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Abstract

In this paper we study Weak Nilpotent Minimum t-norms and their associated algebraic structures, the standard WNM-chains. We classify all the varieties generated by one standard WNM-chain, obtaining all the axiomatic extensions of WNM logic that are complete with respect to the semantics given by a left-continuous t-norm. To this end, we define a set of canonical standard WNM-chains and we prove that they generate pairwise different varieties and there are no other varieties generated by a standard WNM-chain.

Keywords: Fuzzy logics, Left-continuous t-norms, Nilpotent Minimum Logic, MTL-algebras, NM-algebras, Residuated lattices, Varieties, Weak Nilpotent Minimum Logic, Non-classical logics, WNM-algebras.

1 Introduction

In the foundational paper of Fuzzy Sets [7], Zadeh uses the t-norm of the minimum to deal with intersection. Fodor defined in [3] a left-continuous but non-continuous t-norm $*_{NM}$ that he called *Nilpotent Minimum t-norm* (combining minimum with an involutive negation) defined by:

$$a *_{NM} b = \begin{cases} \min\{a, b\} & \text{if } a > 1 - b, \\ 0 & \text{otherwise.} \end{cases}$$

The associated negation function is $n(a) = 1 - a$, i.e. the standard involutive negation. Esteva and Godo generalized Fodor's construction in [2] using any (possibly non-involutive) weak negation function to define what they called *Weak Nilpotent Minimum t-norms*. Namely, given a weak negation function n, they defined the corresponding t-norm $*_n$ as:

$$a *_n b = \begin{cases} \min\{a, b\} & \text{if } a > n(b), \\ 0 & \text{otherwise.} \end{cases}$$

for every $a, b \in [0, 1]$.

This is a generalization of both Zadeh's and Fodor's approaches by allowing any negation function. Moreover, in the same paper new fuzzy logics, Nilpotent Minimum logic and Weak Nilpotent Minimum logic (NM and WNM respectively, for short) were defined in order to capture the semantics given by Fodor's t-norm and by all Weak Nilpotent Minimum t-norms respectively. They also proposed an algebraic semantics for those logics based on the so-called NM-algebras [1] and WNM-algebras, particular kinds of residuated lattices. These classes of algebras contain, of course, those defined by a t-norm, which are called *standard NM-algebras* and *standard WNM-algebras* respectively.

There is up to isomorphism only one standard NM-algebra, the one defined by Fodor's t-norm, but this is not the case of standard WNM-algebras. Indeed there are infinitely-many non-isomorphic WNM-algebras given by Weak Nilpotent t-norms. In this paper we aim to classify them attending to the logic that they define or equivalently to the variety of WNM-algebras that they generate. We begin with some necessary algebraic and logical preliminaries. Then we define several kinds of weak negation functions obtaining thus several families of standard WNM-algebras;

[1]The lattice of varieties of NM-algebras is fully described in [4].

we consider the varieties of WNM-algebras generated by them. This yields a complete classification of all varieties generated by a standard WNM-algebra. Due to the lack of space the results are presented here without proofs. The detailed version will be available in a forthcoming paper.

2 Preliminaries

WNM is the logic introduced by Esteva and Godo in [2] by means of a Hilbert style calculus in the language $\mathcal{L} = \{*, \to, \wedge, 0\}$ of type $(2, 2, 2, 0)$, where the only inference rule is Modus Ponens and the axiom schemata are the following (taking $\to$ as the least binding connective):

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi * \psi \to \varphi$
(A3)	$\varphi * \psi \to \psi * \varphi$
(A4)	$\varphi \wedge \psi \to \varphi$
(A5)	$\varphi \wedge \psi \to \psi \wedge \varphi$
(A6)	$\varphi * (\varphi \to \psi) \to \varphi \wedge \psi$
(A7a)	$(\varphi \to (\psi \to \chi)) \to (\varphi * \psi \to \chi)$
(A7b)	$(\varphi * \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9)	$0 \to \varphi$
(A10)	$\neg(\varphi * \psi) \vee (\varphi \wedge \psi \to \varphi * \psi)$

being $\neg$ and $\vee$ the following defined connectives:

$\neg\varphi := \varphi \to 0;$
$\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi).$

We denote the set of $\mathcal{L}$-formulas built over a countable set of variables by $Fm_{\mathcal{L}}$. Given $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ we write $\Gamma \vdash_{WNM} \varphi$ if, and only if, φ is provable from Γ in the system WNM.

NM logic is the axiomatic extension of WNM by adding the axiom $\neg\neg\varphi \to \varphi$ (involution).

As is proved in [2], algebraic semantics for those logics are given by the classes of WNM-algebras and NM-algebras.

Definition 1 ([2]). *Let $\mathcal{A} = \langle A, *, \to, \wedge, \vee, 0, 1\rangle$ be an algebra of type $(2, 2, 2, 2, 0, 0)$. We define a unary operation by $\neg a := a \to 0$. Then, $\mathcal{A}$ is a WNM-algebra if, and only if, it is a bounded residuated lattice satisfying the following equations:*

$$(x \to y) \vee (y \to x) \approx 1,$$
$$\neg(x * y) \vee (x \wedge y \to x * y) \approx 1.$$

$\mathcal{A}$ is a NM-algebra if, and only if, in addition satisfies the equation of involution: $\neg\neg x \approx x$.

An element $a \in A$ is positive (resp. negative) w.r.t. lattice order if $a > \neg a$ (resp. $a \leq \neg a$). We will say that $\mathcal{A}$ is a WNM-chain (resp. NM-chain) if the lattice order is total, and we will say that the chain is standard if $A = [0, 1]$.

We will denote by $\mathbb{WNM}$ and $\mathbb{NM}$ the classes of WNM-algebras and NM-algebras, respectively. It can proved that both classes are varieties and, of course, $\mathbb{NM} \subseteq \mathbb{WNM}$.

Recall from [2] that the operation $*$ in standard WNM-chains is given by a special kind of left-continuous t-norm. These t-norms are defined in the following way. If n is a negation function (see [1]) and $a, b \in [0, 1]$, the operation $*_n$ is defined as:

$$a *_n b = \begin{cases} \min\{a, b\} & \text{if } a > n(b), \\ 0 & \text{otherwise.} \end{cases}$$

$*_n$ is a left-continuous t-norm and its residuum is given by:

$$a \to_n b = \begin{cases} 1 & \text{if } a \leq b, \\ \max\{n(a), b\} & \text{otherwise.} \end{cases}$$

for every $a, b \in [0, 1]$. It is straightfoward that $[0, 1]_{*_n} := \langle [0, 1], *_n, \to_n, \min, \max, 0, 1\rangle$ is a WNM-chain, and all WNM-chains over $[0, 1]$ are of this form. Notice that a standard WNM-chain given by a negation function n is an NM-chain if, and only if, n is involutive, i.e. $n(n(a)) = a$ for every $a \in [0, 1]$. It follows from the study of such negations in [6] that there is only one standard NM-chain up to isomorphism, namely the one given by the negation $n(x) = 1 - x$. We will refer to it as $[0, 1]_{NM}$.

Standard algebras provide a semantics for WNM and NM logics. This result is known as *strong standard completeness theorem*:

Theorem 1 ([2]). *Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ be a (possibly infinite) set of formulas. Then:*

- *$\Gamma \vdash_{WNM} \varphi$ if, and only if, $\Gamma \models_{[0,1]_{*_n}} \varphi$ for every negation function n.*
- *$\Gamma \vdash_{NM} \varphi$ if, and only if, $\Gamma \models_{[0,1]_{NM}} \varphi$.*

This theorem has these interesting algebraic consequences: $\mathbf{V}([0, 1]_{NM}) = \mathbb{NM}$) and $\mathbf{V}(\{[0, 1]_{*_n} :$ n is a negation function$\}) = \mathbb{WNM}$).

We need also to recall some properties of the negation operation in WNM-chains.[2]

Lemma 1. *Let $\mathcal{A}$ be a WNM-chain. Then for every $a \in A$:*

(i) $\neg a = \neg\neg\neg a$,
(ii) $a \leq \neg\neg a$,
(iii) $a = \neg\neg a$ *if, and only if, there is* $b \in A$ *such that* $a = \neg b$, *and*
(iv) $\neg\neg a = \min\{b \in A : a \leq b \text{ and } b = \neg\neg b\}$.

The last one gives rise to the following notation:

Definition 2. *Let $\mathcal{A}$ be a WNM-chain and $a \in A$ an involutive element. We define $I_a := \{b \in A : \neg\neg b = a\}$ and we call it* the interval associated to a. *We say that a has a trivial associated interval when $I_a = \{a\}$; in this case we also say that a is an* isolated *involutive element.*

3 Main results

Given a class $\mathbb{K}$ of algebras, $\mathbb{K}_{fin}$ will denote the class of its finite members.

Proposition 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be WNM-chains. Then, $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\mathcal{B})$ if, and only if, $\mathbf{IS}(\mathcal{A})_{fin} \subseteq \mathbf{IS}(\mathcal{B})_{fin}$.*

In particular, we obtain that two standard WNM-chains generate the same variety if, and only if, they have the same (up to isomorphism) finite subalgebras. This will be the criterion to give a classification of all varieties generated by a standard WNM-chain. Recall that those chains essentially depend only on a weak negation function. Therefore, taking into account the possible forms of those weak negation functions we will obtain the desired classification.

Definition 3. *Given a weak negation function, f, consider a non-isolated involutive point $a \in [0,1]$ and its non-trivial associated interval I_a. We say that I_a is of type 1 if a is a discontinuity point of f, otherwise we say that I_a is of type 2.*

Lemma 2. *Let f be a weak negation function and $a \in [0,1]$ a non-isolated involutive point. Then the following are equivalent:*

- *I_a is of type 1.*
- *$\neg a$ is non-isolated.*
- *$I_{\neg a}$ is of type 1.*

[2] All the relevant properties of negations in MTL-chains can be found in the appendix of [2].

First we define negation functions with a finite number of intervals:

Definition 4. *(i) Given $n \in \omega \setminus \{0\}$, $m_1, m_2 \in \omega$ (with $n = m_1 + m_2$) and $t_0, \ldots, t_{m_1}, s_0, \ldots, s_{m_2} \in \{0, \infty\}$, we define $h^{n,m_1,m_2}_{t_0,\ldots,t_{m_1},s_0,\ldots,s_{m_2}}$ as a weak negation function such that it has m_1 intervals in the negative part with t_0 isolated involutive elements before the first one, t_m isolated involutive elements after the last one and t_i isolated involutive elements between the i-th and the $(i+1)$-th interval; and there are m_2 intervals in the positive part with s_0 isolated involutive elements before the first one, s_m isolated involutive elements after the last one and s_i isolated involutive elements between the i-th and the $(i+1)$-th interval. The associated standard WNM-chain is $\mathcal{B}^{n,m_1,m_2}_{t_0,\ldots,t_{m_1},s_0,\ldots,s_{m_2}}$.*

And now we define negation functions with an infinite number of intervals:

Definition 5. *Given $m \in \omega$ and $t_0, \ldots, t_m, s_0, \ldots, s_m \in \omega \cup \{\infty\}$, we define the following weak negation functions:*

(ii) $f^{2m}_{t_0,\ldots,t_m,s_0,\ldots,s_m}$ *is a negation function such that it has no fixpoint, it has m intervals of type 1 in the negative part with t_0 intervals of type 2 before the first one, t_m intervals of type 2 after the last one and t_i intervals of type 2 between the i-th and the $(i+1)$-th interval of type 1; and there are m intervals of type 1 in the positive part with s_0 intervals of type 2 before the first one, s_m intervals of type 2 after the last one and s_i intervals of type 2 between the i-th and the $(i+1)$-th interval of type 1, and $I_1 = \{1\}$.*

(iii) $\overline{f}^{2m}_{t_0,\ldots,t_m,s_0,\ldots,s_m}$ *is a negation function such that it has no fixpoint, it has m intervals of type 1 in the negative part with t_0 intervals of type 2 before the first one, t_m intervals of type 2 after the last one and t_i intervals of type 2 between the i-th and the $(i+1)$-th interval of type 1; and there are m intervals of type 1 in the positive part with s_0 intervals of type 2 before the first one, s_m intervals of type 2*

after the last one and s_i intervals of type 2 between the i-th and the $(i+1)$-th interval of type 1, and $I_1 \neq \{1\}$.

(iv) $g^{2m}_{t_0,\ldots,t_m,s_0,\ldots,s_m}$ *is a negation function such that it has a fixpoint which is isolated, it has m intervals of type 1 in the negative part with t_0 intervals of type 2 before the first one, t_m intervals of type 2 after the last one and t_i intervals of type 2 between the i-th and the $(i+1)$-th interval of type 1; and there are m intervals of type 1 in the positive part with s_0 intervals of type 2 before the first one, s_m intervals of type 2 after the last one and s_i intervals of type 2 between the i-th and the $(i+1)$-th interval of type 1, and $I_1 = \{1\}$.*

(v) $\overline{g}^{2m}_{t_0,\ldots,t_m,s_0,\ldots,s_m}$ *is a negation function such that it has a fixpoint which is isolated, it has m intervals of type 1 in the negative part with t_0 intervals of type 2 before the first one, t_m intervals of type 2 after the last one and t_i intervals of type 2 between the i-th and the $(i+1)$-th interval of type 1; and there are m intervals of type 1 in the positive part with s_0 intervals of type 2 before the first one, s_m intervals of type 2 after the last one and s_i intervals of type 2 between the i-th and the $(i+1)$-th interval of type 1, and $I_1 \neq \{1\}$.*

(vi) $g^{2m+1}_{t_0,\ldots,t_m,s_0,\ldots,s_m}$ *is a negation function such that it has a fixpoint c which is non-isolated, there are m intervals of type 1 in the negative part whose right extreme is not c, with t_0 intervals of type 2 before the first one, t_m intervals of type 2 after the last one and t_i intervals of type 2 between the i-th and the $(i+1)$-th interval of type 1; and there are m intervals of type 1 in the positive part with s_0 intervals of type 2 before the first one, s_m intervals of type 2 after the last one and s_i intervals of type 2 between the i-th and the $(i+1)$-th interval of type 1, and $I_1 = \{1\}$.*

(vii) $\overline{g}^{2m+1}_{t_0,\ldots,t_m,s_0,\ldots,s_m}$ *is a negation function such that it has a fixpoint c which is non-isolated, there are m intervals of type 1 in the negative part whose right extreme is not c, with t_0 intervals of type 2 before the first one, t_m intervals of type 2 after the last one and t_i intervals of type 2 between the i-th and the $(i+1)$-th interval of type 1; and there are m intervals of type 1 in the positive part with s_0 intervals of type 2 before the first one, s_m intervals of type 2 after the last one and s_i intervals of type 2 between the i-th and the $(i+1)$-th interval of type 1, and $I_1 \neq \{1\}$.*

(viii) f^{∞} *is a negation function such that it has no fixpoint, it has intervals $[0,a]$ and $[f^{\infty}(a),1]$ of isolated involutive elements, it has ω intervals of type 1 in the negative part and ω intervals of type 1 in the positive part and nothing else.*

(ix) $\overline{f}^{\infty}$ *is a negation function such that it has no fixpoint, it has ω intervals of type 1 in the negative part and ω intervals of type 1 in the positive part and nothing else.*

(x) g^{∞} *is a negation function such that it has a fixpoint c which is isolated, it has intervals $[0,a]$, $[g^{\infty}(a),1]$ and $[d, g^{\infty}(d)]$ (with $d < c < g^{\infty}(d)$) of isolated involutive elements, it has ω intervals of type 1 in the negative part and ω intervals of type 1 in the positive part and nothing else.*

(xi) $\overline{g}^{\infty}$ *is a negation function such that it has a fixpoint c which is isolated, it has an interval $[d, \overline{g}^{\infty}(d)]$ (with $d < c < \overline{g}^{\infty}(d)$) of isolated involutive elements, it has ω intervals of type 1 in the negative part and ω intervals of type 1 in the positive part and nothing else.*

(xii) $g^{\infty+1}$ *is a negation function such that it has a fixpoint c which is non-isolated, it has intervals $[0,a]$ and $[g^{\infty+1}(a),1]$ of isolated involutive elements,it has ω intervals of type 1 in the negative part and ω intervals of type 1 in the positive part and nothing else.*

(xiii) $\overline{g}^{\infty+1}$ *is a negation function such that it has a fixpoint c which is non-isolated, it has ω intervals of type 1 in the negative part and ω intervals of type 1 in the positive part and nothing else.*

The standard WNM-algebras defined by those weak negation functions will be denoted respectively by $\mathcal{C}^{2m}_{t_0,\ldots,t_m,s_0,\ldots,s_m}$, $\overline{\mathcal{C}}^{2m}_{t_0,\ldots,t_m,s_0,\ldots,s_m}$, $\mathcal{D}^{2m}_{t_0,\ldots,t_m,s_0,\ldots,s_m}$, $\overline{\mathcal{D}}^{2m}_{t_0,\ldots,t_m,s_0,\ldots,s_m}$, $\mathcal{D}^{2m+1}_{t_0,\ldots,t_m,s_0,\ldots,s_m}$,

$\overline{\mathcal{D}}^{2m+1}_{t_0,\ldots,t_m,s_0,\ldots,s_m}$, $\mathcal{C}^{\infty}$, $\overline{\mathcal{C}}^{\infty}$, $\mathcal{D}^{\infty}$, $\overline{\mathcal{D}}^{\infty}$, $\mathcal{D}^{\infty+1}$ *and* $\overline{\mathcal{D}}^{\infty+1}$. *We call them* canonical standard WNM-chains.

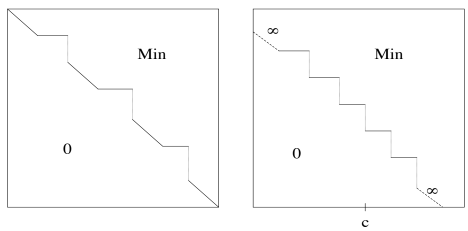

Figure 1: Two examples of canonical standard WNM-chains: $\mathcal{B}^{3,2,1}_{\infty,\infty,0,\infty,\infty}$ and $\overline{\mathcal{D}}^{\infty+1}$.

Theorem 2. *Let $\mathcal{A}$ be a standard WNM-chain. Then:*

1. *If all the elements are involutive, then $\mathcal{A} \cong [0,1]_{NM}$ and $\mathbf{V}(\mathcal{A}) = \mathbb{NM}$.*

2. *Suppose that $\mathcal{A}$ has an finite number $n \in \omega \setminus \{0\}$ of non-isolated involutive elements. Let m_1 be the number of intervals in the negative part and let m_2 be the number of intervals in the positive part. Then, there are $t_0, \ldots, t_{m_1}, s_0, \ldots, s_{m_2} \in \{0, \infty\}$ such that $\mathcal{A} \cong \mathcal{B}^{n,m_1,m_2}_{t_0,\infty,t_1,\infty\ldots,t_{m_1},s_0,\infty,s_1,\infty,\ldots,s_{m_2}}$, and of course, $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\mathcal{B}^{n,m_1,m_2}_{t_0,\ldots,t_{m_1},s_0,\ldots,s_{m_2}})$.*

3. *Suppose that $\mathcal{A}$ has an infinite number of non-isolated involutive elements but the number of intervals of type 1 is finite. Let $m \in \omega$ be the number of intervals of type 1 in the positive part.*

   3.1. *Suppose that $\mathcal{A}$ has no negation fixpoint. It has m intervals of type 1 in the negative part. Then, there are $t_0, \ldots, t_m, s_0, \ldots, s_m \in \omega$ such that:*
      3.1.1. *if $I_1 = \{1\}$, then $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\mathcal{C}^{2m}_{t_0,\ldots,t_m,s_0,\ldots,s_m})$.*
      3.1.2. *if $I_1 \neq \{1\}$, then $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\overline{\mathcal{C}}^{2m}_{t_0,\ldots,t_m,s_0,\ldots,s_m})$.*

   3.2. *Suppose that $\mathcal{A}$ has a negation fixpoint which is isolated. It has m intervals of type 1 in the negative part. Then, there are $t_0, \ldots, t_m, s_0, \ldots, s_m \in \omega$ such that:*
      3.2.1. *if $I_1 = \{1\}$, then $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\mathcal{D}^{2m}_{t_0,\ldots,t_m,s_0,\ldots,s_m})$.*
      3.2.2. *if $I_1 \neq \{1\}$, then $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\overline{\mathcal{D}}^{2m}_{t_0,\ldots,t_m,s_0,\ldots,s_m})$.*

   3.3. *Suppose that $\mathcal{A}$ has a negation fixpoint c which is non-isolated. It has m intervals of type 1 in the negative part whose right extreme is not c. Then, there are $t_0, \ldots, t_m, s_0, \ldots, s_m \in \omega$ such that:*
      3.3.1. *if $I_1 = \{1\}$, then $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\mathcal{D}^{2m+1}_{t_0,\ldots,t_m,s_0,\ldots,s_m})$.*
      3.3.2. *if $I_1 \neq \{1\}$, then $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\overline{\mathcal{D}}^{2m+1}_{t_0,\ldots,t_m,s_0,\ldots,s_m})$.*

4. *Suppose that $\mathcal{A}$ has an infinite number of intervals of type 1.*

   4.1. *Suppose that $\mathcal{A}$ has no negation fixpoint.*
      4.1.1. *if $I_1 = \{1\}$, then $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\mathcal{C}^{\infty})$.*
      4.1.2. *if $I_1 \neq \{1\}$, then $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\overline{\mathcal{C}}^{\infty})$.*

   4.2. *Suppose that $\mathcal{A}$ has a negation fixpoint which is isolated.*
      4.2.1. *if $I_1 = \{1\}$, then $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\mathcal{D}^{\infty})$.*
      4.2.2. *if $I_1 \neq \{1\}$, then $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\overline{\mathcal{D}}^{\infty})$.*

   4.3. *Suppose that $\mathcal{A}$ has a negation fixpoint which is not isolated.*
      4.3.1. *if $I_1 = \{1\}$, then $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\mathcal{D}^{\infty+1})$.*
      4.3.2. *if $I_1 \neq \{1\}$, then $\mathbf{V}(\mathcal{A}) = \mathbf{V}(\overline{\mathcal{D}}^{\infty+1})$.*

Moreover, all canonical standard WNM-chains generate pairwise different varieties.

In [5] the subvarieties $\mathbb{BP}_0$, $\mathbb{BP}_0^{+1}$ of $\mathbb{MTL}$ are studied and axiomatized. Inside $\mathbb{WNM}$ we obtain the following:

- $\mathbb{WNM} \cap \mathbb{BP}_0$ is the variety generated by all WNM-chains without negation fixpoint and it is axiomatized by: $(\neg(\neg x)^2)^2 \leftrightarrow \neg(\neg x^2)^2 \approx 1$.

- $\mathbb{WNM} \cap \mathbb{BP}_0^{+1}$ is the variety generated by all WNM-chains with an isolated negation fixpoint and it is axiomatized by:

   1. $(\neg(\neg x)^2)^2 \leftrightarrow \neg(\neg x^2)^2 \vee (\neg x \leftrightarrow \neg\neg x) \approx 1$,

2. $(x \vee \neg x \to y \vee \neg y) \vee ((y \vee \neg y \to \neg y \wedge \neg\neg y) \to y \vee \neg y) \vee (((x \vee \neg x)^2 \to y \vee \neg y) \to y \vee \neg y) \approx 1$.

Therefore, by Proposition 1 we obtain: $\mathbf{V}(\overline{\mathcal{C}}^{\infty}) = \mathbb{WNM} \cap \mathbb{BP}_0$, $\mathbf{V}(\overline{\mathcal{D}}^{\infty}) = \mathbb{WNM} \cap \mathbb{BP}_0^{+1}$ and $\mathbf{V}(\overline{\mathcal{D}}^{\infty+1}) = \mathbb{WNM}$.

Finally, we study the relations of inclusion between these varieties.

Proposition 2. *Let $\mathcal{A}$ be a canonical standard WNM-chain. Then:*

- *If $\mathcal{A}$ has no fixpoint, then $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\overline{\mathcal{C}}^{\infty})$.*
- *If $\mathcal{A}$ either has no fixpoint or the fixpoint is isolated, then $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\overline{\mathcal{D}}^{\infty})$.*
- *$\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\overline{\mathcal{D}}^{\infty+1})$.*

Proposition 3. *Let $\mathcal{A}$ and $\mathcal{A}'$ be canonical standard WNM-chains both defined from a negation function of type (i), (ii), (iii), (iv), (v), (vi) or (vii). Then, $\mathbf{V}(\mathcal{A}) \subseteq \mathbf{V}(\mathcal{A}')$ if, and only if, $\mathcal{A} \in \mathbf{IS}(\mathcal{A}')$.*

Therefore, given two canonical standard WNM-chains defined from a negation of the same type it is easy to check whether there is some inclusion between the varieties generated by them.

To compare the varieties generated by chains defined from negations of different type, we use again Proposition 1 and the following criteria:

Proposition 4. *Let $\mathcal{A}$ and $\mathcal{A}'$ be canonical standard WNM-chains defined from negations of different type. Then:*

- *If $\mathcal{A}'$ has a finite number of non-isolated involutive elements (i.e. its negation is of type (i)) and $\mathcal{A}$ has an infinite number of non-isolated involutive elements, then $\mathbf{V}(\mathcal{A}) \not\subseteq \mathbf{V}(\mathcal{A}')$.*
- *If $\mathcal{A}'$ has $I_1 = \{1\}$ and $\mathcal{A}$ has $I_1 \neq \{1\}$, then $\mathbf{V}(\mathcal{A}) \not\subseteq \mathbf{V}(\mathcal{A}')$.*
- *If $\mathcal{A}'$ has no fixpoint and $\mathcal{A}$ has fixpoint, then $\mathbf{V}(\mathcal{A}) \not\subseteq \mathbf{V}(\mathcal{A}')$.*
- *If $\mathcal{A}'$ has an isolated fixpoint and $\mathcal{A}$ has a non-isolated fixpoint, then $\mathbf{V}(\mathcal{A}) \not\subseteq \mathbf{V}(\mathcal{A}')$.*

4 Concluding remarks

By defining a collection of canonical standard WNM-chains, we have classified all the varieties generated by one standard WNM-chain. Some of them have been identified and axiomatized. In a forthcoming paper we will give equational bases for all varieties generated by a finite family of WNM-chains with finitely-many non-isolated involutive points, so, in particular, we can axiomatize the varieties of the form $\mathbf{V}(\mathcal{B}^{n,m_1,m_2}_{t_0,\ldots,t_{m_1},s_0,\ldots,s_{m_2}})$. Nevertheless, the axiomatization of the remaining varieties studied here is still an open problem.

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