Trial and error mathematics II: dialectical sets and quasi-dialectical sets, their degrees, and their distribution within the class of limit sets.

TRIAL AND ERROR MATHEMATICS II: DIALECTICAL SETS AND QUASI-DIALECTICAL SETS, THEIR DEGREES, AND THEIR DISTRIBUTION WITHIN THE CLASS OF LIMIT SETS.

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ABSTRACT. This paper is a continuation of [1], where we have introduced the quasi-dialectical systems, which are abstract deductive systems designed to provide, in line with Lakatos' views, a formalization of trial and error mathematics more adherent to the real mathematical practice of revision than Magari's original dialectical systems. In this paper we prove that the two models of deductive systems (dialectical systems and quasi-dialectical systems) have in some sense the same information content, in that they represent two classes of sets (the dialectical sets, and the quasi-dialectical sets, respectively), which have the same Turing degrees (namely, the computably enumerable Turing degrees), and the same enumeration degrees (namely, the Π^0_1 enumeration degrees). Nonetheless, dialectical sets and quasi-dialectical sets do not coincide. Even restricting our attention to the so-called loopless quasi-dialectical sets, we show that the quasi-dialectical sets properly extend the dialectical sets. As both classes consist of Δ^0_2 sets, the extent to which the two classes differ is conveniently measured using the Ershov hierarchy: indeed, the dialectical sets are ω-computably enumerable (close inspection also shows that there are dialectical sets which do not lie in any finite level; and in every finite level $n \geq 2$ of the Ershov hierarchy there is a dialectical set which does not lie in the previous level); on the other hand, the quasi-dialectical sets spread out throughout all classes of the hierarchy (close inspection shows that for every ordinal notation a of a nonzero computable ordinal, there is a quasi-dialectical set lying in $\Sigma^{-1}_a \smallsetminus \bigcup_{b<_O a} \Sigma^{-1}_b$).

1. Introduction

Metamathematics as a natural science: this was Magari's firm belief when he proposed *dialectical systems* [9], as a trial and error model of how mathematics proceeds, and is carried out by the mathematical community. For a detailed analysis of historical and philosophical motivations lying behind Magari's position; for a brief account of his correspondence with Kreisel on the foundational adequacy of dialectical systems; and finally, for a brief survey on dialectical systems, see [1].

In [1], we have enriched dialectical systems with an additional mechanism of revision, trying to capture some of Lakatos' most distinguishing views on the process of revision in mathematics. The new enriched systems have been called *quasi-dialectical systems*.

An obvious interesting problem is to compare the deductive powers of the two approaches. Although every dialectical set (with trivial exceptions) is quasi-dialectical, the converse is not true, as already remarked in [1], since there are peculiar quasi-dialectical systems (those having so-called loops) whose quasi-dialectical sets coincide with the coinfinite, non-simple c.e. sets, whereas a c.e. dialectical set must be decidable. The same holds (and is shown in this paper) if one restricts attention to the so called loopless quasi-dialectical systems, which are far more natural from a

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philosophical point of view: the class of dialectical sets is properly contained in the class of quasi-dialectical sets corresponding to loopless quasi-dialectical systems. In any case, one can argue that the dialectical sets and the quasi-dialectical sets have the same information content, as their Turing degrees coincide, giving exactly the c.e. Turing degrees; and even their enumeration degrees coincide, giving exactly the enumeration degrees of the Π^0_1 sets.

This paper is a continuation of [1]. In section 2 we recall the definitions and some of the basic results about dialectical systems and quasi-dialectical systems. In section 3, we prove that the Turing degrees of the dialectical sets and of the quasi-dialectical sets coincide with the computably enumerable Turing degrees, and we prove that the enumeration degrees of the dialectical sets and of the quasi-dialectical sets coincide with the Π^0_1 enumeration degrees. In Section 4, we observe that all dialectical sets are ω-computably enumerable in the Ershov hierarchy (Theorem 4.5); also, for every $n \geq 2$ there exist dialectical sets that are n-c.e., but not $(n-1)$-c.e.; and there is an ω-c.e. dialectical system, which is not n-c.e., for any finite n. Finally, we show that for every ordinal notation $a \in O$ of a nonzero ordinal, there is a quasi-dialectical set which lies in the level Σ^{-1}_a of the Ershov hierarchy, but not in $\bigcup_{b<_O a} \Sigma^{-1}_b$. From this, it will follow that there are quasi-dialectical sets that are not dialectical, thus concluding that the quasi-dialectical sets do not coincide with the dialectical sets.

1.1. **Background.** This paper uses notations and terminology about dialectical systems and quasi-dialectical systems, which can be found in [1]. Our references for computability theory are the textbooks [4, 12, 14]: in particular, the reader is referred to [12] for Kleene's system O of ordinal notations; to [14] for a clear introduction to Δ^0_2 sets, the least modulus function, and the computably enumerable Turing degrees; finally, [4] contains a clear and succinct account of enumeration reducibility and enumeration degrees; the Ershov hierarchy is excellently treated in a few pages in [2].

2. Dialectical systems and quasi-dialectical systems: the definitions

In this section we recall (almost verbatim) the definition of a dialectical system, and that of a quasi-dialectical system. Our definition of a dialectical system is different from Magari's definition, but equivalent to it, as shown in [1].

In what follows, if f is the so-called proposing function, we will denote $f(i)$ with f_i.

Definition 2.1. A *dialectical system* is a triple $d = \langle H, f, c\rangle$, where f is a computable permutation of ω (called *the proposing function*), $c \in \omega$, and H is an enumeration operator, such that $H(\emptyset) \neq \emptyset$, $H(\{c\}) = \omega$, and H is an algebraic closure operator, i.e., H satisfies, for every $X \subseteq \omega$,

- $X \subseteq H(X)$;
- $H(X) \supseteq H(H(X))$.

Given such a d, and starting from a fixed computable approximation $\{H_s\}_{s\in\omega}$ (see e.g. [1] for the definition of a computable approximation to an enumeration operator) define by induction values for several computable parameters: A_s (a finite set), r_s (a function such that for every x, $r_s(x) = \emptyset$ or $r_s(x) = \{f_x\}$), $m(s)$ (the greatest number m such that $r_s(m) \neq \emptyset$), $h(s)$ (a number). In addition, there are the derived parameters: $L_s(x) = \bigcup_{y<x} r_s(y)$; and, for every i, $\chi_s(i) = \bigcup_{j\leq i} H_s(L_s(j))$.

Stage 0. Define $m(0) = 0$, $h(0) = 0$,

$$r_0(x) = \begin{cases} \{f_0\} & \text{if } x = 0 \\ \emptyset & \text{if } x > 0, \end{cases}$$

and let $A_0 = \emptyset$.

Stage $s+1$. Assume $m(s) = m$. We distinguish the following cases:

(1) there exists no $k \leq m$ such that $c \in \chi_s(k)$: in this case, let $m(s+1) = m+1$, and define

$$r_{s+1}(x) = \begin{cases} r_s(x) & x \leq m \\ \{f_{m+1}\} & \text{if } x = m+1 \\ \emptyset & \text{if } x > m+1; \end{cases}$$

(2) there exists $k \leq m$ such that $c \in \chi_s(k)$: in this case, let z be the least such k, let $m(s+1) = z+1$, and define

$$r_{s+1}(x) = \begin{cases} r_s(x) & \text{if } x < z \\ \{f_{z+1}\} & \text{if } x = z+1 \\ \emptyset & \text{if } x = z \text{ or } x > z+1. \end{cases}$$

Finally define $h(s+1) = m(s+1)$ if Clause (1) applies, otherwise, $h(s+1) = m(s+1) - 1$, and let

$$A_{s+1} = \bigcup_{i < h(s+1)} \chi_{s+1}(i) (= H_{s+1}(L_{s+1}(h(s+1)))).$$

The latter equality is justified by monotonicity with respect to inclusion of H_{s+1}.

We call A_s the set of *provisional theses* of d at stage s. The set A_d defined as

$$A_d = \{f_x : (\exists t)(\forall s \geq t)[f_x \in A_s]\}$$

is called the set of *final theses* of d. We often write $A_s = A_{d,s}$ when we want to specify the dialectical system d. It is easy to see (see [1]) that the set A_d of final theses does not depend on the chosen computable approximation to the enumeration operator H. A set $A \subseteq \omega$ is called *dialectical* if $A = A_d$ for some dialectical system d: in this case, we also say that A is *represented by* d.

Definition 2.2. A *quasi-dialectical system* q is a quintuple $q = \langle H, f, f^-, c, c^- \rangle$, such that $\langle H, f, c \rangle$ is a dialectical system, and in addition the following conditions are satisfied:

(1) $c^- \in \omega$;
(2) f^- is a total computable function and $c^- \notin \text{range}(f^-)$;
(3) f^- is *acyclic*, i.e., for every x, the f^--orbit of x is infinite, where, for any function g and number x, we define the *g-orbit of* x the set

$$\text{orb}_g(x) = \{x, g(x), g(g(x)), \ldots, g^n(x), \ldots\}.$$

If $c \neq c^-$ then q is called a *proper* quasi-dialectical system.

We call f^- the *revising function*, and c^- the *counterexample*.

Let $q = \langle H, f, f^-, c, c^- \rangle$ be a quasi-dialectical system. Having chosen a computable approximation $\alpha = \{H_s\}_{s \in \omega}$, define by induction values for several computable parameters, which depend on α: A_s (a finite set), r_s (a function such that for every x, $r_s(x)$ is a finite string of numbers, viewed as a vertical string, or *stack*), $m(s)$ (the greatest number m such that $r_s(m) \neq \langle\ \rangle$, where the symbol $\langle\ \rangle$ denotes the empty string), $h(s)$ (a number). In addition, there are the derived parameters: $\rho_s(x)$ is the top of the stack $r_s(x)$, $L_s(x) = \{\rho_s(y) : y < x \text{ and } r_s(y) \neq \langle\ \rangle\}$, and, for every i, $\chi_s(i) = \bigcup_{j \leq i} H_s(L_s(j))$.

Stage 0. Define $m(0) = 0$, $h(0) = 0$,

$$r_0(x) = \begin{cases} \langle f_0 \rangle & x = 0 \\ \langle\ \rangle & x > 0, \end{cases}$$

and let $A_0 = \emptyset$.

Stage $s+1$. Assume $m(s) = m$. We distinguish the following cases:

(1) there exists no $k \leq m$ such that $\{c, c^-\} \cap \chi_s(k) \neq \emptyset$: in this case, let $m(s+1) = m+1$, and define

$$r_{s+1}(x) = \begin{cases} r_s(x) & \text{if } x \leq m \\ \langle f_{m+1} \rangle & \text{if } x = m+1 \\ \langle\ \rangle & \text{if } x > m+1; \end{cases}$$

(2) there exists $k \leq m$ such that $c \in \chi_s(k)$, and for all $k' < k$, $c^- \notin \chi_s(k')$: in this case, let z be the least such k, let $m(s+1) = z+1$, and define

$$r_{s+1}(x) = \begin{cases} r_s(x) & x < z \\ \langle f_{z+1} \rangle & x = z+1 \\ \langle\ \rangle & x = z \text{ or } x > z+1; \end{cases}$$

(3) there exists $k \leq m$ such that $c^- \in \chi_s(k)$, and for all $k' \leq k$, $c \notin \chi_s(k')$: in this case, let z be the least such k, let $m(s+1) = z+1$, and define, where $\rho_s(z) = f_y$,

$$r_{s+1}(x) = \begin{cases} r_s(x) & x < z \\ r_s(x)^\frown\langle f^-(f_y) \rangle & x = z \\ \langle f_{z+1} \rangle & x = z+1 \\ \langle\ \rangle & x > z+1. \end{cases}$$

Finally define $h(s+1) = m(s+1)$, if Clause (1) applies, otherwise $h(s+1) = m(s+1) - 1$, and let

$$A_{s+1} = \bigcup_{i < h(s+1)} \chi_{s+1}(i) (= H_{s+1}(L_{s+1}(h(s+1)))).$$

We call A_s the set of *provisional theses* of q with respect to α at stage s. The set A_q^α defined as

$$A_q^\alpha = \{f_x : (\exists t)(\forall s \geq t)[f_x \in A_s]\}$$

is called the set of *final theses* of q with respect to α. We often write $A_s = A_{q,s}^\alpha$ when we want to specify the quasi-dialectical system q and the chosen approximation to the enumeration operator. A pair (q, α) as above is called an *approximated quasi-dialectical system*. A set $A \subseteq \omega$ is called *quasi-dialectical* if $A = A_q^\alpha$ for some approximated quasi-dialectical system, and we say in this case that A is *represented* by the pair (q, α).

We summarize some of the main properties of A_d and A_q^α. In the following, if $g(x,s)$ is a function of numbers, then for every x we say that $\lim_s g(x,s) = \ell$, if there exists t such that $g(x,s) = \ell$, for all $s \geq t$.

We recall from [1], that an approximated quasi-dialectical system does not have a loop over x, if the set $\{\rho^s(x) : s \in \omega\}$ is finite, and is *loopless* if it has no loop over any x. For more information and properties about loopless approximated quasi-dialectical system, the reader is referred to [1].

Theorem 2.3 ([9, 1])**.** *If d and (q,α) are respectively a dialectical system and a loopless approximated quasi-dialectical system, then the following hold:*

(1) A_d and A_q^α are Δ_2 sets;
(2) for every x, $\lim_s r_s(x) = r(x)$ and $\lim_s L_s(x) = L(x)$ exist (whether the functions $r_s(x)$, $L_s(x)$ refer to d, or (q,α)) and

$$A_d = \{f_x : r(x) = \{f_x\}\}$$
$$A_q^\alpha = \{f_x : r(x) = \langle f_x \rangle\},$$

and

$$f_x \in A_d \Leftrightarrow c \notin H(L_x \cup \{f_x\})$$
$$f_x \in A_q \Leftrightarrow \{c, c^-\} \cap H(L_x \cup \{f_x\}) = \emptyset.$$

(In fact, the assumption that (q,α) be loopless is reductant: it is just enough to assume (q,α) has no loop over any $y < x$.)

Proof. The claim that A_d is a Δ_2^0 set comes from [9], where it is proved that $A_d(x) = \lim_s g(x,s)$, with

$$g(x,s) = \begin{cases} 1, & \text{if } x \in A_{d,s} \\ 0, & \text{if } x \notin A_{d,s}. \end{cases}$$

The other claims come from [1, Lemma 3.8, Lemma 3.18]. □

3. Dialectical degrees, quasi-dialectical degrees, Turing degrees, and enumeration degrees

In this section we show that the information content of the dialectical sets coincides with that of the quasi-dialectical sets, by showing that the two classes of sets have the same Turing degrees, and the same enumeration degrees.

Proper quasi-dialectical systems, approximated quasi-dialectical systems with loops, and loopless approximated quasi-dialectical systems are defined in [1]. In the rest of this paper, we will make use of a convention introduced in [1], i.e. when dealing with a loopless approximated quasi-dialectical system we will avoid to specify which approximation we are considering. This way of doing is permitted by the fact – also proved in [1] – that the set of final theses of a loopless approximated quasi-dialectical system is invariant with respect to all the loopless approximations. In this light, we say that a *loopless quasi-dialectical system* is a quasi-dialectical system for which there is a loopless computable approximation, i.e. an approximation α such that the pair (q,α) is a loopless approximated quasi-dialectical system; a *loopless quasi-dialectical* set is a set represented by a loopless approximated quasi-dialectical system. In these cases, we simply write A_q to mean A_q^α, where α is any loopless computable approximation to the enumeration operator of q. We talk about a *proper loopless quasi-dialectical system*, or a *proper loopless quasi-dialectical set*, when the relevant quasi-dialectical system is proper, i.e. $c \neq c^-$.

Definition 3.1. A Turing degree (enumeration degree, respectively) is called *dialectical* if it contains a dialectical set; and it is called *quasi-dialectical* if it contains a quasi-dialectical set.

3.1. **Dialectical sets, quasi-dialectical sets, and Turing degrees.** The following theorem characterizes the dialectical Turing degrees, and the quasi-dialectical Turing-degrees.

Theorem 3.2. *The dialectical degrees and the quasi-dialectical degrees coincide: namely, they coincide with the c.e. Turing degrees.*

Proof. The proof consists of two steps. We show (Lemma 3.3) that every c.e. Turing degree is a dialectical degree; and we show (Lemma 3.4) that every quasi-dialectical degree is a c.e. Turing degree. Since every dialectical set is quasi-dialectical (see [1, Lemma 3.5]; see also [1, Corollary 3.21]), the claim follows immediately. □

Lemma 3.3. *For every c.e. set A there exists a dialectical system $d = \langle H, f, c\rangle$ such that $A_d \equiv_{tt} A$.*

Proof. This is an immediate consequence of the fact that every Π^0_1 set $A \neq \omega$ is dialectical (see [9]; see also [1, Lemma 2.13]). Thus, if A is c.e. then $A \equiv_{tt} A^c$, and A^c is dialectical, where for any given set $X \subseteq \omega$, the symbol X^c denotes the complement of X. □

Lemma 3.4. *If (q, α) is an approximated quasi-dialectical system, then A^α_q has c.e. Turing degree.*

Proof. If (q, α) is an approximated quasi-dialectical system with loops (see [1] for the definition), then A^α_q is c.e., see [1, Lemma 3.10]. Thus, in this case, the claim is trivial.

Let us consider the case when q is loopless. Let us recall the following facts about Δ^0_2 sets. Given a computable function $g(x, s)$ such that, for every x, $g(x, 0) = 0$, and $\lim_s g(x, s)$ exists, recall that the *least modulus function m* for g, is the function

$$m(x) = \mu s. (\forall t \geq s)[g(x, t) = g(x, s)].$$

Notice that if A is a Δ^0_2 set, such that $A(x) = \lim_s g(x, s)$ (where g is a 0-1 valued computable function; here, and in the following, given a set X of numbers, we denote by $X(x)$ the value of the characteristic function of X on x) and m is the least modulus function for g, then $A \leq_T m$. On the other hand, if B is the c.e. set

$$B = \{\langle x, s\rangle : (\exists t > s)[g(x, t) \neq g(x, s)]\}$$

then $B \equiv_T m$. So a least modulus function has always c.e. Turing degree (see e.g. [14]). Therefore, if A is a Δ^0_2 set, $g(x, s)$ is a 0-1 valued computable function such that $A(x) = \lim_s g(x, s)$, for all x, m is the least modulus function for g, and $m \leq_T A$, it follows that A has c.e. Turing degree.

If (q, α) is loopless, then by Corollary 3.17 of [1], we have that the computable sequence of sets $\{A_s\}$,

$$f_y \in A_s \Leftrightarrow \rho_s(y) = f_y,$$

is a Δ^0_2 approximation to A_q.

By [1, Lemma 3.8, Lemma 3.14, Theorem 3.17], for every y, the following hold: there is a least stage t_y such that for all $s \geq t_y$, and $x \leq y$, we have that $\rho_s(x) = \rho_{t_y}(x) = \rho(x)$, and consequently $r_s(x) = r_{t_y}(x) = r(x)$; if $r(x) \neq \langle\ \rangle$ then $r(x) \cap A_q = \{\rho(x)\}$; $f_x \in A_q$ if and only if $r(x) = \{f_x\}$.

Therefore an easy induction shows that, to find such a t_y, given y, it is enough to pick the least s such that for all $x \leq y$ if $\rho_s(x) \neq \langle\ \rangle$ then $\rho_s(x) \in A_q$. In other words,

$$t_y = \begin{cases} \mu s.\, (\forall x < y)([\rho_s(x) \neq \langle\ \rangle \Rightarrow \rho_s(x) \in A_q \,\&\, \rho_s(y) = f_y], & \text{if } f_y \in A_q, \\ \mu s.\, (\forall x < y)([\rho_s(x) \neq \langle\ \rangle \Rightarrow \rho_s(x) \in A_q \,\&\, \rho_s(y) \neq f_y], & \text{if } f_y \notin A_q. \end{cases}$$

Let now m be the least modulus function for

$$g(x, s) = \begin{cases} 1, & \text{if } x \in A_s, \\ 0, & \text{if } x \notin A_s. \end{cases}$$

By induction on y it is easy to see that $m(f_y) \leq t_y$. (Notice that, for $y > 0$, it might be $m(f_y) < t_y$ since at some stage t we could redefine $r_t(y-1)$ through Clause (3) of Stage $s+1$ in the definition of a quasi-dialectical system, and thus $r_t(y) = \langle f_y \rangle$; and at subsequent consecutive stages, we still redefine $r(y-1)$, without touching $r(y)$.) On the other hand, the mapping $y \mapsto t_y$ is $\leq_T A_q$. Therefore, $m \leq_T A_q$.

□

We conclude this section with the following easy consequence of Lemma 3.3.

Corollary 3.5. *Every nonzero dialectical Turing degree contains some immune dialectical set.*

Proof. Let A be a non-decidable dialectical set. By Lemma 3.3 there is a non-decidable c.e. set B such that $A \equiv_T B$. Let c_B be the characteristic function of B, and let

$$S = \{\sigma \in 2^{<\omega} : \sigma < c_B\}$$

where $<$ is the lexicographical order on strings, hence $\sigma < c_B$ means that there is some $i \in \text{domain}(\sigma)$ such that $\sigma(i) < c_B(i)$. Clearly, S is c.e.: to see this, let $\{b_s\}_{s\in\omega}$ be a 1-1 computable enumeration of B; let $B^s = \{b_0, \dots, b_s\}$, and let σ_s to be the longest finite initial segment of the characteristic function of B^s which ends with 1; then it is easy to see that

$$S = \{\sigma \in 2^{<\omega} : (\exists s)[\sigma < \sigma_s]\},$$

where, again, $<$ denotes lexicographical order. At this point (by suitably identifying ω with $2^{<\omega}$), take the dialectical system $d = \langle H, f, c\rangle$, where f enumerates $2^{<\omega}$ in the length-lexicographical order (in which, a string σ precedes a string τ if the length of σ is smaller than the length of τ, or the two strings have the same lengths but $\sigma < \tau$), c is any string, and H is the enumeration operator

$$H = \{\langle x, \{\sigma\}\rangle : x \in \omega \,\&\, \sigma \in S\} \cup \{\langle x, \{\sigma, \tau\}\rangle : x \in \omega \,\&\, |\sigma| = |\tau| \,\&\, \sigma < \tau\} \cup \{\langle \lambda, \emptyset\rangle\}$$

(where $|\ |$ denotes length of strings; notice that the last clause in the definition of H is to comply with the request, in the definition of dialectical systems, that $H(\emptyset) \neq \emptyset$): notice that the enumeration operator H is a closure operator. We can now see that

$$A_d = \{\sigma : \sigma \subset c_B\} :$$

this can easily be proved by induction on x, using (see Theorem 2.3)

$$f_x \in A_d \Leftrightarrow c \notin H(L_x \cup \{f_x\}).$$

Hence $A_d \equiv_T A$, and A_d is immune. □

3.2. **Dialectical sets, quasi-dialectical sets, and enumeration degrees.** To characterize the enumeration degrees of the dialectical sets, and of the quasi-dialectical sets, we first prove the following lemma.

Lemma 3.6. *If A is a loopless quasi-dialectical set then $A^c \leq_e A$.*

Proof. Let $q = \langle H, f, f^-, c, c^- \rangle$ be a loopless quasi-dialectical system, let $\{H_s\}_{s\in\omega}$ be a loopless computable approximation to H, and let $r_s(x), \rho_s(x), L_s(x)$, have the same meaning as in the definition of a quasi-dialectical set, with respect to this approximation. A closer inspection of the proof the second item of Theorem 2.3 easily shows that

$$f_x \in A^c \Leftrightarrow (\exists s)[\{c, c^-\} \cap H_s(L_s(x) \cup \{f_x\}) \neq \emptyset \,\&\, L_s(x) \subseteq A],$$

which provides an algorithm transforming any given enumeration of A into an enumeration of A^c, thus showing that $A^c \leq_e A$. □

Corollary 3.7. *If A is a loopless quasi-dialectical set, then $A \equiv_e A^c \oplus A$, hence the enumeration degree of A is total (i.e. it contains the graph of some total function).*

Proof. The proof is obvious as, for every set X, $X^c \oplus X \equiv_e c_X$, where c_X is (the graph of) the characteristic function of X. □

Lemma 3.8. *If A is a loopless quasi-dialectical set, then there is a c.e. set B such that $A \equiv_e B^c$, hence the enumeration degree of A is Π^0_1.*

Proof. We know that $A \equiv_T m$, where m is the least modulus function for the Δ^0_2 approximation to A, referred to in the proof of Lemma 3.4; on the other hand $m \equiv_T B$, for some c.e. set B, thus

$$A^c \oplus A \equiv_T B^c \oplus B,$$

from which, by totality of the enumeration degrees of $A^c \oplus A$ and $B^c \oplus B$, see for instance [4],

$$A^c \oplus A \equiv_e B^c \oplus B;$$

finally $B^c \equiv_e B^c \oplus B$, since B is c.e., and thus $A \equiv_e B^c$, by the previous corollary. □

We are now ready to characterize the enumeration degrees of the dialectical sets and of the quasi-dialectical sets.

Theorem 3.9. *The enumeration degrees of the dialectical sets and of the quasi-dialectical sets coincide with the Π^0_1 enumeration degrees.*

Proof. If A is a loopless quasi-dialectical set (and this includes also the case when A is dialectical), then its enumeration degree is Π^0_1 by Lemma 3.8. If A is represented by an approximated quasi-dialectical system with loops, then A is c.e., and thus $A \equiv_e B$, for every decidable set B: but every decidable set is Π^0_1.

On the other hand, if B is c.e., then by Lemma 3.4 there is a dialectical set A such that $A \equiv_T B$, hence, as in the proof of Lemma 3.8, $A^c \oplus A \equiv_e B^c \oplus B$. But as B is c.e., we have $B^c \oplus B \equiv_e B^c$, and by Corollary 3.7 we have that $A \equiv_e A^c \oplus A$, thus $A \equiv_e B^c$. □

The following corollary parallels Magari's observation in [9] that every c.e. dialectical set is decidable:

Corollary 3.10. *If A is a loopless quasi-dialectical c.e. set then A is decidable.*

Proof. If A is represented by a loopless quasi-dialectical system, then $A^c \leq_e A$ by Lemma 3.6: thus, if A is c.e., so is A^c. □

4. The distribution of dialectical sets, and of quasi-dialectical sets, within the class of limit sets

A result due to Jockusch [8], states that there is no completion of Peano Arithmetic PA that is a Boolean combination of c.e. sets, i.e. there is no completion of PA in any finite level of the Ershov hierarchy. The result has been more recently generalized by Schmerl [13], to any essentially undecidable theory. Since, given a formal theory T, and any pair f, c where f is a computable permutation of ω, and c is a number, it is possible to associate to T a dialectical system $d = \langle H, f, c\rangle$ such that A_d is, by coding, a completion of T (see [9]), a natural question is then to characterize the levels of the Ershov hierarchy that contain dialectical, or quasi dialectical sets. We show in this section that in every finite level $n \geq 2$ of the Ershov hierarchy lies a dialectical set that does not lie in any smaller level of the hierarchy; there exist dialectical sets that do not lie in any finite level; however, no dialectical set can lie outside of the class of the so-called ω-c.e. sets. As regards quasi-dialectical sets, we show that in every level of the Ershov hierarchy lies a proper quasi-dialectical set, that does not lie in any smaller level. We use these results to conclude that there are proper loopless quasi-dialectical sets that are not dialectical. This section is organized as follows: in Subsection 4.1 we recall the basic definitions and results concerning the Ershov hierarchy of Δ^0_2 sets. Subsection 4.2 shows that the dialectical sets are ω-c.e., and presents a priority-free proof of the fact that for every $n \geq 2$ there is a dialectical set which is properly Σ^{-1}_n. Subsection 4.3 contains a priority-free proof of the fact that for every notation a of an infinite ordinal there is a proper loopless quasi-dialectical set which is properly Σ^{-1}_a. Both the proofs in Subsections 4.2 and 4.3 build sets, which although lying in the appropriate levels of the Ershov hierarchy, are nonetheless introduced through dialectical or quasi-dialectical approximations (i.e., the approximations given by the sets of provisional theses) which in general make "too many" changes and do not directly witness memberships of these sets in the desired levels of the Ershov hierarchy. Finally, in Subsection 4.4, straightforward priority arguments are introduced in these proofs, to show that one can also build sets which are witnessed to lie in the appropriate levels of the Ershov hierarchy by their dialectical approximations (however, if n is odd, the dialectical approximation makes in general one more change than desired), or their quasi-dialectical approximations.

4.1. The Ershov hierarchy.

We now give precise definitions, and a few basic facts, about the Ershov hierarchy. As is known, the Ershov hierarchy classifies the Δ^0_2 sets, through the classes Σ^{-1}_a, where a is the Kleene ordinal notation of a computable ordinal. We use standard notations and terminology for Kleene's system O of ordinal notations: in particular, for $a \in O$, the symbol $|a|_O$ represents the ordinal of which a is a notation; the symbol $<_O$ denotes the Kleene partial ordering relation on O. The Ershov hierarchy of sets was originally introduced in [5, 6, 7]; our presentation is based on [2].

Definition 4.1. *If $a \in O$ is a notation for a nonzero computable ordinal, then a set of numbers A is said to be Σ^{-1}_a if there are computable functions $g(x, s)$ and $h(x, s)$ such that, for all x, s,*

(1) $A(x) = \lim_s g(x, s)$, with $g(x, 0) = 0$;
(2) (a) $h(x, 0) = a$ and $h(x, s + 1) \leq_O h(x, s)$;
(b) $g(x, s + 1) \neq g(x, s) \Rightarrow h(x, s + 1) \neq h(x, s)$.

Without loss of generality, we may assume that at each stage s, $\{x : g(x,s) = 1\}$ is finite.

We recall ([6]) that if $a <_O b$ then Σ^{-1}_a is properly contained in Σ^{-1}_b.

Definition 4.2. *If $a \in O$, a set A is said to be* properly Σ^{-1}_a *if*

$$A \in \Sigma^{-1}_a \smallsetminus \bigcup_{b <_O a} \Sigma^{-1}_b.$$

In order to build a set A which is properly Σ^{-1}_a, one could distinguish the two cases whether $|a|_O$ is a successor ordinal, or a limit ordinal:

(1) if $|a|_O$ is a successor, say $a = 2^b$, with $|a|_O = |b|_O + 1$, then it is enough to build $A \in \Sigma^{-1}_a \smallsetminus \Sigma^{-1}_b$;
(2) if $|a|_O$ is a limit, say $a = 3 \cdot 5^e$, then it is enough to build $A \in \Sigma^{-1}_a$ such that, for every n, $A \notin \Sigma^{-1}_{\varphi_e(n)}$.

However, in the proof of Theorem 4.14 for simplicity the construction of such an A is kept uniform, relying on the following lemma. Recall that if $a \in O$ is a given notation of a non-zero ordinal, then the set $P_a = \{b \in O : b <_O a\}$ is c.e. (see for instance [2]), and thus there exists a computable bijection $p : \omega \times P_a \to \omega$.

Lemma 4.3. *The following hold:*

(1) For every $a \in O$, there is an indexing $\{V_e\}_{e\in\omega}$ of the family of all Σ^{-1}_a-sets, such that $\{\langle e, x\rangle : x \in V_e\} \in \Sigma^{-1}_a$. Moreover, from e one can effectively find a pair $\langle g_e, h_e\rangle$ of computable functions, witnessing that V_e is in Σ^{-1}_a, as in Definition 4.1.
(2) Given $a \in O$, let $p : \omega \times P_a \to \omega$: be a computable bijection: there is an indexing $\{Z_{p(e,b)} : e \in \omega, b \in P_a\}$, of all sets in $\bigcup_{b<_O a} \Sigma^{-1}_b$. Moreover, from e, b one can effectively find a pair $\langle g_{p(e,b)}, h_{p(e,b)}\rangle$ of computable functions, witnessing that $Z_{p(e,b)}$ is in Σ^{-1}_b, as in Definition 4.1.

Proof. Item (1) can be worked out from [2]. For item (2), see [11]. $\square$

4.1.1. *The finite levels of the Ershov hierarchy, and the ω-c.e. sets.* Since finite ordinals have only one notation, one usually writes Σ^{-1}_n instead of Σ^{-1}_a, if a is the notation of $n \in \omega$, and we say that a set A is *n-c.e.* if $A \in \Sigma^{-1}_n$, or equivalently, there is a computable function $g(x,s)$ such that

(1) $A(x) = \lim_s g(x,s)$, and $g(x,0) = 0$;
(2) $|\{s : g(x,s+1) \neq g(x,s)\}| \leq n$.

We may assume that at each stage s, $\{x : g(x,s) = 1\}$ is finite. Moreover,

Definition 4.4. A set A is *ω-c.e.* if there are computable functions $g(x,s)$ and $h(x)$ such that, for every x,

(1) $A(x) = \lim_s g(x,s)$ and $g(x,0) = 0$;
(2) $|\{s : g(s+1) \neq g(s)\}| \leq h(x)$, where the symbol $|X|$ denotes the cardinality of a given set X.

As in Definition 4.1, we may assume that at each stage s, $\{x : g(x,s) = 1\}$ is finite.

4.2. **Dialectical sets and the Ershov hierarchy.** We are now ready to characterize the levels $a \in O$ of the Ershov hierarchy containing properly Σ^{-1}_a dialectical sets. The first claim of Theorem 4.5 is essentially due to Bernardi [3].

Theorem 4.5. *The following hold:*

(1) if A_d is a dialectical set, then A_d is ω-c.e.;
(2) for every n with $2 \leq n \leq \omega$, there exists a properly n-c.e. dialectical set.

Proof. Let us show item (1). The claim follows from the fact that if A_d is dialectic then $A_d \leq_{tt} \emptyset'$ ([3]), and on the other hand, every set $B \leq_{tt} \emptyset'$ is ω-c.e. (see [10]). A direct proof that A_d is ω-c.e. is as follows, where we refer to the approximation $\{A_{d,s}\}_{s\in\omega}$ to A_d, given by the sets of provisional theses. Let $\sigma(y, s)$ be the string of length $y + 1$,

$$\sigma(y,s)(x) = \begin{cases} 1, & \text{if } f_x \in L_s(y+1) \\ 0 & \text{if } f_x \notin L_s(y+1). \end{cases}$$

We claim that for every y, $\sigma(y, s)$ can change at most 2^y times. The claim is true of $y = 0$. If t_0 is that least stage at which $\sigma(y, s)$ stops changing, then after t_0, $\sigma(y + 1, s)$ may additionally change because of additional changes of $A_s(f_{y+1})$. But this can occur at most two more times, yielding that $\sigma(y + 1, s)$ may change at most 2^{y+1} times. From this, it trivially follows that $A_s(f_y)$, which is the y-th bit of $\sigma(y, s)$, may change at most 2^y times. This ends the proof of item (1).

Let as now show (2). Let $2 \leq n < \omega$, and let $\{V_e : e \in \omega\}$ be a computable listing of the $(n-1)$-c.e. sets in the sense of Lemma 4.3(1), and correspondingly let $\{V_{e,s} : e, s \in \omega\}$ be a computable sequence of finite sets such that, for every e, $\{V_{e,s} : s \in \omega\}$ is an $(n-1)$-approximation to V_e: for this, take

$$V_{e,s} = \{x : g_e(x, s) = 1\},$$

where we refer to a pair $\langle g_e, h_e\rangle$ of computable functions witnessing that V_e is in Σ^{-1}_{n-1}, as in Lemma 4.3(1); notice that, for every x,

$$|\{s : V_{e,s}(x) \neq V_{e,s+1}(x)\}| \leq n - 1.$$

We build a dialectical system d such that $A_d \neq V_e$, for all e, and $A_d \in \Sigma^{-1}_n$. Our dialectical system will be of the form $d = \langle H, f, c\rangle$, where we build H, whereas f is the identity function, i.e. $f_x = x$, and $c = 1$. To make the construction simpler to describe, the enumeration operator H that we are going to build will not be a closure operator: we will however argue in Lemma 4.11 that $A_d = A_{d'}$ where $d' = \langle H^\omega, f, c\rangle$, and H^ω is the enumeration operator such that, for every X, $H^\omega(X)$ is the smallest fixed point Y of H, such that $Y \supseteq X$: it is known, see e.g. [1], that H^ω is a closure operator.

Informal description of the construction. The construction is by stages. At stage s we define

(1) an approximation H_s to the enumeration operator H; (H_0 is a decidable set, $H_s \subseteq H_{s+1}$, $H_{s+1} \smallsetminus H_s$ is finite, and the relation $x \in H_s$ is decidable;)
(2) values $g(x, s)$ of a computable function; the construction will guarantee that for every x, $\lim_s g(x, s)$ exists, and in fact $|\{s : g(x, s) \neq g(x, s+1)\}| \leq n$ (thus $A = \{x : \lim_s g(x, s) = 1\}$ is in Σ^{-1}_n), and $A \neq V_e$, for every e.

In other words, we build a set A with the desired property that A be n-c.e., but not $(n-1)$-c.e.; simultaneously, we build H, by defining stage by stage a computable approximation to H; eventually we observe that $A = A_d$, where $d = \langle H, f, c\rangle$.

Remark 4.6. The reader who likes to consider only computable approximations to enumeration operators, consisting of finite sets, could object that H_0, as defined below, is infinite. (This does cause any problem, since, for every decidable X, one easily sees that H_0 satisfies that $H_0(X)$ is decidable, so the construction is computable.) However, one could easily remedy to this, by putting $H_0 = \emptyset$, and delay the enumeration of our infinite H_0 (as given below), by adding step by step a suitable finite portion of it: for instance, by adding $\{\langle 0, \emptyset\rangle, \langle c, \{c\}\rangle\} \in H_1$, and by adding to our H_{s+1} below, the finite set

$$\{\langle x, \{c\}\rangle : x \leq s\} \cup \{\langle x, \{x\}\rangle : x \leq s\}.$$

This remark applies to similar cases in the proofs of Theorems 4.14,4.25,4.29.

Requirements. In addition to the overall requirements that $A = A_d$, and A be n-c.e., the requirements to meet are, for every $e \in \omega$:

$$P_e : A \neq V_e.$$

Strategy to meet P_e. If we were not concerned with eventually getting $A = A_d$, the strategy would be the usual strategy to build an n-c.e. set which is not $(n-1)$-c.e.: we appoint a witness b_e, with initially $b_e \in A$ (so initially, we change $A(b_e)$ (or, rather, the current value $A_s(b_e)$ of $A(b_e)$) from the value 0 to the value 1); then, every time we see that $A(b_e) = V_e(b_e)$, we respond with changing $A(b_e)$, so as to have $A(b_e) \neq V_e(b_e)$. Since $V_e(b_e)$ can change at most $n-1$ times, we have that $A(b_e)$ can change at most n times, both sets A and V_e ending up with final values $A(b_e) \neq V_e(b_e)$, as desired.

Towards getting $A = A_d$. So, what we really need to explain is how to simultaneously construct H, so that eventually we get $A = A_d$. To this end, a *witness* for P_e is in fact a closed interval $I(e) = [a_e, a_e + n - 1]$, where we put $b_e = a_e + n - 1$. We suppose that for every e, $a_{e+1} = a_e + n$, so that the sets $I(e)$ are pairwise disjoint. We suppose also $a_0 = 2 = c + 1$.

When we appoint $I(e)$, we momentarily put $I(e) \subseteq A$, and we go through the following module, where we count the number of cycles by the counter i_e:

(1) set $i_e := n - 1$;
(2) if $b_e \in V_e$, then extract b_e from A and add the axiom $\langle c, \{a_e + j, b_e : j < i_e\}\rangle \in H$; let $i_e := i_e - 1$; go to (2);
(3) if $b_e \notin V_e$, then put back b_e into A; extract $a + i_e$ from A and add the axiom $\langle c, \{a_e + i_e\}\rangle \in H$ (by which $a_e + i(e)$ ends up to be out of A_d); let $i_e := i_e - 1$; go to (2).

Analysis of outcomes of the strategy for P_e. We analyze in more detail the outcomes of the strategy for P_e, with reference to how we get $A = A_d$, where $d = \langle H, f, c\rangle$.

If $i_e = n - 1$ is the final value of i_e, then we do not add any axiom in H which involves elements of $I(e)$: then clearly $b_e \in A_d$, and $a_e + j \in A_d$, for all $j < n - 1$; these values of A_d on the elements of $I(e)$ coincide with those of A;

Suppose that the value of i_e decreases to $i_e = i$ from $i_e = i + 1$. We use Theorem 2.3(2), an easy inductive argument on i, and the definition of H: assume by induction that up to now there is no axiom $\langle c, \{a_e + j\}\rangle \in H$, for any $j < i$; no axiom $\langle c, \{a_e + j, b_e : j < i\}\rangle \in H$; and there are already axioms $\langle c, \{a_e + j\}\rangle \in H$, for all $i < j < n - 1$.

(1) if b_e is extracted from A, then we add the axiom $\langle c, \{a_e + j, b_e : j < i\}\rangle \in H$; we conclude that if this is the final value of i_e, then $b_e \notin A_d$, since $\{a_e + j : j < i\} \subseteq A_d$, and thus $c \in H(L(b_e) \cup \{b_e\})$; moreover $a_e + j \notin A_d$, for all $i \leq j < n - 1$; these values of A_d on the elements of $I(e)$ coincide with those of A;

(2) if b_e is put back into A, then we add the axiom $\langle c, \{a_e + i\}\rangle \in H$, by which $a_e + i$ will be out of A_d; hence the axiom $\langle c, \{a_e + j, b_e : j < i + 1\}\rangle \in H$ does not apply, and if i is the final value of i_e, then $b_e \in A_d$, since $c \notin H(L(b_e) \cup \{b_e\})$; moreover we also have $\{a_e + j : j < i\} \subseteq A_d$, and $a_e + j \notin A_d$, for all $i < j < n - 1$; these values of A_d on the elements of $I(e)$ coincide with those of A.

The construction. The construction is by stages. We make use of the parameter $i_{e,s}$, approximating at stage s the number i_e as in the section "Strategy to meet P_e".

Definition 4.7. A requirement P_e *requires attention* at s, if $s > 0$, and (in the order) either $i_{e,s} =\uparrow$, or $b_e \in V_{e,s}$ if and only if $b_e \in A_{s-1}$.

Stage 0. Let
$$H_0 = \{\langle x, \{c\}\rangle : x \in \omega\} \cup \{\langle 0, \emptyset\rangle\} \cup \{\langle x, \{x\}\rangle : x \in \omega\}.$$
(The reason for having $0 \in H(\emptyset)$ is to comply with the definition of a dialectical system, which requires $H(\emptyset) \neq \emptyset$.) Let also $g(x, 0) = 0$, for all x. Define $i_{e,0} =\uparrow$, for every e.

Stage $s + 1$. Consider all $e \leq s$ such that P_e requires attention at $s + 1$.

(1) If $i_{e,s} =\uparrow$, then set $i_{e,s+1} = n - 1$. We put $I(e) \subseteq A_{s+1}$, by defining $g(x, s + 1) = 1$, for all numbers $x \in I(e)$.
(2) Otherwise:
   (a) if $b_e \in V_{e,s+1}$ (necessarily, $i_{e,s} > 0$), then add the axiom $\langle c, \{a_e + j, b_e : j \leq i_{e,s}\}\rangle \in H$, define $g(b_e, s + 1) = 0$, and define $i_{e,s+1} = i_{e,s} - 1$;
   (b) if $b_e \notin V_{e,s+1}$ (necessarily, $i_{e,s} > 0$), then add the axiom $\langle c, \{a_e + i_{e,s}\}\rangle \in H$, define $g(a_e + i_{e,s}, s + 1) = 0$, $g(b_e, s + 1) = 1$, and define $i_{e,s+1} = i_{e,s} - 1$.

Let H_{s+1} be H_s plus the axioms for H added at stage $s + 1$. Let also $g(0, s + 1) = 1$. Unless explicitly redefined during stage $s + 1$, all remaining parameters and values maintain the same value as at stage s. In particular $g(c, s + 1) = 0$. Go to Stage $s + 2$.

Verification. The verification relies on the following lemmata.

Lemma 4.8. *A is n-c.e.*

Proof. If a number x lies in some $I(e)$, then it is clear that $A_s(x)$ can change at most n times, as has been already discussed in the section "Strategy to meet P_e". Otherwise, $x \in \{0, 1\}$: then $A_s(x)$ changes from 0 to 1 exactly once, if $x = 0$, and $A_s(x)$ never changes, if $x = 1 = c$. □

Lemma 4.9. *For every e, A satisfies P_e.*

Proof. We change the value $A_s(b_e)$ as many times as are necessary to diagonalize against the final value $V_e(b_e)$. □

Lemma 4.10. $A = A_d$.

Proof. Let us consider any x. If $x \in I(e)$ for some e, then it is clear by the way we update H, and the discussion in the section with title "Analysis of the outcomes of the strategy for P_e", that $A(x) = A_d(x)$. If x does not lie in any such $I(e)$, then $x \in \{0, 1\}$, and the claim is trivial. □

Lemma 4.11. *$A_d = A_{d'}$, where $d' = \langle H^\omega, f, c\rangle$.*

Proof. The claim follows from the following easy observation: $H^\omega = H^2$, and obviously $c \in H(H(X))$ if and only if $c \in H(X)$, by the way we have defined the axioms of H involving c. □

Finally we sketch how to prove claim (2) of the statement of the theorem, when $n = \omega$.

We start with an effective listing of all n-c.e. sets, for the various $n \geq 1$: for instance, take $Z_{\langle e,n\rangle} = V^n_e$, where $\{V^n_e\}_{e\in\omega,n\geq 1}$ is an effective listing of all n-c.e. sets.

A witness for the requirement $P_{\langle e,n\rangle}$ (with $e \geq 0$ and $n \geq 1$) is now a closed interval $I(\langle e, n\rangle) = [a_{\langle e,n\rangle}, a_{\langle e,n\rangle} + n]$. The rest of the proof is exactly as before, with the only difference that witnesses are now closed intervals of variable length. □

Remark 4.12. It should be noted that the proof of item (2) of the previous theorem makes use of no priority feature. Each requirement keeps its own witness forever, and there is no interference between the different strategies for the various requirements.

Remark 4.13. Item (2) of Theorem 4.5 can not be extended to include the case $n = 1$, because every c.e. dialectical set is decidable ([9]), and thus, every 1-c.e. dialectical set is also 0-c.e.

4.3. **Quasi-dialectical sets and the Ershov hierarchy.** The goal of this section is to prove that for every notation $a \in O$ of a nonzero computable ordinal there is a proper quasi-dialectical set, which is properly Σ^{-1}_a . The claim should be more precisely stated according to the following distinction: if $|a|_O = 1$ then there is a quasi-dialectical set A, represented by an approximated quasi-dialectical system with loops, such that A is properly Σ^{-1}_a, hence A is c.e. but not decidable; if $|a|_O \geq 2$ then there is a proper loopless quasi-dialectical set which is properly Σ^{-1}_a. It will follow from this, that there are proper loopless quasi-dialectical sets that are not dialectical.

Theorem 4.14. *For every notation $a \in O$, with $|a| \geq 2$, there is a proper loopless quasi-dialectical set which is properly Σ^{-1}_a.*

Proof. We rely on the possibility of building, for any given a as in the statement of the theorem, a proper quasi-dialectical system $q = \langle H, f, f^-, c, c^-\rangle$, together with a suitable loopless computable approximation α to H, which enables us to pick, when needed, pairs of numbers $y < x$ (with $f_x \neq c, c^-$), so as to satisfy the following two desiderata:

(i) no occurrences of f_x is ever permitted to the left of x, i.e., for all $z < x$, at every stage s we have that $\rho_s(z) \neq f_x$;
(ii) at no stage s do we have $c \in H_s(L_s(y + 1))$.

The elimination/recovery mechanism. If so, suppose that at some stage $s+1$, we have $f_x \in A_{q,s}$ (set of provisional theses at stage s) but we want to remove f_x from the provisional theses: we can do so, by defining at $s+1$ the axiom $\langle c, \{\rho_s(y), f_x\}\rangle \in H$. If at some bigger stage $t+1 > s+1$, we want to restore f_x in the provisional theses, it will be enough to define at $t+1$ the axiom $\langle c^-, \{\rho_s(y)\}\rangle \in H$: this has the effect of immediately getting $\rho_s(y)$ out of $A_{q,t+1}$, so that the axiom $\langle c, \{\rho_s(y), f_x\}\rangle \in H$ does not apply any more; thus, the quasi-dialectical procedure (i.e., the procedure through which the sets of provisional theses are constructed) will propose f_x again, and put it back into the set of provisional theses.

It is then clear that, by this mechanism (called the *elimination/recovery mechanism*), using the quasi-dialectical procedure, we can move f_x in and out of A_q as many times as we want.

With reference to the elimination/recovery mechanism, we fix the following terminology:

(1) we call the number y the *fellow* of f_x;
(2) we say that y *eliminates* f_x at stage s if $c \in H_s(\{\rho_s(y), f_x\})$,
(3) we say that y *recovers* f_x at stage s, if $c^- \in H_s(\{\rho_s(y)\})$.

If $a \in O$ is a given notation, with $|a| \geq 2$, then fix a computable bijection $p : \omega \times P_a \to \omega$. Thus, by Lemma 4.3(2), we may refer to an indexing $\{Z_{p(e,b)} : e \in \omega, b \in P_a\}$ of all sets in $\bigcup_{b<_O a} \Sigma^{-1}_b$, such that from e, b one can effectively find a pair $\langle g_{p(e,b)}, h_{p(e,b)}\rangle$ of computable functions, witnessing that $Z_{p(e,b)}$ is in Σ^{-1}_b, as in Definition 4.1.

Informal description of the construction. We build a proper quasi-dialectical system $q = \langle H, f, f^-, c, c^-\rangle$, together with a suitable loopless computable approximation $\alpha = \{H_s\}_{s\in\omega}$ to H, such that $A_q \neq Z_n$, for all $n = p(e, b)$, $e \in \omega$ and $b <_O a$. Our quasi-dialectical system will be of the form $q = \langle H, f, f^-, c, c^-\rangle$, where we build H through α, whereas f is the identity function, $f^-(x) = 3x$, $c = 1$, and $c^- = 2$. To make the construction simpler to describe, the enumeration operator H that we are going to build will not be a closure operator. We will however argue in Lemma 4.19 that $A_q = A_{q'}$ where $q' = \langle H^\omega, f, f^-, c, c^-\rangle$: this is similar to what we have done in the proof of Theorem 4.5. Hopefully, q and α will allow us to pick, as needed, pairs y, x, where y is a fellow of f_x, so that we can play the above described elimination/recovery game. The construction is by stages. At stage s we define

(1) an approximation H_s to the enumeration operator H;
(2) values $g(x, s)$, and $h(x, s)$ of computable functions, guaranteeing that for every x, $\lim_s g(x, s)$ exists, and in fact the pair $\langle g, h\rangle$ witnesses that $A = \{x : \lim_s g(x, s) = 1\}$ is in Σ^{-1}_a, and $A \neq Z_n$, for every n. Throughout the construction, we define

$$A_s = \{x : g(x, s) = 1\}.$$

We build a set A with the desired property that $A \in \Sigma^{-1}_a \smallsetminus \bigcup_{b<_O a} \Sigma^{-1}_b$; simultaneously, we define H through a loopless $\alpha = \{H_s\}_{s\in\omega}$; eventually we observe that $A = A^\alpha_q$. Although there is no reason to conclude that H is a closure operator, nonetheless we can still construct the sets $A^\alpha_{q,s}$ of provisional theses, and thus the set A^α_q, using the approximation α to H built in the construction. For simplicity we will write $A_{q,s} = A^\alpha_{q,s}$, and $A_q = A^\alpha_q$ (also justified by the fact that α will turn out to be loopless, and easily yields a loopless approximation to the closure operator H^ω of Lemma 4.19).

Requirements. The requirements to meet are, for all $n = p(e, b)$, with $e \in \omega$ and $b <_O a$:

$$S : A \in \Sigma^{-1}_a$$
$$P_n : A \neq Z_n.$$

Strategy to meet P_n. As for the case of dialectical systems, the strategy to achieve $A \neq Z_n$ is obvious: we pick a witness x_n; initially we put $x_n \in A$ (notice that $f_{x_n} = x_n$); then we keep extracting and putting back x_n, responding to the movements of x_n in and out of Z_n, so that each time we diagonalize $A(x_n)$ against $Z_n(x_n)$. We keep track of changes of $A(x_n)$ by updating g and h: initially we set $g(x_n, 0) = 0$ and $h(x_n, 0) = a$; if at stage $s + 1$ we change $A(x_n)$, we correspondingly change $g(x_n, s + 1)$, and we decrease $h(x_n, s + 1) <_O h(x_n, s)$, so that we do not end up at $h(x_n, t) = 1$ (recall that $|1|_O = 0$) before $h_n(x_n, t)$ does.

Towards getting $A = A_q$. So, what we really need to explain is again how to simultaneously construct H and $\alpha = \{H_s\}_{s\in\omega}$, so that eventually we get $A = A_q$. A *witness* for P_n, with $n = p(e, b)$, is now the two-element interval $I(n) = [y_n, x_n]$ where $y_n = 3(n+1)+1$, $x_n = 3(n+1)+2$, thus $x_n = y_n+1$, and $y_n, x_n \notin \text{range}(f^-)$. We must ensure that in the limit, the values $A(x_n)$ and $A_q(x_n)$ are equal.

We go through the following module, where we use a counter i_n to count the number of cycles; for simplicity, we use the notation $z^i = f^{-(i)}(z)$:

(1) set $i_n := 0$; put y_n and x_n into A;
(2) if $x_n \in Z_n$, then extract x_n from A, and add the axiom $\langle c, \{y^i_n, x_n\}\rangle \in H$; define $i_n := i_n+1$;
(3) if $x_n \notin Z_n$, then we put back x_n in A, extract y^i_n from A, put y^{i+1}_n into A, and add the axiom $\langle c^-, \{y^i_n\}\rangle \in H$; define $i_n := i_n + 1$.

For A_q to catch up with A, the idea here is to have q and α play the elimination/recovery mechanism with y_n as a fellow of x_n, so that there is a sequence of stages $s_0 < s_1 < \cdots < s_{i_n}$ (where i_n is the final value of the counter), and a sequence $0 = j_0 \leq j_1 \leq \cdots \leq j_n$ (where j_n is the greatest i such that $i = 0$ or at some stage the construction has passed from y^{i-1}_n to y^i_n) such that, for every $i \leq n$, $y^{j_i}_n = \rho_{s_i}(y_n)$, and

(a) if we need to extract x_n from A at s_i, then y_n eliminates x_n at s_i;
(b) if we need to put back x_n in A at s_i, then y_n recovers x_n at s_i.

If we succeed in relating in this way the basic strategy for P_n, with the elimination/recovery mechanism, then by the discussion of this mechanism in the section dealing with this topic at the beginning of the proof, it is clear that for all $z \in \{y^i_n : i \leq j_n\} \cup \{x_n\}$ involved in the strategy for P_n, we get the same limit value $A(z) = A_q(z)$.

Analysis of outcomes of the strategy for $P_{p(e,b)}$. As in the analogous case of a P-requirement in the proof of Theorem 4.5, the above informal discussion regarding the movements of y^i_n and x_n, shows that we are eventually able to diagonalize $A(x_n)$ against $Z_n(x_n)$, as long as we do not exhaust the quota of allowable changes compatible with having $A \in \Sigma^{-1}_a$, i.e. as long as $h(x_n, t)$ does not reach, as a notation, the ordinal 0, before $h_n(x_n, t)$ does. Here is where we need to combine the strategy for P_n, with a suitable strategy for S, as we describe in the next paragraph.

Strategy to meet S. As promised, we define by stages two computable functions $g(x, s), h(x, s)$, witnessing that $A \in \Sigma^{-1}_a$. When, working to satisfy P_n, with $n = p(e, b)$, we first put x_n into A at a stage, say, s_0, and we define $h(x_n, s_0) = b$: up to this stage, we had $h(x_n, s) = a$. Following this stage, whenever we move x_n as above at, say, stage $s+1$, we change the value of $g(x_n, s+1)$, and decrease $h(x_n, s+1)$, by defining

$$h(x_n, s+1) = h_n(x_n, s+1):$$

since the action is taken because there has been a change in $g_n(x_n, s)$ which has occurred between the last stage t, for which we have $h(x_n, s) = h_n(x_n, t)$, and $s+1$, then $h(x_n, s+1)$ does decrease with respect to $<_O$, following the decrease of $h_n(x_n, s+1)$. Therefore, a simple inductive argument shows that, for all s,

$$h(x_n, s) \geq h_n(x_n, s).$$

This shows that, compared with Z_n, the approximation $\{A_s\}_{s\in\omega}$ to the defined set A allows on x_n for one more change than Z_n does, so that we can get to the desired diagonalization. As regards y_n, and the other potential numbers y_n^i, which enter the strategy for P_n, we have no problem here to meet S, since we will see that each number y_n^i moves at most twice, namely it is enumerated into A, and then it may be extracted again: therefore, when y_n^i is enumerated into A, at say stage s, it will be enough to set $h(y_n^i, s) = 2$, ordinal notation of 1. (This is where the assumption that $|a|_O \geq 2$ is being used, as $h(y_n^i, s) = 2$ has to drop to 2 from a bigger notation.)

Construction. The construction is by stages. For every n, s, let

$$Z_{n,s} = \{z : g_n(z, s) = 1\}.$$

For every n, we approximate the counter i_n, with $i_{n,s}$.

Definition 4.15. We say that P_n *requires attention* at s, if $s > 0$, and (in the order) either $i_{n,s} =\uparrow$, or $x_{n,s} \in Z_{n,s+1}$ if and only if $x_{n,s} \in A_{s-1}$.

It will be understood that, at the end of stage $s+1$, parameters and values (including values for $g(x, s+1)$ and $h(x, s+1)$) that have not been explicitly redefined, retain the same value as at the end of stage s.

Stage 0. Let

$$H_0 = \{\langle x, \{c\}\rangle : x \in \omega\} \cup \{\langle 0, \emptyset\rangle\} \cup \{\langle x, \{x\}\rangle : x \in \omega\}.$$

Let $g(x, 0) = 0$, and $h(x, 0) = a$, for all x. For every n, let $i_{n,0} =\uparrow$.

Stage $s+1$**.** Consider all $n \leq s$ such that P_n requires attention. Then consider two cases (where $n = \langle e, b\rangle$):

(1) if $i_{n,s} =\uparrow$ then set $g(y_n, s+1) = 1$, $h(y_n, s+1) = 2$, $g(x_n, s+1) = 1$, $h(x_n, s+1) = b$;
(2) otherwise:
   (a) If $x_n \in Z_{n,s+1}$ then add the axiom $\langle c, \{y_n^{i_{n,s}}, x_n\}\rangle \in H$. Define $g(x_n, s+1) = 0$, and $h(x_n, s+1) = h_n(x_n, s+1)$; set $i_{n,s+1} = i_{n,s} + 1$;
   (b) If $x_n \notin Z_{n,s+1}$ then add the axiom $\langle c^-, \{y_n^{i_{n,s}}\}\rangle \in H$. Define $g(x_n, s+1) = 1$, and $h(x_n, s+1) = h_n(x_n, s+1)$; define also $g(y_n^{i_{n,s}}, s+1) = 0$, and $h(y_n^{i_{n,s}}, s+1) = 1$; set $i_{n,s+1} = i_{n,s} + 1$.

Let H_{s+1} be H_s plus the axioms for H added at stage $s+1$. Finally, define $g(0, s+1) = 1$, $h(0, s+1) = 1$, $g(c, s+1) = g(c^-, s+1) = 0$, $h(c, s+1) = h(c^-, s+1) = 1$. For all other $z \leq s$ such that z is in the range of $f^-(x) = 3x$, and $h(z, s) = a$, set $g(z, s+1) = 1$ and $h(z, s+1) = 2$.

Verification. The verification relies on the following lemmata.

Lemma 4.16. $A \in \Sigma_a^{-1}$.

Proof. We have defined by stages a pair $\langle g, h \rangle$ of computable functions that witness that $A \in \Sigma_a^{-1}$, as is argued in the section with the title "Strategy to meet S". □

Lemma 4.17. *For every n, P_n is satisfied, i.e. $A \neq Z_n$; $i_n = \lim_s i_{n,s}$ exists.*

Proof. Let n be given. It is clear that actions relative to different requirements do not interfere with each other, and thus we are able to keep changing the value of $g(x_n, s)$ (i.e., of $A_s(x_n)$) as (finitely) many times as we need in order eventually to diagonalize $A(x_n)$ against $Z_n(x_n)$, thus getting $A \neq Z_n$. It is also clear from this, that there is a stage at which we stop to change $i_{n,s}$. □

Lemma 4.18. $A = A_q$.

Proof. We claim that the limit value $\lim_s g(x, s)$ that the construction demands for each x, is also achieved by the sequence $\{A_{q,s}\}_{s \in \omega}$, i.e., $\lim_s g(x, s) = \lim_s A_{q,s}(x)$.

On $0, c, c^-$, the sets A and A_q clearly agree in the limit.

Let us recall that j_n is the greatest i such that $i = 0$ or at some stage the construction has passed from y_n^{i-1} to y_n^i. We now show by induction that for every n, $r(y_n) = \lim_s r_s(y_n) = \langle y_n, y_n^{j_1}, \dots, y_n^{j_n} \rangle$; and for all $u \in \mathrm{range}(r(y_n)) \cup \{x_n\}$, $\lim_s g(u, s) = \lim_s A_{q,s}(u)$. Suppose that the claim is true of every $i < n$. Clearly, not only for $z \in I(i)$, $i < n$, can we assume that $r(z) = \lim_s r_s(z)$ exists: indeed, if z does not lie in any such $I(i)$, then $z \in \{0, 1, 2\}$, but then the claim is trivially true, or $z = 3u$, for some u: in this latter case, by definition of H, $r(z) = \langle z \rangle$, or $\rho(z) = \rho(y_i)$, for some $i < n$.

First of all, notice that neither fellows y_j, nor elements of the forms x_j chosen in witnesses $I(j)$, belong to the range of the function $f^-(x) = 3x$: therefore sets of the form $\{y_j^i : i \in \omega\}$ and $\{x_j\}$, for different j's, do not overlap, and we never define axioms for the enumeration operator H, which involve elements belonging to such sets relative to different j's. In the rest of the proof we repeatedly apply Theorem 2.3(2), easy inductive arguments, and the definition of H. Let t_n be the least stage at which all $r_s(z)$ for $z < y_n$ have reached limit. Starting from now on, q and α start to build the final stack on y_n, which never becomes $\langle\ \rangle$ by definition of H (no axiom of the form $\langle c, \{y_n^i\} \rangle \in H$ is ever added). By [1, Corollary 3.9], there is a least stage s_0 after t_n at which $r_{s_0}(y_n) = \langle y_n \rangle$, and $r_{s_0}(x_n) = \langle x_n \rangle$: and if $i_n = 0$, then due to the absence of axioms in H involving y_n and x_n, this value $r_{s_0}(y_n)$ is clearly the last value of $r(y_n)$; moreover $y_n, x_n \in A_q$; these values of A_q on the elements of $I(n)$ coincide with those of A.

Suppose that at a stage $s_u + 1$, we have that $i_{n,s_u+1} = i_{n,s_u} + 1$, and let $i_{n,s_u} = i$; let also $r_{s_u}(y_n) = \langle y_n, y_n^{j_1}, \dots, y_n^{j_i} \rangle$. Assume by induction that up to s_u there are no axioms $\langle c^-, \{y_n^{j_i}\} \rangle \in H$, $\langle c, \{y_n^{j_i+1}, x_n\} \rangle \in H$, but there are already axioms $\langle c^-, \{y_n^j\} \in H$, for all $j < j_i$. There are two possibilities:

(1) at $s_u + 1$ we extract x_n from $A :$ in this case our action introduces the axiom $\langle c, \{y_n^{j_i}, x_n\} \rangle \in H$. The stack does not change, with value $r_{s_u+1}(y_n) = \langle y_n, y_n^{j_1}, \dots, y_n^{j_i} \rangle$. If $i_{n,s_u+1} = i_n$ (thus $j_i = j_n$) then we would permanently get $y_n^{j_i} \in A_q$ and $x_n \notin A_q$, as $\{c, c^-\} \cap H(L(y_n) \cup \{y_n^{j_i}\}) = \emptyset$ and $c \in H(L(x_n) \cup \{x_n\})$; moreover $y_n^j \notin A_q$, for all $j < j_i$; these values of A_q

on the elements used by P_n coincide with those of A; the final value of the stack would be $r(y_n) = \langle y_n, y_n^{j_1}, \dots, y_n^{j_n} \rangle$.

(2) at $s_u + 1$ we put x_n back into A: in this case we introduce the axiom $\langle c^-, \{y_n^{j_i}\}\rangle \in H$. The new stack is $r_{s_u+1}(y_n) = \langle y_n, y_n^{j_1}, \dots, y_n^{j_i}, y_n^{j_i+1}\rangle$. If $i_{n,s_u+1} = i_n$, then $x_n \in A_q$, as $\{c, c^-\} \cap H(L(x_n) \cup \{x_n\}) = \emptyset$, $y_n^{j_i+1} \in A_q$, and $y_n^j \notin A_q$, for all $j \leq j_i$; these values of A_q on the elements used by P_n coincide with those of A; $j_n = j_i + 1$, and the final stack would be $r(y_n) = \langle y_n, y_n^1, \dots, y_n^{j_i}, y_n^{j_n}\rangle$.

On the other numbers, i.e. those z in the range of $f^-(x) = 3x$ which have not participated in the actions taken by any strategy, we have $A(z) = A_q(z) = 1$, thanks to the last clause at each stage $s+1$, demanding to put into A, all such $z \leq s$ such that $h(z, s) = a$: in absence of any axiom in H involving these numbers, they will be proposed and put in A_q by the quasi-dialectical procedure. □

Lemma 4.19. *There is a loopless quasi-dialectical system $q' = \langle H', f, f^-, c, c^-\rangle$, where H' is a closure operator, such that $A_q = A_{q'}$.*

Proof. We have to be more careful here than in the proof of Lemma 4.11, since quasi-dialectical sets may depend on the chosen computable approximation to the enumeration operator. So take again $H' = H^\omega$, and take the approximation $\{H^\omega_s\}_{s\in\omega}$ obtained in the following way: we enumerate in H^ω_s all axioms enumerated into H_s; moreover, whenever at stage s we add an axiom $\langle c, \{x\}\rangle \in H$, then we add also the decidable set of axioms $\langle y, \{x\}\rangle \in H^\omega$: the important thing is that we do not enumerate axioms of the form $\langle c^-, \{\rho_s(y_n), x_n\}\rangle \in H$ *strictly before* enumerating $\langle c, \{\rho_s(y_n), x_n\}\rangle \in H$, so that there is no danger of building a stack on some x_n which is different from $\langle\ \rangle$ or $\langle x_n\rangle$. It is easy to see that we do get a loopless computable approximation α' to H^ω, such that $A^{\alpha'}_{q'} = A_q$. (The reader sensible to the problem raised in Remark 4.6 should easily find a way to approximate H^ω through finite sets: instead of enumerating at once an infinite set of axioms like the previous one, one can just enumerate, stage by stage, finite pieces of it at future stages.) By the proof of the previous lemma, it follows that $A_{q'}$ is loopless. □

This concludes the proof of the theorem. □

Remark 4.20. As for the proof of Theorem 4.5 (see Remark 4.12), it should be noted that the proof of the previous theorem is priority-free.

Remark 4.21. By Corollary 3.10, we can not include the case $|a|_O = 1$ in the statement of Theorem 4.14, since every c.e. set A represented by a loopless quasi-dialectical system is decidable.

Corollary 4.22. *For every $a \in O$ such that $|a|_O \geq 1$, there is a quasi-dialectical set*

$$A \in \Sigma^{-1}_a \smallsetminus \bigcup_{b<_O a} \Sigma^{-1}_b.$$

Proof. If $|a|_O > 1$ this follows from Theorem 4.14. Assume $|a|_O = 1$: we know from [1, Theorem 3.12] that every coinfinite and not simple c.e. set can be represented by a quasi-dialectical system with loops: therefore there are c.e. quasi-dialectical sets which are not decidable. □

A consequence of Theorem 4.14 is:

Theorem 4.23. *There are proper loopless quasi-dialectical sets that are not dialectical.*

Proof. It is well known, and in any case easy to see, that if $a, b \in O$, and $|a|_O = |b|_O = \omega$, then $\Sigma_a^{-1} = \Sigma_b^{-1}$: for this reason, if $|a|_O = \omega$, we usually write $\Sigma_a^{-1} = \Sigma_\omega^{-1}$. On the other hand, the ω-c.e. sets are included in the Σ_ω^{-1} sets, see e.g. [10]. The claim is then immediate by Theorem 4.5 and Theorem 4.14: for instance, it is enough to take a proper loopless quasi-dialectical set $A \in \Sigma_a^{-1} \smallsetminus \Sigma_\omega^{-1}$, where $|a|_O = \omega + 1$. □

Theorem 4.23 can be obtained also as a consequence of the following:

Corollary 4.24. *If $\mathcal{X} = \{V_e : e \in \omega\}$ is an indexing of some class of Δ_2^0 sets, i.e. the predicate $x \in V_e$ is Δ_2^0, then there is a proper loopless quasi-dialectical set A such that $A \notin \mathcal{X}$.*

Proof. Similar to the proof of Theorem 4.14: in fact the proof is much easier, in that we do not have to keep track of the number of changes in the function g, giving A as a limit, since we do not have to worry about making A a Σ_a^{-1} set, for some $a \in O$. □

Theorem 4.23 follows from the previous corollary, by the fact that the ω-c.e. sets can be indexed as a Δ_2^0 class as in the statement of the corollary, see [10].

4.4. **Stretching the proofs of Theorem 4.5 and Theorem 4.14.** A legitimate curiosity is to know whether one can stretch the proofs of Theorem 4.5 and Theorem 4.14, to obtain dialectical sets A_d or quasi-dialectical sets A_q, for which the Δ_2^0 approximations $\{A_{d,s}\}_{s\in\omega}$ or $\{A_{q,s}\}_{s\in\omega}$ yielded by the sets of provisional theses (taken with respect to the computable approximation $\alpha = \{H_s\}_{s\in\omega}$ to H, defined during the construction), already witness that the sets lie in the appropriate level of the Ershov hierarchy.

Recall that by Theorem 4.5(1), for every dialectical system d the Δ_2^0 approximation $\{A_{d,s}\}_{s\in\omega}$ (taken with respect to any computable approximation to the enumeration operator of d) already witnesses that A_d is ω-c.e.

Dialectical approximations. We start up with dialectical sets, and we briefly discuss the difficulties inherent in building a suitable dialectical system $d = \langle H, f, c\rangle$, together with a suitable computable approximation to H, such that for every x, the value $A_{d,s}(x)$ does not make too many changes.

With reference to the construction described in the proof of Theorem 4.5 (claim (2), case of n finite), there is an evident conflict arising by interactions between different strategies. Consider P_e, P_i with $e < i$. We limit our analysis to the components b_i and b_e of the respective witnesses $I(i)$ and $I(e)$, but similar considerations hold for the other components $a_i + j$ and $a_e + k$, with $j, k \le n - 1$. It could happen that we act first to satisfy P_i, so the dialectical procedure (following our definition of H and its approximations) moves b_i in and out of A_d a certain number n' of times. Then we must act for P_e. Now, following the dialectical procedure, when at a stage s we move an element b out of $A_{d,s}$, it happens that we have to keep out of $A_{d,s}$ also the elements $b' > b$: so when the dialectical procedure follows up our action for P_e it may happen that it moves again b_i. Suppose that this happens n'' times: so altogether we would have to move b_i, $n' + n''$ times, with possibly $n' + n'' > n$: too many changes!

The solution consists of course in introducing some priority within the construction, so that when we act for P_e we discard the current witness for P_i which can start afresh, and thus having the possibility of moving n times the components of the new witness, if necessary.

In this new setting, we need to approximate not only $i_{e,s}$, but also $a_{e,s}, b_{e,s}$, and therefore $I(e,s) = [a_{e,s}, a_{e,s} + n - 1]$.

When we choose $I(e,s)$, we choose it *new*, i.e., its members are bigger than all numbers so far mentioned in the construction. In particular, $b_{e,s}$ has never been a provisional thesis, and it may take a while for it to become a provisional thesis, since the dialectical procedure has to propose first a bunch of numbers and to decide on them, before proposing and momentarily accepting $b_{e,s}$; the same may happen when the dialectical procedure has momentarily discarded $b_{e,s}$, but then wants it back. (Notice that on the contrary, when we want out an element a, which is currently in the provisional theses, then we add to H a suitable axiom involving c and a, and this action takes effect immediately: for instance, we add $\langle c, \{a\}\rangle \in H$, and at this stage a is out of the provisional theses.) When in the construction below, we act to put $b_{e,s}$ back and we just need that the dialectical procedure makes it a provisional thesis, the we say that P_e is in "standby": the rigorous definition is given in the construction.

A requirement P_e is *initialized* if we set all of its parameters to be undefined. We say that P_e *requires attention* at s, if $s > 0$, and (in the order) either P_e is initialized, or P_e is in standby, or $b_{e,s} \in V_{e,s}$ if and only if $b_{e,s} \in A_{d,s-1}$.

At stage $s+1$ we act on behalf of the least e, such that P_e requires attention, and we initialize all P_i with $i > e$, by discarding their witnesses and forcing each such P_i to use a new witness when its turn to act comes again. In order to avoid that the components of the discarded witness of some P_i with $i > e$ make more moves than it is allowed, in and out of the sets of provisional theses, we *freeze* them out of the future sets $A_{d,s}$ of provisional theses, by adding the axiom $\langle c, \{a\}\rangle \in H$, for each member a of the discarded witness. Now notice that this may add an additional change for the value $A_d(a)$ with respect to the approximation $\{A_{d,s}\}_{a\in\omega}$, and, if we want this approximation to witness that $A \in \Sigma^{-1}_n$, this may not be allowed if we have already made all available n changes, and we have ended up with $A_d(a) = 1$ (necessarily, in this case, $a = b_i$). Notice however that this can not happen if n is even: in this case, if we have exhausted all allowed changes, then we have acted n times to satisfy P_i, hence $V_i(b_i)$ has changed $n-1$ times, and its final value is 1, so the final value for A_d is $A_d(b_i) = 0$, and thus freezing does not introduce any new change for $A_d(b_i)$.

So, we can state the following:

Theorem 4.25. *For every $n \geq 2$ we can build a dialectical system $d = \langle H, f, c\rangle$, and a computable approximation $\alpha = \{H_s\}_{s\in\omega}$ to H, such that A_d is not $(n-1)$-c.e., and if $\{A_{d,s} : s \in \omega\}$ is the approximation to A_d given by the sets of provisional theses (corresponding to α), then*

(1) if n is even then for every y,

$$|\{s : A_{d,s}(f_y) \neq A_{d,s+1}(f_y)\}| \leq n;$$

(2) if n is odd then for every y,

$$|\{s : A_{d,s}(f_y) \neq A_{d,s+1}(f_y)\}| \leq n+1.$$

Proof. We build $d = \langle H, f, c\rangle$ by building H, whereas f is the identity function and $c = 1$. Given any even $n > 0$, construct H by stages as follows:

Stage 0**.** Initialize all P_e. Let

$$H_0 = \{\langle x, \{c\}\rangle : x \in \omega\} \cup \{\langle 0, \emptyset\rangle\} \cup \{\langle x, \{x\}\rangle : x \in \omega\}.$$

Stage $s+1$**.** Let e be the least number such that P_e requires attention: notice that there always is such an e, since at every stage almost all requirements are initialized.

(1) If P_e is initialized at the beginning of stage $s+1$, then let $a_{e,s+1} > 1$ be the least unused number, let $I(e, s+1) = [a_{e,s+1}, a_{e,s+1}+n-1]$, $b_{e,s+1} = a_{e,s+1}+n-1$; declare $i_{e,s+1} = n-1$; *put P_e in standby*;
(2) if P_e is in standby, and $b_{e,s} \notin A_{d,s}$, then keep P_e in standby; if $b_{e,s} \in A_{d,s}$ then P_e *ceases to be in standby*;
(3) otherwise:
   (a) if $b_{e,s} \in V_{e,s}$ (necessarily, $i > 0$), then add $\langle c, \{a_{e,s} + j, b_{e,s} : j \leq i\}\rangle \in H$; declare $i_{e,s+1} = i_{e,s} - 1$;
   (b) if $b_{e,s} \notin V_{e,s}$ (necessarily, $i > 0$), then add $\langle c, \{a_{e,s} + i\}\rangle \in H$; declare $i_{e,s+1} = i_{e,s} - 1$; *put P_e in standby.*

(Notice that thanks to the standby procedure, there is now a perfect synchronism between the action of P_e and the way the dialectical procedure moves the elements of $I(e, s)$, if P_e is no longer initialized.) After acting for P_e, initialize all P_i with $i > e$; for every $a > b_e$ such that a has been used in the construction (for instance $a \in I(i, s)$ with $i > e$) then add the axiom $\langle c, \{a\}\rangle \in H$: we call the addition of these axioms the *freezing procedure*. Let H_{s+1} be H_s plus the axioms added for H at stage $s+1$. Go to stage $s+2$.

The verification easily follows from:

Lemma 4.26. *For every e, there is a least stage s_e such that, for every $s \geq s_e$, $a_{e,s} = a_{e,s_e}$ (consequently, $I(e, s) = I(e, s_e)$ and $b_{e,s} = b_{e,s_e}$), P_e does not receive attention at stage s, and P_e is satisfied.*

Proof. By induction on e. Let t_e be the least stage after which all parameters relative to any P_i, with $i < e$, have settled down, and P_i does not require attention after t_e. So at stage $t_e + 1$, P_e requires attention, we choose the final value $[a_e, a_e + n - 1]$ of its witness. After this stage, P_e may require attention at most finitely many times. Therefore, the existence of s_e has been demonstrated. Let us call I_e, a_e, and b_e the limit values of the parameters $I(e, s), a_{e,s}, b_{e,s}$. We can repeat for the final values $I(e), a_e$ and b_e the same argument as for the witnesses for P_e in the proof of Theorem 4.5: in particular, as explained in the section on analysis of outcomes for the strategy for P_e in the proof of Theorem 4.5, the axioms which we have placed in H enable us to move $b(e)$ in and out of $A_{d,s}$, as many times we need to get eventually diagonalization of $A_d(b_e)$ against $V_e(b_e)$. □

Lemma 4.27. *If n is even then for every x, $A_{d,s}(x)$ can change at most n-times; if n is odd then for every x, $A_{d,s}(x)$ can change at most $n + 1$-times.*

Proof. This is clear by the discussion on interactions between strategies, which precedes the theorem. Notice that if x lies in some final value $I(e)$, then $A_{d,s}(x)$ can change at most n-times, as the components of I_e make at most the same number of moves as the components of the corresponding set $I(e)$ in the proof of Theorem 4.5. If $x \in I(e, s_0)$, for some e, s_0 such that $I(e, s_0)$ is later discarded, then x can move at most n times before $I(e, s_0)$ is discarded, and then x is frozen, which may bring to $n + 1$ the final number of changes, if n is odd. Otherwise $A_{d,s}(x)$ can change from 0 to 1 if x is not frozen, or from 0 to 1 and back to 0 if x is frozen. $A_{d,s}(c)$ never changes. □

Lemma 4.28. *$A_d = A_{d'}$, where $d' = \langle H^\omega, f, c\rangle$.*

Proof. As in Lemma 4.11. □

This concludes the proof of Theorem 4.25. □

Quasi-dialectical approximations. Let us now tackle the case of quasi-dialectical sets. Since every dialectical set is a quasi-dialectical set ([1, Lemma 3.6]), Theorem 4.25 ipso facto extends to quasi-dialectical sets. We now consider the issue of whether we can stretch the proof of Theorem 4.14 to get proper loopless quasi-dialectical sets whose membership in the appropriate level of the Ershov hierarchy is witnessed by a quasi-dialectical approximation.

We start with the case of the infinite levels of the Ershov hierarchy.

Theorem 4.29. *For every notation $a \in O$, with $|a|_O \geq \omega$, there is a proper loopless quasi-dialectical system $q = \langle H, f, f^-, c, c^- \rangle$ such that A_q is properly Σ^{-1}_a, and if $g(x,s)$ is the approximation to A_q given by the sets of provisional theses, then there is a computable $h(x,s)$ such that the pair $\langle g, h \rangle$ witnesses the fact that $A_q \in \Sigma^{-1}_a \smallsetminus \bigcup_{b <_O a} \Sigma^{-1}_b$.*

Proof. As in the case of Theorem 4.25 we basically insert priority in the proof of Theorem 4.14, with the addition of the "freezing procedure" at the end of each stage, for all discarded witnesses. Throughout the rest of the proof, we refer to notations and terminology as in Theorem 4.14: in particular $n = p(e,b)$, and in order to satisfy P_n, we must diagonalize A_q against $Z_{p(e,b)} \in \Sigma^{-1}_b$.

We build $q = \langle H, f, f^-, c, c^- \rangle$ by building H by stages, whereas f is the identity function, $f^-(x) = 3x$, $c = 1$, $c^- = 2$. We construct H by stages, and the quasi-dialectical procedure that we have in mind for the system $q = \langle H, f, f^-, c, c^- \rangle$ refers to the computable approximations to H, defined during the construction.

We say that a requirement is *initialized* if all parameters relative to P_n are undefined. Similarly to the proof of Theorem 4.25, a requirement P_n may be in *standby* if it has acted to put the component $x_{n,s}$ of its witness in the set of provisional theses and it is just waiting for the quasi-dialectical procedure to comply with this action: the main difference, compared to the proof of Theorem 4.25, (assuming that we work at stages after which P_n will no longer be initialized) is that when now P_n is put in standby for the first time then (as in Theorem 4.25) we may have to wait several stages to see x_n proposed and put into the set of provisional theses; on the other hand for future cycles of the standby procedure we have to wait only one stage for the quasi-dialectical procedure to propose a previously extracted x_n and put it back in the provisional theses.

We say that P_n *requires attention* at s, if $s > 0$, and (in the order) either P_n is initialized, or P_n is in standby, or $x_{n,s} \in A_{q,s-1}$ if and only if $x_{n,s} \in Z_{n,s}$.

Compared to the proof of Theorem 4.14, there is an additional parameter to consider: for every n, s, with $n = p(e,b)$, let

$$k_n(x,s) = \begin{cases} 2, & \text{if } |b|_O \text{ is finite or } (\exists u <_O b)[|u|_O \text{ limit } \& \, h_n(x,s) <_O u] \\ 1, & \text{otherwise.} \end{cases}$$

(Recall that 2 is the notation of the ordinal 1, and 1 is the notation of the ordinal 0.) It is not difficult to see that the function $k_n(x,s)$ is computable. Indeed, to compute $k_n(x,s)$, if $|b|_O$ is not finite, one checks the values $h_n(x,t)$, for $t \leq s$: if one finds the least $t < s$ such that $h_n(x,t) \geq_O u$, for some $u \in O$ with $|u|_O$ limit, and $h(x,t) <_O u$, then $k(x,s) = 2$; otherwise $k(x,s) = 1$.

Stage 0. Initialize all P_n. Let

$$H_0 = \{\langle x, \{c\} \rangle : x \in \omega\} \cup \{\langle 0, \emptyset \rangle\} \cup \{\langle x, \{x\} \rangle : x \in \omega\}.$$

For every x, n let $h(x, 0) = a$, $k_n(x, 0) = 1$, $i_{n,0} =\uparrow$.

Stage $s+1$**.** Let $n = p(e, b)$ be the least number such that P_n requires attention: notice that there always is such an n.

(1) If P_n is initialized at the beginning of stage $s+1$, then let $y_{n,s+1}, x_{n,s+1} > 0$ be the least unused pair of numbers, such that $x_{n,s+1} = y_{n,s+1} + 1$ and $\{y_{n,s+1}, x_{n,s+1}\} \cap \mathrm{range}(f^-) = \emptyset$; *put* P_n *in standby*; set $i_{n,s+1} = 0$;
(2) if P_n is standby, and $x_{n,s} \notin A_{q,s+1}$ then keep P_n in standby; if $x_{n,s} \in A_{q,s+1}$ then P_n *ceases to be in standby*; we have in this case $x_{n,s} \in A_{q,s+1} \smallsetminus A_{q,s}$: if $h(x_n, s) = a$ then define $h(x_{n,s}, s+1) = b$; otherwise define $h(x_{n,s}, s+1) = h_n(x_{n,s}, s+1) +_O k_n(x_{n,s}, s)$; set $i_{n,s+1} = i_{n,s} + 1$;
(3) if $x_{n,s} \in Z_{n,s+1}$, then eliminate $x_{n,s}$ by $y_{n,s}$, i.e. add the axiom $\langle c, \{\rho_s(y_{n,s}), x_{n,s}\}\rangle \in H$. This has the effect of immediately having $x_{n,s} \in A_{q,s} \smallsetminus A_{q,s+1}$. Define $h(x_{n,s}, s+1) = h_n(x_{n,s}, s+1) +_O k_n(x_{n,s}, s+1)$;
(4) if $x_{n,s} \notin Z_{n,s+1}$ then recover $x_{n,s}$ by $y_{n,s}$, i.e. add $\langle c^-, \{\rho_s(y_{n,s})\}\rangle \in H$; *put* P_n *in standby*; set $i_{n,s+1} = i_{n,s} + 1$.

(Notice that thanks to the standby procedure, there is now a perfect synchronism between the action of P_n and the elimination/recovery mechanism for $y_{n,s}, x_{n,s}$, if P_n is no longer initialized.) After acting for P_n, initialize all P_i with $i > n$: for each $a > x_{n,s}$ such that a has been used in the construction (so that $h(a, s) \neq a$) we *freeze* a out of A_q, by adding the axiom $\langle c, \{a\}\rangle \in H$, and defining $h(a, s+1) = 1$. If a is any number such that $a \in A_{q,s+1}$ and $h(a, s) = a$ then define $h(a, s+1) = 2$. Define also $h(0, s+1) = 1$, and $h(c, s+1) = h(c^-, s+1) = 1$. Let H_{s+1} be H_s plus the axioms added to H at stage $s+1$. All parameters that have not been explicitly redefined maintain the same values as at the previous stage. Go to stage $s+2$.

The verification easily follows from the following lemmata:

Lemma 4.30. *For every* n*, there is a least stage* s_n *such that, for every* $s \geq s_n$*,* $x_{n,s} = x_{n,s_n}$ *(consequently,* $I(n, s) = I(n, s_n)$ *and* $y_{n,s} = y_{n,s_n}$*),* P_n *does not receive attention at stage* s*, and* P_n *is satisfied.*

Proof. By induction on n. Let t_n be the least stage after which all parameters relative to any P_i, with $i < n$, have settled down, and P_i does not require attention anymore. So at stage $t_n + 1$, P_n requires attention, we choose the final value $I(n) = [y_n, x_n]$ of its witness.

After this stage, P_n may require attention only finitely many times. Therefore, the existence of s_n has been demonstrated. After the least stage at which $I(n)$ has reached its limit, the witness $I(n)$ behaves exactly as the witness $I(n)$ in the proof of Theorem 4.14, except for the delaying effect of the "standby" feature. Thus P_n is eventually satisfied. □

Lemma 4.31. *Let* g *be the approximation to* A_q *given by the sets of provisional theses, i.e.,*

$$g(x, s) = \begin{cases} 1 & \text{if } x \in A_{q,s} \\ 0 & \text{if } x \notin A_{q,s}. \end{cases}$$

Then, the pair $\langle g, h\rangle$ *witnesses the fact that* A_q *is properly in* Σ^{-1}_a*.*

Proof. The claim has been achieved by synchronizing the changes of g with corresponding decreases of h. Indeed, consider first the case of $h(x, s)$ where $x = x_{i,s_0} \in I(i, s_0)$, for some i, s_0, and s_0 is the least stage at with $I(i, s_0)$ is appointed as witness. We claim that whenever $g(x, s+1) \neq g(x, s)$ then $h(x, s+1) <_O h(x, s)$, and, until $I(i, s_0)$ is discarded, for all s, $h(x, s) \geq_O h_i(x, s)$, and if $h(i, s)$ is $<_O$ a notation $u <_O b$, such that $|u|_O$ is limit, then $h(x, s) >_O h_i(x, s)$. To see this, first of all notice that $h(x, 0) = a >_O h_i(x, 0) = b$. Next change of $g(x, s)$ is at, say, s_0, when we put $h(x, s_0) = b \geq_O h_i(x, s_0)$. Suppose now by induction that the claim is true up to stage s_1, and suppose that $g(x, s_1 + 1) \neq g(x, s_1)$: this is due to the fact that the strategy has responded to a change of $g_i(x_i, s)$ which has taken place between the last stage t, for which we have $h(x_i, s_1) = h_i(x_n, t)$, and s_1+1, and thus we redefine $h(x, s_1+1) = h_i(x, s_1+1) +_O k_i(x, s_1+1)$. If $h_i(x, s_1+1)$ has not dropped below a notation of a limit ordinal, then trivially $h(x, s_1 + 1) \geq_O h_i(x, s_1 + 1)$; otherwise $k_i(x, s_1 + 1) = 2$, and thus $h(x, s_1 + 1) >_O h_i(x, s_1 + 1)$. From now on, until $I(i, s_0)$ is discarded, it is easy to see that $h(x, s + 1) >_O h_i(x, s + 1)$. Moreover in the case $k_i(x, s + 1) = 2$, whether or not $k_i(x, s) = 1$ or $k_i(x, s) = 2$, it is easy to see that $h(x, s) >_O h(x, s + 1)$.

If and when $I(i, s_0)$ is discarded at, say $s_2 + 1$, then we have room for freezing x, with an extra change of $h(x, s_2 + 1)$. Indeed, up to that moment either $h(x, s + 1) \in \{a, b\}$, or $k_i(x, s_2) = 1$, and thus $h(x, s_2) >_O 1$ (in fact $h(x, s_2) \geq_O g_i(x, s_2) \geq_O u$, where $u <_O b$ is the notation of the greatest limit ordinal below $|b|_O$); or, $h(x, s_2) = g_i(x, s_2) +_O k_i(x, s_2)$, with $k_i(x, s_2) = 2$, and thus $h(x, s_2) >_O 1$.

As to numbers x which are never appointed as $x = x_{i,s_0}$, for any i, s_0, the claim is easy to show. Indeed, for any such x, one of the following holds: either x never enters a set of provisional theses, and thus there is no problem for a possible freezing action; or (and this is the case for instance, for numbers of the form $\rho_s(y_n)$ that enter elimination/recovery activities) x enters some set of provisional theses, at say $t_0 + 1$, at which point we set $h(x, t_0 + 1) = 2$, and thus there is room for a possible future freezing action. □

Lemma 4.32. *There are a proper loopless quasi-dialectical system $q' = \langle H', f, f^-, c, c^- \rangle$, where H' is a closure operator, such that $A_q = A_{q'}$.*

Proof. As in Lemma 4.19. □

This concludes the proof of the theorem. □

Finally, we prove:

Corollary 4.33. *For every finite $n \geq 2$ we can build a proper loopless quasi-dialectical system $q = \langle H, f, f^-, c, c^- \rangle$, and a computable approximation $\{H_s\}_{s\in\omega}$ to H, such that A_q is not $(n-1)$-c.e., and if $\{A_{q,s} : s \in \omega\}$ is the approximation to A_q given by the sets of provisional theses (corresponding to the built approximation to H), then*

(1) if n is even then for every y,

$$|\{s : A_{q,s}(f_y) \neq A_{q,s+1}(f_y)\}| \leq n;$$

(2) if n is odd then for every y,

$$|\{s : A_{q,s}(f_y) \neq A_{q,s+1}(f_y)\}| \leq n + 1.$$

Proof. The proof goes as the proof of the previous theorem, by taking $b = n - 1$, $\{Z_e\}_{e\in\omega}$ an effective listing of the $(n-1)$-c.e. sets, and

$$a = \begin{cases} n, & \text{if } n \text{ is even,} \\ n+1, & \text{if } n \text{ is odd.} \end{cases}$$

Of course for all e, x, s, we have in this case $k_e(x, s) = 1$. □

5. Conclusions

This paper has been mainly concerned with comparing dialectical and quasi-dialectical systems with respect to both their information content and their deductive power. We have shown that dialectical sets and quasi-dialectical sets have the same Turing-degrees, and the same enumeration degrees. Nonetheless, the class of dialectical sets is properly contained in the class of quasi-dialectical sets, and in fact the latter is much larger than the former.

Of course many interesting problems remain untouched. In particular, recall that Magari introduced dialectical systems in order to provide a simple - yet expressive - logical model for representing the (dynamic) behavior of mathematical theories. Hence, it comes naturally to ask if such a relationship between (quasi-)dialectical systems and formal theories can be better clarified. In this regard, let us conclude by hinting at two possible directions of research – first introduced in [9] and [3] – one can take to investigate this problem. Firstly, given a system S (that could be either dialectical or quasi-dialectical) it is possible to dismiss some pieces of the generality of its deduction operator H, by adding particular constraints that aim at mimicking logical connectives, thus making the behavior of S somewhat closer to the one expressed by classical deduction rules. Secondly, we have already mentioned that it is possible to associate to each formal theory T, dialectical systems $d = \langle H, f, c\rangle$ such that A_d is a completion of T (see the introduction of Section 4). Thus, one could try to study completions of (essentially undecidable) theories in terms of dialectical and quasi-dialectical sets. These lines of research will be pursued in a forthcoming work.

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