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ON GENERIC IDENTIFIABILITY OF SYMMETRIC TENSORS OF SUBGENERIC RANK

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Abstract. We prove that the general symmetric tensor in $S^d\mathbb{C}^{n+1}$ of rank r is identifiable, provided that r is smaller than the generic rank. That is, its Waring decomposition as a sum of r powers of linear forms is unique. Only three exceptional cases arise, two of which were known classically. The third exceptional case is given by cubic tensors in 6 variables of rank 9, whose proof of nonidentifiability we could not find in the literature. Our original contribution regards only the case of cubics ($d = 3$), while for $d \geq 4$ we rely on known results on weak defectivity by Ballico, Ciliberto, Chiantini, and Mella.

1. Introduction

We denote by $S^d\mathbb{C}^{n+1}$ the space of symmetric tensors on $\mathbb{C}^{n+1}$; such tensors can be identified with homogeneous polynomials of degree d in $n+1$ variables, which are also referred to as *forms*. In this symmetric setting, the most natural tensor rank decomposition is the classical Waring decomposition, which expresses a symmetric tensor as a sum of powers of linear forms. Precisely, every form $f \in S^d\mathbb{C}^{n+1}$ has a minimal expression

$$f = \sum_{i=1}^{r} l_i^d, \tag{1}$$

where $l_i \in \mathbb{C}^{n+1}$ are linear forms [21]; the minimal number of summands r is called the symmetric rank of f, since in the correspondence between forms and symmetric tensors, powers of linear forms correspond to tensors of rank 1. A natural question concerns the number of summands required for representing a general form in $S^d\mathbb{C}^{n+1}$. The problem is elementary for $d = 2$, which corresponds to the case of symmetric matrices. For $d \geq 3$ the question was answered by Alexander and Hirschowitz in [2]. Letting

$$r_{d,n} = \frac{\binom{n+d}{d}}{n+1}, \tag{2}$$

they proved that the general $f \in S^d\mathbb{C}^{n+1}$ with $d \geq 3$ has rank $\lceil r_{d,n} \rceil$, which is called the *generic rank*, unless the space $S^d\mathbb{C}^{n+1}$ is one of the so-called defective cases $S^4\mathbb{C}^{n+1}$ for $n = 2, 3, 4$ and $S^3\mathbb{C}^5$, where the generic rank is $\lceil r_{d,n} \rceil + 1$[1]. When the rank of a Waring decomposition is strictly smaller than $r_{d,n}$, we say that this decomposition is of *subgeneric rank*. It could be worthy noticing that, in our

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[1] $r_{4,3}$ **is not an integer**

notation, being of *subgeneric rank* is not always equivalent to being of *rank smaller than the one of a general tensor*, because in the defective cases above a general tensor has rank strictly bigger than $r_{d,n}$.

The Alexander–Hirschowitz theorem implies that the generic tensor of subgeneric rank admits only a finite number of alternative Waring decompositions [22]. In this paper, we shall be concerned with proving that the generic tensor of subgeneric rank admits precisely one Waring decomposition, modulo permutations of the summands and scaling by d-roots of unity. [2] More precisely, the main result of this paper is the following theorem.

Theorem 1.1. *Let $d \geq 3$. The general tensor in $S^d\mathbb{C}^{n+1}$ of subgeneric rank $r < r_{d,n}$ with $r_{d,n}$ as in (2) has a* unique *Waring decomposition, i.e., it is* identifiable*, unless it is one of the following cases:*

(1) $d = 6$, $n = 2$, and $r = 9$;
(2) $d = 4$, $n = 3$, and $r = 8$;
(3) $d = 3$, $n = 5$, and $r = 9$.

In all of these exceptional cases, there are exactly two Waring decompositions.

The first two cases are well known: see Remark 4.4 in [24] and Remark 6.5 in [13]. Contrariwise, we are not aware of any direct reference for the third case, although the fundamental geometric method to understand it follows from previous works of Veneroni, Coble, Room and Fisher, which we summarize in Proposition 2.2.

Even though the general symmetric tensor is not of subgeneric rank, the setting considered in this paper is nevertheless important in practice: in most applications, one is interested in the identifiability of symmetric tensors of subgeneric rank. For instance, Anandkumar, Ge, Hsu, Kakade, and Telgarsky [3] consider statistical parameter inference algorithms based on decomposing symmetric tensors for a wide class of latent variable models. The identifiability of the Waring decomposition then ensures that the recovered parameters, which correspond with the individual symmetric rank-1 terms in Waring's decomposition, are unique, and, thus, admit an interpretation in the application domain. The rank of the Waring decomposition, in these applications, is invariably much smaller than the generic rank. As general sources on tensor decomposition, we refer to [13, 16, 21, 22, 29]. [3]

Symmetric tensors of general rank are not expected to admit only a finite number of Waring decompositions, because the expected dimension $\lceil r_{d,n} \rceil (n + 1)$ of the $\lceil r_{d,n} \rceil$-secant variety of the Veronese variety $v_d(\mathbb{P}^n)$ exceeds the dimension $\binom{n+d}{d}$ of the embedding space $S^d\mathbb{C}^{n+1}$. Therefore, at least a curve's worth of alternative Waring decompositions of a general symmetric tensor is anticipated. However, if $r_{d,n} = \lceil r_{d,n} \rceil$ is integer, then a general symmetric tensor is still expected to admit only a finite number of Waring decompositions. The approach pursued in this paper, i.e., proving not tangential weak defectivity, cannot handle tensors of the generic rank. Other approaches need to be considered, in this setting. In fact, Mella [25]

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[2] **Giorgio: I have modified the sentence about scaling indeterminacies, which is correct in the unsymmetric case, but redundant here. Indeed in (1) there are no λ_i and the only indeterminacies are by d roots of unity. Do you agree? Nick: Yes. Luca: ok.**

[3] **Giorgio: I resurrected [29], because it is the first source where generic identifiability was studied in a large range. At the same time, I would prefer not to quote [Kolda2009], which is a controversal reference and it received already extra credit for what actually they exposed. Nick: All fine for me. Luca: ok.**

formulated a conjecture about the cases where the expression in (1) is still expected to be unique even for general symmetric tensors. In [20], further evidence for this conjecture was given; in addition, the analogous problem for nonsymmetric tensors was also considered.

In analogy to Theorem 1.1, we mention that the results in [6, 11, 12] give broad evidence to the analogous problem in the setting of nonsymmetric tensors, i.e., that a general nonsymmetric tensor of subgeneric rank admits a unique tensor rank decomposition, unless it is one of the exceptional cases that have already been proved in [1, 5, 6, 10, 11].

The proof of Theorem 1.1 is based on the study of the geometric concepts of *weak defectivity*, developed in [8], and *tangentially weak defectivity*, developed in [6]. Indeed, from this point of view, the theorem can be reformulated in the following way, which is a result of independent interest.

Theorem 1.2. *Let $d \geq 3$, $r_{d,n}$ as in (2), and $r < r_{d,n}$. Then, the common singular locus of the space of hypersurfaces of degree d in $\mathbb{P}^n$ that are singular at r general points, consists of exactly these r points, except in the following cases:*

(1) *$d = 6$, $n = 2$, and $r = 9$. The unique sextic plane curve singular at 9 general points is a double cubic, so that its singular locus is an elliptic cubic curve;*
(2) *$d = 4$, $n = 3$, and $r = 8$. The net of quartic surfaces singular at 8 points consists of reducible quadrics, so that the common singular locus is the base locus of the pencil of quadrics through 8 general points, which is an elliptic normal curve of degree 4;*
(3) *$d = 3$, $n = 5$, and $r = 9$. The common singular locus of the pencil of cubic 4-folds singular at 9 general points is the unique elliptic normal curve of degree 6 through these 9 points.*

Furthermore, the above exceptional cases are the only instances where there exists a unique elliptic normal curve of degree $n + 1$ in $\mathbb{P}^n$ through r general points.

In this formulation, the theorem was already partially proved: the case $n \leq 2$ was proved by Chiantini and Ciliberto [9]; for $d \geq 4$ it was proved by Ballico [4, Theorem 1.1]; and for $d = 3$ with $r < r_{d,n} - \frac{n+2}{3} + 1$ it was proved by Mella [24, Theorem 4.1]. Consequently, the original contribution of this paper concerns the case of cubics, i.e., $d = 3$, which we solve completely in the subgeneric case.

We notice that Ballico [4] proved an even stronger result for $d \geq 4$. Namely, he showed that a general hypersurface of degree $d \geq 4$ in $\mathbb{P}^n$ that is singular in r general points, is singular only at these r points (except for the exceptional cases (1) and (2) of Theorem 1.2). This is equivalent to showing that the Veronese variety $v_d(\mathbb{P}^n)$ is not r-weakly defective, while our result only says that it is not r-*tangentially* weakly defective. We wonder whether the above list also gives the classification of all r-weakly defective Veronese varieties $v_d(\mathbb{P}^n)$, even for $d = 3$.

The content of the paper is the following. In section 2, we describe the third exceptional case appearing in Theorems 1.1 and 1.2. Remark 2.6 also discusses the origin of this example and, by extension, of this paper. Section 3 contains the proof of the main theorem. Thereafter, the connection between weakly defective varieties and the dual varieties to secant varieties, including a description of the dual varieties of all weakly defective examples appearing in Theorem 1.2, is explored in section 4. In particular, Theorem 4.2 contains the description of cubic hypersurfaces in $\mathbb{P}^5$ which can be written as the determinant of a 3×3 matrix with linear entries. In section 5, we give an explicit criterion allowing to check if a given Waring decomposition is unique. This algorithm is an extension to the symmetric case of the one provided in [12] for general tensors.

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2. The new example of cubics in $\mathbb{P}^5$ singular in 9 points

The following classical result shows that the values $n = 2, 3, 5$, which appear in Theorems 1.1 and 1.2, have a special role for elliptic normal curves.

Proposition 2.1 (Coble [15])**.** *Assume that there are only finitely many elliptic normal curves passing through k general points in $\mathbb{P}^n$. Then, $n = 2$, 3, or 5 and, correspondingly, $k = \frac{(n+1)^2}{n-1}$. In these three cases, there is a unique elliptic normal curve in $\mathbb{P}^n$ passing through $\frac{(n+1)^2}{n-1}$ general points.*

Proof. Elliptic normal curves of degree $(n+1)$ in $\mathbb{P}^n$ depend on $(n+1)^2$ parameters, which is the dimension of the space of sections of the normal bundle. The passage of the curve through a point in $\mathbb{P}^n$ imposes $n-1$ conditions, which is the codimension of the curve. Therefore, we may expect finitely many elliptic normal curves through k *general* points in $\mathbb{P}^n$ only if $k(n-1) = (n+1)^2$. This implies that $(n-1)$ divides $(n+1)^2 = (n-1)(n+3) + 4$, hence $(n-1)$ divides 4, which gives the values $n = 2, 3, 5$. Moreover, $k = (n+1)^2/(n-1)$.

In case $n = 2$ and $k = 9$, the elliptic curve is a plane cubic, and it is unique.

In case $n = 3$ and $k = 8$, an elliptic normal curve is a complete intersection of two quadrics. Thus, if $\langle Q_1, Q_2\rangle$ is the pencil of quadrics through 8 general points $p_1, \ldots, p_8$, then $C = Q_1 \cap Q_2$ is the unique elliptic normal curve through the p_i's.

In case $n = 5$ and $k = 9$, the existence and the uniqueness of the curve was found by Coble [15, Theorem 19] by applying a Gale transform—see [18] for a nice review—and reducing to the case $n = 2$ and $k = 9$; a modern treatment was given by Dolgachev [17, Theorem 5.2]. □

Next, we analyze the case of cubic hypersurfaces in $\mathbb{P}^5$ that are singular at 9 general points; we could not find this analysis in the literature. We will demonstrate, in particular, that the general symmetric tensor of rank 9 in $S^3\mathbb{C}^6$ admits exactly two distinct Waring decompositions.

We begin by summarizing the essential geometrical argument that was already known in the literature and which underlies the proof of the last exceptional case in Theorems 1.1 and 1.2.

Proposition 2.2 (Veneroni [31, Section 1], Coble [15, p. 16], Room [28, Sections 9–22], Fisher [19, Lemma 2.9])**.** *We have the following two results.*

- *(i) The 2-minors of a 3×3 matrix with linear entries on $\mathbb{P}^5$ define a (sextic) elliptic normal curve in $\mathbb{P}^5$.*
- *(ii) If $\mathcal{C}$ is a (sextic) elliptic normal curve in $\mathbb{P}^5$, then the variety of secant lines $\sigma_2(\mathcal{C})$ is a complete intersection of two cubic hypersurfaces on $\mathbb{P}^5$, each one being the determinant of a 3×3 matrix with linear entries on $\mathbb{P}^5$.*

Proof. Part (i) is well known: The curve is obtained by cutting the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^9$, i.e., the variety of 3×3 matrices of rank 1, with a linear space $\mathbb{P}^5$. Claim (ii) follows by [19, Lemma 2.9]. □

Theorem 2.3. *Let $p_1, \ldots, p_9$ be general points in $\mathbb{P}^5$. Let $\mathcal{C}$ be the elliptic normal sextic curve through these points. A cubic that is singular at $p_1, \ldots, p_9$ contains $\sigma_2(\mathcal{C})$ and is singular on $\mathcal{C}$.*

Proof. By Proposition 2.2, in the pencil of cubics containing $\sigma_2(\mathcal{C})$, the general element is singular along $\mathcal{C}$. This pencil fills the space of cubics that are singular at $p_1, \ldots, p_9$, which is 2-dimensional by the Alexander–Hirschowitz theorem [2]. □

Proposition 2.4. *The general tensor in $S^3\mathbb{C}^6$ of rank 9 has exactly two Waring decompositions as sum of 9 powers of linear forms.*

Proof. In the language of [9], we have to prove that the secant order of $\sigma_9(v_3(\mathbb{P}^5))$ is 2. By [9, Theorem 2.4], this is equal to the secant order of the 9-contact locus $\mathcal{C}$, which corresponds to the third Veronese embedding of an elliptic normal sextic curve in $\mathbb{P}^5$, by Theorem 2.3. Thus, $\mathcal{C}$ is an elliptic curve of degree 18 in $\mathbb{P}^{17}$, whose secant order is 2 by [9, Proposition 5.2]. □

Corollary 2.5. *Let $n = 2$, 3, or 5. The unique elliptic normal curve that is mentioned in Proposition 2.1), which passes through $\frac{(n+1)^2}{n-1}$ general points in $\mathbb{P}^n$, can be constructed as the singular locus of a general hypersurface of degree $\frac{2(n+1)}{n-1}$ that is singular in the $\frac{(n+1)^2}{n-1}$ points.*

Proof. The cases $n = 2, 3$ have already been considered in the proof of Proposition 2.1. The case $n = 5$ follows from Theorem 2.3. □

Remark 2.6. *The third case in Theorems 1.1 and 1.2 was suggested by a computational analysis performed by the third author, who ran the algorithm that we present in section 5, for hypersurfaces of degree d in $\mathbb{P}^n$ singular at the maximal number of random points, i.e., $r = r_{d,n} - 1$ with $r_{d,n}$ as in (2), for all reasonably small values of d, n. It took a while to realize what happened, because this third case was missing in [23, Theorem 6.1.2]. Actually, Theorem 6.1.2 of [23] only intended to collect previous results by Ballico [4], Ciliberto and Chiantini [8], and Mella [24, 25], which are individually correct. The second author takes the responsibility to have first overlooked the assumption $d \geq 4$ in summarizing and reporting the results of [4, 24]. For $d = 3$, Theorem 1.1 was known with the additional assumption $r < r_{3,n} - \frac{n+2}{3} + 1$; see [24, Theorem 4.1]. From the theoretical proof that we present in section 3, we can conclude that the third case was the last exception. Therefore, Theorem 6.1.2 in [23] remains true if the third case $(k, d, n) = (9, 5, 3)$ is added to the list of exceptions. Exactly the same remark applies to the formulation of Theorem 2.3 in [26] and Theorem 12.3.4.3 in [22]. This conclusion is corroborated by the coauthors of [23, 26].* [4]

[4]**Giorgio: Any alternative to save the meaning? Nick: I would prefer "corroborated" or "confirmed" rather than "shared," because it indicates more clearly that they agree with us.**

3. Cubics singular at the maximum number of points.

We turn our attention to the proof of Theorem 1.1 in the case of cubics, i.e., $d = 3$. Given n, we define

$$k_n = \left\lceil \frac{\binom{n+3}{3}}{n+1} \right\rceil = \left\lceil \frac{(n+3)(n+2)}{6} \right\rceil; \tag{3}$$

it is the generic rank for cubic polynomials for $n \neq 4$. In other words, a cubic polynomial on $\mathbb{P}^n$ singular at k_n general points vanishes identically for $n \neq 4$ [2]. Some elementary algebra shows that $k_n = \frac{(n+3)(n+2)}{6}$ if $n \not\equiv 2 \mod 3$, while $k_n = \frac{(n+3)(n+2)}{6} + \frac{2}{3} = \frac{(n+4)(n+1)}{6} + 1$ if $n \equiv 2 \mod 3$.

For the sake of future reference, let us state explicitly the following consequence of the Alexander–Hirschowitz theorem [2].

Theorem 3.1 (Alexander–Hirschowitz [2])**.** *The space of cubic hypersurfaces on $\mathbb{P}^n$ that are singular at $k_n - 1$ general points has dimension*

(i) $n + 1$ *if* $n \not\equiv 2 \mod 3$*, or*
(ii) $\frac{n+1}{3}$ *if* $n \equiv 2 \mod 3$*.*

To complete the proof of the main theorems, it remains to show the following result, which refines Theorem 3.1.

Theorem 3.2. *The space of cubic hypersurfaces in $\mathbb{P}^n$ that are singular at $k_n - 1$ general points has dimension*

(i) $n + 1$ *if* $n \not\equiv 2 \mod 3$*, or*
(ii) $\frac{n+1}{3}$ *if* $n \equiv 2 \mod 3$*,*

and, in addition, its common singular locus consists only of these $k_n - 1$ points, provided that $n \neq 5$.

In order to prove Theorem 3.2, we may assume $n \geq 6$, since the cases with $n \leq 4$ (as well as the case of cubics in $\mathbb{P}^5$ singular at 8 points) can be checked separately. The proof of Theorem 3.2 will follow the following steps. In section 3.1, we will prove case (i) by induction on subspaces of codimension 3, following an approach that is mainly inspired by [7, Section 5], where an alternative proof of Theorem 3.1 was given. To prove case (ii), the aforementioned technique needs a modification. We will construct an inductive proof on subspaces of codimension 3 and 4; in the inductive step, we will rely, additionally, on the argument of case (i). This strategy will be presented in section 3.2.

In the rest of this section, if S is a set of simple points in $\mathbb{P}^n$ and $P \subset \mathbb{P}^n$ is a linear subspace, we denote with $I_{S,P}(d)$ the space of degree d polynomials in P vanishing at all of the points in S. Moreover, if $\mathbf{X}$ is a a set of double (singular) points, we denote by $I_{\mathbf{X} \cup S,P}(d)$ the space of degree d polynomials in P vanishing on all of the points in $S \cup \mathbf{X}$ and whose derivatives vanish on all of the points in $\mathbf{X}$.

3.1. **Proof of Theorem 3.2 (i) by induction on codimension 3.**

We start by proving three auxiliary results.

Proposition 3.3. *Let $n \geq 6$, and let $L, M, N \subset \mathbb{P}^n$ be general subspaces of codimension 3. Let l_i, respectively m_i, with $i = 1, 2, 3$ be three general points on L, respectively M. Let n_i with $i = 1, 2$ be two general points on N. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain $L \cup M \cup N$ and that are singular at the* *eight points* $\mathbf{X} = \{l_1, l_2, l_3, m_1, m_2, m_3, n_1, n_2\}$ *has dimension* 3. *Furthermore, the common singular locus is contained in* $L \cup M \cup N$.

Proof. The base cases $n = 6$, 7 and 8 can be proved with the Macaulay2 script `section3-symmetric-identifiability.m2` that is an ancillary file accompanying the arXiv version of this paper. Using this software, we may compute the following dimensions:

$$\dim I_{\mathbf{X}\cup L\cup N\cup M,\mathbb{P}^6}(2) = 0, \qquad \dim I_{\mathbf{X}\cup L\cup N\cup M,\mathbb{P}^6}(3) = 3, \text{ and}$$
$$\dim I_{\mathbf{X}\cup L\cup N\cup M,\mathbb{P}^7}(2) = 0, \qquad \dim I_{\mathbf{X}\cup L\cup N\cup M,\mathbb{P}^7}(3) = 3,$$

so that the claim on the codimension follows. The code proves also the statement on the singular locus.

For $n \geq 9$, the statement follows by induction on n. Indeed, we may choose coordinates such that $L = (x_0 \dots x_2)$, $M = (x_3 \dots x_5)$, $N = (x_6 \dots x_8)$. In this setting it is clear that there are no quadrics that contain $L \cup M \cup N$, and moreover every cubic containing $L \cup M \cup N$ is a cone with vertex in $L \cap M \cap N$ (in the classical terminology, see e.g. [Terracini] or [Zak]???, a *cone* is a subvariety that can be defined, in a suitable set of coordinates, by equations missing one or more variables). Thus, for a general hyperplane $H \subset \mathbb{P}^n$, the Castelnuovo sequence (see [7, Equation (1)]) induces an inclusion

$$0 \longrightarrow I_{L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{(L\cup M\cup N)\cap H,H}(3)$$

Hence, if we specialize the eight points to the hyperplane H, we get an inclusion

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L\cup M\cup N)\cap H,H}(3).$$

Then, our statement follows by induction. The singular locus is a cone with vertex $L \cap M \cap N$ over the singular locus of the base case $n = 8$. □

Remark 3.4. *Following the output of the software for the case* $n = 8$, *we can describe more precisely the common singular locus of the cubic hypersurfaces in* $I_{\mathbf{X}\cup L\cup M\cup N,\mathbb{P}^n}(3)$, *for* $n \geq 8$.

It turns out that (at least in some example, but we believe in general) it is given by the union of the three linear subspaces $L \cap M$, $L \cap N$, $M \cap N$ *and by* 8 *linear subspaces of codimension* 7, *each containing one of the* 8 *points, and three of them contained in* L, *three of them contained in* M, *two of them contained in* N.

Proposition 3.5. *Let* $n \geq 5$, *and let* $L, M \subset \mathbb{P}^n$ *be subspaces of codimension three. Let* l_i, *respectively* m_i, *with* $i = 1, \dots, n-2$ *be general points on* L, *respectively* M. *Let* $p_1, p_2 \in \mathbb{P}^n$ *be general points. Then, the space of cubic hypersurfaces in* $\mathbb{P}^n$ *containing* $L \cup M$ *and singular along the set of* $2n-2$ *points* $\mathbf{X} = \{l_1, l_2, \dots, l_{n-2}, m_1, m_2, \dots, m_{n-2}, p_1, p_2\}$ *has dimension* $n+1$. *Its common singular locus contains the linear space* $L \cap M$ *and is* 0*-dimensional at the points* p_1 *and* p_2.

Proof. The base cases $n = 5$, 6, and 7 can be proved with the Macaulay2 script `section3-symmetric-identifiability.m2`. Running the software, we find the following dimensions

$$\dim I_{\mathbf{X}\cup L\cup M,\mathbb{P}^5}(2) = 0, \qquad \dim I_{\mathbf{X}\cup L\cup M,\mathbb{P}^5}(3) = 6,$$
$$\dim I_{\mathbf{X}\cup L\cup M,\mathbb{P}^6}(2) = 0, \qquad \dim I_{\mathbf{X}\cup L\cup M,\mathbb{P}^6}(3) = 7, \text{ and}$$
$$\dim I_{\mathbf{X}\cup L\cup M,\mathbb{P}^7}(2) = 0, \qquad \dim I_{\mathbf{X}\cup L\cup M,\mathbb{P}^7}(3) = 8.$$

These values indeed correspond to the claimed dimensions.

For $n \geq 8$, the statement follows by induction from $n-3$ to n. Indeed, given a third general subspace N of codimension 3, we get the exact sequence

$$0 \longrightarrow I_{L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{(L\cup M)\cap N,N}(3),$$

where the dimensions of the three spaces in the sequence are respectively 27, $9(n-1)$, and $9(n-4)$. Let us specialize $n-5$ of the points $l_i \in L$ to $L\cap N$, $n-5$ of the points $m_i \in M$ to $M \cap N$, and the two points p_1, p_2 to N. Then, we obtain a sequence

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{\mathbf{X}\cup L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L\cup M)\cap N,N}(3),$$

where the trace $(\mathbf{X}\cup L\cup M)\cap N$ satisfies the assumptions on $N = \mathbb{P}^{n-3}$, so that we can apply the induction. Notice that the residual (left) space satisfies the hypotheses of Proposition 3.3 and has dimension 3. Since the common singular locus of the cubics containing $L\cup M$ and singular at $\mathbf{X}$ must be contained in the common singular locus of the leftmost 3-dimensional space, it follows by Proposition 3.3 that its components through p_1 and p_2 must be contained in N. After the degeneration, the space of cubics $I_{\mathbf{X}\cup L\cup M,\mathbb{P}^n}(3)$ still has dimension at most $3+(n-2) = n+1$ by induction. Hence, by semicontinuity it follows that its dimension is indeed equal to $n+1$. The common singular locus cannot be positive dimensional at points p_1 and p_2, because otherwise it should be of positive dimension in the trace (right space), where by induction we know that it is 0-dimensional. ☐

Proposition 3.6. *Let $n \geq 6$, and let $L \subset \mathbb{P}^n$ be a subspace of codimension 3. If $n \not\equiv 2 \mod 3$, then the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain L and that are singular at $\frac{n(n-1)}{6}$ general points $l_i \in L$ and at n general points $p_i \in \mathbb{P}^n$ has dimension $n+1$. Moreover, its common singular locus is 0-dimensional at the n points p_i.*

Proof. The statement can be checked for $n = 6, 7$ with the Macaulay2 script `section3-symmetric-identifiability.m2`. [5]

Let $n \geq 9$. Consider the sequence

$$0 \longrightarrow I_{L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{L,\mathbb{P}^n}(3) \longrightarrow I_{L\cap M,M}(3),$$

where M is a general subspace of codimension 3. Denoting by $\mathbf{X}$ the union of the double points supported at the points l_i and p_i, we get

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{\mathbf{X}\cup L,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L)\cap M,M}(3).$$

We specialize $\frac{(n-3)(n-4)}{6}$ of the points l_i to $L\cap M$ and $n-2$ of the points p_i to M. Let's note that we do not apply induction from $n-3$ to n here, otherwise we should specialize $n-3$ of the points p_i to M, so losing the control on the singular locus. We can assume that at least one point p_i that is a contained in a positive dimensional component of the singular locus is not specialized. Thus, we left $n-2$ general points on L and 2 general points in $\mathbb{P}^n$. Now, we can use Proposition 5.4 of [7] on the trace (right space), which turns out to be empty, and Proposition 3.5 on the residual (left space), which has dimension $n+1$. If the singular locus would have

[5] **Really there is no induction here! It is tricky, the points are specialized to fill the trace (right space), so we specialize $n-2$ points on L instead of the $n-3$'s needed in the induction procedure. Otherwise we cannot argue on the singular locus. Do you agree ? Luca: I do.**

a positive dimensional component, then, since the dimension of the space of cubics is constant along the specialization (it equals $n+1$), we would get a deformation of the singular locus, which should be of positive dimension at every point. This, however, contradicts Proposition 3.5, hereby concluding the proof. □

We are now ready to prove the first part of Theorem 3.2.

Proof of Theorem 3.2, case (i). We fix a linear subspace $L \subset \mathbb{P}^n$ of codimension 3 and consider the exact sequence

$$0 \longrightarrow I_{L,\mathbb{P}^n}(3) \longrightarrow S_{\mathbb{P}^n}(3) \longrightarrow S_L(3),$$

where $S_{\mathbb{P}^n}(3)$ is the space of cubic polynomials on $\mathbb{P}^n$ and the quotient space $S_L(3)$ is isomorphic to the space of cubic polynomials on L. Then, we specialize to L as many points as possible in such a way that the trace with respect to L imposes independent conditions on the cubics of L. To be precise, we have $k_n - 1 = \frac{(n+3)(n+2)}{6} - 1$ double points and we specialize $k_{n-3} = \frac{n(n-1)}{6}$ of them to L[6], leaving n points outside. Then, the result follows from Theorem 5.1 of [7] on the trace (right space), which turns out to be empty, and by Proposition 3.6 on the residual (left space), which has dimension $n+1$. If the contact locus has positive dimension, then, since the dimension of the space of cubics is constant and equal to $(n+1)$ in the degeneration, we get a deformation of the singular locus with a positive dimension at every point, contradicting Proposition 3.6 and concluding the proof. □

3.2. **Proof of Theorem 3.2 (ii) by induction on codimension 3 and 4.** For proving the second case in Theorem 3.2, we need to introduce several other auxiliary results on configurations that involve subspaces of codimension three and four. These configurations are covered in Propositions 3.7 through 3.12.

3.2.1. *Codimension 4, 3, 3.*

Proposition 3.7. *Let $n \geq 6$, and let $L, M, N \subset \mathbb{P}^n$ be general subspaces of codimension 4, 3, and 3, respectively. Let l_1, l_2, l_3 be general points on L. Let m_i, respectively, n_i with $i = 1, \ldots, 4$ be four general points on M, respectively N. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain $L \cup M \cup N$ and are singular at the 11 points $\mathbf{X} = \{l_1, l_2, l_3, m_1, \ldots, m_4, n_1, \ldots, n_4\}$ is empty.*

Proof. The proof is similar to the proof of Proposition 3.3. The Macaulay2 code proves the base cases $n = 6$, 7, 8 and 9. For $n \geq 9$, we may choose coordinates such that $L = (x_0 \ldots x_3)$, $M = (x_4 \ldots x_6)$, $N = (x_7 \ldots x_9)$ and the statement follows by induction on n. Indeed, as in the proof of Proposition 3.3, the space $I_{L\cup M\cup N,\mathbb{P}^n}(2)$ is empty, thus for a general hyperplane $H \subset \mathbb{P}^n$, the Castelnuovo sequence induces an embedding

$$0 \longrightarrow I_{L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{(L\cup M\cup N)\cap H,H}(3)$$

and moreover every cubic in the left space is a cone with vertex at $L \cap M \cap N$. Hence, by specializing the 11 points to the hyperplane H, we get:

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L\cup M\cup N)\cap H,H}(3).$$

Then, the statement follows by induction. □

[6] **here is similar, we do not play induction!**

Proposition 3.8. *Let $n \geq 7$, let $n \equiv 1 \mod 3$, and let $L, M \subset \mathbb{P}^n$ be subspaces of codimension 4 and 3, respectively. Let l_i with $i = 1, \ldots, n-3$ be general points on L. Let m_i with $i = 1, \ldots, \frac{4n-10}{3}$ be general points on M. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain $L \cup M$ and are singular at all the l_i's and m_i's and at four general points $p_1, p_2, p_3, p_4 \in \mathbb{P}^n$, is empty.*

Proof. The Macaulay2 script proves the base case $n = 7$. For $n = 3k+1$ with $k \geq 3$, the statement follows by induction from $n-3$ to n. Indeed, given a third general subspace N of codimension 3, we get the exact sequence

$$0 \longrightarrow I_{L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{(L\cup M)\cap N,N}(3),$$

where the dimensions of the three spaces in the sequence are respectively 36, $12n-18$ and $12n-54$. Let $\mathbf{X}$ denote the union of the double points supported at the p_i's, l_i's and m_i's. Assume that we specialize $n-6$ of the points $l_i \in L$ to $L \cap N$, $\frac{4n-22}{3}$ of the points $m_i \in M$ to $M \cap N$, and the four points $p_1, \ldots, p_4$ to N. Then, we obtain a sequence

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{\mathbf{X}\cup L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L\cup M)\cap N,N}(3),$$

where the trace $(\mathbf{X} \cup L \cup M) \cap N$ satisfies the assumptions on $N = \mathbb{P}^{n-3}$, so that we can apply induction. Then, we may conclude, as the residual (left space) satisfies the hypotheses of Proposition 3.7 and consequently it is empty. □

Proposition 3.9. *Let $n \geq 7$, $n \equiv 1 \mod 3$, and $L \subset \mathbb{P}^n$ be a subspace of codimension four. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that are singular at $k_{n-4} = \frac{(n-1)(n-2)}{6}$ general points l_i on L (and, thus, contain L, by Theorem 3.1* [7] *) and at $\frac{4n+2}{3}$ general points $p_i \in \mathbb{P}^n$ is empty.*

Proof. The Macaulay2 script proves the base case $n = 7$. For $n = 3k+1$ with $k \geq 3$, the statement follows by the sequence

$$0 \longrightarrow I_{L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{L,\mathbb{P}^n}(3) \longrightarrow I_{L\cap M,M}(3),$$

where M is a general subspace of codimension 3. If we denote by $\mathbf{X}$ the union of the double points supported at the points l_i and p_i, then we get the sequence

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{\mathbf{X}\cup L,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L)\cap M,M}(3).$$

Then, we specialize k_{n-7} of the points l_i to $L \cap M$ and $\frac{4n-10}{3}$ of the points p_i to M. The trace (right space) contains exactly k_{n-3} double points and turns out to be empty by induction. Thus, there remain $n-3$ general points on L and 4 general points on $\mathbb{P}^n$; we can then use Proposition 3.8 on the residual (left space) to conclude.

□

3.2.2. *Codimension 4, 4, 3.*

Proposition 3.10. *Let $n \geq 8$, and let $L, M, N \subset \mathbb{P}^n$ be general subspaces of codimension respectively 4, 4, and 3. Let l_i, respectively m_i, with $i = 1, \ldots, 4$ be general points on L, respectively M. Finally, let n_i with $i = 1, \ldots, 5$ be general points on N. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain $L \cup M \cup N$ and that are singular at the 13 points $\mathbf{X} = \{l_1, \ldots, l_4, m_1, \ldots, m_4, n_1, \ldots, n_5\}$ has dimension 1. In other words, there is a unique cubic hypersurface W through $L \cup$*

[7] BEWARE: as it is formulated, Theorem 3.1 does not state this

$M \cup N$ and singular at $\mathbf{X}$. Furthermore, the singular locus of W is contained in $L \cup M \cup N$.

Proof. The proof is similar to the proof of Proposition 3.3. The Macaulay2 code proves the base cases $n = 8$, 9, and 10. For $n \geq 10$, we may choose coordinates such that $L = (x_0 \ldots x_3)$, $M = (x_4 \ldots x_7)$, $N = (x_8 \ldots x_{10})$ and the statement follows by induction on n.

Indeed, as in the proofs of Proposition 3.3 and Proposition 3.7, letting $H \subset \mathbb{P}^n$ be a general hyperplane, then the Castelnuovo sequence induces the inclusion

$$0 \longrightarrow I_{L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{(L\cup M\cup N)\cap H,H}(3).$$

because the space $I_{L\cup M\cup N,\mathbb{P}^n}(2)$ is empty. Hence, by specializing the 13 points on the hyperplane H, we get an exact sequence:

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L\cup M\cup N)\cap H,H}(3).$$

Now the statement follows by induction. □

Proposition 3.11. *Let $n \geq 8$, $n \equiv 2 \mod 3$, and $L, M \subset \mathbb{P}^n$ be subspaces of codimension 4. Let l_i and m_i, where $i = 1, \ldots, \frac{4n-14}{3}$, be general points on L and M, respectively. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain $L \cup M$ and are singular at the $\frac{8n-28}{3}$ points l_i, m_i, $i = 1, \ldots, \frac{4n-14}{3}$, and at an additional set of five general points $p_i \in \mathbb{P}^n$, $i = 1, \ldots, 5$, has dimension $\frac{n+1}{3}$. Furthermore, its common singular locus, which contains the linear space $L \cap M$, is 0-dimensional at each of the points $p_1, \ldots, p_5$.*

Proof. The base case $n = 8$ can be proved with the Macaulay2 script. For $n = 3k+2$ with $k \geq 3$, the statement follows by induction on k. Given a third general subspace N of codimension 3, we get the exact sequence

$$0 \longrightarrow I_{L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{(L\cup M)\cap N,N}(3),$$

where the dimensions of the three spaces in the sequence are respectively 48, $16(n-2)$ and $16(n-5)$.

Let $\mathbf{X}$ denote the union of the double points supported at $p_1, \ldots, p_5$, l_i and m_i with $i = 1, \ldots, \frac{4n-14}{3}$. Then, we specialize $\frac{4n-26}{3}$ of the points $l_i \in L$ to $L \cap N$, $\frac{4n-26}{3}$ of the points $m_i \in M$ to $M \cap N$, and the points $p_1, \ldots, p_5$ to N. We thus obtain a sequence

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M\cup N,\mathbb{P}^n}(3) \longrightarrow I_{\mathbf{X}\cup L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L\cup M)\cap N,N}(3),$$

where the trace $(\mathbf{X} \cup L \cup M) \cap N$ satisfies the assumptions on $N = \mathbb{P}^{n-3}$, so that we can apply induction. Then, the residual (left space) satisfies the hypotheses of Proposition 3.10 and has dimension one. Moreover, the common singular locus has to be contained in the common singular locus of the left 1-dimensional space. After the degeneration, the space $I_{\mathbf{X}\cup L\cup M,\mathbb{P}^n}(3)$ still has dimension less than or equal to $1 + \frac{n-2}{3} = \frac{n+1}{3}$, by induction, and, therefore, its dimension equals $\frac{n+1}{3}$, by semicontinuity. The common singular locus cannot be positive dimensional at the points $p_1, \ldots, p_5$, because otherwise it should be positive dimensional in the trace (right space), while we know that it is 0-dimensional there by induction. □

Proposition 3.12. *Let $n \geq 8$, $n \equiv 2 \mod 3$, and $L \subset \mathbb{P}^n$ be a subspace of codimension 4. Then, the space of cubic hypersurfaces in $\mathbb{P}^n$ that contain L and are singular at $k_{n-4} = \frac{(n-1)(n-2)}{6}$ general points $l_i \in L$ and at $\frac{4n+1}{3}$ general points*

$p_i \in \mathbb{P}^n$ has dimension $\frac{n+1}{3}$. Furthermore, its singular locus is of dimension 0 at the points p_i. [8]

Proof. The statement follows by the sequence[9]

$$0 \longrightarrow I_{L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{L,\mathbb{P}^n}(3) \longrightarrow I_{L\cap M,M}(3),$$

where M is a general subspace of codimension 4. Denoting by $\mathbf{X}$ the union of the double points supported at the points l_i's and p_i's, we get

$$0 \longrightarrow I_{\mathbf{X}\cup L\cup M,\mathbb{P}^n}(3) \longrightarrow I_{\mathbf{X}\cup L,\mathbb{P}^n}(3) \longrightarrow I_{(\mathbf{X}\cup L)\cap M,M}(3).$$

We assume by contradiction that the singular locus has positive dimension at some point p_i. Then, we specialize k_{n-8} of the points l_i to $L \cap M$ and $\frac{4n-14}{3}$ of the points p_i to M. We choose the specialization so that at least one point p_i that is a contained in a positive dimensional component of the singular locus is not specialized to M. The trace (right space) contains exactly k_{n-4} double points and is empty because of Proposition 3.9. There remain $\frac{4n-14}{3}$ general points on L and 5 general points on $\mathbb{P}^n$. Thus, we can use Proposition 3.11 on the residual (left space), which has dimension $\frac{n+1}{3}$. If the singular locus has positive dimension at the points p_i, since the dimension of the space of cubics is constant and equal to $\frac{n+1}{3}$ through the degeneration, we get a deformation of the singular locus, which is of positive dimension at every point, hereby contradicting Proposition 3.11. $\square$

Proof of Theorem 3.2, part (ii). We fix a codimension four linear subspace $L \subset \mathbb{P}^n$ and we use the exact sequence

$$0 \longrightarrow I_{L,\mathbb{P}^n}(3) \longrightarrow S_{\mathbb{P}^n}(3) \longrightarrow S_L(3),$$

where, as above, $S_{\mathbb{P}^n}(3)$ is the space of cubic polynomials on $\mathbb{P}^n$ and the quotient space $S_L(3)$ is isomorphic to the space of cubic polynomials on L. We specialize k_{n-4} points on L, leaving $\frac{4n+1}{3}$ points outside. Then, the result follows from Theorem 5.1 of [7] on the trace (right space), which turns out to be empty and by Proposition 3.12 on the residual (left space). If the contact locus would have a positive dimension, then, since the dimension of the space of cubics is constant and equal to $\frac{n+1}{3}$ in the degeneration, we would get a deformation of the singular locus, which should be of positive dimension at every point; however, this contradicts Proposition 3.12, hereby concluding the proof. $\square$

4. Dual varieties to the relevant secant varieties

Denote by $\mathrm{T}_x\mathcal{X}$ the tangent space to the projective variety $\mathcal{X} \subset \mathbb{P}^n$ at the point $x \in \mathcal{X}$. Following the notation of [9] we say that $\mathcal{X}$ is *not k-weakly defective* if the general hyperplane H containing the linear span of the tangent spaces at k general points $x_1, \ldots, x_k \in \mathcal{X}$, i.e., $\langle \mathrm{T}_{x_1}\mathcal{X}, \ldots, \mathrm{T}_{x_k}\mathcal{X}\rangle \subset H$, is tangent to $\mathcal{X}$ only at finitely many points. This is equivalent with saying that the k-contact locus with respect to $x_1, \ldots, x_k$ and H is zero-dimensional.

---

[8] Do we prove that it is 0-dimensional at *any* point p_i, or just at *some* point p_i (which, by the way, is sufficient for our purposes?)

[9] **I think there is no need of induction, neither to prove any base case. The M2 script contains it just for a double-check. Luca: ok.**

For any projective variety $\mathcal{X}$, we will denote by $\mathcal{X}^\vee$ the dual variety to $\mathcal{X}$. Note that the dual of the secant variety $\sigma_k(v_d(\mathbb{P}^n))^\vee$ contains the points corresponding to hypersurfaces of degree d in $\mathbb{P}^n$ with k general singular points, and it has codimension $\geq k$, where k is the expected value for the codimension.

Proposition 4.1. *Let $\mathcal{X} \subset \mathbb{P}^N$ and let $\sigma_k(\mathcal{X})$ be the k-secant variety of $\mathcal{X}$. Then, the following are equivalent:*

(i) the general hyperplane H containing $\langle \mathrm{T}_{x_1}\mathcal{X}, \dots, \mathrm{T}_{x_k}\mathcal{X}\rangle$ for general $x_1, \dots, x_k$ is tangent to $\mathcal{X}$ only at $x_1, \dots, x_k$, i.e., the k-contact locus with respect to $x_1, \dots, x_k$ and H consists exactly of the points $x_1, \dots, x_k$,

(ii) $\mathcal{X}$ is not k-weakly defective, and

(iii) $\dim[\sigma_k(\mathcal{X})]^\vee = N-k$, that is a general hyperplane tangent to $\sigma_k(\mathcal{X})$ is tangent along a linear space of projective dimension $k-1$.

Proof. (i) $\iff$ (ii) follows from [8, Theorem 1.4]. (ii) $\iff$ (iii) follows from Terracini's Lemma. $\square$

It is interesting to describe the dual varieties of $\sigma_k(v_d(\mathbb{P}^n))$ in the exceptional cases of Theorems 1.1 and 1.2. They have dimension smaller than expected.

Theorem 4.2. *The following dual varieties correspond to the exceptional cases appearing in Theorems 1.1 and 1.2.*

(i) $\sigma_9(v_6(\mathbb{P}^2))^\vee$ contains the plane sextics which are double cubics. It has codimension 18.

(ii) $\sigma_8(v_4(\mathbb{P}^3))^\vee$ contains the quartic surfaces which are reducible in a pair of quadrics. It has codimension 16.

(iii) $\sigma_9(v_3(\mathbb{P}^5))^\vee$ contains the cubic 4-folds which can be written as the determinant of a 3×3 matrix with linear entries. It has codimension 18.

To compute the dimension in third case, note that the Hilbert scheme of elliptic normal sextic curves in $\mathbb{P}^5$ has dimension 36. So the cubic hypersurfaces coming from this construction have dimension 37, and $37 + 18 = 55 = \binom{8}{3} - 1$.

We remark that the defective Veronese varieties according to the classification of Alexander and Hirschowitz [2] (see [27] for the equations of the defective secant varieties) yield the following dual varieties

(i) $\sigma_{n(n+3)/2}(v_4(\mathbb{P}^n))^\vee$, for $n = 2, 3, 4$, contains quartic hypersurfaces which are double quadrics. It has codimension $\binom{n+2}{3}\frac{n+7}{4}$.

(ii) $\sigma_7(v_3(\mathbb{P}^4))^\vee$ contains cubic 3-folds which can be written as the determinant of a 3×3 symmetric matrix with linear entries. It has codimension 13. Indeed, it is birational to the Hilbert scheme of quartic rational normal curves which has dimension 21.

5. Specific identifiability of symmetric tensors

While the generic symmetric tensor is expected to admit a unique Waring decomposition, *specific* tensors, whose Waring decomposition is assumed to be known, may admit multiple decompositions. Therefore, we proceed by presenting an approach for certifying specific identifiability of symmetric tensors of small rank by checking not tangential weak defectivity of the r-secant variety of a Veronese variety in the given symmetric tensor. The strategy is an adaption of the algorithm from [12] to the setting of identifiability with respect to the Veronese variety $\mathcal{V} = v_d(\mathbb{P}^n)$.

As such, the presented condition will only be a sufficient condition; that is, if the criterion does not apply, then the outcome of the test is inconclusive. On the other hand, if the criterion applies, then the given input tensor is r-identifiable and of symmetric rank r. Throughout this section, it is assumed that we are handed a Waring decomposition

$$p = p_1 + \cdots + p_r \in \sigma_r(\mathcal{V}) \subset S^d\mathbb{C}^{n+1} \subset (\mathbb{C}^{n+1})^{\otimes d},$$

wherein the point $p_i \in \mathcal{V}$ is the degree d Veronese embedding of a vector $\mathbf{x}_i \in \mathbb{C}^{n+1}$. That is, we know the points p_i appearing in the decomposition. The goal consists of certifying that p is r-identifiable. To this end, the strategy in [12] suggests a two-step procedure: Prove that p is a smooth point, and verify the Hessian criterion.

We will restrict our attention to nondefective r-secants of $\mathcal{V}$, because identifiability will not hold for general tensors on a defective r-secant variety. This is the interesting setting, because the Alexander–Hirschowitz theorem [2] stipulates that most $\sigma_r(\mathcal{V})$ are nondefective.

In this section, the following constants will be employed regularly:

$$\Gamma = \dim S^d\mathbb{C}^{n+1} = \binom{n+d}{d} \text{ and } \Pi = \dim(\mathbb{C}^{n+1})^{\otimes d} = (n+1)^d.$$

5.1. **The Hessian criterion.** We recall the main proposition from [12], and adapt it to the present context of symmetric tensors.

Lemma 5.1. *Let $\mathcal{V} = v_d(\mathbb{P}^n)$ be a nondefective Veronese variety let $r \le \lceil r_{d,n} \rceil - 1$ with $r_{d,n}$ as in (2). Let $p = \sum_{i=1}^r p_i \in \sigma_r(\mathcal{V})$ be a nonsingular point. If the r-*tangential contact locus

$$\mathcal{C}_r = \{p \in \mathcal{V} \mid \mathrm{T}_p\mathcal{V} \subset \mathrm{M} = \langle \mathrm{T}_{p_1}\mathcal{V}, \ldots, \mathrm{T}_{p_r}\mathcal{V}\rangle\} \subset \mathcal{V}. \tag{4}$$

is zero-dimensional at every $p_1, p_2, \ldots, p_r$, then p is r-identifiable and $p = \sum_{i=1}^r p_i$ is its unique decomposition.

Proof. The proof is obtained as in the proof of [12, Theorem 4.5] and [12, Lemma 4.4], substituting the Segre $\mathcal{S}$ with the Veronese variety $\mathcal{V}$. All arguments employed in those proofs are valid for the Veronese variety. □

The only part of the discussion in [12, Section 2] that requires some significant modifications concerns the check that the r-tangential contact locus is zero-dimensional at $p_i \in \mathcal{V}$, $i = 1, \ldots, r$. We derive the appropriate Hessian criterion as follows. The Veronese embedding is given explicitly by

$$\begin{aligned} v_d : \mathbb{C}^{n+1} &\to \mathbb{C}^\Gamma \\ \mathbf{x} &\mapsto [x_{i_1}x_{i_2}\cdots x_{i_d}]_{0\le i_1\le i_2\le\cdots\le i_d\le n} \end{aligned}$$

Its formal derivative is readily found to be

$$\begin{aligned} \frac{\partial}{\partial x_u} v_d &= \left[\tfrac{\partial}{\partial x_u} x_{i_1}x_{i_2}\cdots x_{i_d}\right]_{0\le i_1\le i_2\le\cdots\le i_d\le n} \\ &= \left[z_{i_1,i_2,\ldots,i_d,u} \cdot \textstyle\prod_{i_k\ne u} x_{i_k}\right]_{0\le i_1\le i_2\le\cdots\le i_d\le n} \end{aligned}$$

where $z_{i_1,i_2,\ldots,i_d,u}$ is the number of indices equal to u:

$$z_{i_1,i_2,\ldots,i_d,u} = m \text{ with } i_{k-1} < u = i_k = i_{k+1} = \cdots = i_{k+m-1} < i_{k+m}.$$

The span of the tangent space $\mathrm{T}_{p_i}\mathcal{V}$ to the Veronese variety $\mathcal{V}$, evaluated at $p_i = v_d(\mathbf{x}_i)$, is given by the column span of

$$T_i = \begin{bmatrix}(\frac{\partial}{\partial x_0}\phi)(\mathbf{x}_i) & (\frac{\partial}{\partial x_1}\phi)(\mathbf{x}_i) & \cdots & (\frac{\partial}{\partial x_n}\phi)(\mathbf{x}_i)\end{bmatrix}. \tag{5}$$

By Terracini's lemma [30, 32], the tangent space $\mathrm{T}_p\sigma_r(\mathcal{V})$ to the r-secant variety of $\mathcal{V}$ at a smooth point $p \in \sigma_r(\mathcal{V})$ is given by the concatenation of the matrix representations of the tangent space:

$$T = \begin{bmatrix}T_1 & T_2 & \cdots & T_r\end{bmatrix}, \tag{6}$$

where we assumed $p = \sum_{i=1}^r v_d(\mathbf{x}_i)$. We then construct the Cartesian equations for this tangent space; practically, this can be done through Gaussian elimination on the extended system $[\,T\ I\,]$, or by computing the orthogonal complement of the column span of T by using the full singular value decomposition. Let K be a $\Gamma \times (\Gamma - r(n+1))$ matrix with linearly independent columns such that $K^T T = 0$; that is, the columns of K give the desired Cartesian equations. Then, letting $\ell = \Gamma - r(n+1)$ be the expected codimension of $\sigma_r(\mathcal{V})$, we have

$$q_l(y_1, y_2, \ldots, y_\Gamma) = \sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\cdots\sum_{i_d=i_{d-1}}^{n} k_{m(i_1,i_2,\ldots,i_d),l}\cdot y_{m(i_1,i_2,\ldots,i_d)} = 0,$$

$l = 1, 2, \ldots, \ell$, where y_i are the variables in $\mathbb{C}^\Gamma$. Herein, $m(\cdot)$ is essentially a map between the symmetric multilinear indices $(i_1, i_2, \ldots, i_d)$, $0 \le i_1 \le i_2 \le \cdots \le i_d \le n$, and a linear index $m(i_1, i_2, \ldots, i_d)$, which can be defined by requiring that

$$(v_d(\mathbf{x}))_{m(i_1,i_2,\ldots,i_d)} = x_{i_1}x_{i_2}\cdots x_{i_d}.$$

Plugging in a parameterization of the Veronese variety and letting M be as in (4), we can parameterize the intersection $\mathcal{V} \cap \mathrm{M}$ explicitly by exploiting the above observation

$$q_l(x_0, x_1, \ldots, x_n) = \sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\cdots\sum_{i_d=i_{d-1}}^{n} k_{m(i_1,i_2,\ldots,i_d),l}\cdot x_{i_1}x_{i_2}\cdots x_{i_d} = 0, \tag{7}$$

$l = 1, 2, \ldots, \ell$, with x_i the variables in $\mathbb{C}^{n+1}$. Deriving this expression with respect to the variables gives us ℓ Cartesian equations of $\mathcal{C}_r = \mathrm{T}\mathcal{V} \cap M$. The dimension of this algebraic variety can then be determined by computing the dimension of its tangent space at each of the points p_1, p_2, …, p_r. Thus, as in [12], we derive each of the ℓ equations in (7) twice to the variables. From straightforward computations, we find, for $l = 1, 2, \ldots, \ell$, that

$$\frac{\partial^2}{\partial x_u \partial x_v} q_l = \sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\cdots\sum_{i_d=i_{d-1}}^{n} k_{m(i_1,i_2,\ldots,i_d),l}\cdot z_{i_1,i_2,\ldots,i_d,u,v}\prod_{\substack{i_k\neq u\\ i_k\neq v}} x_{i_k},$$

where we have

$$z_{i_1,i_2,\ldots,i_d,u_1,u_2} = \begin{cases} m_1 m_2 & \text{if } u_1 \neq u_2 \\ m_1(m_1-1) & \text{if } u_1 = u_2 \end{cases}$$

with

$$i_{k_j-1} < u_j = i_{k_j} = \cdots = i_{k_j+m_j-1} < i_{k_j+m_j}, \quad j = 1, 2.$$

We thus obtain the stacked Hessian evaluated at $p_k \in \mathcal{V}$, precisely as in [12], i.e.,

$$H(p_k) = [H_i]_{i=1}^{\ell} \ \text{ with } \ H_i = \left[\left(\frac{\partial^2}{\partial x_u \partial x_v} q_i\right)(\mathbf{x}_k) \right]_{u,v=0}^{n}.$$

If the rank of this stacked Hessian matrix is maximal, i.e., $\dim \mathcal{V} = n$, at every point $p_1, \ldots, p_r \in \mathcal{V}$, then the r-contact locus is zero-dimensional through these points, so that the symmetric tensor p is r-identifiable, by Lemma 5.1, provided that p is a smooth point of $\sigma_r(\mathcal{V})$. Note that the general point is smooth, so that the criterion may be applied with probability one to a randomly sampled point of $\sigma_r(\mathcal{V})$, imposing any reasonable continuous probability distribution.

5.2. **The smoothness criterion.** The Hessian criterion in Lemma 5.1 may only be applied to smooth points of $\sigma_r(\mathcal{V})$. One approach for proving smoothness consists of verifying that the local equations of $\sigma_r(\mathcal{V})$ are of the expected degree. Such equations are known in the case when the number of terms r in the symmetric decomposition is sufficiently small. A standard nontrivial set of local equations is generated by the $(r+1)$-minors of the usual symmetric flattenings; see [22, Theorem 7.3.3.3] and [21, Theorems 4.10A and 4.5A]. For Veronese embeddings of odd degree, the Young flattenings from [23, Section 4] apply in a wider range than the standard symmetric flattenings; however, they are more involved to explain and implement. Our discussion will focus on the simple symmetric flattenings, which can still handle a respectable number of cases for Veronese embeddings of degree at least four.

The strategy that we adopt for proving that p corresponds to a smooth point consists of obtaining simultaneously a lower and an upper bound on the dimension of the [10] r-secant variety $\sigma_r(\mathcal{V})$ in p, which, additionally, remains valid in an open set around p. A lower bound that remains valid in a Euclidean-open set containing p is readily obtained from an application of Terracini's lemma; namely, by verifying that the rank of the matrix T in (6) equals $r(n+1)$, i.e., the expected dimension of $\sigma_r(\mathcal{V})$ when the rank r is subgeneric. An upper bound can be obtained by exploiting the property

$$\mathrm{N}_p \sigma_r(\mathcal{V}) \supset \mathrm{N}_p \mathcal{Y} \ \text{ whenever } \ \sigma_r(\mathcal{V}) \subset \mathcal{Y}$$

is an inclusion of varieties and $p \in \sigma_r(\mathcal{V})$; in the above, $\mathrm{N}_x \mathcal{X}$ denotes the normal space to the variety $\mathcal{X}$ in the point x. The inclusions we consider arise from inspecting the standard $(k, d-k)$-symmetric flattenings of $p \in S^d \mathbb{C}^{n+1}$, which we can regard as a linear map $p_{(k,d-k)} : (S^{d-k} \mathbb{C}^{n+1})^\vee \to S^k \mathbb{C}^{n+1}$ for every $1 \le k \le d-1$. The map can be expressed explicitly in coordinates as the matrix

$$p_{(k,d-k)} = \mathbf{x}_1^{\otimes k} (\mathbf{x}_1^{\otimes d-k})^T + \cdots + \mathbf{x}_r^{\otimes k} (\mathbf{x}_r^{\otimes d-k})^T;$$

note that we regard this matrix as living in $(\mathbb{C}^{n+1})^{\otimes k} \otimes (\mathbb{C}^{n+1})^{\otimes d-k}$, rather than in $S^k \mathbb{C}^{n+1} \otimes S^{d-k} \mathbb{C}^{n+1}$, i.e., we consider the Veronese embedding in $(\mathbb{C}^{n+1})^{\otimes d}$ rather than $S^d \mathbb{C}^{n+1} \cong \mathbb{C}^\Gamma$, for reasons that will become clear shortly. Clearly, $p \in \sigma_r(v_d(\mathbb{P}^n)) \subset \sigma_r(v_k(\mathbb{P}^n) \times v_{d-k}(\mathbb{P}^n))$, so that

$$\mathrm{N}_p \sigma_r(\mathcal{V}) \supset \mathrm{N}_p \sigma_r(v_k(\mathbb{P}^n) \times v_{d-k}(\mathbb{P}^n)), \quad k = 1, 2, \ldots, d-1.$$

By [23, Proposition 2.5.1], the conormal space is given explicitly by

$$\mathrm{N}_p \sigma_r(v_k(\mathbb{P}^n) \times v_{d-k}(\mathbb{P}^n)) = \ker(p_{(k,d-k)}) \otimes (\mathrm{Im}(p_{(k,d-k)})^\perp,$$

[10] **tangent space to the ?**

subject to the condition that $\operatorname{rank}(p_{(k,d-k)}) = r$. As these spaces are naturally embedded in $(\mathbb{C}^{n+1})^{\otimes k} \otimes (\mathbb{C}^{n+1})^{\otimes d-k} \cong (\mathbb{C}^{n+1})^{\otimes d}$, we can consider their span, which is still contained in the normal space to the r-secant variety of $\mathcal{V}$:

$$\mathrm{N}_p\sigma_r(\mathcal{V}) \supset \sum_{k=1}^{\lfloor \frac{d}{2} \rfloor} \big(\ker(p_{(k,d-k)}) \otimes (\operatorname{Im}(p_{(k,d-k)})^{\perp} + (\operatorname{Im}(p_{(k,d-k)})^{\perp} \otimes \ker(p_{(k,d-k)})\big),$$

where all of the sums should be interpreted as (nondirect) sums of vector spaces. Note that we exploited the duality of the maps $p_{(k,d-k)} = p_{(d-k,k)}^T$ in the above expression. In coordinates, a basis for $\mathrm{N}_p\sigma_r(v_k(\mathbb{P}^n) \times v_{d-k}(\mathbb{P}^n))$ can be obtained readily from the singular value decomposition (SVD) of $p_{(k,d-k)}$; namely, if $r < \min\{(n+1)^k, (n+1)^{d-k}\}$, then we can write the SVD as

$$p_{(k,d-k)} = USV^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T, \tag{8}$$

where $S \in \mathbb{R}^{(n+1)^k \times (n+1)^{d-k}}$ is diagonal, $S_1 \in \mathbb{R}^{r \times r}$, $U_1 \in \mathbb{C}^{(n+1)^k \times r}$, $V_1 \in \mathbb{C}^{(n+1)^{d-k} \times r}$, and $U \in \mathbb{C}^{(n+1)^k \times (n+1)^k}$ and $V \in \mathbb{C}^{(n+1)^{d-k} \times (n+1)^{d-k}}$ are orthonormal with respect to the Hermitian inner product. Then, provided that $p_{(k,d-k)}$ is of maximal rank, i.e., r, we have

$$\mathrm{N}_p\sigma_r(v_k(\mathbb{P}^n) \times v_{d-k}(\mathbb{P}^n)) = \operatorname{range}(U_2 \otimes V_2) = N_k, \text{ and} \tag{9a}$$

$$\mathrm{N}_p\sigma_r(v_{d-k}(\mathbb{P}^n) \times v_k(\mathbb{P}^n)) = \operatorname{range}(V_2 \otimes U_2) = N_{d-k}, \tag{9b}$$

where $\otimes$ should be interpreted as the Kronecker product in the above expression. Defining

$$N = \begin{bmatrix} N_1 & N_2 & \cdots & N_{d-1} \end{bmatrix}, \tag{10}$$

we can state the following sufficient condition for smoothness.

Lemma 5.2 (Sufficient condition for smoothness)**.** *Let $v_d(\mathbb{P}^n)$ be the dth degree Veronese embedding of $\mathbb{P}^n$, let $p = p_1 + \cdots + p_r \in \sigma_r(\mathcal{V})$, let the tangent space be represented by T as in (6), and let the normal space matrix N be given as in (10). If*

$$\operatorname{rank}(T) = r(n+1) \quad \textit{and} \quad \operatorname{rank}(N) = \Pi - r(n+1),$$

then p is a smooth point of $\sigma_r(\mathcal{V})$.

Proof. If T is of maximal rank, i.e., $r(n+1)$, then p will be a singular point only if the tangent cone and tangent space do not coincide. Then, there should exist a vector $\mathbf{v} \in \mathbb{C}^{\Pi}$ not contained in the column span of T, so that the normal space in p will be of maximal dimension $\Pi - r(n+1) - 1$. This contradicts the assumption on the rank of N, concluding the proof. □

5.3. **An elementary algorithm.** We present an elementary algorithm implementing the steps outlined in the previous subsections for verifying the Hessian and smoothness criteria. As performance is not the main concern, in this paper, we have chosen to present a simpler algorithm that evaluates the Hessian criterion in the enlarged space $(\mathbb{C}^{n+1})^{\otimes d} \cong \mathbb{C}^{\Pi}$, rather than $S^d\mathbb{C}^{n+1} \cong \mathbb{C}^{\Gamma}$, for the sake of a more straightforward computer implementation in typical programming languages. For a constant d, the asymptotic time and space complexities in function of n are not influenced by this choice. Explicit expressions in coordinates for the tangent space matrix T and the stacked Hessian H are readily obtained in $\mathbb{C}^\Pi$. It is not difficult to see that T_i in (5) becomes

$$T_i = I_{n+1} \otimes \mathbf{x}_i \otimes \cdots \otimes \mathbf{x}_i + \cdots + \mathbf{x}_i \otimes \cdots \otimes \mathbf{x}_i \otimes I_{n+1}, \quad i = 1, 2, \ldots, r, \tag{11}$$

where the sum is a sum of matrices; note that this is essentially the symmetrization of the tangent space to the Segre variety in p_i. Similarly, every individual component in the stacked Hessian $H(p_k)$ can be found to equal, for $l = 1, 2, \ldots, \ell$,

$$H_l(p_k) = \sum_{0 \le p \ne q \le n} (\mathbf{x}_i, \ldots, \mathbf{x}_i, I^p, \mathbf{x}_i, \ldots, \mathbf{x}_i, I^q, \mathbf{x}_i, \ldots, \mathbf{x}_i)^T \cdot \mathfrak{K}_l, \tag{12}$$

where I^p is the $n+1$ identity matrix, which appears at position p in the tuple, and I^q is the $n+1$ identity matrix, which appears at position q in the tuple. In the above expression, $\mathfrak{K}_l \in (\mathbb{C}^{n+1})^{\otimes d}$ is the nonsymmetric tensor that satisfies

$$(\mathfrak{K}_l)_{i_1, i_2, \ldots, i_d} = k_{\hat{m}(i_1, i_2, \ldots, i_d), l},$$

where the columns of the matrix $K \in \mathbb{C}^{\Pi \times \ell}$, where $\ell = \Pi - r(n+1)$, contain a basis for the kernel of T and where $\hat{m}(\cdot)$ is a map from multilinear indices $(i_1, i_2, \ldots, i_d)$, $0 \le i_j \le n$, $j = 1, 2, \ldots, d$, to a linear index $\hat{m}(i_1, i_2, \ldots, i_d)$, which can be defined as a map that satisfies

$$(\mathbf{v}^1 \otimes \mathbf{v}^2 \otimes \cdots \otimes \mathbf{v}^d)_{\hat{m}(i_1, i_2, \ldots, i_d)} = v^1_{i_1} v^2_{i_2} \cdots v^d_{i_d}$$

for every set of vectors $\mathbf{v}^k \in \mathbb{C}^{n+1}$, and where the tensor product is embedded naturally into $\mathbb{C}^\Pi$, i.e., it may be interpreted as the Kronecker product. Note that (12) is essentially the symmetrization of the Hessian matrices appearing in [12, Remark 4.6].

Based on the foregoing discussion, we may now present an algorithm for testing specific r-identifiability of a point on $\sigma_r(\mathcal{V})$; it is given as Algorithm 5.1.

TODO: Fix smoothness test

TODO: Write Octave/Matlab implementation.

6. Conclusions

We adapted and extended the approach of Brambilla and Ottaviani [7] for proving nondefectivity so that identifiability of third-order symmetric tensors could be proved. By combining this new result with Ballico's result [4] concerning the identifiability of Veronese embeddings of degree at least four, we were able to conclude the classification of the r-tangentially weakly defective secant varieties of Veronese varieties in the case of subgeneric ranks r. For $d \ge 3$, only three Veronese varieties admit tangentially weakly defective r-secant varieties. The dual variety to these exceptional r-secant varieties were considered, all of which had a dimension strictly less than the expected dimension. Finally, we presented an algorithm for testing whether a specific symmetric tensor is identifiable, by adapting the algorithm from [12] to the symmetric setting.

It is interesting to compare the list of exceptional cases in Theorem 1.1 with the list of defective Veronese varieties in the Alexander–Hirschowitz theorem [2]. The tangentially weakly defective cases have numerical invariants that are close to the defective cases. In particular, the final nonidentifiable case that we proved in this paper is close to the defective case of cubics in $\mathbb{P}^4$, which was studied in [14]. We suspect a fundamental connection.

Algorithm 5.1 Certifying specific identifiability of symmetric tensors

S0. Let N be the empty matrix.

S1. For every $k \in \{1, 2, \ldots, \lfloor \frac{d}{2} \rfloor\}$ do:

S1a. Construct the symmetric flattening $p_{(k,d-k)} \in \mathbb{C}^{n^k \times n^{d-k}}$ and compute its full singular value decomposition as in (8).

S1b. If $\operatorname{rank}(S) = r$, then compute N_k and N_{d-k} as in (9) and append them to N. Otherwise, do nothing.

S2. If $\operatorname{rank}(N) \neq \Pi - r(n+1)$, then the algorithm halts and claims that it cannot certify r-identifiability.

S3. Construct the tangent space matrix $T \in \mathbb{C}^{\Pi \times r(n+1)}$ as in (11) and (6).

S4. Compute the full singular value decomposition of T:

$$T = \begin{bmatrix} U_T & K \end{bmatrix} \begin{bmatrix} S_T & 0 \\ 0 & 0 \end{bmatrix} V_T^T,$$

where $S_T \in \mathbb{R}^{r(n+1) \times r(n+1)}$ is a diagonal matrix, $V_T \in \mathbb{C}^{\Pi \times \Pi}$ and $[U_T \;\; K] \in \mathbb{C}^{\Pi \times \Pi}$ both are orthogonal matrices with respect to the Hermitian inner product.

S6. If $\operatorname{rank}(S_T) \neq r(n+1)$, then the algorithms halts and claims that it cannot certify r-identifiability.

H1. For every $k \in \{1, 2, \ldots, r\}$ do:

H1a. Construct the Hessian matrices $H_l(p_k) \in \mathbb{C}^{(n+1)\times(n+1)}$ as in (12) for $l = 1,2,\ldots,\ell$.

H1b. Let H be the stacked Hessian obtained by appending all matrices $H_l(p_k)$.

H1c. If $\operatorname{rank}(H) \neq n$, then the algorithm halts and claims that it cannot certify r-identifiability.

H2. The algorithm halts and certifies that $p = p_1 + \cdots + p_r$ is the unique Waring decomposition.

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