Initial segments of the Σ0 2 enumeration degrees

INITIAL SEGMENTS OF THE Σ^0_2 ENUMERATION DEGREES

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Abstract. Using properties of $\mathcal{K}$-pairs of sets, we show that every nonzero enumeration degree $\mathbf{a}$ bounds a nontrivial initial segment of enumeration degrees whose nonzero elements have all the same jump as $\mathbf{a}$. Some consequences of this fact are derived, that hold in the local structure of the enumeration degrees, including: There is an initial segment of enumeration degrees, whose nonzero elements are all high; there is a nonsplitting high enumeration degree; every noncappable enumeration degree is high; every nonzero low enumeration degree can be capped by degrees of any possible local jump (i.e., any jump that can be realized by enumeration degrees of the local structure); every enumeration degree that bounds a nonzero element of strictly smaller jump, is bounding; every low enumeration degree below a non low enumeration degree $\mathbf{a}$ can be capped below $\mathbf{a}$.

1. Introduction

Turing reducibility formalizes the notion of relative computability of sets, for which, given sets A and B of numbers, A is computable relatively to B if there is an algorithm by means of which any decision procedure for B can be transformed into some decision procedure for A. Enumeration reducibility is, in turn, a formalization of the idea of relative enumerability of sets: A is enumerable relatively to B if there is an algorithm that transforms any enumeration of B into some enumeration of A. Following [12], this intuitive notion is made precise by defining A to be *enumeration reducible to* B (or, simply, A *e-reducible to* B, notation: $A \leq_e B$) if there is a computably enumerable (or, simply, c.e.) set Φ, such that

$$A = \{x : (\exists u)[\langle x, u\rangle \in \Phi \,\&\, D_u \subseteq B]\}$$

(D_u is the finite set with canonical index u). We usually write in this case $A = \Phi(B)$: thus, every c.e. set defines in this way what is called an *enumeration operator* (or, simply, *e-operator*), i.e. a mapping Φ, from sets of numbers to sets of numbers, taking a set B to the set $\Phi(B)$. Enumeration reducibility gives rise in the usual way to a degree structure $\mathcal{D}_e$, a poset whose elements are called *enumeration degrees* (or, simply, *e-degrees*): we use the symbol $\leq$ to denote the partial ordering relation on the e-degrees, whereas the symbol $\leq_e$ is reserved for e-reducibility on sets of numbers; the equivalence relation induced by $\leq_e$ will be denoted by $\equiv_e$. The poset $\mathcal{D}_e$ turns out to be an upper semilattice with least element $\mathbf{0}_e$, consisting of the c.e. sets. On $\mathcal{D}_e$, one can define a jump operation $'$, which maps any e-degree $\mathbf{a}$ to a strictly bigger e-degree $\mathbf{a}'$. Interest in enumeration reducibility is motivated by the fact that the e-degrees provide a wider context for the Turing degrees: the mapping sending the Turing degree of a set A to the e-degree of the characteristic function of A (or equivalently, to the e-degree of $A \oplus \overline{A}$, where $\overline{A}$ denotes the complement of A, and $\oplus$ is the usual operation of disjoint union of sets of numbers), is an embedding of upper semilattices, preserving the least element and the jump operation. Recent developments have shown that, under this embedding, the Turing degrees are first order definable in $\mathcal{D}_e$ (Cai, Ganchev, Lempp, Miller and Soskova [3]), and the Turing degress below the first Turing jump are first order definable

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in the *local structure* $\mathcal{G}_e$, i.e. the poset of the e-degrees below the first e-jump $\mathbf{0}'_e$ (Ganchev and Soskova [16]). The proofs of these results use the so-called Kalimullin pairs ($\mathcal{K}$-pairs), introduced by Kalimullin [20].

Definition 1. A pair of sets $\{A, B\}$ is a *$\mathcal{K}$-pair* of sets if there is a c.e. set W, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. Moreover if both A and B are not c.e. we shall say that the $\mathcal{K}$-pair is *nontrivial.* A pair of e-degrees $\{\mathbf{a}, \mathbf{b}\}$ is a *$\mathcal{K}$-pair* of e-degrees, if there are sets $A \in \mathbf{a}$ and $B \in \mathbf{b}$, such that $\{A, B\}$ is a $\mathcal{K}$-pair; the $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is *nontrivial* if the $\mathcal{K}$-pair $\{A, B\}$ of sets is nontrivial.

$\mathcal{K}$-pairs of e-degrees are the first nontrivial example of a class of e-degrees that is both arithmetically and degree-theoretically definable. Indeed, it turns out (Kalimullin [20]) that $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair *if and only if* for every e-degree $\mathbf{x}$,

$$\mathbf{x} = (\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}).$$

The simplicity of the degree-theoretic definition of $\mathcal{K}$-pairs is not a an isolated phenomenon. In fact all degree-theoretic definitions involving $\mathcal{K}$-pairs turn out to be quite simple and understandable. For example, the degree-theoretic definition of the jump, given by Kalimullin [20], says that the jump of an e-degree, say $\mathbf{u}$, is the least degree bounding a triple $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, such that each pair $\{\mathbf{a}_i, \mathbf{a}_j\}$, for $i \neq j$, is a *nontrivial $\mathcal{K}$-pair relative to* $\mathbf{u}$: in other words, $\mathbf{u}'$ is the least e-degree bounding a triple $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, such that $\mathbf{a}_i, \mathbf{a}_j \not\leq \mathbf{u}$ and

$$\forall \mathbf{x} \geq \mathbf{u} \big(\mathbf{x} = (\mathbf{x} \vee \mathbf{a}_i) \wedge (\mathbf{x} \vee \mathbf{a}_j)\big),$$

for $1 \leq i < j \leq 3$.

A further analysis by Ganchev and Soskova revealed a degree-theoretic definition of the jump operation (on nonzero e-degrees), not requiring the relativisation of the notion of $\mathcal{K}$-pair. This definition relies on the following simple facts. Firstly, the jump of an e-degree $\mathbf{x}$ is the biggest degree containing a set of the form $A \oplus \overline{A}$, for which the e-degree $\mathbf{a}$ of A satisfies $\mathbf{a} \leq \mathbf{x}$. Secondly, for each nontrivial $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ and every $A \in \mathbf{a}$, the e-degree $\overline{\mathbf{a}}$ of $\overline{A}$ satisfies $\mathbf{b} \leq \overline{\mathbf{a}}$. Finally, for every non c.e. set X, with e-degree $\mathbf{x}$, there is a $\mathcal{K}$-pair $\{\mathbf{a}, \overline{\mathbf{a}}\}$, for which $\mathbf{a} \leq \mathbf{x}$ and $\mathbf{a} \vee \overline{\mathbf{a}}$ is the degree of $X \oplus \overline{X}$. Thus the jump of a nonzero e-degree $\mathbf{x}$ *is the biggest degree, which is the least upper bound of a nontrivial $\mathcal{K}$-pair* $\{\mathbf{a}, \mathbf{b}\}$*, such that* $\mathbf{a} \leq \mathbf{x}$.

Note that the second property used above yields that for every nontrivial $\mathcal{K}$-pair $\{\mathbf{a}, \overline{\mathbf{a}}\}$, if $A \in \mathbf{a}$ and $\overline{A} \in \overline{\mathbf{a}}$ for some set A, then the pair $\{\mathbf{a}, \overline{\mathbf{a}}\}$ is maximal, i.e. for every $\mathcal{K}$-pair $\{\mathbf{b}_1, \mathbf{b}_2\}$, if $\mathbf{a} \leq \mathbf{b}_1$ and $\overline{\mathbf{a}} \leq \mathbf{b}_2$, then $\mathbf{a} = \mathbf{b}_1$ and $\overline{\mathbf{a}} = \mathbf{b}_2$. Now, this, combined with the third property, yields that the images of the nonzero Turing degrees under the standard embedding of the Turing degrees $\mathcal{D}_T$ in $\mathcal{D}_e$ are least upper bounds of nontrivial maximal $\mathcal{K}$-pairs. On the other hand it turns out that the least upper bounds of nontrivial $\mathcal{K}$-pairs are images of Turing degrees (Cai, Ganchev, Lempp, Miller and Soskova [3], and Ganchev and Soskova [16]), so that an e-degree $\mathbf{x}$ *is the image of a nonzero Turing degree under the standard embedding of* $\mathcal{D}_T$ *in* $\mathcal{D}_e$ if and only if $\mathbf{x} = \mathbf{a} \vee \overline{\mathbf{a}}$ *for some nontrivial maximal $\mathcal{K}$-pair* $\{\mathbf{a}, \overline{\mathbf{a}}\}$.

Kalimullin pairs turn out to be very useful also in the context of the local structure $\mathcal{G}_e$ of the e-degrees. Most of the literature on the subject of e-degrees is dedicated to the investigation of $\mathcal{G}_e$. A lot of results have been obtained so far, including but not only: the degrees in $\mathcal{G}_e$ are exactly the e-degrees of the Σ^0_2 sets (Cooper [5]); the degrees in $\mathcal{G}_e$ are dense (Cooper [5]); there are noncuppable nonzero degrees (Cooper, Sorbi and Yi [10]); every nonzero degree containing a Δ^0_2 set is cuppable (Cooper, Sorbi and Yi [10]); there are noncappable degrees strictly below $\mathbf{0}'_e$ (Cooper and Sorbi [9]); every finite lattice is embeddable preserving both 0 and 1 (Lempp and Sorbi [22]); a decidable necessary and sufficient condition for extension of embeddings of finite partial orders (Lempp, Slaman and Sorbi [23]); every nonzero degree bounds a nonsplitting degree (Kent and Sorbi [21]); $\mathbf{0}'_e$ is splittable over every degree $\mathbf{a} < \mathbf{0}'_e$ containing a Δ^0_2 set (Arslanov and Sorbi [1]); there is a degree $\mathbf{a} < \mathbf{0}'_e$, such that $\mathbf{0}'_e$ is not splittable over $\mathbf{a}$ (Soskova [28]).

Typically these results are proven by very complex and sophisticated priority arguments. They shed light on the structural properties of $\mathcal{G}_e$, but nevertheless they do not answer any natural question about the definability in $\mathcal{G}_e$ of arithmetically definable classes of degrees, such as: the degrees containing Δ^0_2 sets (usually referred to as Δ^0_2 e-degrees); the degree not containing Δ^0_2 sets (usually referred to as *properly* Σ^0_2 e-degrees); the degrees that do not bound any nonzero Δ^0_2 e-degree (usually referred to as *downwards properly* Σ^0_2 e-degrees); the degrees that are not bounded by any Δ^0_2 e-degree different from $\mathbf{0}'_e$ (usually referred to as *upwards properly* Σ^0_2 e-degrees); the degrees whose n-th jump is equal to the n-th jump of $\mathbf{0}_e$ (usually referred to as *low*$_n$ e-degrees; the *low*$_1$ e-degrees are usually called *low*); the degrees whose n-th jump is equal to the $n+1$-st jump of $\mathbf{0}_e$ (usually referred to as *high*$_n$ e-degrees; the *high*$_1$ e-degrees are usually called *high*).

In this context, $\mathcal{K}$-pairs once again play a very important role. $\mathcal{K}$-pairs turn out to be definable in $\mathcal{G}_e$ (Ganchev and Soskova [14]) by the very formula defining them in the global structure of the e-degrees (Cai, Miller, Lempp and Soskova [4]). Thus, once again, $\mathcal{K}$-pairs are the first example of an arithmetically definable class of degrees which is first order definable in $\mathcal{G}_e$. From the definability of $\mathcal{K}$-pairs in $\mathcal{G}_e$ and the fact that the images of the Turing degrees under the standard embedding of $\mathcal{D}_T$ in $\mathcal{D}_e$ are exactly the least upper bounds of maximal $\mathcal{K}$-pairs, it follows that the class of the images of Turing degrees in $\mathcal{G}_e$ is definable in $\mathcal{G}_e$.

Further, according to a result by Ganchev and Soskova [13], every nonzero Δ^0_2 degree bounds a nontrivial $\mathcal{K}$-pair, so that the downwards properly Σ^0_2 e-degrees are exactly *the degrees that do not bound nontrivial $\mathcal{K}$-pairs*. On the other hand every low e-degree does not bound any downwards properly Σ^0_2 degree, whereas every non low e-degree, which is the image of a Turing degree, bound s a downwards properly Σ^0_2 degree (Giorgi, Sorbi and Yang [17]). Thus $\mathbf{x} \in \mathcal{G}_e$ is the image of a low Turing degree *if and only if* $\mathbf{x}$ *is the least upper bound of a nontrivial maximal $\mathcal{K}$-pair, and further for every* $\mathbf{0}_e < \mathbf{y} < \mathbf{x}$, $\mathbf{y}$ *bounds a nontrivial $\mathcal{K}$-pair*. Note that for the moment it is not known whether the class of low Turing degrees is definable in the local structure of the Turing degrees. Thus the above definition is a clear evidence of the advantages of considering the Turing degrees in the wider context of the e-degrees.

Finally let us mention that, using the structural properties of $\mathcal{K}$-pairs, Ganchev and Soskova [15] have managed to prove that true arithmetic is interpretable in $\mathcal{G}_e$ and hence the first order theory of the upper semilattice $\mathcal{G}_e$ is as complex as possible. This said, it is clear that Kalimullin pairs have turned out to be one of the most powerful tools for studying the e-degrees, and their local structure.

In this paper, we use a simple theorem on $\mathcal{K}$-pairs (Theorem 5), to derive as straightforward and immediate corollaries, some new and somewhat unexpected results on the local structure of the e-degrees, including: Theorem 6 (every level of the high/low hierarchy of the local structure of the e-degrees contains an interval of the form $(\mathbf{0}_e, \mathbf{a}]$, for some nonzero $\mathbf{a} \in \mathcal{G}_e$: in particular, with the exception of the level of the low e-degrees, every such interval consists entirely of downwards properly Σ^0_2 e-degrees); Corollary 7 (there is a nontrivial initial segment of e-degrees, whose nonzero elements are all high); Corollary 8 (there is a nonsplitting high e-degree); Corollary 9 (every noncappable e-degree is high); Corollary 13 (every e-degree that bounds a nonzero e-degree of strictly smaller jump, is bounding; hence the nonzero degrees below a nonbonding degree have all the same jump as the nonbonding e-degree itself). In other cases, we obtain new and very simple proofs of *nontrivial* extensions of known results, whose original proofs were very complicated and sometimes used complex priority arguments, including (references to the original papers containing the results that have been extended are given in the text): Corollary 14 (for every possible local jump, every low e-degree caps with some e-degree having that jump); Corollary 16 (every low e-degree below a non low $\mathbf{a}$ can be capped below $\mathbf{a}$).

2. Background

Our terminology and notations are standard. For an excellent introduction to the e-degrees and their local structure, the reader is referred to Cooper's survey paper [6]. We only recall the definition of the jump operation, and the basic properties of $\mathcal{K}$-pairs of sets. More technical definitions are postponed to Section 4, where and when are specifically needed.

For every set of numbers A let

$$E_A = \{e : e \in \Phi_e(A)\},$$

where $\{\Phi_e : e \in \omega\}$ is the standard listing of the e-operators, and define

$$A' = E_A \oplus \overline{E_A}.$$

This allows us to introduce a well defined jump operation in the e-degrees, namely if $\mathbf{a}$ is the e-degree of A, then one defines $\mathbf{a}'$ to be the e-degree of A'. The local structure $\mathcal{G}_e = \{\mathbf{a} : \mathbf{a} \leq \mathbf{0}_e'\}$ partitions, under $\equiv_e$, the Σ^0_2 sets. The classes of high$_n$ and low$_n$ e-degrees, with $n \geq 1$, introduced in Section 1, form the so-called *high/low hierarchy* of the e-degrees in the local structure, which parallels the much studied high/low hierarchy in the Turing degrees, see e.g. Cooper's textbook [7, Definition 12.1.1]. A *left-open interval* of e-degrees is a set of the form $(\mathbf{a}, \mathbf{b}] = \{\mathbf{c} \in \mathcal{D}_e : \mathbf{a} < \mathbf{c} \leq \mathbf{b}\}$; an initial segment I of e-degrees is *nontrivial* if $I \neq \emptyset$, and $I \neq \{\mathbf{0}_e\}$.

2.1. Properties of $\mathcal{K}$-pairs.

$\mathcal{K}$-pairs of sets (see Definition 1) have the following basic properties, proved by Kalimullin [20] (item (5) comes from [20, Theorem 2.6]; (3) and (4) are in [20, Proposition 2.7]; (2) follows from (5)):

(1) If A is a c.e. set, then $\{A, B\}$ is a $\mathcal{K}$-pair for every set B;
(2) Given a set A, the sets B for which $\{A, B\}$ is a $\mathcal{K}$-pair form an ideal with respect to the preorder relation $\leq_e$. In other words, if

$$\mathcal{I}_A = \{B \mid \{A, B\} \text{ is a } \mathcal{K}\text{-pair}\},$$

then
   (a) $B \in \mathcal{I}_A \mathbin{\&} C \leq_e B \Longrightarrow C \in \mathcal{I}_A$,
   (b) $B \in \mathcal{I}_A \mathbin{\&} C \in \mathcal{I}_A \Longrightarrow B \oplus C \in \mathcal{I}_A$;
(3) If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair, then $A \leq_e \overline{B}$ and $B \leq_e \overline{A}$;
(4) If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair and W is the c.e. set from the definition, then $\overline{A} \leq_e \overline{W} \oplus B$ and $\overline{B} \leq_e \overline{W} \oplus A$;
(5) $\{A, B\}$ is a $\mathcal{K}$-pair if and only if

$$(\forall X)(\forall Y)[Y \leq_e X \oplus A \mathbin{\&} Y \leq_e X \oplus B \Longrightarrow Y \leq_e X].$$

Property (5) of $\mathcal{K}$-pairs of sets shows that the property of being a $\mathcal{K}$-pair of e-degrees is first order definable in the structure of the e-degrees $\mathcal{D}_e$, as remarked in Section 1.

3. The theorem

In this section we state and prove (Theorem 5) the main result of the paper, namely, every nonzero e-degree $\mathbf{a}$ bounds a nontrivial principal ideal whose nonzero elements have all the same jump as $\mathbf{a}$.

We begin by using properties (2a), (3), and (4), of $\mathcal{K}$-pairs of sets, in order to prove that if $\{\mathbf{a}, \mathbf{b}\}$ form a nontrivial $\mathcal{K}$-pair of e-degrees, then $\mathbf{a}' = \mathbf{b}'$.

Proposition 2. *Let $\{A, B\}$ be a nontrivial $\mathcal{K}$-pair of sets. Then $A' \equiv_e B'$.*

Proof. Fix a nontrivial $\mathcal{K}$-pair $\{A, B\}$. Since $A \equiv_e E_A$ and $B \equiv_e E_B$, by property (2a) we have that $\{E_A, E_B\}$ is a nontrivial $\mathcal{K}$-pair. Let W be a c.e. set such that $E_A \times E_B \subseteq W$ and $\overline{E_A} \times \overline{E_B} \subseteq \overline{W}$. According to properties (3) and (4) we have respectively $E_A \leq_e \overline{E_B}$ and $\overline{E_A} \leq_e \overline{W} \oplus E_B$. Hence

$$A' = E_A \oplus \overline{E_A} \leq_e \overline{W} \oplus (E_B \oplus \overline{E_B}) \leq_e \emptyset' \oplus B' \leq_e B'.$$

Analogously, $B' \leq_e A'$, so that $A' \equiv_e B'$. □

Corollary 3. *Let $\mathbf{a}$ be a half of a nontrivial $\mathcal{K}$-pair of e-degrees. Then for every nonzero e-degree $\mathbf{c} \leq \mathbf{a}$, we have $\mathbf{c}' = \mathbf{a}'$.*

Proof. Let $\{\mathbf{a}, \mathbf{b}\}$ be a nontrivial $\mathcal{K}$-pair of e-degrees and let us fix a nonzero e-degree $\mathbf{c} \leq \mathbf{a}$. According to property (2a), $\{\mathbf{c}, \mathbf{b}\}$ is also a nontrivial $\mathcal{K}$-pair. Now applying Proposition 2 we obtain

$$\mathbf{a}' = \mathbf{b}' = \mathbf{c}'.$$

□

Next, we recall the following result from Ganchev and Soskova [16], and Cai, Ganchev, Lempp, Miller and Soskova [3]:

Theorem 4. *For every nonzero e-degree $\mathbf{x}$ there is an e-degree $\mathbf{a}$, which is a half of a nontrivial $\mathcal{K}$-pair and such that $\mathbf{a}' = \mathbf{x}'$.*

Proof. We sketch the proof: for full details, see Ganchev and Soskova [16] and Cai, Ganchev, Lempp, Miller and Soskova [3]. For a given set A let us denote by L_A the collection of the codes of all finite binary strings that are lexicographically less than the characteristic function of A, denoted by χ_A, i.e.

$$L_A = \{\sigma \in 2^{<\omega} \mid \sigma <_L \chi_A\}.$$

Further, by R_A we shall denote the complement of L_A. The sets L_A and R_A have the following properties:

(i) $L_A \leq_e A$, $R_A \leq_e \overline{A}$;
(ii) $L_A \oplus R_A \equiv_e A \oplus \overline{A}$;
(iii) $\{L_A, R_A\}$ is a $\mathcal{K}$-pair; moreover $L_A \times R_A \subseteq W_L$ and $\overline{L_A} \times \overline{R_A} \subseteq \overline{W_L}$, where

$$W_L = \{\langle \sigma_1, \sigma_2 \rangle \in 2^{<\omega} \times 2^{<\omega} \mid \sigma_1 <_L \sigma_2 \vee \sigma_2 \subset \sigma_1\}.$$

Coming back to the proof of the theorem, suppose first that $\mathbf{x}$ is low. Then $\mathbf{x}$ is a Δ^0_2 degree so that it bounds a nontrivial $\mathcal{K}$-pair (this has been proved by Kalimullin [20, Theorem 4.3]), each side of which is low.

Now let $\mathbf{x}' > \mathbf{0}'_e$. Fix a set $X \in \mathbf{x}$. Consider the sets L_{E_X} and R_{E_X} as defined above. Applying property (ii), we obtain

$$X' = E_X \oplus \overline{E_X} \equiv_e L_{E_X} \oplus R_{E_X} \leq_e L'_{E_X}$$

On the other hand by property (i) $L_{E_X} \leq_e X$, so that $L'_{E_X} \leq_e X'$. Thus

$$X' \equiv_e L'_{E_X}.$$

Finally, neither L_{E_X} nor R_{E_X} is c.e., for otherwise we would have $L_{E_X} \oplus R_{E_X} \leq_e \emptyset'$, contradicting $\emptyset' \not\leq_e X'$.

Thus $\{L_{E_X}, R_{E_X}\}$ is a nontrivial $\mathcal{K}$-pair, such that $L_{E_X} \leq_e X$ and $L'_{E_X} \equiv_e X'$. □

Theorem 5. *Every nonzero e-degree* $\mathbf{a}$ *bounds a nontrivial initial segment of e-degrees whose nonzero elements have all the same jump as* $\mathbf{a}$.

Proof. A direct application of Theorem 4 and Corollary 3. □

4. Applications to the local structure

Now we turn to some applications of Theorem 5 to the local theory of the e-degrees.

4.1. **The jump hierarchy.** Our first application shows that for every possible jump of a local e-degree, there is a nonempty left-open interval $(\mathbf{0}_e, \mathbf{a}]$ of e-degrees, all having that jump. Thus, every level of the high/low hierarchy of the local theory of the e-degrees contains a nonempty left-open interval of e-degrees. In particular, with the exception of the level L_1 of the high/low hierarchy (i.e., the level of the low e-degrees), every such left-open interval consists entirely of downwards properly Σ^0_2 e-degrees. (As a corollary of a stronger result, the fact that every non low e-degree bounds a downwards properly Σ^0_2 degree, was observed also by Harris [18].) More precisely we have the following:

Theorem 6. *Let* $\mathbf{x} \in \mathcal{G}_e$ *be a nonzero e-degree. Then there is an e-degree* $\mathbf{0}_e < \mathbf{a} \leq \mathbf{x}$*, such that, for every* $\mathbf{0}_e < \mathbf{c} \leq \mathbf{a}$*,* $\mathbf{c}' = \mathbf{x}'$*. Moreover, if* $\mathbf{x}$ *is not low, then* $\mathbf{a}$ *is downwards properly* Σ^0_2*.*

Proof. Fix an e-degree $\mathbf{x} \in \mathcal{G}_e$. The first part of the claim of the theorem is a paraphrase of Theorem 5. For the second part, recall that every nonzero Δ^0_2 e-degree bounds a nonzero low e-degree (see McEvoy and Cooper [24, Theorem 7]): so, if $\mathbf{x}$ is not low, then no nonzero $\mathbf{a} \leq \mathbf{x}$ can contain Δ^0_2 sets. □

We would like to emphasize the following somewhat surprising corollary:

Corollary 7. *There is a nontrivial initial segment of e-degrees whose nonzero elements are high e-degrees.*

Proof. Take $\mathbf{x}$ to be high. Then by Theorem 5 there is $\mathbf{a} \leq \mathbf{x}$, such that $(\mathbf{0}_e, \mathbf{a}]$ consists of high e-degrees. □

Recall that an e-degree $\mathbf{a}$ is said to be *nonsplitting*, if there is no pair of smaller e-degrees whose join is $\mathbf{a}$. Then the following results holds:

Corollary 8. *There is a high nonsplitting e-degree.*

Proof. This follows from Corollary 7, together with the fact that every nonzero Σ^0_2 e-degree bounds a nonzero nonsplitting e-degree (see Kent and Sorbi [21]). □

Notice that this gives an elementary difference between the high e-degrees and the high Turing degrees, for which it is known, that every high Turing degree is join-reducible, see Posner [25], in fact the join of two minimal degrees, see Ellison and Lewis [11].

4.2. **Capping and noncapping.** Recall that a pair $\{\mathbf{a}, \mathbf{b}\}$ of nonzero e-degrees in $\mathcal{G}_e$ is a *minimal pair* if the infimum $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}_e$. An e-degree $\mathbf{a}$ *caps* (or, is *capping*) if $\mathbf{a} = \mathbf{0}_e$, or there exists $\mathbf{b}$ such that $\{\mathbf{a}, \mathbf{b}\}$ is a minimal pair. Finally, a degree $\mathbf{x} \in \mathcal{G}_e$ is *noncappable* if it is not half of a minimal pair. The existence of incomplete (i.e., $\neq \mathbf{0}'_e$) noncappable e-degrees was proved by Cooper and Sorbi [9].

As another application of Theorem 6 we obtain the following property of noncappable e-degrees.

Corollary 9. *Let $\mathbf{x} \in \mathcal{G}_e$ be a noncappable degree. Then $\mathbf{x}$ bounds an e-degree of every possible local jump. In particular $\mathbf{x}$ is high.*

Proof. Let $\mathbf{x} \in \mathcal{G}_e$ be a noncappable e-degree. Fix any jump $\mathbf{b}'$ of an element $\mathbf{b} \in \mathcal{G}_e$, and by Theorem 6 let $\mathbf{a} \in \mathcal{G}_e$ be such that the interval $(\mathbf{0}_e, \mathbf{a}]$ consists entirely of e-degrees having jump $\mathbf{b}'$. Since $\mathbf{x}$ is noncappable, the pair $\{\mathbf{x}, \mathbf{a}\}$ is not minimal and hence there is a nonzero e-degree $\mathbf{c}$ such that $\mathbf{c} \leq \mathbf{a}$ and $\mathbf{c} \leq \mathbf{x}$. By the choice of $\mathbf{a}$, one has that $\mathbf{c}$ has the same jump as $\mathbf{b}$. □

In particular, we have the following corollary.

Corollary 10. *If $\mathbf{x} \in \mathcal{G}_e$ is not high, then $\mathbf{x}$ caps.*

Proof. Immediate, by contraposition. □

The inclusion of the noncappale e-degrees in the high e-degrees is a proper inclusion, since it is known that there exist capping high e-degrees, in fact minimal pairs of high e-degrees, see e.g. Sorbi, Wu and Yang [26], or diamonds formed by a low and a high e-degree, see Sorbi, Wu and Yang [27]. However this proper inclusion is also an immediate consequence of Theorem 6, as shown in the following corollary:

Corollary 11. *There exists a high $\mathbf{a}$ such that all $\mathbf{c} \in (\mathbf{0}_e, \mathbf{a}]$ are high and cappable.*

Proof. Consider a high $\mathbf{x}$, and let $\mathbf{a} \leq \mathbf{x}$ be as in Theorem 6. Then all nonzero $\mathbf{c} \leq \mathbf{a}$ are high. Now, if $\mathbf{0}_e < \mathbf{c} \leq \mathbf{a}$, then $\mathbf{c}$ caps with every $\mathbf{d}$ that is not high, as $\mathbf{c}$ and $\mathbf{d}$ can not bound a nonzero e-degree. □

For the following corollary, recall that an e-degree $\mathbf{a}$ is *noncuppable* if there is no incomplete e-degree $\mathbf{b}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$. The existence of nonzero noncuppable e-degrees was proved by Cooper, Sorbi and Yi [10].

Corollary 12. *Every noncappable e-degree bounds a noncuppable e-degree.*

Proof. This follows from Corollary 9, and the fact that every high e-degree bounds a noncuppable e-degree, see Giorgi, Sorbi and Yang [17]. □

4.3. **Bounding and nonbounding minimal pairs.** An e-degree $\mathbf{a}$ *bounds a minimal pair* if there exist $\mathbf{b}, \mathbf{c} \leq \mathbf{a}$ such that $\{\mathbf{b}, \mathbf{c}\}$ is a minimal pair: an e-degree is *bounding*, if it bounds a minimal pair. Bounding e-degrees have been studied in McEvoy and Cooper [24], and in Cooper, Li, Sorbi and Yang [8]: in this latter paper it is proved that every nonzero Δ^0_2 e-degree bounds a minimal pair.

Corollary 13. *In $\mathcal{G}_e$, if $\mathbf{a}' \not\leq \mathbf{b}'$ and $\mathbf{b} \neq \mathbf{0}_e$ then there exists $\mathbf{c} \leq \mathbf{a}$ such that $\mathbf{c}' = \mathbf{a}'$ and $\{\mathbf{b}, \mathbf{c}\}$ is a minimal pair.*

Proof. Given $\mathbf{a}, \mathbf{b}$ as in the statement of the corollary, by Theorem 6 let $\mathbf{0}_e < \mathbf{c} \leq \mathbf{a}$ be such that all the nonzero elements below $\mathbf{c}$ have the same jump as $\mathbf{a}$. Then $\{\mathbf{b}, \mathbf{c}\}$ is clearly a minimal pair, as $\{\mathbf{b}, \mathbf{c}\}$ can not bound a nonzero e-degree. □

The following particular case of the previous corollary can be seen as a generalization of a result in Badillo and Harris [2], stating that there is a nonzero low e-degree that caps with a nonzero low e-degree and a high e-degree:

Corollary 14. *For every possible jump of an element in $\mathcal{G}_e$, every low e-degree caps with some nonzero e-degree having that jump.*

Proof. Let $\mathbf{a} \in \mathcal{G}_e$, and let $\mathbf{b}$ be a low e-degree: we may assume that $\mathbf{b}$ is nonzero, as by definition $\mathbf{0}_e$ caps with every e-degree. We distinguish two cases: if $\mathbf{a}$ is low, then by Kalimullin [19, Theorem 2], $\mathbf{b}$ caps with a nonzero Δ^0_2 e-degree, and hence (as every nonzero Δ^0_2 e-degree bounds a nonzero low e-degree, see [24]), caps with a nonzero low e-degree, which has the same jump as $\mathbf{a}$. On the other hand, if $\mathbf{a}$ is not low, then $\mathbf{a}' \not\leq \mathbf{b}'$: by Corollary 13, we conclude that there is some $\mathbf{c}$ having the same jump as $\mathbf{a}$, and forming a minimal pair with $\mathbf{b}$. □

Corollary 15. *In $\mathcal{G}_e$, if a nonzero $\mathbf{a}$ is either low, or bounds an e-degree of strictly smaller jump, then $\mathbf{a}$ is bounding.*

Proof. Let $\mathbf{a}$ be nonzero. If $\mathbf{a}$ is low then we can use the fact that every nonzero low e-degree is bounding, see [8]. If $\mathbf{a}$ bounds an e-degree $\mathbf{b}$ such that $\mathbf{b}' < \mathbf{a}'$, then by Corollary 13 there is a $\mathbf{c} \leq \mathbf{a}$ such that $\{\mathbf{b}, \mathbf{c}\}$ form a minimal pair. □

An immediate consequence of Corollary 15 is the following extension of a theorem in McEvoy and Cooper [24], stating that if $\mathbf{a}$ is Δ^0_2 and high, then for every nonzero low $\mathbf{b} \leq \mathbf{a}$ there exists $\mathbf{c} \leq \mathbf{a}$ such that $\{\mathbf{b}, \mathbf{c}\}$ form a minimal pair.

Corollary 16. *In $\mathcal{G}_e$, if $\mathbf{a}$ is not low, then for every nonzero low $\mathbf{b} \leq \mathbf{a}$ there exists $\mathbf{c} \leq \mathbf{a}$ such that $\{\mathbf{b}, \mathbf{c}\}$ form a minimal pair.*

Our last corollary shows that the nonzero e-degrees below a nonbounding one have all the same jump: the existence of nonzero nonbounding e-degrees (i.e., e-degrees bounding no minimal pairs) was proved by Cooper, Li, Sorbi and Yang [8].

Corollary 17. *In $\mathcal{G}_e$, if $\mathbf{a}$ is nonbounding, then the left-open interval $(\mathbf{0}_e, \mathbf{a}]$ consists of elements having all the same jump.*

Proof. Suppose that $\mathbf{a}$ is a nonzero nonbounding e-degree, and there exists $\mathbf{0}_e < \mathbf{b} \leq \mathbf{a}$, such that $\mathbf{b}' \neq \mathbf{a}'$: then $\mathbf{a}' \not\leq \mathbf{b}'$. By Corollary 13, there exists $\mathbf{c} \leq \mathbf{a}$ such that $\{\mathbf{b}, \mathbf{c}\}$ is a minimal pair, so $\mathbf{a}$ is bounding, contrary to the assumptions. □

5. Conclusion

Kalimullin pairs have turned out to be a powerful tool for obtaining simple and elegant proofs of nontrivial strucural properties and definability results in the structure of the e-degrees. A clear evidence for this is the proof of the existence of a nontrivial initial segment whose nonzero elements are high e-degrees, given in Corollary 7. Although there are already several papers investigating $\mathcal{K}$-pairs, in our opinion, the full potential of the Kalimullin pairs has not yet been achieved. Indeed, the results obtained so far suggest that $\mathcal{K}$-pairs might be used in order to define in $\mathcal{G}_e$ the high e-degrees, the Δ^0_2 e-degrees and the images of the c.e. Turing degrees under the standard embedding of the Turing degrees in $\mathcal{D}_e$. The last class mentioned is the most likely to be definable in $\mathcal{G}_e$, since it is definable in the global structure of the e-degrees (Cai, Ganchev, Lempp, Miller and Soskova [3]). We strongly encourage the researchers interested in Turing and enumeration reducibility to study thoroughly Kalimullin pairs and to introduce them in their toolbox.

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