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Neighborhood Semantics for Modal Many-Valued Logics[☆]

Petr Cintula

Institute of Computer Science, Czech Academy of Sciences
Pod Vodárenskou věží 2, 182 07 Prague, Czech Republic

Carles Noguera[*]

Institute of Information Theory and Automation, Czech Academy of Sciences
Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic

Abstract

The majority of works on modal many-valued logics consider Kripke-style possible worlds frames as the principal semantics despite their well-known axiomatizability issues when considering non-Boolean accessibility relations. The present work explores a more general semantical picture, namely a many-valued version of the classical neighborhood semantics. We present it in two levels of generality. First, we work with modal languages containing only the two usual unary modalities, define neighborhood frames over algebras of the logic $\mathrm{FL_{ew}}$ with operators, and show their relation with the usual Kripke semantics (this is actually the highest level of generality where one can give a straightforward definition of the Kripke-style semantics). Second, we define generalized neighborhood frames for arbitrary modal languages over a given class of algebras for an arbitrary protoalgebraic logic and, assuming certain additional conditions, axiomatize the logic of all such frames (which generalizes the completeness theorem of the classical modal logic E with respect to classical neighborhood frames).

Keywords: mathematical fuzzy logic, modal fuzzy logics, neighborhood frames, Kripke semantics, many-valued logics
2010 MSC: 03B45, 03B52, 03G27, 03B47, 03B50

1. Introduction

The study of many-valued propositional logics expanded with modal operators was started by Melvin Fitting in [15, 16] and later continued by Petr Hájek and others in the field of Mathematical Fuzzy Logic [19, 8] resulting in an active field research (see e.g., [3, 4, 7, 6, 5, 20, 21, 23, 24, 29, 30]). In many of these works, since the initial propositional logic may lack an involutive negation, the extended modal system is endowed with two non-interdefinable modal operators, $\Box$ and $\Diamond$, or alternatively one may restrict to the fragment given by only one of these operators. Another peculiarity of the syntax of these systems is that, for technical reasons related to the proof of completeness already encountered in Fitting's seminal papers, it often includes truth-constants to denote each element of the intended algebraic semantics. On the other hand, modal fuzzy logics are typically endowed with a relational semantics that generalizes the classical Kripke semantics by allowing a many-valued scale for either (or for both) the truth-values of propositions at each possible world and for the degree of accessibility from one world to another. However,

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[*] Corresponding author
Email addresses: `cintula@cs.cas.cz` (Petr Cintula), `noguera@utia.cas.cz` (Carles Noguera)
URL: `www.cs.cas.cz/cintula` (Petr Cintula), `www.carlesnoguera.cat` (Carles Noguera)

despite its very natural definition, such semantics brings forth serious technical difficulties. Indeed, axiomatizing the Kripke-style semantics over a given algebra (or class of algebras) of truth-values can be in general a complex problem (for instance, no simple axiomatic presentation is known for modal extensions of product logic and one has to resort to the use of truth-constants, $\triangle$ projection, and infinitary rules [30], or for the modal logic over the standard finitary Łukasiewicz logic, which has been axiomatized with an infinitary rule [21]). Conversely, already in the classical case, proof systems with natural syntactic conditions may fail to be complete with any such Kripke-style semantics.

In modal extensions of classical logic, the Scott–Montague neighborhood semantics [25, 28] has been used as a more general framework than Kripke frames where, instead of using an accessibility relation, each world is mapped to a set of sets of worlds known as its *neighborhood*. It allows to prove completeness for non-normal modal logics, where the Kripke-style semantics would not work. For analogous reasons, recently some authors have started introducing some notions of neighborhood semantics for modal fuzzy logics. It has been studied in particular settings in [26, 27] and in a general framework of fuzzy logics extending MTL (the basic t-norm-based logic [14, 22]) in the conference paper [11].

The aim of this paper is to introduce neighborhood semantics for the widest possible class of modal many-valued logics (building on the partial results of [11]) to fulfill the following goals: (1) show the exact relation between the new neighborhood semantics and the usual Kripke-style semantics used so far in modal many-valued logics, (2) assume only the necessary conditions to obtain a semantics that naturally generalizes the classical Scott–Montague semantics and the previous particular proposals for a neighborhood semantics of modal fuzzy logics, and (3) obtain an axiomatization, and the corresponding completeness theorem, of the *global* consequence given by the neighborhood frames defined over an arbitrary class of algebras. Unlike in classical logic, there is no straightforward relationship between the global and the local consequence and, hence, the study of the latter is left for a future investigation.

To achieve the first goal it suffices to formulate our new notions in the usual framework of modal many-valued logics with Kripke frames, that is, modal extensions of logics with an algebraic counterpart composed by a class of (expansions of) bounded complete lattice-ordered residuated commutative integral monoids, that is, $\mathrm{FL_{ew}}$-algebras (possibly with operators). In this setting each frame, be it neighborhood or Kripke, is defined over a fixed algebra used as scale to measure both degrees of truth in each possible world and degrees of accessibility.[1] We show that, as in classical logic, Kripke frames correspond to a particular kind of neighborhood frames, namely, the *augmented* frames. Then, a natural question arises: how can one axiomatize the (global) logic of *all* neighborhood frames? We propose an axiomatization and obtain a corresponding completeness theorem for finitary expansions of the logic $\mathrm{FL_{ew}}$. However, in order to prove such a result, we move to a higher level of abstraction, capable of including possible future developments of modal non-classical logics with much more general algebraic semantics. To this end, we consider arbitrary classes of algebras, arbitrary sets of designated elements in these algebras, and arbitrary modalities of arbitrary arities in the language. In this general setting, neighborhood frames are allowed to use different algebras of truth-values in each world to evaluate propositions. We demonstrate that such level of abstraction not only does not add much conceptual difficulty, but it actually simplifies the presentation and reduces the proof of the completeness theorem to its essential components.

The paper is organized as follows. Section 2 recalls the usual algebraic framework for many-valued logics based on $\mathrm{FL_{ew}}$-algebras, introduces some useful notation for fuzzy sets evaluated on these algebras, recalls the Kripke-style semantics of modal many-valued logics and the classical Scott–Montague semantics. In Section 3 we introduce our neighborhood semantics for modal many-valued logics based on $\mathrm{FL_{ew}}$-algebras, we describe its relationship with the usual Kripke-style semantics, and formulate an axiomatization for the global consequence of all neighborhood frames based on a $\mathrm{FL_{ew}}$-algebra. Finally, Section 4 generalizes the neighborhood semantics to arbitrary classes of algebras and arbitrary modal languages, proposes a simple axiomatization and proves a completeness theorem for the global consequence relation that, in particular, entails the completeness results of the previous section.

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[1] For the sake of subsuming the previous works, we define the semantics for a language with both of the usual modal operators $\Box$ and $\Diamond$ (the description of the relationship between the neighborhood and the Kripke style semantics for a language with only one of these modalities can be easily obtained by restricting all the notions to the corresponding fragment of the language).

2. Preliminaries

2.1. $\mathrm{FL_{ew}}$*-algebras with operators and finitary expansions of* $\mathrm{FL_{ew}}$

We start by recalling a common algebraic and logical framework that covers most modal many-valued logics studied in the literature. We use the same notation for equivalent algebraic and logical notions (e.g., algebraic type = propositional language, operations = connectives, propositional atoms = object variables, terms = formulas). Our basic language, denoted as $\mathcal{L}_{\mathrm{FL_{ew}}}$, is that of the *Full Lambek logic with exchange and weakening* (see e.g., [18]), which contains binary connectives $\wedge$, $\vee$, $\&$, and $\to$, and two constants $\overline{0}$ and $\overline{1}$. Throughout the paper we make use of the following derived connectives: $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$ and $\neg\varphi = \varphi \to \overline{0}$. As mentioned in the introduction, we want to prove our results not only for logics/algebras in the language of $\mathrm{FL_{ew}}$, but also for those with greater expressive power. Therefore, we define:

Definition 1. *Let $\mathcal{L}$ be an algebraic type extending $\mathcal{L}_{\mathrm{FL_{ew}}}$. We say that an algebra $\boldsymbol{A}$ of type $\mathcal{L}$ with domain A is an* $\mathrm{FL_{ew}}$-algebra with operators, *(if $\mathcal{L} = \mathcal{L}_{\mathrm{FL_{ew}}}$ we drop the suffix 'with operators') whenever*

- $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ *is a bounded lattice*
- $\langle A, \&^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}} \rangle$ *is a commutative monoid*
- $\&^{\boldsymbol{A}}$ *and* $\to^{\boldsymbol{A}}$ *form a residuated pair, i.e.,* $a \;\&^{\boldsymbol{A}}\; b \leq c$ *iff* $a \leq b \to^{\boldsymbol{A}} c$*, for all* $a, b, c \in A$*, where* $\leq$ *is the induced lattice order.*

We say that an $\mathrm{FL_{ew}}$*-algebra with operators* $\boldsymbol{A}$ *is* complete *if its lattice reduct is a complete lattice, i.e.,* $\bigvee B$ *and* $\bigwedge B$ *exist in* A*, for each subset* $B \subseteq A$*.*

The two-element Boolean algebra can be seen as an $\mathrm{FL_{ew}}$-algebra: $\mathbf{2} = \langle \{0, 1\}, \wedge, \vee, \wedge, \to, 0, 1 \rangle$, where $\wedge$, $\vee$, and $\to$ are the usual Boolean operations. Other special cases are the so-called *t-algebras*, i.e., $\mathrm{FL_{ew}}$-algebras of the form $\langle [0, 1], \min, \max, *, \Rightarrow, 0, 1 \rangle$, where $*$ is a left-continuous t-norm and $\Rightarrow$ its residuum. Note that $\mathrm{FL_{ew}}$-algebras are also known under a systematic name: integral commutative bounded residuated lattices; let $\mathbb{FL}_{\mathrm{ew}}$ denote the class of *all* $\mathrm{FL_{ew}}$-algebras.

Next, we list a few simple and well-known properties of $\mathrm{FL_{ew}}$-algebras that we need throughout this paper.

Lemma 2. *The following properties hold in all* $\mathrm{FL_{ew}}$*-algebras:*

- $x \leq y \to z$ *iff* $y \leq x \to z$
- $\overline{0} \;\&\; x = \overline{0}$
- $x \to y = \overline{1}$ *iff* $x \leq y$
- $\overline{1} \to x = x.$

Let us now introduce the necessary logical notions. First, we fix a propositional language L extending $\mathcal{L}_{\mathrm{FL_{ew}}}$. We denote by $Fm_{\mathcal{L}}$ the set of formulas (terms) in $\mathcal{L}$ and by $\boldsymbol{Fm}_{\mathcal{L}}$ the absolutely free algebra of type $\mathcal{L}$. Given any class $\mathbb{K}$ of $\mathrm{FL_{ew}}$-algebras with operators of type $\mathcal{L}$ we define a structural consequence relation $\models_{\mathbb{K}}$ on $Fm_{\mathcal{L}}$ in the following way: if $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$,

$$\Gamma \models_{\mathbb{K}} \varphi \text{ iff for each } \boldsymbol{A} \in \mathbb{K} \text{ and each homomorphism } e\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{A} \text{ we have:}$$
$$\text{if } e[\Gamma] \subseteq \{\overline{1}^{\boldsymbol{A}}\}, \text{ then } e(\varphi) = \overline{1}^{\boldsymbol{A}}.$$

Then $\models_{\mathbb{FL}_{\mathrm{ew}}}$ is a finitary logic, i.e., a structural consequence relation on $Fm_{\mathcal{L}}$ such that if $\Gamma \models_{\mathbb{FL}_{\mathrm{ew}}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models_{\mathbb{FL}_{\mathrm{ew}}} \varphi$. Let us denote this logic as $\mathrm{FL_{ew}}$. It is well known that $\mathrm{FL_{ew}}$ is axiomatizable by several axioms and one deduction rule of *modus ponens* (from φ and $\varphi \to \psi$ infer ψ), see [18] for details.

Definition 3. *Let* $\mathcal{L}$ *be a propositional language extending* $\mathcal{L}_{\mathrm{FL_{ew}}}$*. A* finitary expansion of $\mathrm{FL_{ew}}$ *is any logic* L *axiomatizable by adding axioms and finitary rules to the axiomatic system of* $\mathrm{FL_{ew}}$ *such that for any additional n-ary connective* $c \in \mathcal{L} \setminus \mathcal{L}_{\mathrm{FL_{ew}}}$ *and formulas* $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n \in Fm_{\mathcal{L}}$ *we have:*

$$\varphi_1 \leftrightarrow \psi_1, \ldots, \varphi_n \leftrightarrow \psi_n \vdash_{\mathrm{L}} c(\varphi_1, \ldots, \varphi_n) \leftrightarrow c(\psi_1, \ldots, \psi_n).$$

We say that $\boldsymbol{A}$ *is an* L*-algebra if* $\vdash_{\mathrm{L}} \subseteq \models_{\{\boldsymbol{A}\}}$*, i.e., for each* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ *and each homomorphism* $e\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{A}$ *we have: if* $\Gamma \vdash_{\mathrm{L}} \varphi$ *and* $e[\Gamma] \subseteq \{\overline{1}^{\boldsymbol{A}}\}$*, then* $e(\varphi) = \overline{1}^{\boldsymbol{A}}$*. We denote by* $\mathbb{L}$ *the class of all* L*-algebras.*

The following theorem summarizes the well-known relationship between quasivarieties $\mathrm{FL_{ew}}$-algebras and finitary expansions of $\mathrm{FL_{ew}}$.

Theorem 4. *Let* $\mathbb{Q}$ *be a quasivariety of* $\mathrm{FL_{ew}}$*-algebras with operators and* L *a finitary expansion of* $\mathrm{FL_{ew}}$.

- $\mathrm{L} = \models_{\mathbb{L}}$ *and* $\mathbb{L}$ *is a quasivariety.*
- $\models_{\mathbb{Q}}$ *is a finitary expansion of* $\mathrm{FL_{ew}}$.

Actually, the connection is much stronger but the formulation above is sufficient for our needs.

2.2. *$\boldsymbol{A}$-valued sets and their notation*

The formulation of Kripke and neighborhood semantics for usual many-valued logics is obtained by substituting the two-element Boolean algebra by an $\mathrm{FL_{ew}}$-algebra with operators. To this end, we need to refer to many-valued sets of worlds and many-valued sets of many-valued sets of worlds. We introduce a convenient notation inspired by the syntax of fuzzy class theory (FCT), see e.g., in [1].

Given a complete $\mathrm{FL_{ew}}$-algebra with operators $\boldsymbol{A}$ and a non-empty set of worlds W, we use upper case letters ($X, Y, Z, \ldots$) to denote $\boldsymbol{A}$-valued sets of worlds (i.e., mappings $W \to A$ or elements of A^W) and calligraphic letters ($\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$) to denote the $\boldsymbol{A}$-valued sets of $\boldsymbol{A}$-valued sets of worlds (i.e., mappings $A^W \to A$ or elements of A^{A^W}).

Given an $\boldsymbol{A}$-valued set X we sometimes follow the usual set-theoretic notation and write $X = \{w \mid X(w)\}$ (and analogously with $\boldsymbol{A}$-valued sets of $\boldsymbol{A}$-valued sets). This notation is useful when X is described in a complex way; for example, consider an $\boldsymbol{A}$-valued binary relation R (i.e., a mapping $W \times W \to A$ or an element of $A^{W \times W}$) and define for any $w \in W$ the $\boldsymbol{A}$-valued set of worlds $R[w]$ to which each $v \in W$ belongs to the degree Rwv, in symbols:

$$R[w] = \{v \in W \mid Rwv\}.$$

We denote by $\{w\}$ the $\boldsymbol{A}$-valued set to which w belongs in degree $\overline{1}$ and all other worlds belong in degree $\overline{0}$. The next subsection contains more illustrations of this kind of definition.

We also use the set theoretic notation $w \in X$ (instead of $X(w)$) to denote the degree to which w belongs to X, and analogously for $X \in \mathcal{Y}$. This convention makes the following two crucial notions syntactically identical to their classical analogues:

$$X \subseteq Y = \bigwedge_{w \in W} (w \in X \to w \in Y) \qquad \text{degree of subsethood}$$

$$X \between Y = \bigvee_{w \in W} (w \in X \,\&\, w \in Y) \qquad \text{degree of overlap}$$

Note the above defined notions can be seen as functions assigning to each pair of $\boldsymbol{A}$-valued sets an element of $\boldsymbol{A}$. We conclude this subsection by a list of simple and well-known properties of $\boldsymbol{A}$-valued sets that we need throughout this paper.

Lemma 5. *For each* $w \in W$ *and* $\boldsymbol{A}$*-valued sets* X, Y *we have:*

- $X \subseteq X = \overline{1}$
- $X \subseteq Y = \overline{1}$ *and* $Y \subseteq X = \overline{1}$ *iff* $X = Y$
- $\{w\} \between X = w \in X$.

2.3. Kripke semantics for modal many-valued logics

Let us fix a propositional language $\mathcal{L}$, recall that we denote by $Fm_{\mathcal{L}}$ the corresponding set of formulas. We denote by $Fm_{\mathcal{L}}^{\Box,\Diamond}$ the set of formulas in the language $\mathcal{L}$ expanded with two unary modalities $\Box$ and $\Diamond$. Analogously we define the sets $Fm_{\mathcal{L}}^{\Box}$ and $Fm_{\mathcal{L}}^{\Diamond}$ when we consider only one modality.

Definition 6. *Let $\boldsymbol{A}$ be a complete* $\mathrm{FL_{ew}}$*-algebra with operators. An $\boldsymbol{A}$-*Kripke frame *(*$\mathrm{K}(\boldsymbol{A})$-frame*, for short) is a pair $\langle W, R\rangle$ such that W is a non-empty (classical) set of worlds while R is a binary $\boldsymbol{A}$-valued relation.*

*An $\boldsymbol{A}$-*Kripke model *(*$\mathrm{K}(\boldsymbol{A})$-model*, for short) is a triple $\mathcal{M} = \langle W, R, V\rangle$, where $\langle W, R\rangle$ is a* $\mathrm{K}(\boldsymbol{A})$*-frame and V is an evaluation $V\colon Var \to A^W$, i.e., a mapping assigning to each variable an $\boldsymbol{A}$-valued set to which each world belongs to the degree to which the given variable is true in that world. The evaluation is then extended to all formulas, i.e., it is extended to a mapping $V^{\mathcal{M}}\colon Fm_{\mathcal{L}}^{\Box,\Diamond} \to A^W$ inductively defined in the following way:*

$$
\begin{aligned}
V^{\mathcal{M}}(p) &= V(p) \\
V^{\mathcal{M}}(c(\varphi_1, \dots, \varphi_n)) &= \{w \mid c^{\boldsymbol{A}}(w \in V^{\mathcal{M}}(\varphi_1), \dots, w \in V^{\mathcal{M}}(\varphi_n))\} && \text{for any } n\text{-ary } c \in \mathcal{L} \\
V^{\mathcal{M}}(\Box\varphi) &= \{w \mid R[w] \subseteq V^{\mathcal{M}}(\varphi)\} \\
V^{\mathcal{M}}(\Diamond\varphi) &= \{w \mid R[w] \between V^{\mathcal{M}}(\varphi)\}.
\end{aligned}
$$

Note that, when $\boldsymbol{A} = \mathbf{2}$, this yields the classical definition of Kripke semantics. Let us now introduce the notions of validity and global consequence for many-valued Kripke semantics.

Definition 7. *Given a complete* $\mathrm{FL_{ew}}$*-algebra with operators $\boldsymbol{A}$ and a* $\mathrm{K}(\boldsymbol{A})$*-model $\mathcal{M} = \langle W, R, V\rangle$, a formula $\varphi \in Fm_{\mathcal{L}}^{\Box,\Diamond}$ is* valid in *$\mathcal{M}$, $\mathcal{M} \models \varphi$ in symbols, if $V^{\mathcal{M}}(\varphi)$ contains each world in degree $\overline{1}$.*

Let $\mathbb{F}$ be a class of Kripke frames (possibly over different algebras). For a subset $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}^{\Box,\Diamond}$, we say that φ is an $\mathbb{F}$-consequence *of Γ, $\Gamma \models_{\mathbb{F}} \varphi$ in symbols, if for each model $\mathcal{M}$ over any frame from $\mathbb{F}$:*

$$\text{if } \mathcal{M} \models \psi \text{ for each } \psi \in \Gamma, \text{ then also } \mathcal{M} \models \varphi.$$

Let us denote by $\mathrm{K}(\boldsymbol{A})$ the class of all $\mathrm{K}(\boldsymbol{A})$-frames. Note that $\models_{\mathrm{K}(\mathbf{2})}$ is the global variant of the classical modal logic K.

2.4. Classical neighborhood semantics

Introduced independently by Scott [28] and Montague [25], neighborhood semantics is a kind of possible worlds semantics for modal logics, similar in spirit to the well-known Kripke semantics, but resulting in a weaker logic. A good overview of these semantics can be found in [13].

A *neighborhood frame*, or shortly $\mathrm{SM}(\mathbf{2})$*-frame*, is a tuple $\mathfrak{M} = \langle W, N\rangle$, where W is a non-empty set of *worlds* while N is a function $N\colon W \to 2^{2^W}$ ($2 = \{0, 1\}$ denotes the domain of the two-element Boolean algebra $\mathbf{2}$) that assigns to each world w a set of subsets of W, called the *neighborhood* of w.

A *neighborhood model*, or shortly $\mathrm{SM}(\mathbf{2})$*-model*, is a triple $\mathfrak{M} = \langle W, N, V\rangle$, where $\langle W, N\rangle$ is an $\mathrm{SM}(\mathbf{2})$-frame and V is an *evaluation* $V\colon Var \to 2^W$ that is extended to all formulas similarly to the Kripke case, defining the value of formulas starting with modalities in the following way:

$$
\begin{aligned}
V^{\mathfrak{M}}(\Box\varphi) &= \{x \mid V^{\mathfrak{M}}(\varphi) \in N(x)\} \\
V^{\mathfrak{M}}(\Diamond\varphi) &= \{x \mid V^{\mathfrak{M}}(\neg\varphi) \notin N(x)\}.
\end{aligned}
$$

Observe that, thanks to the classical interdefinability of modalities, one neighborhood function is enough to define their semantics.

It is not hard to see that, given any $\mathrm{K}(\mathbf{2})$-model $\mathcal{M} = \langle W, R, V\rangle$, we obtain an $\mathrm{SM}(\mathbf{2})$-model $\mathfrak{M} = \langle W, N_R, V\rangle$ by setting for all $w \in W$,

$$N_R(w) = \{X \mid R[w] \subseteq X\},$$

and the truth values of formulas are preserved in all worlds.

Conversely, given any SM($\mathbf{2}$)-model $\mathfrak{M} = \langle W, N, V\rangle$, we can define a K($\mathbf{2}$)-model $\mathcal{M} = \langle W, R_N, V\rangle$ by setting for all $w, v \in W$,

$$R_N wv \qquad \text{iff} \qquad \text{for each } X \in N(w), \text{ we have } v \in X.$$

Note that this entails that $R_N[w] = \bigcap_{X \in N(w)} X$. However, in order to preserve the truth of all formulas in each world, we need the original SM-model $\mathfrak{M}$ to satisfy the following two additional conditions for each $w \in W$:

- $N(w)$ contains its core, i.e., the set $\bigcap_{X \in N(w)} X$,
- $N(w)$ is closed under taking supersets, i.e., if $X \in N(w)$ and $X \subseteq Y$, then $Y \in N(w)$.

In this case, $\mathfrak{M}$ (or more precisely, its underlying SM($\mathbf{2}$)-frame) is called *augmented*. Note that we could use the following equivalent definition: for each $w \in W$ there is a set C_w such that, for each $X \in N(w)$, $X \in N(w)$ iff $C_w \subseteq X$.

The following results about these translations can be found for example in [13].

Theorem 8.

(a) Let $\mathcal{M} = \langle W, R, V\rangle$ be a K($\mathbf{2}$)*-model. Then, $R_{N_R} = R$, $\mathfrak{M} = \langle W, N_R, V\rangle$ is an augmented* SM($\mathbf{2}$)*-model, and $V^{\mathcal{M}} = V^{\mathfrak{M}}$.*

(b) Let $\mathfrak{M} = \langle W, N, V\rangle$ be an augmented SM($\mathbf{2}$)*-model. Then, $N_{R_N} = N$, $\mathcal{M} = \langle W, R_N, V\rangle$ is a* K($\mathbf{2}$)*-model, and $V^{\mathfrak{M}} = V^{\mathcal{M}}$.*

Let us denote by SM($\mathbf{2}$) (or ASM($\mathbf{2}$) resp.) the class of all (resp. augmented) neighborhood frames. Validity and global consequence, w.r.t. a class of frames, is defined as in the Kripke case. Therefore, from the previous theorem, we know that $\models_{\mathrm{ASM}(\mathbf{2})}$ coincides with $\models_{\mathrm{K}(\mathbf{2})}$, i.e., the global variant of the logic given by augmented neighborhood frames is the classical global modal logic K.

Finally, let CL denote any Hilbert-style axiomatization of classical propositional logic in a language $\mathcal{L}$.

Theorem 9. *Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}^{\Box,\Diamond}$. Then, the following are equivalent*

- $\Gamma \models_{\mathrm{SM}} \varphi$
- *there is a proof of φ from Γ using axioms and rules of* CL *plus the rule E:*

$$\varphi \leftrightarrow \psi \vdash \Box\varphi \leftrightarrow \Box\psi.$$

3. Neighborhood semantics for modal many-valued logics

In this section we introduce a neighborhood semantics for modal many-valued logics and show its relationship with the Kripke-style semantics as a natural generalization to the many-valued setting of the constructions and the results seen in Theorem 8. A previous investigation in [26, 27] addresses the same problem in a more restricted framework, focusing on the relationship between models of the two kinds. In constrast, in this section we study the relation between frames and only later we add evaluations and obtain the desired result for models (Theorem 17).

Let us start with the definition of neighborhood frame where, unlike in the classical case, we need two neighborhood functions to take care of the two non-interdefinable modalities. If one is interested in the fragment with only one of these modalities, then the whole section should be read disregarding the notions for the excluded modality and all results would still hold.

Throughout this section we fix $\boldsymbol{A}$ to be a complete FL$_{\mathrm{ew}}$-algebra with operators.

Definition 10. *An* $\boldsymbol{A}$-neighborhood frame *(*SM($\boldsymbol{A}$)-frame*, for short) is a tuple $\langle W, N^{\Box}, N^{\Diamond}\rangle$ such that*

- *W is a non-empty (classical) set of worlds*
- *$N^{\Box}, N^{\Diamond} \colon W \to A^{A^W}$, i.e., functions that assigns to each world $w \in W$ an $\boldsymbol{A}$-valued set of $\boldsymbol{A}$-valued subsets of W.*

Furthermore, an $\boldsymbol{A}$-neighborhood model *(*$\mathrm{SM}(\boldsymbol{A})$-model*, for short) is a tuple* $\mathfrak{M} = \langle W, N^\Box, N^\Diamond, V\rangle$*, where* $\langle W, N^\Box, N^\Diamond\rangle$ *is an* $\mathrm{SM}(\boldsymbol{A})$*-frame and* V *is an* evaluation $V\colon Var \to A^W$ *that is extended to all formulas similarly to the Kripke case, defining the value of formulas starting with modalities in the following way:*

$$V^{\mathfrak{M}}(\Box\varphi) = \{w \mid V^{\mathfrak{M}}(\varphi) \in N^\Box(w)\}$$
$$V^{\mathfrak{M}}(\Diamond\varphi) = \{w \mid V^{\mathfrak{M}}(\varphi) \in N^\Diamond(w)\}.$$

Let us now introduce the notions of validity and global consequence for the neighborhood semantics.

Definition 11. *Given an* $\mathrm{SM}(\boldsymbol{A})$*-model* $\mathfrak{M} = \langle W, N^\Box, N^\Diamond, V\rangle$*, a formula* $\varphi \in Fm^{\Box,\Diamond}_{\mathcal{L}}$ *is* valid in $\mathfrak{M}$*,* $\mathfrak{M} \models \varphi$ *in symbols, if* $V^{\mathfrak{M}}(\varphi)$ *contains each world in degree* $\overline{1}$.

Let $\mathbb{F}$ *be a class of neighborhood frames (possibly over different algebras). For a subset* $\Gamma \cup \{\varphi\} \subseteq Fm^{\Box,\Diamond}_{\mathcal{L}}$*, we say that* φ *is an* SM-consequence *of* Γ*,* $\Gamma \models_{\mathbb{F}} \varphi$ *in symbols, if for each model* $\mathfrak{M}$ *over any frame from* $\mathbb{F}$*:*

$$\textit{if } \mathfrak{M} \models \psi \textit{ for each } \psi \in \Gamma\textit{, then also } \mathfrak{M} \models \varphi.$$

We denote by $\mathrm{SM}(\boldsymbol{A})$ *the class of all* $\mathrm{SM}(\boldsymbol{A})$*-frames.*

As in the classical case one could define as well a notion of local consequence for both the neighborhood and the Kripke-style semantics, but in this paper we keep the focus on the global consequence.

Our next goal is to demonstrate that the relationship between the neighborhood and the Kripke-style semantics is analogous to the classical case. In particular, we need a suitable notion of augmented frame, for which we use the following lemma.

Lemma 12. *Let* $\langle W, N^\Box, N^\Diamond\rangle$ *be an* $\mathrm{SM}(\boldsymbol{A})$*-frame,* $w \in W$ *a world, and* $C, C' \in A^W$ *such that one of the following two conditions holds for each* $X \in A^W$*:*

$$C \subseteq X = X \in N^\Box(w) = C' \subseteq X \tag{1}$$
$$C \between X = X \in N^\Diamond(w) = C' \between X. \tag{2}$$

Then, $C = C'$.

Proof. Assume the first condition. From $C \subseteq C = \overline{1}$, we obtain that $C \in N^\Box(w) = \overline{1}$ and, analogously, we get $C' \in N^\Box(w) = \overline{1}$. Thus also $C \subseteq C' = \overline{1}$ and $C' \subseteq C = \overline{1}$, i.e., $C = C'$.

Assume now the second condition. Then, for each $v \in W$, we have: $v \in C = C \between \{v\} = \{v\} \in N^\Diamond(w) = C' \between \{v\} = v \in C'$, and hence $C = C'$. ☐

This lemma allows to define the following notion of core of a frame.

Definition 13. *Given an* $\mathrm{SM}(\boldsymbol{A})$*-frame* $\langle W, N^\Box, N^\Diamond\rangle$ *is* augmented *if for each* $w \in W$ *there is (a unique)* $C_w \in A^W$ *such that for each* $X \in A^W$ *the following hold:*

$$C_w \subseteq X = X \in N^\Box(w)$$
$$C_w \between X = X \in N^\Diamond(w).$$

The set C_w *is called the* core *of* $N^\Box$ *and* $N^\Diamond$*. We denote by* $\mathrm{ASM}(\boldsymbol{A})$ *the class of all augmented* $\mathrm{SM}(\boldsymbol{A})$*-frames.*

Observe that we have just generalized the notion of augmented $\mathrm{SM}(\mathbf{2})$-frame seen in the previous section.

Now we are ready to define the general translations between both semantics. First, given $\mathrm{K}(\boldsymbol{A})$-frame $\langle W, R\rangle$, we define an $\mathrm{SM}(\boldsymbol{A})$-frame $\langle W, N^\Box_R, N^\Diamond_R\rangle$, where for each $w \in W$:

$$N^\Box_R(w) = \{X \in A^W \mid R[w] \subseteq X\}$$
$$N^\Diamond_R(w) = \{X \in A^W \mid R[w] \between X\}.$$

Conversely, given an $\mathrm{SM}(\boldsymbol{A})$-frame $\langle W, N^\Box, N^\Diamond\rangle$, we consider two, in principle different, ways to define the accessibility relation of the corresponding Kripke frame:

$$R_{N^\Box}wv = \bigwedge_{X\in A^W} (X \in N^\Box(w) \to v \in X)$$

$$R_{N^\diamond}wv = \bigwedge_{X\in A^W} (v \in X \to X \in N^\Diamond(w)).$$

We start by showing that in augmented frames both definitions coincide.

Lemma 14. *Let $\langle W, N^\Box, N^\Diamond\rangle$ be an augmented* $\mathrm{SM}(\boldsymbol{A})$*-frame. Then, for each $w \in W$, we have:*

$$C_w = R_{N^\Box}[w] = R_{N^\diamond}[w].$$

Therefore, $R_{N^\diamond} = R_{N^\diamond}$.

Proof. It suffices to check the following inequalities for arbitrary $w, v \in W$ (note that we use properties of $\mathrm{FL}_{\mathrm{ew}}$-algebras from Lemmas 2 and 5):

- $v \in C_w \le R_{N^\Box}wv$: From $X \in N^\Box(w) = C_w \subseteq X \le (v \in C_w \to v \in X)$, we obtain $v \in C_w \le X \in N^\Box(w) \to v \in X$.
- $v \in C_w \le R_{N^\diamond}wv$: From $v \in C_w \mathbin{\&} v \in X \le C_w \between X = X \in N^\Diamond(w)$, we obtain $v \in C_w \le v \in X \to X \in N^\Diamond(w)$.
- $R_{N^\Box}wv \le v \in C_w$: Clearly $R_{N^\Box}wv \le C_w \in N^\Box(w) \to v \in C_w = C_w \subseteq C_w \to v \in C_w = v \in C_w$.
- $R_{N^\diamond}wv \le v \in C_w$: Clearly $R_{N^\diamond}wv \le v \in \{v\} \to \{v\} \in N^\Diamond(w) = \{v\} \between C_w = v \in C_w$. ☐

Next, we show that the neighborhood frame built from a Kripke frame is always augmented and, moreover, when we apply both constructions consecutively we retrieve the original Kripke frame.

Lemma 15. *If $\langle W, R\rangle$ is a* $\mathrm{K}(\boldsymbol{A})$*-frame, then the* $\mathrm{SM}(\boldsymbol{A})$*-frame $\langle W, N_R^\Box, N_R^\Diamond\rangle$ is augmented and $R = R_{N_R^\Box} = R_{N_R^\diamond}$.*

Proof. $\langle W, N_R^\Box, N_R^\Diamond\rangle$ is augmented because for each world $w \in W$ we know, from the definition of $N_R^\Box(w)$ and $N_R^\Box(w)$, that we can take $C_w = R[w]$ as the core.

From the previous lemma we know that, for each $w \in W$, $C_w = R_{N^\Box}[w] = R_{N^\diamond}[w]$ and so the claim follows. ☐

Moreover, we can prove that the augmented property is both a sufficient and necessary condition in order for retrieving the original neighborhood frame when consecutively applying both constructions.

Lemma 16. *An* $\mathrm{SM}(\boldsymbol{A})$*-frame $\langle W, N^\Box, N^\Diamond\rangle$ is augmented iff $N^\Box_{R_{N^\Box}} = N^\Box$, $N^\Diamond_{R_{N^\Diamond}} = N^\Diamond$ and $R_{N^\Box} = R_{N^\diamond}$.*

Proof. For the left-to-right direction, for each $X \in A^W$, we check the following:

$$\begin{aligned} X \in N^\Box_{R_{N^\Box}} &= R_{N^\Box}[w] \subseteq X & \qquad X \in N^\Diamond_{R_{N^\Diamond}} &= R_{N^\diamond}[w] \between X \\ &= C_w \subseteq X & &= C_w \between X \\ &= X \in N^\Box(w) & &= X \in N^\Diamond(w). \end{aligned}$$

For the right-to-left direction, for each $w \in W$, we define the set $C_w = R_{N^\Box}[w] = R_{N^\diamond}[w]$ and show that it is the core of $N^\Box(w)$ and $N^\Diamond(w)$. Indeed, for each $X \in A^W$, we know that

$$X \in N^\Box(w) = X \in N^\Box_{R_{N^\Box}}(w) = R_{N^\Box}[w] \subseteq X = C_w \subseteq X,$$

$$X \in N^\Diamond(w) = X \in N^\Diamond_{R_{N^\Diamond}}(w) = R_{N^\diamond}[w] \between X = C_w \between X.$$

☐

After showing this tight connection between $\boldsymbol{A}$-Kripke frames and augmented $\boldsymbol{A}$-neighborhood frames, we can extend it to models.

Theorem 17. *Let* $\boldsymbol{A}$ *be a complete* $\mathrm{FL_{ew}}$*-algebra with operators.*

(a) Given a $\mathrm{K}(\boldsymbol{A})$*-model* $\mathcal{M} = \langle W, R, V\rangle$*, define the* $\mathrm{SM}(\boldsymbol{A})$*-model* $\mathfrak{M} = \langle W, N_R^\square, N_R^\diamond, V\rangle$*. Then,* $V^{\mathcal{M}} = V^{\mathfrak{M}}$*.*

(b) Given an $\mathrm{SM}(\boldsymbol{A})$*-model* $\mathfrak{M} = \langle W, N^\square, N^\diamond, V\rangle$ *over an augmented frame, define the* $\mathrm{K}(\boldsymbol{A})$*-model* $\mathcal{M} = \langle W, R_N, V\rangle$*. Then,* $V^{\mathfrak{M}} = V^{\mathcal{M}}$*.*

Proof. We proceed by induction over the complexity of a formula $\varphi \in Fm_{\mathcal{L}}^{\square,\diamond}$. For (a) and (b), the case where $\varphi \in Var$ or it is a constant follows by the definition of V, while the case where φ is not a formula starting with box or diamond follows trivially from the induction hypothesis (since only formulas starting with a modal operator depend on R or $N^\square$ and $N^\diamond$). Let $\varphi = \square\psi$ for some $\psi \in Fm_{\mathcal{L}}^{\square,\diamond}$ (for $\varphi = \diamond\psi$ it is analogous).

For (a), note that by the induction hypothesis, for any $w \in W$:

$$\begin{aligned} w \in V^{\mathcal{M}}(\square\psi) &= R[w] \subseteq V^{\mathcal{M}}(\psi) \\ &= R[w] \subseteq V^{\mathfrak{M}}(\psi) \\ &= V^{\mathfrak{M}}(\psi) \in N_R^\square(w) \\ &= w \in V^{\mathfrak{M}}(\square\psi)\,. \end{aligned}$$

For (b), using that the frame in $\mathfrak{M}$ is augmented and thus, for any $w \in W$, $R_{N^\square}[w]$ is the core of $N^\square(w)$ by Lemma 14, we have:

$$\begin{aligned} w \in V^{\mathfrak{M}}(\square\psi) &= V^{\mathfrak{M}}(\psi) \in N^\square(w) \\ &= C_w \subseteq V^{\mathfrak{M}}(\psi) \\ &= R_{N^\square}[w] \subseteq V^{\mathfrak{M}}(\psi) \\ &= R_{N^\square}[w] \subseteq V^{\mathcal{M}}(\psi) \\ &= w \in V^{\mathcal{M}}(\square\psi). \end{aligned}$$

$\square$

Therefore, we obtain that the logic given by the global consequence of Kripke frames coincides with that given by augmented neighborhood frames.

Corollary 18.

1. *Let* $\mathbb{F}$ *be a class of augmented neighborhood frames and let* $\mathbb{F}^{\mathrm{K}}$ *the class of their corresponding Kripke frames. Then, for each* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}^{\square,\diamond}$*,*

$$\Gamma \models_{\mathbb{F}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathbb{F}^{\mathrm{K}}} \varphi.$$

2. *Let* $\mathbb{F}$ *be a class of Kripke frames and let* $\mathbb{F}^{\mathrm{SM}}$ *the class of their corresponding augmented neighborhood frames. Then, for each* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}^{\square,\diamond}$*,*

$$\Gamma \models_{\mathbb{F}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathbb{F}^{\mathrm{SM}}} \varphi.$$

In particular, given a complete algebra $\boldsymbol{A}$*, for each* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}^{\square,\diamond}$*,*

$$\Gamma \models_{\mathrm{K}(\boldsymbol{A})} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathrm{ASM}(\boldsymbol{A})} \varphi.$$

It is easy to check that this corollary would also hold if we considered the local instead of the global consequence. However, the proofs of the results in the rest of the paper work only for the global case.

We have established that the Kripke semantics only can capture a part of the neighborhood semantics, namely that given by augmented frames. The next natural step is to investigate weaker modal many-valued logics given by bigger classes of neighborhood frames.

In the rest of this section we axiomatize the weakest logic in this setting, that is, the logic of all neighborhood frames, i.e., we want to formulate and prove an analog of Theorem 9. We have two possible formulations: (1) starting from a complete $\mathrm{FL_{ew}}$-algebra with operators, or (2) starting from a finitary expansion $\mathrm{FL_{ew}}$. The classical formulation was based on the fact that $\mathrm{CL} = \models_{\mathbf{2}}$. Accordingly, our two formulations will be determined by possible answers to the following two questions:

Q1 Given a complete $\mathrm{FL_{ew}}$-algebra $\boldsymbol{A}$ with operators, is there a finitary expansion L of $\mathrm{FL_{ew}}$ such that $\mathrm{L} = \models_{\boldsymbol{A}}$?

Q2 Given a finitary expansion L of $\mathrm{FL_{ew}}$, is there a complete $\mathrm{FL_{ew}}$-algebra $\boldsymbol{A}$ with operators such that $\mathrm{L} = \models_{\boldsymbol{A}}$?

Using Theorem 4 it is easy to see that the answer to the first question is YES, whenever $\models_{\boldsymbol{A}} = \models_{\mathbf{Q}(\boldsymbol{A})}$, where $\mathbf{Q}(\boldsymbol{A})$ is the quasivariety generated by $\boldsymbol{A}$ (because we know that $\models_{\mathbf{Q}(\boldsymbol{A})}$ is always a finitary expansion of $\mathrm{FL_{ew}}$). Interestingly enough, the equality always holds if we restrict to derivations from finite sets of premises i.e., for each *finite* set $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ we have:

$$\Gamma \models_{\boldsymbol{A}} \varphi \qquad \text{iff} \qquad \Gamma \models_{\mathbf{Q}(\boldsymbol{A})} \varphi.$$

Thus given a complete $\mathrm{FL_{ew}}$-algebra $\boldsymbol{A}$ with operators, let us denote by $\mathrm{L}_{\boldsymbol{A}}$ the logic $\models_{\mathbf{Q}(\boldsymbol{A})}$. Now we are ready to formulate the promised analogs of Theorem 9: the former is formulated semantics-first, the latter is logic-first. Both theorems will be obtained as corollaries of Theorem 25 which we prove in the next section in a much wider syntactical and semantical framework.

Theorem 19. *Let $\boldsymbol{A}$ be a complete* $\mathrm{FL_{ew}}$*-algebra with operators and* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}^{\Box,\Diamond}$ *be a* finite *set. Then, the following are equivalent:*

- $\Gamma \models_{\mathrm{SM}(\boldsymbol{A})} \varphi$
- *there is a proof of φ from Γ using axioms and rules of* $\mathrm{L}_{\boldsymbol{A}}$ *plus the following rules:*

$$\varphi \leftrightarrow \psi \vdash \Box\varphi \leftrightarrow \Box\psi$$

$$\varphi \leftrightarrow \psi \vdash \Diamond\varphi \leftrightarrow \Diamond\psi.$$

If furthermore $\mathrm{L}_{\boldsymbol{A}} = \models_{\boldsymbol{A}}$*, the equivalence holds for* all *sets of formulas.*

Theorem 20. *Let* L *be a finitary expansion of* $\mathrm{FL_{ew}}$ *and* $\boldsymbol{A} \in \mathbb{L}$ *such that* $\mathrm{L} = \mathrm{L}_{\boldsymbol{A}}$*. Then, the following are equivalent for each finite* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}^{\Box,\Diamond}$*:*

- $\Gamma \models_{\mathrm{SM}(\boldsymbol{A})} \varphi$
- *there is a proof of φ from Γ using axioms and rules of* L *plus the following rules:*

$$\varphi \leftrightarrow \psi \vdash \Box\varphi \leftrightarrow \Box\psi$$

$$\varphi \leftrightarrow \psi \vdash \Diamond\varphi \leftrightarrow \Diamond\psi.$$

If furthermore $\mathrm{L} = \models_{\boldsymbol{A}}$*, the equivalence holds for* all *sets of formulas.*

Observe that the classical Completeness Theorem 9 follows as a corollary when $\boldsymbol{A} = \mathbf{2}$.

4. An axiomatization of the global logic of neighborhood frames

The goal of this section is to prove the last two theorems of the previous section about the axiomatization of the global modal logic of all neighbourhood frames over a given $\mathrm{FL_{ew}}$-algebra with operators. Without much extra effort we can prove a more general result that entails the desired two theorems. In this way, we manage to cover a natural wider class of logics, arbitrary sets of modalities of arbitrary arity, and a more general notion of frame.

First, we need to recall a few notions of algebraic logic (see e.g., [17]). We no longer assume languages $\mathcal{L}$ to contain $\mathcal{L}_{\mathrm{FL_{ew}}}$. Recall that we denote by $Fm_{\mathcal{L}}$ the set of all formulas and by $\boldsymbol{Fm}_{\mathcal{L}}$ the absolutely free algebra of type $\mathcal{L}$. An *$\mathcal{L}$-matrix* is a tuple $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is algebra of type $\mathcal{L}$ and $F \subseteq A$ is called the *filter* of the matrix (a set of designated elements used to define logical consequence). Each matrix has the largest congruence compatible with F (i.e., such that no element from F is congruent with an element outside F); it is called the *Leibniz congruence*. A matrix is *reduced* if its Leibniz congruence is the identity. Given any class $\mathbb{K}$ of (reduced) $\mathcal{L}$-matrices we define a structural consequence relation $\models_{\mathbb{K}}$ on $Fm_{\mathcal{L}}$:

$$\begin{aligned}\Gamma \models_{\mathbb{K}} \varphi \text{ iff for each } \langle \boldsymbol{A}, F\rangle \in \mathbb{K} \text{ and each homomorphism } e\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{A} \text{ we have:}\\ \text{if } e[\Gamma] \subseteq F, \text{ then } e(\varphi) \in F.\end{aligned}$$

A *logic* L in a language $\mathcal{L}$ is a structural consequence relation on $Fm_{\mathcal{L}}$. We write $\Gamma \vdash_{\mathrm{L}} \varphi$ to signify that the formula φ follows from the set of formulas Γ in the logic L. We say that L is *finitary* if, whenever $\Gamma \vdash_{\mathrm{L}} \varphi$, there is a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{L}} \varphi$. For each logic L there is the largest class of reduced matrices, denoted as $\mathbf{MOD}^*(\mathrm{L})$, such that $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^*(\mathrm{L})}$. We say that L is *protoalgebraic* if there is a set of formulas $\Leftrightarrow$ (called an *equivalence*) in variables $p, q, r_1, r_2, \dots$, such that for each n-ary $c \in \mathcal{L}$:[2]

$$\vdash_{\mathrm{L}} \varphi \Leftrightarrow \varphi \qquad \varphi, \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} \psi \qquad \varphi \Leftrightarrow \psi, \psi \Leftrightarrow \chi \vdash_{\mathrm{L}} \varphi \Leftrightarrow \chi \qquad \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} \psi \Leftrightarrow \varphi$$

$$\varphi_1 \Leftrightarrow \psi_1, \dots, \varphi_n \Leftrightarrow \psi_n \vdash_{\mathrm{L}} c(\varphi_1, \dots, \varphi_n) \Leftrightarrow c(\psi_1, \dots, \psi_n).$$

For each $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$ and each $a, b \in A$, we have $a = b$ iff $\Leftrightarrow^{\boldsymbol{A}}(a, b) \subseteq F$.

It is easy to see that any (finitary) expansion L of $\mathrm{FL_{ew}}$ is protoalgebraic, $\leftrightarrow$ is the equivalence, and $\mathbf{MOD}^*(\mathrm{L}) = \{\langle \boldsymbol{A}, \overline{1}^{\boldsymbol{A}}\rangle \mid \boldsymbol{A} \in \mathbb{L}\}$.

In order to distinguish modalities from the remaining connectives, we start from a propositional language $\mathcal{L}$ of connectives that are not regarded as modalities and add a disjoint set Λ of modalities of arbitrary arities. The set of all formulas is denoted by $Fm_{\mathcal{L}}^{\Lambda}$ (which is actually the same as $\mathrm{Fm}_{\mathcal{L}\cup\Lambda}$, but keeping the intended distinction).

We work with a generalized notion of $\boldsymbol{A}$-valued set that allows us to define a more general notion of neighborhood frame using different algebras (from different matrices) to evaluate formulas at each world. To this end, instead of elements of $\boldsymbol{A}^W$ we consider elements of $\prod_{w\in W} \boldsymbol{A}_W$, where A_ws are $\mathcal{L}$-algebras. We call these objects $\langle \boldsymbol{A}_w\rangle_{w\in W}$-valued sets. As before we write $w \in X$ instead of $X(w)$ and use comprehension terms $\{w \mid w \in X\}$.

Definition 21. *Given a class $\mathbb{K}$ of $\mathcal{L}$-matrices and a set of modalities Λ, we define an* $\mathrm{SM}(\mathbb{K}, \Lambda)$-frame *as a tuple $\langle W, \langle \mathbf{A}_w\rangle_{w\in W}, \langle N^{\heartsuit}\rangle_{\heartsuit\in\Lambda}\rangle$ such that*

- *$W \neq \emptyset$ (worlds)*
- *$\mathbf{A}_w = \langle \boldsymbol{A}_w, F_w\rangle \in \mathbb{K}$ for each $w \in W$ (scales)*
- *for each n-ary $\heartsuit \in \Lambda$, $N^{\heartsuit}$ is a neighborhood function assigning to each world w an A_w-valued set of n-tuples of $\langle \boldsymbol{A}_w\rangle_{w\in W}$-valued sets, in symbols: $N^{\heartsuit}(w)\colon (\prod\limits_{v\in W} A_v)^n \to A_w$.*

Furthermore, we define an $\mathrm{SM}(\mathbb{K}, \Lambda)$-model *as a tuple $\mathfrak{M} = \langle W, \langle \boldsymbol{A}_w\rangle_{w\in W}, \langle N^{\heartsuit}\rangle_{\heartsuit\in\Lambda}, V\rangle$, where $\langle W, \langle \boldsymbol{A}_w\rangle_{w\in W}, \langle N^{\heartsuit}\rangle_{\heartsuit\in\Lambda}\rangle$ is an* $\mathrm{SM}(\mathbb{K}, \Lambda)$*-frame and $V\colon Var \to \prod\limits_{w\in W} A_w$ (evaluation), i.e., a mapping assigning to each variable an $\langle \boldsymbol{A}_w\rangle_{w\in W}$-valued set to which each world belongs to the degree to which the given variable is true in that world. The evaluation is extended to all formulas, i.e., it is extended to a mapping $V^{\mathfrak{M}}\colon Var \to \prod\limits_{w\in W} A_w$ inductively defined in the following way:*

$$\begin{aligned}
V^{\mathfrak{M}}(p) &= V(p) \\
V^{\mathfrak{M}}(c(\varphi_1, \dots, \varphi_n) &= \{w \mid c^{\boldsymbol{A}}(w \in V^{\mathcal{M}}(\varphi_1), \dots, w \in V^{\mathcal{M}}(\varphi_n))\} && \textit{for n-ary } c \in \mathcal{L} \\
V^{\mathfrak{M}}(\heartsuit(\varphi_1, \dots, \varphi_n)) &= \{w \mid \langle V^{\mathfrak{M}}(\varphi_1), \dots V^{\mathfrak{M}}(\varphi_n)\rangle \in N^{\heartsuit}(w)\} && \textit{for n-ary } \heartsuit \in \Lambda.
\end{aligned}$$

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[2] We write $\Gamma \vdash_{\mathrm{L}} \Delta$ if $\Gamma \vdash_{\mathrm{L}} \psi$ for each $\psi \in \Delta$. Also we define $\varphi \Leftrightarrow \psi = \{\chi(\varphi, \psi, \chi_1, \dots, \chi_n) \mid \chi(p, q, r_1, \dots, r_n) \in \Leftrightarrow$ and $\chi_1, \dots, \chi_n \in Fm_{\mathcal{L}}\}$.

This semantics gives rise to its corresponding notion of global consequence, in which we need to refer to filters of the matrices to represent truth in each world.

Definition 22. *Given an* SM($\mathbb{K}, \Lambda$)*-model* $\mathfrak{M} = \langle W, \langle\langle \boldsymbol{A}_w, F_w\rangle\rangle_{w\in W}, \langle N^\heartsuit\rangle_{\heartsuit\in\Lambda}, V\rangle$*, a formula* $\varphi \in Fm^\Lambda_\mathcal{L}$ *is* valid *in* $\mathfrak{M}$*,* $\mathfrak{M} \models \varphi$ *in symbols, if* $V^\mathfrak{M}(\varphi)(w) \in F_w$ *for each* $w \in W$*. Let* $\mathbb{F}$ *be a class of* SM($\mathbb{K}, \Lambda$)*-frames. For a subset* $\Gamma \cup \{\varphi\} \subseteq Fm^\Lambda_\mathcal{L}$*, we say that* φ *is an* SM-consequence *of* Γ*,* $\Gamma \models_\mathbb{F} \varphi$ *in symbols, if for each model* $\mathfrak{M}$ *over any frame from* $\mathbb{F}$*:*

$$\textit{if } \mathfrak{M} \models \psi \textit{ for each } \psi \in \Gamma, \textit{ then also } \mathfrak{M} \models \varphi.$$

To fulfill our aim of describing syntactically the logic given all the neighborhood frames over a given class of matrices, we introduce the following simple axiomatization for the expansion of an arbitrary protoalgebraic logic with arbitrary modalities requiring only that they preserve the congruence property with respect to $\Leftrightarrow$. This axiomatization generalizes that shown in Theorem 9 for the expansion of classical logic with $\Box$ and $\Diamond$.

Definition 23. *Let* L *be a protoalgebraic logic in a language* $\mathcal{L}$ *and let* Λ *be a disjoint language (modalities). We define* L_Λ *as the expansion of* L *obtained by adding the following rule for each* $\heartsuit \in \Lambda$*:*

$$(E^\heartsuit) \qquad \varphi_1 \Leftrightarrow \psi_1, \dots, \varphi_n \Leftrightarrow \psi_n \vdash \heartsuit(\varphi_1, \dots, \varphi_n) \Leftrightarrow \heartsuit(\psi_1, \dots, \psi_n).$$

Observe that the expanded logic remains protoalgebraic with the same equivalence set $\Leftrightarrow$. Moreover, this logic always enjoys completeness with respect to a semantics of neighborhood frames, in a rather trivial way, if we consider frames with only one world over any reduced model of the initial logic.

Proposition 24. *Let* L *be a protoalgebraic logic in a language* $\mathcal{L}$ *and let* Λ *be a disjoint language (modalities). Then, for each* $\Gamma \cup \{\varphi\} \subseteq Fm^\Lambda_\mathcal{L}$*, we have:*

$$\Gamma \vdash_{\mathrm{L}_\Lambda} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathrm{SM}(\mathbf{MOD}^*(\mathrm{L}),\Lambda)} \varphi.$$

The same result holds when restricting the semantics to frames with only one world.

Proof. For the soundess, we only need to check the validity of the rules $E^\heartsuit$. Let us assume that, for an SM($\mathbf{MOD}^*$(L), Λ)-model $\mathfrak{M}$ and formulas $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_n \in Fm^\Lambda_\mathcal{L}$, we have $\mathfrak{M} \models \varphi_i \Leftrightarrow \psi_i$ for each i. Then, $V^\mathfrak{M}(\varphi_i) = V^\mathfrak{M}(\psi_i)$ for each i and hence, for each $w \in W$, we have: $\langle V^\mathfrak{M}(\varphi_1), \dots, V^\mathfrak{M}(\varphi_n)\rangle \in N^\heartsuit(w) = \langle V^\mathfrak{M}(\psi_1), \dots, V^\mathfrak{M}(\psi_n)\rangle \in N^\heartsuit(w)$. Therefore, $\mathfrak{M} \models \heartsuit(\varphi_1, \dots, \varphi_n) \Leftrightarrow \heartsuit(\psi_1, \dots, \psi_n)$, as we wanted.

To prove completeness, assume that $\Gamma \nvdash_{\mathrm{L}_\Lambda} \varphi$. Since we can see L_Λ as a protoalgebraic logic in the language $\mathcal{L} \cup \Lambda$, we know that there exist $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L}_\Lambda)$ and an $\boldsymbol{A}$-evaluation e such that $e[\Gamma] \subseteq F$ and $e(\varphi) \notin F$. We define the following SM($\mathbf{MOD}^*$(L), Λ)-model: $\mathfrak{M} = \langle \{w\}, \langle \boldsymbol{A}, F\rangle, \langle N^\heartsuit\rangle_{\heartsuit\in\Lambda}, V\rangle$, where

- $N^\heartsuit(w)\colon \langle \{a_1\}, \dots, \{a_n\}\rangle \mapsto \heartsuit^{\boldsymbol{A}}(a_1, \dots, a_n)$
- $V(p) = \{e(p)\}$.

It is easy to see that for each $\psi \in Fm^\Lambda_\mathcal{L}$, we have $V^\mathfrak{M}(\psi) = \{e(\psi)\}$. Thus $\mathfrak{M} \models \psi$ for each $\psi \in \Gamma$, while $\mathfrak{M} \not\models \varphi$. $\square$

A more interesting question is whether one can restrict the completeness to a more meaningful class of neighborhood frames based on a family of matrices that already provides a complete semantics for the initial logic. This is achieved in the following theorem. The completeness properties of the starting logic are typically found in the literature in at least two different versions, namely, given a logic L and a class of models $\mathbb{K} \subseteq \mathbf{MOD}^*$(L), we say that L has the property of:

- *Strong* $\mathbb{K}$*-completeness*, S$\mathbb{K}$C for short, if L and $\models_\mathbb{K}$ coincide, i.e, for every set of formulas $\Gamma \cup \{\varphi\}$: $\Gamma \vdash_\mathrm{L} \varphi$ if, and only if, $\Gamma \models_\mathbb{K} \varphi$.
- *Finite strong* $\mathbb{K}$*-completeness*, FS$\mathbb{K}$C for short, if finitary companions of L and $\models_\mathbb{K}$ coincide, i.e., when for every *finite* set of formulas $\Gamma \cup \{\varphi\}$: $\Gamma \vdash_\mathrm{L} \varphi$ if, and only if, $\Gamma \models_\mathbb{K} \varphi$.

Theorem 25. *Let* L *be a finitary protoalgebraic logic in a countable language* $\mathcal{L}$*, let* Λ *be a countable language (modalities), and* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$*. Then:*

1. *If* L *has the* S$\mathbb{K}$C*, then for each* $\Gamma \cup \{\varphi\} \subseteq Fm^{\Lambda}_{\mathcal{L}}$ *we have:*
$$\Gamma \vdash_{\mathrm{L}_\Lambda} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathrm{SM}(\mathbb{K},\Lambda)} \varphi.$$
2. *If* L *has the* FS$\mathbb{K}$C *and* $\mathcal{L}$ *and* Λ *are finite, then, for each finite* $\Gamma \cup \{\varphi\} \subseteq Fm^{\Lambda}_{\mathcal{L}}$*, we have:*
$$\Gamma \vdash_{\mathrm{L}_\Lambda} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathrm{SM}(\mathbb{K},\Lambda)} \varphi.$$

Proof. The left-to-right directions follow from Proposition 24.

For the reverse implication in the finite strong completeness case assume that, for a finite set $\Gamma \cup \{\varphi\} \subseteq Fm^{\Lambda}_{\mathcal{L}}$, we have $\Gamma \nvdash_{\mathrm{L}_\Lambda} \varphi$. Since L_Λ is a protoalgebraic logic, we know that there exist $\langle \boldsymbol{B}, F\rangle \in \mathbf{MOD}^*(\mathrm{L}_\Lambda)$ and an evaluation $e\colon \boldsymbol{Fm}^{\Lambda}_{\mathcal{L}} \to \boldsymbol{B}$ such that $e[\Gamma] \subseteq F$ and $e(\varphi) \notin F$. Taking the restriction $\boldsymbol{B}{\restriction}\mathcal{L}$ of the algebra to the original language $\mathcal{L}$ without the modalities and factorizing by the Leibniz congruence, we obtain the reduced model $\langle \boldsymbol{B}{\restriction}\mathcal{L}, F\rangle^* \in \mathbf{MOD}^*(\mathrm{L})$; let π be the projection to such reduction. Since L is finitary, $\langle \boldsymbol{B}{\restriction}\mathcal{L}, F\rangle^*$ is representable as the subdirect product of a family of relatively subdirectly irreducible models $\{\langle \boldsymbol{B}_w, G_w\rangle \mid w \in W\} \subseteq \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ (see e.g., [12, Theorem 1.3.5]); we denote by π_w the projection to the component indexed by w.

Let S be the finite set of the subformulas of $\Gamma \cup \{\varphi\}$. Therefore, for each $w \in W$, the set $(\pi_w \circ \pi \circ e)[S] \subseteq B_w$ is also finite. Since we are assuming that the language $\mathcal{L}$ is finite and L has the FS$\mathbb{K}$C, by [10, Theorem 6], for each $w \in W$ we have a partial embedding $g_w\colon (\pi_w \circ \pi \circ e)[S] \to A_w$ for some $\langle \boldsymbol{A}_w, F_w\rangle \in \mathbb{K}$. For each $w \in W$, we take an arbitrary $\boldsymbol{A}_w$-evaluation e_w such that $e_w(\psi) = (g_w \circ \pi_w \circ \pi \circ e)(\psi)$ for each $\psi \in S$.

Now we are ready to define the needed SM$(\mathbb{K}, \Lambda)$-model: $\mathfrak{M} = \langle W, \langle \boldsymbol{A}_w\rangle_{w\in W}, \langle N^{\heartsuit}\rangle_{\heartsuit\in\Lambda}, V\rangle$, where $V(p) = \{w \mid e_w(p)\}$ and

$$\langle X_1, \dots, X_n\rangle \in N^{\heartsuit}(v) = \begin{cases} e_v(\heartsuit(\psi_1, \dots, \psi_n)) & \text{if there are } \psi_1, \dots, \psi_n \in S \\ & \text{such that for each } i \leq n,\ X_i = \{w \mid e_w(\psi_i)\} \\ b_w \in A_w \setminus F_w & \text{otherwise.} \end{cases}$$

Then, one can prove, by induction on the complexity of the formula, that for each $\psi \in S$ we have $V^{\mathfrak{M}}(\psi) = \{w \mid e_w(\psi)\}$. Therefore, $\mathfrak{M}$ is a model of Γ; indeed for each $\psi \in \Gamma$ we have $e(\psi) \in F$ and so for each $w \in W$: $w \in V^{\mathfrak{M}}(\psi) = e_w(\psi) = (g_w \circ \pi_w \circ \pi \circ e)(\psi)$, which is a value in F_w. But $\mathfrak{M}$ is not a model of φ; indeed $e(\varphi) \notin F$, so there has to be a $w \in W$ such that $(\pi_w \circ \pi \circ e)(\varphi) \notin G_w$ and hence $w \in V^{\mathfrak{M}}(\varphi) = e_w(\varphi) = (g_w \circ \pi_w \circ e)(\varphi)$, which is not a value in F_w.

The proof of the reverse implication in the case of strong completeness is similar and a bit simpler. Since the language is countable we can start from a countable $\langle \boldsymbol{B}, F\rangle \in \mathbf{MOD}^*(\mathrm{L}_\Lambda)$ and, reasoning as before, obtain countable models $\{\langle \boldsymbol{B}_w, G_w\rangle \mid w \in W\} \subseteq \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$. Since L has the S$\mathbb{K}$C we obtain that, by [10, Corollary 4], for each $w \in W$ there is an embedding $g_w\colon \langle \boldsymbol{B}_w, G_w\rangle \to \langle \boldsymbol{A}_w, F_w\rangle$ for some $\langle \boldsymbol{A}_w, F_w\rangle \in \mathbb{K}$. For each $w \in W$, we take the $\boldsymbol{A}_w$-evaluation $e_w = g_w \circ \pi_w \circ \pi \circ e$ and define as before an SM$(\mathbb{K}, \Lambda)$-model: $\mathfrak{M} = \langle W, \langle \boldsymbol{A}_w\rangle_{w\in W}, \langle N^{\heartsuit}\rangle_{\heartsuit\in\Lambda}, V\rangle$, where $V(p) = \{w \mid e_w(p)\}$ and

$$\langle X_1, \dots, X_n\rangle \in N^{\heartsuit}(v) = \begin{cases} e_v(\heartsuit(\psi_1, \dots, \psi_n)) & \text{if there are } \psi_1, \dots, \psi_n \in Fm^{\Lambda}_{\mathcal{L}} \\ & \text{such that for each } i \leq n\colon X_i = \{w \mid e_w(\psi_i)\} \\ b_w \in A_w \setminus F_w & \text{otherwise.} \end{cases}$$

Similarly to the previous case, the proof is concluded by showing that for each $\psi \in Fm^{\Lambda}_{\mathcal{L}}$ we have $V^{\mathfrak{M}}(\psi) = \{w \mid e_w(\psi)\}$ and $\mathfrak{M}$ is a model of Γ but not of φ. □

Theorems 19 and 20 are a corollary of the previous theorem. Indeed, given any complete FL$_{\mathrm{ew}}$-algebra with operators $\boldsymbol{A}$, the logic $\mathrm{L}_{\boldsymbol{A}}$ is finitary and protoalgebraic with equivalence $\leftrightarrow$ in a countable language and, thus, we can apply the theorem with $\mathbb{K} = \{\langle \boldsymbol{A}, \{\overline{1}^{\boldsymbol{A}}\}\rangle\}$ and $\Lambda = \{\Box, \Diamond\}$. In particular, we have obtained an alternative algebraic proof of the classical completeness result (Theorem 9).

5. Conclusion and further work

In this paper we have studied neighborhood semantics for modal many-valued logics. More precisely, we have

- defined it for a very wide class of logics given algebras and matrices,
- described its relation with the Kripke-style semantics,
- axiomatized global consequence relations (w.r.t. all models).

With this proposal, in particular, we have further expanded the realm of fuzzy logics, understood as the logics of chains [2]. A previous proposal in [9] introduced semilinear logics (that is, logics strongly complete w.r.t. linearly ordered matrices) as an attempt to capture this intuition in a mathematical definition. In the present paper we have dealt with modal logics that are not semilinear in that sense, but yet, when built upon a fuzzy logic, they enjoy a neighborhood semantics where in each world truth is evaluated over a chain of truth values.

Future work will focus mainly on other elements of the usual agenda of modal logics: axiomatizing global consequence relations w.r.t. classes of models (i.e., extensions with modal axioms), studying the local consequence relation, canonical models, solving related decidability and complexity issues, etc.

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