebra $\mathcal{B} \subseteq \mathcal{A}$, $F \cap B$ is a maximal filter of $\mathcal{B}$.*

Proof. It is straightforward to check that $F \cap B$ is a filter of $\mathcal{B}$. It is proper because $0 \notin F$. Moreover, we know that for every $a \in A$, $a \notin F$ iff $\exists n\ \neg a^n \in F$. Therefore it is obvious that for every $a \in B$, $a \notin F \cap B$ iff $\exists n\ \neg a^n \in F \cap B$. Thus $F \cap B$ is also maximal. $\square$

Lemma 3.3. *Let $\mathcal{A}$ be an MTL-algebra. Then for any subalgebra $\mathcal{B} \subseteq \mathcal{A}$, $Max(\mathcal{B}) = \{M \cap B : M \in Max(\mathcal{A})\}$. Therefore, $Rad(\mathcal{B}) = Rad(\mathcal{A}) \cap B$.*

Proof. We know by the previous lemma that for every $M \in Max(\mathcal{A})$, $M \cap B \in Max(\mathcal{B})$. Take $F \in Max(\mathcal{B})$. Then, by the CEP, there is a proper filter F' of $\mathcal{A}$ such that $F = F' \cap B$. But then there is a maximal filter $M \in Max(\mathcal{A})$ containing F', so $F' \cap B \subseteq M \cap B$. Hence, since F is maximal in $\mathcal{B}$, we obtain $F = M \cap B$. $\square$

Next we will describe some maximal filters in direct products. To this end we will need some more notation. Given a set of MTL-algebras $\{\mathcal{A}_i : i \in I\}$, $\bar{a} \in \prod_{i\in I} A_i$, $k \in I$ and $b \in A_k$, we define $\sigma_k(\bar{a}, b) \in \prod_{i\in I} A_i$ by:

$$\sigma_k(\bar{a}, b)_i = \begin{cases} a_i & \text{if } i \neq k, \\ b & \text{if } i = k. \end{cases}$$

Lemma 3.4. *Let $\{\mathcal{A}_i : i \in I\}$ be a set of MTL-algebras and consider their direct product $\prod_{i\in I} \mathcal{A}_i$. Then, for every $k \in I$ and every $M_k \in Max(\mathcal{A}_k)$, the set $M_k \times \prod_{i\neq k} A_i$ is a maximal filter of $\prod_{i\in I} \mathcal{A}_i$.*

Proof. It is easy to check that $M_k \times \prod_{i\neq k} A_i$ is a proper filter of $\prod_{i\in I} \mathcal{A}_i$. Moreover, for every $\bar{a} \in \prod_{i\in I} A_i$, $\bar{a} \notin M_k \times \prod_{i\neq k} A_i$ iff $a_k \notin M_k$ iff $\exists n\ \neg a_k^n \in M_k$ iff $\exists n\ \neg \bar{a}^n \in M_k \times \prod_{i\neq k} A_i$. Thus, $M_k \times \prod_{i\neq k} A_i$ is a maximal filter. $\square$

Lemma 3.5. *Given any set of MTL-algebras $\{\mathcal{A}_i : i \in I\}$, $Rad(\prod_{i\in I} \mathcal{A}_i) = \prod_{i\in I} Rad(\mathcal{A}_i)$.*

Proof. By applying the definition of the radical and the previous lemma we obtain:
$Rad(\prod_{i\in I} \mathcal{A}_i) = \bigcap Max(\prod_{i\in I} \mathcal{A}_i) \subseteq \bigcap\{M_k \times \prod_{i\neq k} A_i : k \in I, M_k \in Max(\mathcal{A}_k)\} = \prod_{i\in I} \bigcap Max(\mathcal{A}_i) = \prod_{i\in I} Rad(\mathcal{A}_i)$.
Conversely, take $\overline{a} \in \prod_{i\in I} Rad(\mathcal{A}_i)$ and $M \in Max(\prod_{i\in I} \mathcal{A}_i)$. We must prove that $\overline{a} \in M$. Suppose not. Then, by the maximality of M, there is $\overline{m} \in M$ such $\overline{a} * \overline{m} = \overline{0}$, i.e. $a_i * m_i = 0$, for every $i \in I$. Therefore, $m_i \leq \neg a_i$, for every $i \in I$. Since each $a_i \in Rad(\mathcal{A}_i)$, this implies $m_i \in (A_i)_-$, for every $i \in I$, so $\overline{m}^2 = \overline{0}$, contradicting $\overline{m} \in M$. $\square$

Theorem 3.6. *Let $\mathcal{A}$ be an MTL-algebra. Then:*
$Rad(\mathcal{A}) = \{a \in A : a^n > \neg a \quad \forall n \geq 1\}$.

Proof. $\mathcal{A}$ is representable as a subdirect product of some set of MTL-chains $\{\mathcal{A}_i : i \in I\}$. Using previous lemmas we can compute the radical of $\mathcal{A}$ in following way:
$Rad(\mathcal{A}) = Rad(\prod_{i\in I} \mathcal{A}_i) \cap A = \prod_{i\in I} Rad(\mathcal{A}_i) \cap A = \prod_{i\in I}\{a_i \in A_i : a_i^n > \neg a_i \quad \forall n \geq 1\} \cap A = \{a \in A : a^n > \neg a \quad \forall n \geq 1\}$. $\square$

Corollary 3.7. *For every MTL-algebra $\mathcal{A}$, $Rad(\mathcal{A}) \subseteq A_+$.*

Corollary 3.8. *Let $\mathcal{A}$ be an MTL-algebra. Then:*
A_+ is a filter iff $A_+ = Rad(\mathcal{A})$.

Proof. As in the involutive case (see [29]). $\square$

3.2 Perfect, local and bipartite MTL-algebras

Definition 3.9. *Let $\mathcal{A}$ be an MTL-algebra. We define the order of $a \in A$ as:*

$$ord(a) = \begin{cases} min\{n : a^n = 0\} & \text{if it exists,} \\ \infty & \text{otherwise.} \end{cases}$$

Definition 3.10. *An MTL-algebra $\mathcal{A}$ is perfect if, and only if, $\forall a \in A \quad (ord(a) < \infty$ iff $ord(\neg a) = \infty)$.*

Some easy examples of perfect MTL-algebras are $\mathcal{B}_2$, the Chang algebra defined in [6] (page 474) and all WNM-chains with no negation fixpoint.

Definition 3.11. *An MTL-algebra $\mathcal{A}$ is local if, and only if, for every $a \in A$ $ord(a) < \infty$ or $ord(\neg a) < \infty$.*

It is clear that all the chains are local algebras. Also all perfect algebras are local. In fact, we have this characterization:

Proposition 3.12. *An MTL-algebra is local iff it has a unique maximal filter.*

Proof. As in the involutive case (see [29]). □

Corollary 3.13. *Let $\mathcal{A}$ be an MTL-algebra.*
$\mathcal{A}$ is local iff $Rad(\mathcal{A}) = \{a \in A : ord(a) = \infty\}$.

In order to state a classification theorem of local algebras, we define two new classes of MTL-algebras.

Definition 3.14. *An MTL-algebra $\mathcal{A}$ is locally finite[5] iff for every $a \in A \setminus \{1\}$ $ord(a) < \infty$. $\mathcal{A}$ is peculiar iff is local and $\exists a, b \in A \setminus \{0, 1\}$ such that $ord(a) = \infty$, $ord(b) < \infty$ and $ord(\neg b) < \infty$.*

Theorem 3.15. *Let $\mathcal{A}$ be a local MTL-algebra such that $\mathcal{A} \ncong \mathcal{B}_2$. Then $\mathcal{A}$ satisfies one, and only one, of the following:*

- *$\mathcal{A}$ is perfect.*
- *$\mathcal{A}$ is locally finite.*
- *$\mathcal{A}$ is peculiar.*

Perfect algebras cannot have negation fixpoint, but this is not the case of the other types of local algebras. For instance, on one hand, $[0, 1]_L$ is a locally finite MTL-algebra and, on the other hand, $[0, 1]_{NM}$ and in general all WNM-chains with negation fixpoint are peculiar.

Definition 3.16. *Let $\mathcal{A}$ be an MTL-algebra. Given a filter $F \subseteq A$, we define the set $\overline{F} := \{a \in A : \neg a \in F\}$.*

Notice that if $\mathcal{A}$ is an IMTL-algebra and $F \subseteq A$, then $\overline{F} = \neg F = \{\neg a : a \in F\}$.

[5] We follow here the nomenclature introduced by Chang for MV-algebras. Do not confuse this with the general notion of locally finite algebra in Universal Algebra, i.e. those algebras where the all finitely generated subalgebras are finite.

Proposition 3.17. *Let $F \subseteq A$ be a filter of $\mathcal{A}$. Then the subuniverse of A generated by F is $F \cup \overline{F}$.*

Definition 3.18. *An MTL-algebra $\mathcal{A}$ is bipartite if, and only if, there is a maximal filter $F \subseteq A$ such that $A = F \cup \overline{F}$. In this case we say that $\mathcal{A}$ is bipartite by F.*

Definition 3.19. *Let $\mathcal{A}$ be an MTL-algebra. $\mathcal{A} \in \mathbb{BP}_0$ if, and only if, for every $F \in Max(\mathcal{A})$, $A = F \cup \overline{F}$, i.e. $\mathcal{A}$ is bipartite by all maximal filters.*

As perfect algebras, bipartite algebras do not have negation fixpoint. Notice that all the examples of perfect algebras mentioned before are also in $\mathbb{BP}_0$. $\mathcal{B}_4$ is an example of an algebra in $\mathbb{BP}_0$ which is not perfect. Besides, not all bipartite algebras are in $\mathbb{BP}_0$; for instance, $Ł_3 \times \mathcal{B}_2$ and $G_3 \times \mathcal{B}_2$ are bipartite algebras (involutive and non-involutive, respectively) that are not in $\mathbb{BP}_0$.

However, in MTL-chains, perfect and bipartite algebra and algebras from $\mathbb{BP}_0$ turn out to be the same:

Theorem 3.20. *Let $\mathcal{A}$ be an MTL-chain. The following are equivalent:*

(1) $A = Rad(\mathcal{A}) \cup \overline{Rad(\mathcal{A})}$.

(2) $\mathcal{A}$ *is bipartite.*

(3) $\mathcal{A} \in \mathbb{BP}_0$.

(4) $Rad(\mathcal{A}) = A_+$ *and* $\mathcal{A}$ *has no fixpoint.*

(5) $\mathcal{A}$ *is perfect.*

(6) $\mathcal{A} \models Bp(x) \approx 1$.

(7) $\mathcal{A}/Rad(\mathcal{A}) \cong \mathcal{B}_2$.

where $Bp(x) = (\neg(\neg x)^2)^2 \leftrightarrow \neg(\neg x^2)^2$.

Proof. (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are straightforward.

(5) $\Rightarrow$ (6): If the chain is perfect, then one can check that for every $a \in A_+$, $(\neg(\neg a)^2)^2 = \neg(\neg a^2)^2 = 1$ and for every $a \in A_-$, $(\neg(\neg a)^2)^2 = \neg(\neg a^2)^2 = 0$.

(6) $\Rightarrow$ (7): Suppose that $\mathcal{A}$ satisfies the equation. Notice that in this case the set of positive elements is a proper filter. Indeed, if $a \in A_+$, then $\neg a \in A_-$, so $(\neg a)^2 = 0$. Therefore $(\neg(\neg a)^2)^2 = 1 = \neg(\neg a^2)^2$ and this implies $a^2 \in A_+$. Now, take $a, b \in A_+$ such that $a \leq b$. Then $a^2 \leq a * b$ and $a^2 \in A_+$, so $a * b \in A_+$. Thus $A_+ = Rad(\mathcal{A})$. Consider the algebra $\mathcal{A}/Rad(\mathcal{A})$ and take $a \in A$. If a is positive, then $a \rightarrow 1 = 1 \in Rad(\mathcal{A})$ and $1 \rightarrow a = a \in Rad(\mathcal{A})$, so $a/Rad(\mathcal{A}) = 1/Rad(\mathcal{A})$. If a is negative, then $a \rightarrow 0 = \neg a \in Rad(\mathcal{A})$ and $0 \rightarrow a = 1 \in Rad(\mathcal{A})$, so $a/Rad(\mathcal{A}) = 0/Rad(\mathcal{A})$. Therefore $\mathcal{A}/Rad(\mathcal{A}) \cong \mathcal{B}_2$.

(7) $\Rightarrow$ (1): Suppose that the quotient by the radical is the two element Boolean algebra. Take an arbitrary $a \in A$ and suppose $a \notin Rad(\mathcal{A})$. Then $a/Rad(\mathcal{A}) \neq 1/Rad(\mathcal{A})$, so $a/Rad(\mathcal{A}) = 0/Rad(\mathcal{A})$ and hence $\neg a/Rad(\mathcal{A}) = 1/Rad(\mathcal{A})$, i.e. $\neg a \in Rad(\mathcal{A})$.

□

Theorem 3.21. *Let $\mathcal{A}$ be an MTL-algebra. Then:*
$\mathcal{A}$ is perfect iff $A = Rad(\mathcal{A}) \cup \overline{Rad(\mathcal{A})}$.

Proof. Suppose that $\mathcal{A}$ is perfect. By Corollary 3.13 we know that $Rad(\mathcal{A}) = \{a \in A : ord(a) = \infty\}$ and then the result follows immediately. Conversely, if $A = Rad(\mathcal{A}) \cup \overline{Rad(\mathcal{A})}$ then every $a \in Rad(\mathcal{A})$ has infinite order and every $a \in \overline{Rad(\mathcal{A})}$ has finite order, hence the algebra is perfect. □

Corollary 3.22. *Every perfect algebra is bipartite.*

Proof. If the algebra is perfect, then it is local, so the radical is the only maximal filter and the result is obvious. □

Another easy consequence is the following proposition about perfect subalgebras:

Corollary 3.23. *Given an MTL-algebra $\mathcal{A}$, $Rad(\mathcal{A}) \cup \overline{Rad(\mathcal{A})}$ is a perfect subalgebra and contains all perfect subalgebras.*

Theorem 3.24. *Let $\mathcal{A}$ be an MTL-algebra. Then the following are equivalent:*

(1) $\mathcal{A}$ is perfect.

(2) $\mathcal{A}/Rad(\mathcal{A}) \cong \mathcal{B}_2$.

Proposition 3.25. *Let $\mathcal{A}$ be an MTL-algebra and let $M \subseteq A$ be a prime filter. Then the following are equivalent:*

(1) $A_+ \subseteq M$ and $\mathcal{A}$ has no fixpoint.

(2) M is maximal and $A = M \cup \overline{M}$.

(3) $\mathcal{A}/M \cong \mathcal{B}_2$.

Proof. (1) ⇒ (2): If $a \in A$, then by Proposition 2.9, $a \vee \neg a \in A_+ \subseteq M$, but since M is prime, $a \in M$ or $\neg a \in M$.

(2) ⇒ (3): On one hand, M is prime, so $\mathcal{A}/M$ is a chain. On the other hand, for every $a \in A$, $a/M \vee \neg(a/M) = (a \vee \neg a)/M = 1/M$, hence $\mathcal{A}/M$ is Boolean, so it must be the two element Boolean algebra.

(3) ⇒ (1): Take any $a \vee \neg a \in A_+$. $(a \vee \neg a)/M = a/M \vee \neg(a/M) = 1/M$, so $a \vee \neg a \in M$.

□

Lemma 3.26. *Let $\mathcal{A}$ be an MTL-algebra and let $F \subseteq A$ be a proper filter. Then: $\mathcal{A}/F \in \mathbb{BA}$ iff $\{a \vee \neg a : a \in A\} \subseteq F$.*

Proof. Suppose that the quotient is a Boolean algebra and take $a \in A_+$. Then $a/F \vee \neg(a/F) = (a \vee \neg a)/F = 1/F$. Thus: $a = a \vee \neg a \in F$. Conversely, it is straightforward to check that $\mathcal{A}/F$ satisfies the law of the excluded middle. □

Theorem 3.27. *For every MTL-algebra $\mathcal{A}$ the following are equivalent:*

(1) $\mathcal{A} \in \mathbb{BP}_0$.

(2) $\mathcal{A}/Rad(\mathcal{A}) \in \mathbb{BA}$.

(3) $Rad(\mathcal{A}) = A_+$ *and* $\mathcal{A}$ *has no fixpoint.*

Proof. (1) $\Leftrightarrow$ (2): For every maximal filter M, $A = M \cup \overline{M}$ iff (by Theorem 3.25) A_+ is contained in every maximal filter iff $A_+ \subseteq Rad(\mathcal{A})$. By Lemma 3.26, this is equivalent to $\mathcal{A}/Rad(\mathcal{A}) \in \mathbb{BA}$.

(2) $\Rightarrow$ (3): By Lemma 3.26, we obtain $A_+ \subseteq Rad(\mathcal{A})$ and the other inclusion is always true.

(3) $\Rightarrow$ (2): Also by Lemma 3.26.

□

From (2) of the last theorem and Theorem 3.24 we obviously obtain the following result:

Corollary 3.28. *Every perfect MTL-algebra is in* $\mathbb{BP}_0$.

Theorem 3.29. $\mathbb{BP}_0$ *is a variety. One equational base is obtained by adding the next set of equations to the usual axiomatization for* $\mathbb{MTL}$*:*

$$\{(\neg x \wedge \neg\neg x) \to (x \vee \neg x)^n \approx 1 : n \geq 1\}$$

Proof. Let $\mathcal{A}$ be an MTL-algebra. $\mathcal{A} \in \mathbb{BP}_0$ iff $A_+ \subseteq Rad(\mathcal{A})$ and there is no fixpoint iff for every $a \vee \neg a \in A_+$ and every $n \geq 1$, $(a \vee \neg a)^n \geq \neg(a \vee \neg a) = \neg a \wedge \neg\neg a$. □

Corollary 3.30. $\mathbb{BP}_0$ *is the variety generated by all perfect MTL-algebras.*

Proof. Let $\mathbb{K}$ be the variety generated by all perfect MTL-algebras. By Corollary 3.28, $\mathbb{K} \subseteq \mathbb{BP}_0$. The other inclusion follows from the subdirect representation theorem and Theorem 3.20. □

Corollary 3.31. *There is a simpler axiomatization for* $\mathbb{BP}_0$ *obtained by adding to the axioms of* $\mathbb{MTL}$ *only the equation* $Bp(x) \approx 1$.

Proof. Let $\mathbb{K}$ be the variety of MTL-algebras satisfying this equation. We will prove $\mathbb{K} = \mathbb{BP}_0$. If $\mathcal{A} \in \mathbb{K}$, then by the subdirect representation theorem $\mathcal{A}$ is representable as a subdirect product of chains satisfying the equation. By Theorem 3.20, these chains are in $\mathbb{BP}_0$, so $\mathcal{A} \in \mathbb{BP}_0$. Conversely, take $\mathcal{A} \in \mathbb{BP}_0$. Then $\mathcal{A}$ is isomorphic to a subdirect product of MTL-chains in $\mathbb{BP}_0$, so $\mathcal{A}$ satisfies the equation. □

As in [29], we can prove the following Glivenko-style theorem[6] for the logic BP_0 associated to the variety $\mathbb{BP}_0$:

Theorem 3.32. *Let* $\vdash_{CPC}$ *denote the relation of derivability in the classical propositional calculus. Then, for every* $\varphi \in Fm_{\mathcal{L}}$, $\vdash_{CPC} \varphi$ *iff* $\vdash_{BP_0} (\neg(\neg\varphi)^2)^2$.

[6] For a general study of this type of theorems in the framework of natural expansions of BCK logic see [9].

Proof. Suppose that $\vdash_{CPC} \varphi$. It suffices to prove that for each chain $\mathcal{A} \in \mathbb{BP}_0$, $\mathcal{A} \models (\neg(\neg\varphi)^2)^2 \approx 1$. Let $\mathcal{A}$ be such a chain and $v : Fm_{\mathcal{L}} \to \mathcal{A}$ an evaluation. We know that $\mathcal{A}/Rad(\mathcal{A}) \cong \mathcal{B}_2$, so $v(\varphi)/Rad(\mathcal{A}) = 1/Rad(\mathcal{A})$, i.e. $v(\varphi) \in A_+$, hence $(\neg(\neg v(\varphi))^2)^2 = 1$. Conversely, if $\vdash_{BP_0} (\neg(\neg\varphi)^2)^2$, then $\mathcal{B}_2 \models (\neg(\neg\varphi)^2)^2 \approx 1$, i.e. $\mathcal{B}_2 \models \varphi \approx 1$, hence $\vdash_{CPC} \varphi$. □

Concerning the structure of the class of bipartite MTL-algebras, we obtain the following results:

Proposition 3.33. *The class of bipartite MTL-algebras is closed under subalgebras.*

Theorem 3.34. *Let $\{\mathcal{A}_i : i \in I\}$ be a set of MTL-algebras and take their direct product $\mathcal{A}$. If there is a $j \in I$ such that $\mathcal{A}_j$ is bipartite, then $\mathcal{A}$ is bipartite.*

Proof. Using the same reasoning as in Theorem 4.5 of [11]. □

Corollary 3.35. *The class of bipartite MTL-algebras is closed under direct products.*

Corollary 3.36. *The variety generated by all bipartite MTL-algebras is* $\mathbb{MTL}$.

Proof. Let $\mathcal{A}$ be an arbitrary MTL-algebra. Consider $\mathcal{A}\times\mathcal{B}_2$, that is a bipartite MTL-algebra since $\mathcal{B}_2$ is bipartite. Thus, taking the projection over the first component, we obtain $\mathcal{A}$ as a homomorphic image of a bipartite algebra. Therefore, every MTL-algebra is in the variety generated by all bipartite algebras. □

3.3 Adding the fixpoint to perfect algebras

In this section we will use perfect MTL-algebras to construct new kinds of MTL-algebras and we will study the varieties and the logics that they define.

Definition 3.37. *For every natural number $n \geq 1$, we define a WNM-chain $\mathcal{W}_n = \langle W_n, *, \to, \wedge, \vee, 0^{\mathcal{W}_n}, 1^{\mathcal{W}_n}\rangle$ by taking $W_n = \{1^{\mathcal{W}_n} > a_0 > a_1 > \ldots > a_{n-1} > 0^{\mathcal{W}_n}\}$ and $\neg a_i = a_0$ for every $i < n$. As in every WNM-chain, the operations $*$ and $\to$ are defined as:*

$$a * b = \begin{cases} a \wedge b & \text{if } a > \neg b, \\ 0^{\mathcal{W}_n} & \text{otherwise.} \end{cases}$$

$$a \to b = \begin{cases} 1^{\mathcal{W}_n} & \text{if } a \leq b, \\ \neg a \vee b & \text{otherwise.} \end{cases}$$

for every $a, b \in W_n$.

Moreover, we define the WNM-chain $\mathcal{W}_\omega = \langle W_\omega, *, \to, \wedge, \vee, 0^{\mathcal{W}_\omega}, 1^{\mathcal{W}_\omega}\rangle$ by taking an infinite set $\{a_k : k < \omega\}$, letting $W_\omega = \{1^{\mathcal{W}_\omega} > a_0 > a_1 > \ldots > a_n > a_{n+1} > \ldots > 0^{\mathcal{W}_\omega}\}$ and defining the operations in the same way.

Notice that $Ł_3 \cong \mathcal{W}_1$.

Definition 3.38. *Let $\mathcal{A}$ be a perfect MTL-algebra and $1 \leq n \leq \omega$. We define an MTL-algebra $\mathcal{A} \oplus \mathcal{W}_n$, whose carrier is $A \cup (W_n \setminus \{0^{\mathcal{W}_n}, 1^{\mathcal{W}_n}\})$, the orderings in A and W_n are extended by letting $a < b < c$ for every $a \in A_-$, $b \in W_n \setminus \{0^{\mathcal{W}_n}, 1^{\mathcal{W}_n}\}$ and $c \in A_+$, and the operations are defined as $0^{\mathcal{A}\oplus\mathcal{W}_n} = 0^{\mathcal{A}}$, $1^{\mathcal{A}\oplus\mathcal{W}_n} = 1^{\mathcal{A}}$ and:*

$$a *^{\mathcal{A}\oplus\mathcal{W}_n} b := \begin{cases} a *^{\mathcal{A}} b & \text{if} \quad a, b \in A \\ 0^{\mathcal{A}} & \text{if} \quad a, b \in W_n \setminus \{0^{\mathcal{W}_n}, 1^{\mathcal{W}_n}\} \\ b & \text{if} \quad a \in A_+, b \in W_n \setminus \{0^{\mathcal{W}_n}, 1^{\mathcal{W}_n}\} \\ 0^{\mathcal{A}} & \text{if} \quad a \in A_-, b \in W_n \setminus \{0^{\mathcal{W}_n}, 1^{\mathcal{W}_n}\} \end{cases}$$

$$a \to^{\mathcal{A}\oplus\mathcal{W}_n} b := \begin{cases} a \to^{\mathcal{A}} b & \text{if} \quad a, b \in A \\ 1^{\mathcal{C}} & \text{if} \quad a, b \in W_n \setminus \{0^{\mathcal{W}_n}, 1^{\mathcal{W}_n}\}, a \leq b \\ a \to^{\mathcal{W}_n} b & \text{if} \quad a, b \in W_n \setminus \{0^{\mathcal{W}_n}, 1^{\mathcal{W}_n}\}, a > b \\ 1^{\mathcal{A}} & \text{if} \quad a \in A_-, b \in W_n \setminus \{0^{\mathcal{W}_n}, 1^{\mathcal{W}_n}\} \\ 1^{\mathcal{A}} & \text{if} \quad a \in W_n \setminus \{0^{\mathcal{W}_n}, 1^{\mathcal{W}_n}\}, b \in A_+ \\ a_0 & \text{if} \quad a \in W_n, b \in A_- \\ b & \text{if} \quad a \in A_+, b \in W_n \end{cases}$$

It is routine to check that the definition is sound. We will also use the notation $\mathcal{A}^{+n}$ for this algebra and we will call it a *perfect algebra plus n points*. Notice that one of the added points, namely a_0, is a negation fixpoint; so when $n = 1$ we will call $\mathcal{A}^{+1}$ a *perfect algebra plus fixpoint*.

We must be careful to avoid any misunderstanding here. We know that perfect algebras cannot have fixpoint. Therefore, we are not saying that $\mathcal{A}^{+n}$ is a perfect algebra with fixpoint; this would not make sense. On the contrary, we are just saying that $\mathcal{A}^{+n}$ is a perfect algebra plus n points, in the sense that it is obtained by adding n new points to a given perfect algebra $\mathcal{A}$. Thus, $\mathcal{A}^{+n}$ is not perfect and it has negation fixpoint.

Notice that if we start with an IMTL-algebra, this definition is only preserving the involution when $n = 1$. Moreover, the construction of $\mathcal{A}^{+1}$ is canonical in the sense that it is the only possible way to add the fixpoint to a perfect algebra:

Theorem 3.39. *Let $\mathcal{A}$ be an MTL-algebra with negation fixpoint such that $A = A_+ \cup A_-$ and $Rad(\mathcal{A}) = A_+$. Let a be the fixpoint. Then, $a * b = a$ for every $b \in A_+$ and $a * b = 0$ for every $b \in A_-$.*

Proof. Take $b > a$. We know that $a * b \leq a$. Suppose $a * b < a$. Then, $\neg(a * b) \in A_+$, hence $b * \neg(a * b) \in A_+$. This implies $a > \neg(b * \neg(a * b))$, in contradiction with $a * (b * \neg(a * b)) = (a * b) * \neg(a * b) = 0$. If $b \in A_-$, then $\neg b \in A_+$, so $a \leq \neg b$ and this is equivalent to $a * b = 0$. $\square$

Besides, the construction of adding the fixpoint to perfect algebras can be characterized in terms of coproducts:[7]

Definition 3.40. *Let $\{\mathcal{A}_j \mid j \in J\} \cup \{\mathcal{A}\} \subseteq \mathbb{MTL}$ be MTL-algebras. We say that $\mathcal{A}$ is the coproduct of the family $\{\mathcal{A}_j \mid j \in J\}$ iff there exist homomorphisms $\{h_j : \mathcal{A}_j \to \mathcal{A} \mid j \in J\}$ such that for every $\mathcal{B} \in \mathbb{MTL}$ and every family of homomorphisms $\{f_j : \mathcal{A}_j \to \mathcal{B} \mid j \in J\}$, there is a unique $f : \mathcal{A} \to \mathcal{B}$ such that $f \circ h_j = f_j$ for every $j \in J$. In this case we write $\mathcal{A} = \coprod_{j\in J} \mathcal{A}_j$, or $\mathcal{A} = \mathcal{A}_1 \amalg \ldots \amalg \mathcal{A}_n$ when the family is finite.*

[7] We assume the basic knowlegde on category theory. It can be found in [28]. We thank professor Antonio Di Nola for leading our attention to this categorial aspect of the construction.

The existence of the coproduct for every family of MTL-algebras and its construction when it exists is an open problem. Nevertheless, we can prove that the coproduct of any perfect MTL-algebra with $\mathcal{W}_n$ exists for every $1 \leq n \leq \omega$ and we can even describe it:

Theorem 3.41. *If $\mathcal{A}$ is a perfect MTL-algebra and $1 \leq n \leq \omega$, then $\mathcal{A}^{+n} = \mathcal{A} \amalg \mathcal{W}_n$.*

Proof. Let $h_1 : \mathcal{A} \to \mathcal{A}^{+n}$ the identity mapping and $h_2 : \mathcal{W}_n \to \mathcal{A}^{+n}$ the identity mapping on $\{a_k : k < \omega\}$ extended with $h_2(0^{\mathcal{W}_n}) = 0^{\mathcal{A}}$ and $h_2(1^{\mathcal{W}_n}) = 1^{\mathcal{A}}$. Given an arbitrary $\mathcal{B} \in \mathbb{MTL}$ and arbitrary homomorphisms $f_1 : \mathcal{A} \to \mathcal{B}$ and $f_2 : \mathcal{W}_n \to \mathcal{B}$, the homomorphism $f : \mathcal{A}^{+n} \to \mathcal{B}$ defined by $f(a) = f_1(a)$ for every $a \in A$ and $f(a) = f_2(a)$ for every $a \in W_n \setminus \{0^{\mathcal{W}_n}, 1^{\mathcal{W}_n}\}$, does the job. □

The class of perfect IMTL-algebras plus fixpoint coincides with the class of all connected rotations of MTL-algebras without zero divisors:

Theorem 3.42. *Let $\mathcal{A}$ be an IMTL-algebra. The following are equivalent:*

(1) $\mathcal{A}$ is a perfect algebra plus fixpoint.

(2) $\mathcal{A}$ is isomorphic to the connected rotation of an MTL-algebra without zero divisors.

Proof. (1) $\Rightarrow$ (2): Let a be the fixpoint of the algebra. Consider the MTL-algebra $\mathcal{B}$ defined by $Rad(\mathcal{A}) \cup \{a\}$ such that $0^{\mathcal{B}} = a$. Since the radical is closed under $*$, $\mathcal{B}$ is an MTL-algebra without zero divisors. Thus $\mathcal{A} \cong \mathcal{B}^{\star}$.

(2) $\Rightarrow$ (1): If $\mathcal{A} \cong \mathcal{B}^{\star}$ for some MTL-algebra $\mathcal{B}$ without zero divisors, then is clear that all the positive elements have infinite order and all the negative elements have finite order, so it is a perfect algebra plus the fixpoint.

□

Proposition 3.43. *Let $\mathcal{A}$ be a perfect MTL-algebra and n any ordinal number such that $1 \leq n \leq \omega$. Then, $\mathcal{A}^{+n}/Rad(\mathcal{A}^{+n}) \cong \mathcal{W}_n$.*

Proof. Recall that $Rad(\mathcal{A}^{+n}) = A_+$. So, on one hand, it is clear that $1^{\mathcal{A}^{+n}}/Rad(\mathcal{A}^{+n}) = A_+$ and $0^{\mathcal{A}^{+n}}/Rad(\mathcal{A}^{+n}) = A_-$. On the other hand, for every $a, b \in W_n \setminus \{0^{\mathcal{W}_n}, 1^{\mathcal{W}_n}\}$ such that $a < b$, we have $b \to a = a_0 \notin Rad(\mathcal{A}^{+n})$, thus $a/Rad(\mathcal{A}^{+n}) \neq b/Rad(\mathcal{A}^{+n})$. Therefore, the function defined by:

$$f(x/Rad(\mathcal{A}^{+n})) := \begin{cases} 1^{\mathcal{W}_n} & \text{if} \quad x = 1^{\mathcal{A}^{+n}} \\ 0^{\mathcal{W}_n} & \text{if} \quad x = 0^{\mathcal{A}^{+n}} \\ x & \text{if} \quad x \in W_n \setminus \{0^{\mathcal{W}_n}, 1^{\mathcal{W}_n}\} \end{cases}$$

is an isomorphism from $\mathcal{A}^{+n}/Rad(\mathcal{A}^{+n})$ to $\mathcal{W}_n$. □

However, in this case the quotient by the radical does not characterize perfect algebras plus fixpoint. This is false even for MV-algebras. Take for instance the MV-algebra $Ł_3^{\omega}$. Indeed, $Ł_3^{\omega}/Rad(Ł_3^{\omega}) \cong \mathcal{W}_1$ but $Ł_3^{\omega}$ is not a perfect algebra plus fixpoint.

Definition 3.44. *For every $1 \leq n \leq \omega$, let $\mathbb{BP}_0^{+n}$ be the variety generated by all perfect MTL-algebras plus n points.*

Obviously, $\mathbb{BP}_0 \cap \mathbb{IMTL} \subsetneq \mathbb{BP}_0^{+1} \cap \mathbb{IMTL}$, since $\text{Ł}_3 \in (\mathbb{BP}_0^{+1} \cap \mathbb{IMTL}) \setminus (\mathbb{BP}_0 \cap \mathbb{IMTL})$.

It is also clear that we have the following chain of strict inclusions:

$\mathbb{BP}_0 \subsetneq \mathbb{BP}_0^{+1} \subsetneq \ldots \subsetneq \mathbb{BP}_0^{+n} \subsetneq \mathbb{BP}_0^{+(n+1)} \subsetneq \ldots \subsetneq \mathbb{BP}_0^{+\omega}$.

Proposition 3.45. *$\mathbb{BP}_0^{+\omega}$ is the minimum variety containing $\mathbb{BP}_0^{+n}$ for every finite n, i.e. $\mathbb{BP}_0^{+\omega} = \bigvee_{1 \leq n < \omega} \mathbb{BP}_0^{+n}$.*

Proof. It is obvious that $\bigvee_{1 \leq n < \omega} \mathbb{BP}_0^{+n} \subseteq \mathbb{BP}_0^{+\omega}$. To prove the other inclusion, consider any equation $\varphi \approx \psi \in Eq_{\mathcal{L}}$ such that is not verified by all algebras in $\mathbb{BP}_0^{+\omega}$. We must show that $\varphi \approx \psi$ is not verified by all algebras in $\bigvee_{1 \leq n < \omega} \mathbb{BP}_0^{+n}$. Suppose that $\{x_1, \ldots, x_n\}$ is the set of variables appearing in $\varphi \approx \psi$. There is a chain $\mathcal{C} \in \mathbb{BP}_0^{+\omega}$ and an evaluation v in $\mathcal{C}$ such that $v(\varphi) \neq v(\psi)$. If $\mathcal{C}$ is perfect or $\mathcal{C} = \mathcal{A}^{+k}$ for some $k < \omega$ and some perfect algebra $\mathcal{A}$, the proof finishes. Suppose that $\mathcal{C} = \mathcal{A}^{+\omega}$ for some perfect algebra $\mathcal{A}$. Then the subalgebra generated by the set $A_+ \cup \overline{A_+} \cup \{v(x_1), \ldots, v(x_n)\}$ is also not satisfying the equation and it belongs to the variety $\mathbb{BP}_0^{+(n+1)}$. $\square$

Theorem 3.46. *We can obtain an equational base for $\mathbb{BP}_0^{+\omega}$ by adding to the axioms of $\mathbb{MTL}$ the following:*

1. $Bp(x) \vee (\neg x \leftrightarrow \neg\neg x) \approx 1$
2. $(x \vee \neg x \to y \vee \neg y) \vee ((y \vee \neg y \to \neg y \wedge \neg\neg y) \to y \vee \neg y) \vee (((x \vee \neg x)^2 \to y \vee \neg y) \to y \vee \neg y) \approx 1$
3. $Bp(x) \vee (\neg y \leftrightarrow \neg\neg y) \vee (x \to x * p(y)) \approx 1$

Proof. Let $\mathbb{K}$ be the variety of MTL-algebras where these equations are valid. Let $\mathcal{A}$ be a perfect MTL-algebra plus ω points. One can easily check that $\mathcal{A} \models ((\neg(\neg x)^2)^2 \leftrightarrow \neg(\neg x^2)^2) \vee (\neg x \leftrightarrow \neg\neg x) \approx 1$. Let's prove that also the second equation is valid in $\mathcal{A}$. Take $a, b \in A$. If $\neg a$ is the fixpoint, then $a \vee \neg a \to b \vee \neg b = 1$ and the equation is satisfied. Suppose now that $\neg b$ is the fixpoint and $\neg a \neq \neg\neg a$. $a \vee \neg a > b \vee \neg b$, so $(a \vee \neg a)^2 > b \vee \neg b$. Thus, $(a \vee \neg a)^2 \to b \vee \neg b = b \vee \neg b$ and the equation is satisfied too. Finally, suppose that neither $\neg a$ nor $\neg b$ are the fixpoint. Then $b \vee \neg b \to \neg b \wedge \neg\neg b \in A_-$, hence $(b \vee \neg b \to \neg b \wedge \neg\neg b) \to b \vee \neg b = 1$. Finally, let's prove that also the third equation is valid in $\mathcal{A}$. Take $a, b \in A$. Suppose $Bp(a) \neq 1$ and $\neg\neg b \neq \neg b$ (otherwise the equation is clearly satisfied). Then, we have $a \notin Rad(\mathcal{A}) \cup \overline{Rad(\mathcal{A})}$ and $p(b) \in A_+$, so $a * p(b) = a$ and the equation is also satisfied. Therefore, $\mathbb{BP}_0^{+\omega} \subseteq \mathbb{K}$.

In order to prove the other inclusion and taking into account the representation theorem in subdirect products of chains, we only need to check that all chains in $\mathbb{K}$ are either perfect or perfect plus some points. Let $\mathcal{C}$ be such a chain and take $a \in C_+$; we will see that $a^2 \in C_+$. Suppose that it is not true. Then there are two possibilities: either a^2 is the fixpoint or it is smaller than its negation. If $a^2 = b = \neg b$, then the second equation would imply $(a \to b) \vee ((b \to b) \to b) \vee ((a^2 \to b) \to b) = (a \to b) \vee b \vee b = a \to b = 1$, so $a \leq b$, a contradiction. Suppose now, that $a^2 < \neg a^2$. By the first equation $\neg(\neg a^2)^2 = (\neg(\neg a)^2)^2 = (\neg 0)^2 = 1$, so $(\neg a^2)^2 = 0$, i. e. $\neg a^2 \leq \neg\neg a^2$. This means that $a^2 < \neg a^2 \leq \neg\neg a^2$, so $\neg a^2 = \neg\neg a^2$. Therefore $\neg a^2$ is the fixpoint. Using values a and $\neg a^2$ in the second equation we obtain: $(a \to \neg a^2) \vee ((\neg a^2 \to \neg a^2) \to \neg a^2) \vee ((a^2 \to \neg a^2) \to \neg a^2) = (a \to \neg a^2) \vee \neg a^2 \vee ((a^2 \to \neg a^2) \to \neg a^2) = (a \to \neg a^2) \vee ((a^2 \to \neg a^2) \to \neg a^2) = 1$, so one of the two disjuncts must be 1. $a > \neg a^2$, thus $a^2 \to \neg a^2 \leq \neg a^2$, but this is absurd since $a^2 \to \neg a^2 = 1$. Thus, given $a, b \in C_+$ such that $a \leq b$, we have $a * b \geq a^2 \in C_+$; therefore, C_+ is closed under $*$. If $C = Rad(\mathcal{C}) \cup \overline{Rad(\mathcal{C})}$, the chain is perfect. Suppose not. Then for every $a \notin Rad(\mathcal{C}) \cup \overline{Rad(\mathcal{C})}$, we have $\neg a = \neg\neg a$. Indeed, $a \leq \neg a$ (because $a \notin Rad(\mathcal{C}) = C_+$), and $\neg a \leq \neg\neg a$ (because $\neg a \notin Rad(\mathcal{C}) = C_+$). Thus, $a \leq \neg a \leq \neg\neg a$, and this implies $\neg a = \neg\neg a$. Moreover, given $a \notin Rad(\mathcal{C}) \cup \overline{Rad(\mathcal{C})}$, and $b \in C_+$, the third equation implies $a * b = b$, so $\mathcal{C}$ is perfect plus some points. □

Theorem 3.47. *If $1 \leq n < \omega$, we can obtain an equational base for $\mathbb{BP}_0^{+n}$ by adding to the axioms of $\mathbb{MTL}$ the following:*

1. $Bp(x) \vee (\neg x \leftrightarrow \neg\neg x) \approx 1$
2. $(x \vee \neg x \to y \vee \neg y) \vee ((y \vee \neg y \to \neg y \wedge \neg\neg y) \to y \vee \neg y) \vee (((x \vee \neg x)^2 \to y \vee \neg y) \to y \vee \neg y) \approx 1$
3. $Bp(x) \vee (\neg y \leftrightarrow \neg\neg y) \vee (x \to x * p(y)) \approx 1$
4. $\bigvee_{0 \leq i \leq n} Bp(x_i) \vee \bigvee_{0 \leq i < j \leq n} (x_i \leftrightarrow x_j) \approx 1$

Proof. Let $\mathbb{K}$ be the variety defined by these equations. Let $\mathcal{A}^{+n}$ be a perfect algebra plus n points. By the previous theorem the first three equations are valid in this algebra. Let's check the fourth one. Consider $a_0, \ldots, a_n \in A^{+n}$. If there is some i such that $a_i \in Rad(\mathcal{A}^{+n}) \cup \overline{Rad(\mathcal{A}^{+n})}$, then $Bp(a_i) = 1$. If for every i $a_i \notin Rad(\mathcal{A}^{+n}) \cup \overline{Rad(\mathcal{A}^{+n})}$, then there must be some i, j such that $a_i = a_j$, since there are only n elements in these conditions, so $a_i \leftrightarrow a_j = 1$ and the equation is also satisfied. Therefore, $\mathbb{BP}_0^{+n} \subseteq \mathbb{K}$. Conversely, take any chain $\mathcal{C} \in \mathbb{K}$. On one hand, by the proof of the previous theorem we know that $\mathcal{C}$ is perfect or perfect plus some points. On the other hand, the fourth equation implies that there are at most n points not belonging to $Rad(\mathcal{C}) \cup \overline{Rad(\mathcal{C})}$. Therefore, we obtain $\mathcal{C} \in \mathbb{BP}_0^{+n}$, hence $\mathbb{K} \subseteq \mathbb{BP}_0^{+n}$. □

Corollary 3.48. *An equational base for $\mathbb{BP}_0^{+1} \cap \mathbb{IMTL}$ is obtained by adding to the axioms of $\mathbb{IMTL}$ the following:*

1. $Bp(x) \vee (x \leftrightarrow \neg x) \approx 1$
2. $(x \vee \neg x \to y \vee \neg y) \vee ((y \vee \neg y \to y \wedge \neg y) \to y \vee \neg y) \vee (((x \vee \neg x)^2 \to y \vee \neg y) \to y \vee \neg y) \approx 1$

Notice that the equation $Bp(x) \vee (\neg y \leftrightarrow \neg\neg y) \vee (x \to x * p(y)) \approx 1$ of the last two theorems is strictly necessary. Indeed, if $\mathcal{A}$ is any perfect MTL-algebra, we can define an MTL-algebra $\mathcal{B}$ whose carrier is $A \cup \{a, b\}$, the ordering in A is extended by letting $x < b < a < y$ for every $x \in A_-$, $y \in A_+$, and the operations are defined as $0^{\mathcal{B}} = 0^{\mathcal{A}}$, $1^{\mathcal{B}} = 1^{\mathcal{A}}$ and:

$$x *^{\mathcal{B}} y := \begin{cases} x *^{\mathcal{A}} y & \text{if} \quad x, y \in A \\ 0^{\mathcal{A}} & \text{if} \quad x, y \in \{a, b\} \\ b & \text{if} \quad x \in A_+, y \in \{a, b\} \\ 0^{\mathcal{A}} & \text{if} \quad x \in A_-, y \in \{a, b\} \end{cases}$$

and $\to$ is its residuum. Then, $\mathcal{B}$ is neither in $\mathbb{BP}_0^{+\omega}$ nor in $\mathbb{BP}_0^{+2}$, and it does not satisfy the equation $Bp(x) \vee (\neg y \leftrightarrow \neg\neg y) \vee (x \to x * p(y)) \approx 1$, even though it satisfies the remaining equations.

If BP_0^{+1}, IBP_0^{+1}, BP_0^{+n} and W_n are respectively the logics associated to the varieties $\mathbb{BP}_0^{+1}$, $\mathbb{BP}_0^{+1} \cap \mathbb{IMTL}$, $\mathbb{BP}_0^{+n}$ and $\mathbf{V}(\mathcal{W}_n)$ for every $2 \leq n \leq \omega$, and L_3 is the three-valued logic of Lukasiewicz, i.e. the logic associated to the variety $\mathbf{V}(\text{Ł}_3)$, we can prove the following Glivenko-style theorems for these logics:

Theorem 3.49. *For every $\varphi \in Fm_{\mathcal{L}}$, we have:*

(i) $\vdash_{L_3} \varphi$ *if, and only if,* $\vdash_{BP_0^{+1}} t(\varphi) \vee t(\varphi \leftrightarrow \neg\varphi) * \varphi$.

(ii) $\vdash_{L_3} \varphi$ *if, and only if,* $\vdash_{IBP_0^{+1}} t(\varphi) \vee t(\varphi \leftrightarrow \neg\varphi) * \varphi$.

(iii) $\vdash_{W_n} \varphi$ *if, and only if,* $\vdash_{BP_0^{+n}} t(\varphi) \vee t(\neg\varphi \leftrightarrow \neg\neg\varphi) * \varphi$*, for every* $2 \leq n \leq \omega$.

where $t(x) = \neg(\neg x^2)^2$.

Proof. We will prove the first case as an example. The remaining ones are analogous. Suppose that $\vdash_{L_3} \varphi$ and take any chain $\mathcal{A} \in \mathbb{BP}_0^{+1}$. We must prove that $\mathcal{A} \models t(\varphi) \vee t(\varphi \leftrightarrow \neg\varphi) * \varphi \approx 1$. Let $v : Fm_{\mathcal{L}} \to \mathcal{A}$ be an evaluation. We know that $\mathcal{A}/Rad(\mathcal{A}) \cong \text{Ł}_3$, so $v(\varphi)/Rad(\mathcal{A}) = 1/Rad(\mathcal{A})$, i.e. $v(\varphi) \in A_+ = Rad(\mathcal{A})$, hence $t(v(\varphi)) = 1$. Conversely, if $\vdash_{BP_0^{+1}} t(\varphi) \vee t(\varphi \leftrightarrow \neg\varphi) * \varphi$, then in particular $\text{Ł}_3 \models t(\varphi) \vee t(\varphi \leftrightarrow \neg\varphi) * \varphi \approx 1$. Let v be any evaluation on Ł_3. We have $t(v(\varphi)) \vee t(v(\varphi) \leftrightarrow \neg v(\varphi)) * v(\varphi) = 1$. The assumptions $v(\varphi) = 0$ and $v(\varphi) = \frac{1}{2}$ lead to contradiction, so it must be $v(\varphi) = 1$, and this finishes the proof. □

Finally, we will discuss which of those varieties define new fuzzy logics with strong standard completeness theorem. We will prove the theorem for $\mathbb{BP}_0^{+\omega}$, $\mathbb{BP}_0$, $\mathbb{BP}_0^{+1} \cap \mathbb{IMTL}$ and $\mathbb{BP}_0^{+1}$. For $\mathbb{BP}_0^{+\omega}$ the original method of Jenei and Montagna [26], that we have sketched in Theorem 2.6, will be enough to prove it, and for $\mathbb{BP}_0$, $\mathbb{BP}_0^{+1} \cap \mathbb{IMTL}$ and $\mathbb{BP}_0^{+1}$ we will need some modifications of the method. For the remaining varieties, $\mathbb{BP}_0^{+n}$ (for every $1 < n < \omega$) and $\mathbb{BP}_0 \cap \mathbb{IMTL}$ we will prove that there is no standard completeness.

Theorem 3.50. *The logic associated to the variety $\mathbb{BP}_0^{+\omega}$ is strong standard complete.*

Proof. Let $\mathcal{A} \in \mathbb{BP}_0^{+\omega}$ be a countable perfect chain plus infinitely many points. Using the method of Jenei and Montagna we obtain an MTL-chain $\mathcal{B}$ over $[0,1]$ and an embedding $h : \mathcal{A} \to \mathcal{B}$. It is easy to check that $Rad(\mathcal{B}) = B_+$, so $\mathcal{B} \in \mathbb{BP}_0^{+\omega}$. □

Theorem 3.51. *The logic associated to $\mathbb{BP}_0$ is strong standard complete.*

Proof. Let $\mathcal{A} \in \mathbb{BP}_0$ be a countable chain. We know that $\mathcal{A}$ is perfect. If A_- has no maximum, the method of Jenei and Montagna would not work. Indeed, the resulting chain over $[0,1]$ would have a negation fixpoint, so it would not belong to $\mathbb{BP}_0$. To avoid this problem and make sure that A_- has a maximum element, we add a couple of new elements $a, b \notin A$ requiring:

- $a < x$ for each $x \in A_+$,
- $b < a$,
- $x < b$ for each $x \in A_-$,

- $\neg a = b$,
- $\neg b = a$,
- $a * x = x * a = a$ for each $x \in A_+ \cup \{a\}$,
- $a * x = x * a = 0$ for each $x \in A_- \cup \{b\}$,
- $b * x = x * b = b$ for each $x \in A_+$ and
- $b * x = x * b = 0$ for each $x \in A_- \cup \{b\}$.

$\mathcal{A}$ is a subalgebra of this extended chain. Therefore, we can suppose without losing generality that A_- has a maximum, say b. Now we apply the usual method of Jenei and Montagna. First we obtain a densely ordered countable monoid $\mathcal{B}$ over the set $\{\langle 0^{\mathcal{A}}, 1\rangle\} \cup \{\langle a, q\rangle : a \in A \backslash \{0^{\mathcal{A}}\}, q \in Q \cap (0, 1]\}$, with the lexicographical order and the following monoidal operation:

$$\langle a, q\rangle \circ \langle c, r\rangle := \begin{cases} min\{\langle a, q\rangle, \langle c, r\rangle\} & \text{if } a * c = min\{a, c\} \\ \langle a * c, 1\rangle & \text{otherwise.} \end{cases}$$

Notice that for every $\langle c, r\rangle > \langle b, 1\rangle$ (i.e. $c > b$) we have $\langle c, r\rangle^n > \langle b, 1\rangle$ for every $n \geq 1$. Notice also that given $\langle c, r\rangle \leq \langle b, 1\rangle$ we can define $\neg\langle c, r\rangle := max\{\langle a, q\rangle : \langle a, q\rangle \circ \langle c, r\rangle = \langle 0^{\mathcal{A}}, 1\rangle\}$ and we get $\neg\langle c, r\rangle > \langle b, 1\rangle$.

$\mathcal{B}$ is isomorphic to a monoid over $Q \cap [0, 1]$ and it is completed to $[0, 1]$ by defining $\alpha \otimes \beta := sup\{p \circ q : p, q \in Q, p \leq \alpha, q \leq \beta\}$ and we obtain an MTL-chain $\mathcal{C}$ over $[0, 1]$ and an embedding $h : \mathcal{A} \to \mathcal{C}$. It is easy to check that $\mathcal{C}$ is perfect. □

Theorem 3.52. *The logic associated to the variety $\mathbb{BP}_0^{+1} \cap \mathbb{IMTL}$ is strong standard complete.*

Proof. Let $\mathcal{A} \in \mathbb{BP}_0^{+1} \cap \mathbb{IMTL}$ be a countable chain. As we have seen, $\mathcal{A}$ is either perfect or perfect plus fixpoint. It is enough to suppose that $\mathcal{A}$ is a countable perfect chain plus fixpoint and show that it can be embedded in a standard chain of $\mathbb{BP}_0^{+} \cap \mathbb{IMTL}$ over $[0, 1]$. Let $a \in A$ be the fixpoint. If we use the usual method we first obtain an algebra over a densely ordered set B, as we have described in the preliminaries. For every $q \in Q \cap (0, 1)$ the element $\langle a, q\rangle \in B$ is such that $\neg\langle a, q\rangle = \langle a, 1\rangle$, so the resulting standard chain will not be perfect plus fixpoint.

In order to solve this problem, we consider the construction of Jenei and Montagna applied to the prelinear semihoop defined by $Rad(\mathcal{A})$, but giving an algebra $\mathcal{C}$ over $[0.6, 1]$ instead of being over $[0, 1]$ as usual. We have an embedding $h : Rad(\mathcal{A}) \to [0.6, 1]$ such that is a homomorphism with respect to $*$, is monotonic and $h(1^{\mathcal{A}}) = 1$. We extend h to $\hat{h} : A \to [0, 1]$ in the following way:

- $\hat{h}(x) = h(x)$, if $a \in Rad(\mathcal{A})$,
- $\hat{h}(\neg x) = 1 - h(x)$, if $\neg a \in \neg Rad(\mathcal{A})$, and
- $\hat{h}(a) = \frac{1}{2}$.

Consider now the algebra $\mathcal{B}$ over $[0.5, 1]$ given by the ordinal sum of the G-chain over $[0.5, 0.6]$ and $\mathcal{C}$. $\mathcal{B}$ is an MTL-algebra without zero divisors. Consider its connected rotation $\mathcal{B}^\star$ defined over $[0, 1]$. Then, $\mathcal{B}^\star \in \mathbb{BP}_0^{+} \cap \mathbb{IMTL}$ and $\hat{h}$ is an embedding from $\mathcal{A}$ into $\mathcal{B}^\star$, so the theorem holds. □

Theorem 3.53. *The logic associated to the variety $\mathbb{BP}_0^{+1}$ is strong standard complete.*

Proof. Let $\mathcal{A} \in \mathbb{BP}_0^{+1}$ be a countable perfect chain plus fixpoint. Let $a \in A$ be the fixpoint. The usual method would produce the same problem as in the previous proof, so we will modify it again. If A_+ has minimum or $A_- \setminus \{a\}$ has maximum we embed $\mathcal{A}$ into a new countable perfect chain plus fixpoint in the following way. Let $\mathcal{B}$ be the disconnected rotation of a countable cancellative hoop such that $A \cap B = \emptyset$. We will define a new chain over $C := (A \setminus \{a\}) \cup (B \setminus \{0^{\mathcal{B}}, 1^{\mathcal{B}}\})$ by extending the operations and the order of $\mathcal{A}$ and $\mathcal{B}$ in this way:

- $x < y$ for each $x \in B \setminus \{0^{\mathcal{B}}, 1^{\mathcal{B}}\}$ and each $y \in A_+$,
- $x < y$ for each $x \in A_- \setminus \{a\}$ and each $y \in B \setminus \{0^{\mathcal{B}}, 1^{\mathcal{B}}\}$,
- $x * y = y * x = y$ for each $x \in A_+$ and each $y \in B \setminus \{0^{\mathcal{B}}, 1^{\mathcal{B}}\}$,
- $x * y = y * x = x *^{\mathcal{B}} y$ for each $x, y \in B \setminus \{0^{\mathcal{B}}, 1^{\mathcal{B}}\}$ such that $x > \neg y$,
- $x * y = y * x = 0^{\mathcal{A}}$ for each $x, y \in B \setminus \{0^{\mathcal{B}}, 1^{\mathcal{B}}\}$ such that $x \leq \neg y$, and
- $x * y = y * x = 0^{\mathcal{A}}$ for each $x, y \in A_- \setminus \{a\}$.

Let $\rightarrow$ be the residuum of $*$. With this order and these operations $\mathcal{C}$ is a countable perfect MTL-algebra. Then, considering $\mathcal{C}^{+1}$ we obtain a countable perfect chain plus fixpoint where it is possible to embed $\mathcal{A}$ and with no minimum in the set of positives and no maximum in the set of negatives minus the fixpoint. Thus we can suppose without losing generality that $\mathcal{A}$ is such that A_+ has no minimum and $A_- \setminus \{a\}$ has no maximum.

Now we will use the construction of Jenei and Montagna slightly modified. Indeed, we define a densely ordered countable monoid with the lexicographical order and the usual operations, but over the set $\{\langle 0^{\mathcal{A}}, 1\rangle, \langle a, 1\rangle\} \cup \{\langle b, q\rangle : b \in A \setminus \{0^{\mathcal{A}}, a\}, q \in Q \cap (0, 1]\}$. To be sure that this also works we only need to check the left-continuity of the monoidal operation on $\langle a, 1\rangle$. Let $\{\langle b_i, q_i\rangle : i \in \omega\}$ be such that $sup\{\langle b_i, q_i\rangle : i \in \omega\} = \langle a, 1\rangle$ and take an arbitrary element $\langle c, p\rangle$. We must prove $sup\{\langle b_i, q_i\rangle \circ \langle c, p\rangle : i \in \omega\} = \langle a, 1\rangle \circ \langle c, p\rangle$. If $c \leq a$ then $\langle a, 1\rangle \circ \langle c, p\rangle = \langle 0^{\mathcal{A}}, 1\rangle$ and $\langle b_i, q_i\rangle \circ \langle c, p\rangle = \langle 0^{\mathcal{A}}, 1\rangle$ for every $i \in \omega$, so it holds. Suppose that $c > a$. $\langle a, 1\rangle \circ \langle c, p\rangle = \langle a, 1\rangle$ and for every $i \in \omega$ we have:

$$\langle b_i, q_i\rangle \circ \langle c, p\rangle := \begin{cases} \langle b_i * c, q_i\rangle & \text{if } b_i * c = b_i \\ \langle b_i * c, 1\rangle & \text{otherwise.} \end{cases}$$

Now using that $sup\{b_i * c : i \in \omega\} = a * c$ the proof finishes. □

Finally, we prove that the remaining varieties do not define a logic with standard completeness:

Theorem 3.54. *The logic associated to $\mathbb{BP}_0 \cap \mathbb{IMTL}$ has no standard completeness.*

Proof. This is clear because all IMTL-chains over $[0, 1]$ have negation fixpoint, so there are no perfect standard IMTL-chains. □

Theorem 3.55. *For every $1 < n < \omega$, the logic associated to $\mathbb{BP}_0^{+n}$ has no standard completeness.*

Proof. Observe that the only standard chains in $\mathbb{BP}_0^{+n}$ are perfect chains plus fixpoint, hence if the standard completeness was true we would have $\mathbb{BP}_0^{+n} = \mathbb{V}(\{\text{standard } \mathbb{BP}_0^{+n}\text{-chains}\}) = \mathbb{BP}_0^{+1}$, a contradiction. $\square$

Acknowledgements: The authors acknowledge partial support of the Spanish projects TIN2004-07933-C03-01, TIN2004-07933-C03-02 and MTM 2004-03102 and the Catalan project 2001SGR-0017 of DGR. We are indebted with the anonymous referee, who found a mistake in the former proof of Lemma 3.5 and made a lot of useful remarks.

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