Splittings in GBL-algebras I: The general case

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Abstract. We study and characterize splitting algebras in varieties of integral residuated (semi)lattices; the main result is a complete characterization of the splitting algebras in the variety of GBL_{ew}-algebras, i.e. integral, bounded, commutative and divisible residuated lattices.

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1. Introduction

The concept of splitting algebra has a lattice theoretic origin: in [36] P.M. Whitman introduced splitting lattices and proved the first basic results about them. Splitting algebras have the property to split the lattice of subvarieties $\Lambda(\mathsf{V})$ of a variety V into the disjoint union of a principal filter and a principal ideal; if $\mathbf{A}$ is splitting for V then there is a subvariety $\mathsf{W}_{\mathbf{A}}$ such that for any $\mathsf{U} \subseteq \mathsf{V}$ either $\mathbf{A} \in \mathsf{U}$ or $\mathsf{U} \subseteq \mathsf{W}_{\mathbf{A}}$.

R. McKenzie in [31] showed that splitting lattices are useful for investigating non modular varieties of lattices, thus putting the concept in the forefront; a classification of all *finite* lattices that are splitting for the variety of all lattices followed [32]. In the same period R. McKenzie was proving results about splitting lattices, in another part of the world V. Jankov was studying intermediate logics, equivalently varieties of Heyting algebras; what he did, in our language, was to show that any finite subdirectly irreducible Heyting algebra is splitting for any variety of Heyting algebras to which it belongs [23]. He did that by mean of the so-called *Jankov formulas*, a concept that has been recycled many times; a crucial part of his proof used the well known characterization of subdirectly irreducible Heyting algebras, i.e. that any such Heyting algebra has a unique maximal lower cover of the top element 1. Jankov's results were later generalized by Blok and Pigozzi; in [11] they showed that in a variety with *equationally definable principal congruences* (EDPC) every *finitely presentable* subdirectly irreducible algebra is splitting. It follows that in a variety having both the EDPC and the finite model property all the finite subdirectly irreducible algebras are splitting;

the connection here is that Heyting algebras do enjoy both properties. For a more detailed account of other connections between varieties with EDPC and Jankov formulas the interested reader may want to look at the first section in [3].

Here we deal with residuated semilattices or lattices that may or may not be lower bounded. We will see that a proper description of the (finite) subdirectly irreducible algebras in a variety is a crucial ingredient for the characterization of splitting algebras. Our techniques build on the one employed in [3], where we characterized splitting algebras in some proper variety of GBL-algebras.

2. Preliminary results

2.1. Residuated semilattices

An algebra $\mathbf{A} = \langle A, \vee, \wedge, \cdot, \rightarrow, \leftarrow, 0, 1\rangle$ is an **FL-algebra**[1] if

(1) $\langle A, \vee, \wedge\rangle$ is a lattice;
(2) $\langle A, \cdot, 1\rangle$ is a monoid;
(3) $\rightarrow$ and $\leftarrow$ are the left and right residuation w.r.t. $\cdot$ i.e. for all $a, b, c \in A$

$$ab \leq c \quad \text{if and only if} \quad a \leq b \rightarrow c \quad \text{if and only if} \quad b \leq a \leftarrow c;$$

(4) 0 is an element of A.

A **residuated lattice** is a subreduct of an FL-algebra to the type without 0; a **residuated semilattice** is a subreduct of an FL-algebra to the type without 0 and $\vee$. Residuated (semi)lattices have a very long history (see [27] and the bibliography therein) and are very rich algebraic structures. An FL-algebra or a residuated (semi)lattice $\mathbf{A}$ is **commutative** if it satisfies $xy \approx yx$, **integral** if it satisfies $x \leq 1$ and **zero-bounded** if it satisfies $0 \leq x$; the zero-bounded FL-algebras are sometimes called FL_o-algebras. From now on we will assume with no further mention that every FL-algebra we consider is zero-bounded. In a commutative FL-algebra left and right residuation coincide and we will denote both by $\rightarrow$. The variety of commutative FL-algebras is denoted by FL_e, that of integral FL-algebras by FL_w and that of commutative and integral FL-algebras by FL_{ew} (the reason for this choice of symbols is clearly explained in Chapter 2 of [19]). Here is a (very) little sample of the equations holding in a residuated (semi)lattice; for a rather exhaustive list the reader can consult [13] and [18].

[1]Although / and \ are the standard notations for left and right residuals in the case of non-commutative logics, in this paper (and in its follow-up version [2]) we shall replace them by the arrows $\rightarrow$ and $\leftarrow$. The translation matrix is $a \leftarrow b = a\backslash b$ and $a \rightarrow b = b/a$.

Lemma 2.1. *If* $\mathbf{S} = \langle S, \wedge, \rightarrow, \leftarrow, \cdot, 1\rangle$ *is a residuated integral semilattice, then for any* $a, b, c \in S$ *the following hold:*

$$\begin{aligned} &a \rightarrow (b \leftarrow c) = b \leftarrow (a \rightarrow c) \\ &a \leq (a \rightarrow b) \leftarrow b \\ &a \leq (a \leftarrow b) \rightarrow b \\ &a \leq b \rightarrow a \\ &a \leq b \leftarrow a. \end{aligned}$$

2.2. Filters and normality

Residuated semilattices are congruence permutable; this was implicit in [1] but the reader can check directly that

$$m(x, y, z) = [((x \rightarrow y) \wedge 1) \leftarrow z] \wedge [((z \rightarrow y) \wedge 1) \leftarrow x)$$

is a Mal'cev term (see [14], Theorem 12.2, for a textbook explanation). They are also congruence point-regular at 1 since they satisfy the implication

$$(x \rightarrow y) \wedge 1 \approx 1 \quad \text{and} \quad (y \rightarrow x) \wedge 1 \approx 1 \quad \text{implies} \quad x \approx y;$$

this means that for any residuated semilattice $\mathbf{S}$ and any $\theta, \varphi \in \mathrm{Con}(\mathbf{S})$, $1/\theta = 1/\varphi$ implies $\theta = \varphi$. Next a **filter** F of $\mathbf{S}$ is a subset $F \subseteq S$ with the following properties:

(1) $1 \in F$;
(2) if $a \in F$ and $a \leq b$ then $b \in F$;
(3) if $a, b \in F$ then $a \wedge b, ab \in F$.

So a filter is just a semilattice filter containing 1 and closed under multiplication; if $\nabla_{\mathbf{S}} = \{a \in S : a \geq 1\}$ and $\theta \in \mathrm{Con}(\mathbf{S})$, then $\nabla_{\mathbf{S}}/\theta$ is a filter and the filters of that form will be called **congruence filters**. Both filters and congruence filters form algebraic lattices of closed sets, with the obvious closure operators; note also that congruence filters are the *normal ideals* of $\mathbf{S}$ in the sense of [7]. Congruence permutability and point regularity at 1 imply that the congruence filter lattice and the congruence lattice of $\mathbf{S}$ are isomorphic [7]. This isomorphism has been described in [1]; if $\theta \in \mathrm{Con}(\mathbf{S})$ we define $F_\theta = \bigcup\{a/\theta : a \geq 1\}$ that is clearly a filter. Then the two mappings

$$\theta \longmapsto F_\theta \qquad\qquad F \longmapsto \theta_F = \{(u, v) : u \rightarrow v, v \rightarrow u \in F\}$$

are mutually inverse lattice isomorphisms.

From now on we will deal only with integral residuated semilattices and we will state our results accordingly; note that in this case a congruence filter of $\mathbf{S}$ is of the form $1/\theta$ for $\theta \in \mathrm{Con}(\mathbf{S})$. The closure operator defining the filters is easy to describe; if $\mathbf{S}$ is an integral residuated semilattice and $X \subseteq S$ then the filter $\mathrm{Fil}_{\mathbf{S}}(X)$ generated by X is

$$\mathrm{Fil}_{\mathbf{A}}(X) = \{b : \text{there are } a_1, \ldots, a_n \in X \text{ with } a_1 \cdots a_n \leq b\}.$$

It also possible to define the congruence filter generated by a subset X (see [18]), however there is a nicer characterization of congruence filters in term of filters.

Lemma 2.2. [34] *Let* $\mathbf{S}$ *be an integral residuated semilattice and let* F *be a filter of* $\mathbf{S}$*. Then the following are equivalent:*

(1) *F is a congruence filter;*
(2) *for all $a, b \in S$, $a \in F$ implies $b \to ba, b \leftarrow ab \in F$;*
(3) *for all $a, b, c \in S$, $a \in F$ implies $c \to c(b \leftarrow ab) \in F$;*
(4) *for all $a, b, c \in S$, $a \in F$ implies $c \leftarrow (b \to ba)c \in F$.*

An (integral) residuated semilattice is **normal** if every filter is a congruence filter and a variety is normal if each of its members is normal; commutative residuated semilattices are normal [1] but there are non commutative normal varieties of residuated semilattices. Section 4 of [20] contains an implicit characterization of integral and normal residuated semilattices. Let's make it explicit:

Lemma 2.3. *For an integral residuated semilattice* $\mathbf{S}$ *the following are equivalent:*

(1) $\mathbf{S}$ *is normal;*
(2) *every principal filter of* $\mathbf{S}$ *is a congruence filter;*
(3) *if $a \in S$ then for all $b \in S$ there exist $n, m \in \mathbb{N}$ (possibly depending on b) such that $a^n b \leq ba$ and $ba^m \leq ab$.*

Proof. (1) clearly implies (2). Assume then (2), and let $a \in S$; since $\mathrm{Fil}_{\mathbf{S}}(a)$ is normal, by Lemma 2.2 for any $b \in S$ $b \to ba, b \leftarrow ab \in \mathrm{Fil}_{\mathbf{S}}(a)$. This implies that there are $n, m \in \mathbb{N}$ such that $a^n \leq b \to ba$ and $a^m \leq b \leftarrow ab$, from which (3) follows. Finally assume (3) and let F be a filter of $\mathbf{S}$; if $a \in F$ and $b \in S$ then there is an $n \in \mathbb{N}$ with $a^n b \leq ba$. It follows that $a^n \leq b \to ba$ and so $b \to ba \in F$ and, by similar argument, $b \leftarrow ab \in F$ as well. By Lemma 2.2 F is a congruence filter and (1) holds. □

Of course there might be no bound on n in the above lemma; but if there is one (for instance, if $\mathbf{S}$ is finite), then $\mathbf{V}(\mathbf{S})$ is a normal variety of integral residuated semilattices. On the other hand if an entire variety is normal then we can say more.

Corollary 2.4. *Let* V *be a normal variety of integral residuated semilattices; then there exists and $n \in \mathbb{N}$ such that the equations $x^n y \leq yx$ and $yx^n \leq xy$ hold in* V*.*

Proof. If V is normal then the free algebra $\mathbf{F}_{\mathsf{V}}(x, y)$ is normal, so by Lemma 2.3 there is an n with $x^n y \leq yx$ and $yx^n \leq xy$. But since $\mathbf{F}_{\mathsf{V}}(x, y)$ is free the two equations must hold in V. □

Note that it is easily seen that the congruences (and the filters) of an FL-algebra or a residuated lattice are the same as the congruences of its residuated semilattice reduct; so everything we said in this section holds for FL-algebras and residuated semilattices as well. For instance:

Corollary 2.5. *Let FL_w^n be the variety of integral FL-algebras axiomatized by the equations $x^n y \le yx$ and $yx^n \le xy$; then*
1. $\mathsf{FL}_w^1 = \mathsf{FL}_{ew}$;
2. *if $n < m$, then $\mathsf{FL}_w^n \subseteq \mathsf{FL}_w^m$;*
3. *a variety of integral FL-algebras is normal if and only if $\mathsf{V} \subseteq \mathsf{FL}_w^n$ for some n.*

Proof. (1) and (2) are obvious; for (3) if V is normal, then Corollary 2.4 applied to FL-algebras implies $\mathsf{V} \subseteq \mathsf{FL}_w^n$ for some n. Conversely if $\mathsf{V} \subseteq \mathsf{FL}_w^n$ for some n, then V is normal by Lemma 2.3. □

2.3. Subreducts

There are also two useful consequences of normality; if $\mathbf{S}$ is a residuated semilattice we denote as usual by $\theta_{\mathbf{S}}(a, b)$ the smallest congruence of $\mathbf{S}$, containing the pair (a, b). If $\mathbf{S}$ is normal and $a, b, c, d \in S$ then

$$(c, d) \in \theta_{\mathbf{S}}(a, b) \qquad \text{if and only if} \qquad \text{there is an } n \text{ with } (a \leftrightarrow b)^n \le c \leftrightarrow d.$$

This was observed in [1] for integral and commutative residuated semilattices, but the reader can check that the only thing that was used was normality (that one gets for free because of commutativity). Thus, following [15], any normal and integral residuated semilattice $\mathbf{S}$ has the *congruence extension property* (CEP): the congruences of a subalgebra of $\mathbf{S}$ are exactly the restriction to the subalgebra of the congruences of $\mathbf{S}$. This in turn implies that for any normal residuated semilattice $\mathbf{S}$, $\mathbf{HS}(\mathbf{S}) = \mathbf{SH}(\mathbf{S})$ (this in fact holds for every algebra having the CEP, see [14] Ch.II, §9, Exercise 5).

Next for a variety V of integral FL-algebras we denote by $\mathbf{S}^{rl}(\mathsf{V})$ and $\mathbf{S}^{rs}(\mathsf{V})$ the class of subreducts of V to the type of residuated lattices and of residuated semilattices respectively.

Theorem 2.6. *For any normal variety V of integral FL-algebras, $\mathbf{S}^{rl}(\mathsf{V})$ is a variety of integral residuated lattices and $\mathbf{S}^{rs}(\mathsf{V})$ is a variety of integral residuated semilattices.*

Proof. Since the congruences of an FL-algebra are the congruences of its residuated semilattice reduct, it is enough to prove the second claim. The class $\mathbf{S}^{rs}(\mathsf{V})$ is clearly closed for subalgebras and direct products; so let $\mathbf{S} \in \mathbf{S}^{rs}(\mathsf{V})$ and let $\mathbf{A} \in \mathsf{V}$, where $\mathbf{S}$ is a subreduct of $\mathbf{A}$. Note that since $\mathbf{A}$ is normal, so is $\mathbf{S}$, since normality depends only on the residuated semilattice reduct. Take $\alpha \in \mathrm{Con}(\mathbf{S})$; then $1/\alpha = F$ is a congruence filter; let G be the filter of $\mathbf{A}$ generated by F and let θ_G be the associated congruence of $\mathbf{A}$. By the CEP $\alpha = \theta_G \cap S^2$; this implies that $\mathbf{S}/\alpha$ is isomorphic with a subreduct of $\mathbf{A}/\theta_G$. Hence $\mathbf{S}/\alpha \in \mathbf{S}^{rs}(\mathsf{V})$ and the proof is concluded. □

2.4. Jónsson Lemma

An algebra $\mathbf{A}$ is *finitely subdirectly irreducible* if whenever it is isomorphic with a subdirect product of a finite family of algebras, then it is already isomorphic to one of them. Clearly any subdirectly irreducible algebra is finitely subdirectly irreducible, but the converse fails nontrivially. Note that any totally ordered FL-algebra is finitely subdirectly irreducible, in that the principal congruence filters form a chain.

FL-algebras have lattice terms so they are congruence distributive; residuated semilattices are also congruence distributive, as observed by P. Jipsen in [24], and for congruence distributive varieties there is a strong result available. It was proved by B. Jónsson in [28] for subdirectly irreducible algebras, but a quick examination of the textbook exposition in [14] p. 148 ff. reveals that the conclusion holds for finitely subdirectly irreducible algebras as well.

Lemma 2.7. *(Jónsson's Lemma) Let* K *be a class of algebras such that* $\mathbf{V}(\mathsf{K})$ *is congruence distributive; then*

(1) *if* $\mathbf{A}$ *is a finitely subdirectly irreducible algebra in* $\mathbf{V}(\mathsf{K})$*, then* $\mathbf{A} \in \mathbf{HSP}_u(\mathsf{K})$*;*
(2) *if* $\mathbf{A}, \mathbf{B}$ *are finite subdirectly irreducible algebras in* $\mathbf{V}(\mathsf{K})$ *then* $\mathbf{V}(\mathbf{A}) = \mathbf{V}(\mathbf{B})$ *if and only if* $\mathbf{A} \cong \mathbf{B}$*.*

2.5. Splitting algebras

If V is any variety, a **splitting pair** in V is a pair $(\mathsf{W}_1, \mathsf{W}_2)$ of subvarieties of V such that if U is a subvariety of V either $\mathsf{U} \subseteq \mathsf{W}_1$ or $\mathsf{W}_2 \subseteq \mathsf{U}$.

Lemma 2.8. *If* $(\mathsf{W}_1, \mathsf{W}_2)$ *is a splitting pair for* V *then*

(1) W_1 *is axiomatized by a single equation;*
(2) W_2 *is generated by a single finitely generated subdirectly irreducible algebra;*
(3) W_1 *and* W_2 *are respectively completely meet prime and completely join prime in the lattice of subvarieties of* V*.*

Proof. (1) and (2) were proved by R. McKenzie in [31] (but see [19] Chapter 10.2, for a textbook proof); point (3) is in [36]. □

The equation in (1) above is called the **splitting equation** of the pair and the algebra in (2) is called a **splitting algebra for** V; so a splitting algebra for V is a finitely generated subdirectly irreducible algebra $\mathbf{A} \in \mathsf{V}$ such that there exists a $\mathsf{W} \subseteq \mathsf{V}$ such that $(\mathsf{W}, \mathbf{V}(\mathbf{A}))$ is a splitting pair. In other words W is the largest subvariety of V to which $\mathbf{A}$ does not belong. This variety, called the **conjugate variety** of $\mathbf{A}$, admits a better description: for any algebra $\mathbf{A} \in \mathsf{V}$ let

$$\mathsf{W}_{\mathbf{A}}^{\mathsf{V}} = \bigvee \{\mathbf{V}(\mathbf{B}) : \mathbf{B} \in \mathsf{V}, \mathbf{A} \notin \mathbf{V}(\mathbf{B})\}.$$

Lemma 2.9. *Let* V *be any variety and let* $\mathbf{A} \in \mathsf{V}$*. The following are equivalent:*

(1) $\mathbf{A}$ *is splitting for* V*;*
(2) $\mathbf{A}$ *is splitting for* V *with conjugate variety* $\mathsf{W}_{\mathbf{A}}^{\mathsf{V}}$*;*

Proof. Clearly (2) implies (1). Assume that $\mathbf{A}$ is splitting for V with conjugate variety W; if $\mathbf{B} \in \mathsf{V}$ and $\mathbf{A} \notin \mathbf{V}(\mathbf{B})$, then $\mathbf{V}(\mathbf{B}) \subseteq \mathsf{W}$ and thus $\mathsf{W}_{\mathbf{A}}^{\mathsf{V}} \subseteq \mathsf{W}$. Conversely if $\mathbf{B} \in \mathsf{W}$, then $\mathbf{V}(\mathbf{B}) \subseteq \mathsf{W}$ and thus $\mathbf{V}(\mathbf{A}) \not\subseteq \mathbf{V}(\mathbf{B})$; it follows that $\mathbf{A} \notin \mathbf{V}(\mathbf{B})$ and hence $\mathbf{B} \in \mathsf{W}_{\mathbf{A}}^{\mathsf{V}}$. So $\mathsf{W} = \mathsf{W}_{\mathbf{A}}^{\mathsf{V}}$ and (1) implies (2). □

As a consequence we get:

Corollary 2.10. *Let V be any variety and let $\mathbf{A} \in \mathsf{V}$; then*

(1) *if $\mathbf{A} \in \mathsf{W}^{\mathsf{V}}_{\mathbf{A}}$, then $\mathbf{A}$ cannot be splitting for V;*
(2) *if $\mathbf{A}$ is splitting for V then it is splitting for any subvariety of V to which it belongs;*
(3) *if $\mathsf{V} \subseteq \mathsf{V}'$ and and $\mathbf{A}$ is not splitting for V, then it is not splitting for V'.*

Proof. Suppose that $\mathbf{A}$ is splitting for V; then by Lemma 2.9 it is splitting with conjugate variety $\mathsf{W}^{\mathsf{V}}_{\mathbf{A}}$ that is also completely join prime. But if $\mathbf{A} \in \mathsf{W}^{\mathsf{V}}_{\mathbf{A}} = \bigvee\{\mathbf{V}(\mathbf{B}) : \mathbf{B} \in \mathsf{V}, \mathbf{A} \notin \mathbf{V}(\mathbf{B})\}$ then complete join primeness would force $\mathbf{A} \notin \mathbf{V}(\mathbf{A})$ a clear contradiction. So $\mathbf{A} \notin \mathsf{W}^{\mathsf{V}}_{\mathbf{A}}$ and (1) holds. (2) and (3) follow straight from the definition. $\square$

If $\mathbf{A}$ is splitting for V and V is generated by its finite algebras, then $\mathbf{A}$ is finite and if moreover V is also congruence distributive, then by Jónsson Lemma, $\mathbf{A}$ is uniquely determined (this was observed by McKenzie in [31]). Note that generation by finite algebras is just the *finite model property* for the equational theory of V; if an equation fails in V, then it fails in a finite algebra in V.

Example 2.11. If there is a single atom U in $\Lambda(\mathsf{V})$ then (T, U) is a splitting pair in $\Lambda(\mathsf{V})$ (T is the trivial variety). It follows that any finitely generated subdirectly irreducible algebra in U is splitting, with splitting equation $x \approx y$ and conjugate variety T. It is obvious that the two element Boolean algebra $\mathbf{2}$ is contained in any nontrivial variety of FL_{ew}-algebras, so the variety BA of Boolean algebras is the only atom in any subvariety of FL_{ew}. Hence $\mathbf{2}$ is splitting for any variety in $\Lambda(\mathsf{FL}_{ew})$; in [30] T. Kowalski and H. Ono proved that it is the only splitting algebra in FL_{ew}, a rather deep result.

Example 2.12. If there is a single coatom U in $\Lambda(\mathsf{V})$ then (U, V) is a splitting pair in $\Lambda(\mathsf{V})$; it follows that V must be generated by a finitely generated subdirectly irreducible algebra. Now there is a standard way to look at ℓ-groups as (non integral) residuated lattices; more precisely the variety LG of ℓ-groups is the subvariety of residuated lattices axiomatized by $x(x \to 1) \approx 1$ (see [18]). It is well-known (see for instance [9] p. 47 ff) that if we regard $\mathbb{Z}$ as a totally ordered group, then $\mathbf{V}(\mathbb{Z})$ is the only atom in $\Lambda(\mathsf{LG})$, while the only coatom in $\Lambda(\mathsf{LG})$ is the variety N of *normal-valued* ℓ-groups. It follows that $\mathbf{V}(\mathbb{Z})$ is splitting for LG and that any finitely generated subdirectly irreducible ℓ-group $\mathbf{G}$ such that $\mathbf{V}(\mathbf{G}) = \mathsf{LG}$ is splitting for LG (there is at least one such ℓ-group by the above consideration). If we want a similar but integral example we can consider LG^-, i.e. the variety of *negative cones* of ℓ-groups (see [10] for a detailed explanation).

3. Ordinal sums and splittings

Let $\mathbf{F}, \mathbf{S}$ be two integral residuated semilattices such that $F \cap S = \{1\}$; the **ordinal sum** $\mathbf{F} \oplus \mathbf{S}$ is an integral residuated lattice whose universe is $F \cup S$

and the operations are defined in the following way. If x, y both belong to F or S then the operations are defined as those in each algebra; otherwise

$$x \to y = x \leftarrow y = \begin{cases} y & \text{if } x \in S \text{ and } y \in F \\ 1 & \text{if } x \in F \setminus \{1\} \text{ and } y \in S \end{cases}$$

$$x \cdot y = \begin{cases} y & \text{if } x \in S \text{ and } y \in F \setminus \{1\} \\ x & \text{if } x \in F \setminus \{1\} \text{ and } y \in S \end{cases}$$

$$x \wedge y = \begin{cases} x & \text{if } x \in F \setminus \{1\} \text{ and } y \in S \\ y & \text{if } y \in F \setminus \{1\} \text{ and } x \in S \\ 1 & \text{if } x = y = 1. \end{cases}$$

It is easily seen that $\mathbf{F} \oplus \mathbf{S}$ is always a residuated semilattice and the ordering is the one obtained stacking the two semilattices one over the other.

In this very general setting there are few things we can say; if $\mathbf{O}$ is an operator that is a composition of $\mathbf{H}, \mathbf{S}, \mathbf{I}, \mathbf{P}, \mathbf{P}_u$ and $\mathbf{F}, \mathbf{S}$ are integral residuated semilattices, we define

$$\mathbf{O}(\mathbf{F}) \oplus \mathbf{O}(\mathbf{S}) = \{\mathbf{F}' \oplus \mathbf{S}' : \mathbf{F}' \in \mathbf{O}(\mathbf{F}), \mathbf{S}' \in \mathbf{O}(\mathbf{S})\}.$$

Lemma 3.1. *Let $\mathbf{A}, \mathbf{B}$ be integral residuated semilattices; then*

(1) $\mathbf{S}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{S}(\mathbf{A}) \oplus \mathbf{S}(\mathbf{B})$;
(2) $\mathbf{H}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{H}(\mathbf{A}) \cup \mathbf{A} \oplus \mathbf{H}(\mathbf{B})$;
(3) $\mathbf{P}_u(\mathbf{A} \oplus \mathbf{B}) \subseteq \mathbf{P}_u(\mathbf{A}) \oplus \mathbf{P}_u(\mathbf{A})$;
(4) $\mathbf{ISP}_u(\mathbf{A} \oplus \mathbf{B}) = \mathbf{ISP}_u(\mathbf{A}) \oplus \mathbf{ISP}_u(\mathbf{B})$.

Proof. (1) is obvious. For (2) let h be a homomorphism with domain $\mathbf{A} \oplus \mathbf{B}$ and let $G = \{a : h(a) = 1\}$; then G is a filter of $\mathbf{A} \oplus \mathbf{B}$. If $G \subseteq B$, then for $a, b \in A$, $h(a) = h(b)$ if and only if $h(a \to b) = h(b \to a) = 1$; thus $a \to b, b \to a \in G$ and since $G \subseteq B$ this is possible only if $a \to b = b \to a = 1$. i.e. $a = b$. Thus $h_{|A}$ is an isomorphism and thus $h(\mathbf{A} \oplus \mathbf{B}) = \mathbf{A} \oplus h(\mathbf{B})$. Suppose now that $G \not\subseteq B$, i.e. there is an $a \in A \setminus \{1\}$ with $h(a) = 1$. For all $b \in B$, $a \leq b$ implies $1 = h(a) \leq h(b)$, so $h(b) = 1$ and $B \subseteq G$; clearly $h(\mathbf{A} \oplus \mathbf{B}) = h_{|A}(\mathbf{A})$.

For (3) suppose that $\mathbf{D} \in \mathbf{P}_u(\mathbf{A} \oplus \mathbf{B})$; then there is a set X, an algebra $\mathbf{C} \cong (\mathbf{A} \oplus \mathbf{B})^X$ and and an ultrafilter U on X such that $\mathbf{C}/U \cong \mathbf{D}$; through the isomorphism, we can identify each $a \in C$ with the choice function $(a_x)_{x \in X}$ where $a_x \in A \oplus B$ for all x. Next define

$$Y_a = \{x \in X : a_x \in A \setminus \{1\}\} \qquad Z_a = \{x \in X : a_x \in B\};$$

since for all a, Y_a, Z_a form a partition of X, either $Y_a \in U$ or $Z_a \in U$. Let then

$$A' = \{a/U : Y_a \in U\} \cup \{1\} \qquad B' = \{a/U : Z_a \in U\};$$

it is easily seen that A, B are the universes of two subalgebras of $\mathbf{C}/U$. Now take $a/U \in A' \setminus \{1\}$ and $b/U \in B'$; then $Y_a \cap Z_b \in U$ and for any $x \in Y_a \cap Z_b$, $b_x a_x = a_x$ and $b_x \to a_x = a_x$. By definition of ultraproduct it follows that

$b/U\, a/U = a/U$ and $b/U \to a/U = a/U$, so $\mathbf{C}/U = \mathbf{A}' \oplus \mathbf{B}'$ by definition of ordinal sum. It remains to check that $\mathbf{A}' \in \mathbf{P}_u(\mathbf{A})$ and $\mathbf{B}' \in \mathbf{P}_u(\mathbf{B})$, and we leave it to the reader.

For (4), the left-to-right inclusion follows from (1) and (3); take then $\mathbf{A}' \in \mathbf{ISP}_u(\mathbf{A})$ and $\mathbf{B}' \in \mathbf{ISP}_u(\mathbf{B})$. Let $\mathbf{A}^X/U$ be the ultrapower of $\mathbf{A}$ in which $\mathbf{A}'$ embeds; then it is easily seen that $\mathbf{A}' \oplus \mathbf{B}'$ embeds in $(\mathbf{A} \oplus \mathbf{B}')^X/U$. Similarly if $\mathbf{B}^Y/V$ is the ultrapower in which $\mathbf{B}'$ embeds, then $\mathbf{A} \oplus \mathbf{B}'$ embeds into $(\mathbf{A} \oplus \mathbf{B})^Y/V$. Therefore

$$\mathbf{A}' \oplus \mathbf{B}' \in \mathbf{ISP}_u(\mathbf{SP}_u(\mathbf{A} \oplus \mathbf{B})) \subseteq \mathbf{ISP}_u(\mathbf{A} \oplus \mathbf{B})$$

as wished. □

If $\mathbf{A}$ is an integral residuated semilattice a **cut** of $\mathbf{A}$ is a pair (F, S) of subsets of A with the following properties:

(1) $F \cap S = \{1\}, F \cup S = A$;
(2) F is the universe of a subalgebra of $\mathbf{A}$;
(3) for all $a \in F \setminus \{1\}$, $b \in S$, $a \leq b$;
(4) S is closed under $\cdot$;
(5) for all $a \in F \setminus \{1\}$ and $b \in S$

$$ab = ba = a \qquad\qquad b \to a = b \leftarrow a = a.$$

A cut is trivial if either $F = \{1\}$ or $S = \{1\}$. This concept was foreshadowed in [6], Lemma 3.5, and formally introduced in [17].

Lemma 3.2. *Let $\mathbf{A}$ be an integral residuated semilattice:*

(1) *if (F, S) is a nontrivial cut of $\mathbf{A}$ then S is the universe of a subalgebra $\mathbf{S}$ of $\mathbf{A}$ and $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$.*
(2) *Conversely, if $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$ nontrivially, then (F, S) is a nontrivial cut.*
(3) *If $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$, then $\mathbf{S}$ is a congruence filter of $\mathbf{A}$.*

Proof. (2) is clear from the definition of ordinal sum. For (1), suppose that (F, S) is a nontrivial cut of $\mathbf{A}$; then, by (2) and (3) of the definition, S is upward closed, hence it is closed under $\to$ and $\leftarrow$. It is also closed under $\wedge$ since if $a, b \in S$, then $a \wedge b \geq ab \in S$; everything else follows from the definition.

For (3), by Lemma 2.2, it is enough to prove that for all $a \in S$ and $b \in A$, $b \to ba, b \leftarrow ab \in S$. Since $\mathbf{S}$ is a subalgebra, the conclusion holds if $a, b \in S$. If $a \in S$ and and $b \in F \setminus \{1\}$ then $b \to ba = b \to b = 1 \in S$ and $b \leftarrow ab = b \leftarrow b = 1 \in S$. Finally if $b = 1$ then $b \to ba = 1 \to a = a \in S$ and $b \leftarrow ab = 1 \leftarrow a = a \in S$. □

Corollary 3.3. *If $\mathbf{S} = \mathbf{A} \oplus \mathbf{F}$, then $\mathbf{S}$ is subdirectly irreducible if and only if $\mathbf{F}$ is subdirectly irreducible.*

If we try to define the ordinal sum for integral residuated lattices, then we run into a problem: the ordering that results in stacking two integral residuated lattices one over another without any other provision might not be a lattice ordering. For instance let $\mathbf{A}, \mathbf{B}$ two integral residuated lattices and suppose that 1 is not join irreducible in $\mathbf{A}$ and $\mathbf{B}$ has no minimum. Then if $a, b \in A$ and $a \vee b = 1$, then $a \vee b$ simply does not exist in the new ordering, since the upper bounds of a, b are all the elements of B and B has no minimum. We adopt the solution introduced in [25]; if $\mathbf{A}, \mathbf{B}$ are integral residuated lattices then

type (1): if 1 is join irreducible in $\mathbf{A}$, then $\mathbf{A} \oplus \mathbf{B}$ is defined as they were semilattices and the join is the one induced by the ordering;

type (2): if 1 is not join irreducible in $\mathbf{A}$ but $\mathbf{B}$ has a minimum m then then $\mathbf{A} \oplus \mathbf{B}$ is defined as they were semilattices and

$$x \vee y = \begin{cases} x \vee^{\mathbf{B}} y & \text{if } x, y \in B \\ x \vee^{\mathbf{A}} y & \text{if } x, y \in A \text{ and } x \vee y < 1 \\ m & \text{if } x, y \in A \text{ and } x \vee y = 1 \\ x & \text{if } x \in B \text{ and } y \in A \\ y & \text{if } x \in A \text{ and } y \in B; \end{cases}$$

type (3): if 1 is not join irreducible in $\mathbf{A}$ and $\mathbf{B}$ has no minimum, then $\mathbf{A} \oplus \mathbf{B} = (\mathbf{A} \oplus \mathbf{2}) \oplus \mathbf{B}$, where $\mathbf{2}$ is the two element residuated lattice.

Note that if $\mathbf{B}$ is finite the sum is always of type (1) or (2); the sum of type (3) is introduced only for completeness and, up to our knowledge, has never been really used. Of course if $\mathbf{A}$ as an integral FL-algebra, then $\mathbf{A} \oplus \mathbf{B}$ is an integral FL-algebra. Homomorphic images of a sum of two integral residuated lattices behave as in the semilattice case (and the proof is the same as in Lemma 3.1); if $\mathbf{F}, \mathbf{S}$ are residuated lattices then

$$\mathbf{H}(\mathbf{F} \oplus \mathbf{S}) = \mathbf{H}(\mathbf{F}) \cup \mathbf{F} \oplus \mathbf{H}(\mathbf{S}).$$

Things are slightly different for subalgebras (here we abbreviate by $\mathbf{C} \leq \mathbf{A}$ the sentence "$\mathbf{C}$ is a subalgebra of $\mathbf{A}$"):

Lemma 3.4. *Let $\mathbf{A} \oplus \mathbf{B}$ the ordinal sum of type (1) or (2) of two residuated lattices. Then:*

1. *if $\mathbf{C} \leq \mathbf{A}$, $\mathbf{D} \leq \mathbf{B}$ and $\mathbf{D}$ is non trivial then $\mathbf{C} \oplus \mathbf{D}$ is a subalgebra of $\mathbf{A} \oplus \mathbf{B}$;*
2. *for any subalgebra $\mathbf{C} \leq \mathbf{A}$, if 1 is join irreducible in $\mathbf{C}$, then $\mathbf{C}$ is a subalgebra of $\mathbf{A} \oplus \mathbf{B}$; otherwise $\mathbf{C} \oplus \mathbf{2}$ is a subalgebra of $\mathbf{A} \oplus \mathbf{B}$.*

We leave the description of the situation in sums of type (3) to the reader. We can introduce cuts also in FL-algebras, but we have to divide them according to the type of the ordinal sum. If $\mathbf{A}$ is an integral FL-algebra a **cut of type (1)** is just a cut as defined above; a **cut of type (2)** is a triple (F, S, m), where F, S are subsets of $\mathbf{A}$ and m is the minimum of S, such that

1. $F \cap S = \{1\}, F \cup S = A$;
2. m is idempotent, i.e. $m^2 = m$;
3. $F \cup \{m\}$ is the universe of a subalgebra of $\mathbf{A}$;
4. for all $a \in F \setminus \{1\}$, $b \in S$, $a \leq b$;
5. S is closed under $\cdot$;

(6) for all $a \in F \setminus \{1\}$ and $b \in S$

$$ab = ba = a \qquad b \to a = b \leftarrow a = a.$$

Lemma 3.5. *Let* $\mathbf{A}$ *be an integral FL-algebra;*

(1) *if* (F,S) *is a cut of type (1) then* S *is the universe of a subalgebra* $\mathbf{S}$ *of* $\mathbf{A}$ *and* $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$ *where the ordinal sum is of type (1). Conversely, if* $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$ *nontrivially as a sum of type (1), then* (F,S) *is a nontrivial cut of type (1).*

(2) *if* (F,S,m) *is cut of type (2) then* S *is the universe of a subalgebra* $\mathbf{S}$ *of* $\mathbf{A}$ *and* $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$ *where the ordinal sum is of type (2). Conversely, if* $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$ *nontrivially as a sum of type (2), and* m *is the minimum of* $\mathbf{S}$*, then* (F,S,m) *is a nontrivial cut of type (2).*

In this very general setting the concept of cut is not much more than a rephrasing of the definition of ordinal sum; it will reveal its importance in less generic contexts.

Let now $\mathbf{S}$ be a finite integral residuated semilattice. Note that integrality implies that 1 is meet irreducible; hence if $p_i(\mathbf{x})$, $i = 1, \dots, n$ are terms in the language of integral residuated semilattices, then the equation $\bigwedge_{i=1}^n p_i(\mathbf{x}) \approx 1$ is true in a model if and only if all the equations $p_i(\mathbf{x}) \approx 1$ are true in the model. Since the equality relation is term definable, this makes possible to encode many information about $\mathbf{S}$ into a term. Suppose for instance that $S = \{a_1, \dots, a_n, 1\}$ and let $X_S = \{x_{a_i} : i = 1, \dots, n\}$ be a set of distinct variables; then, roughly speaking, the term

$$p(x_{a_1}, \dots, x_{a_n}) = \bigwedge_{i,j=1}^{n} (x_{a_i \to^{\mathbf{S}} a_j} \leftrightarrow (x_{a_i} \to x_{a_j}))$$

encodes the operation table for $\to$ in $\mathbf{S}$, in the sense that $\mathbf{S} \vDash p^{\mathbf{S}}(a_1, \dots, a_n) = 1$. Similarly the term

$$p'(x_{a_1}, \dots, x_{a_n}) = \bigwedge_{i=1}^{n} (x_{a_i} \to x_{a_n})$$

encodes the fact that a_n is the only coatom (and hence that 1 is join irreducible), while

$$p''(x_{a_1}, \dots, x_{a_n}) = \bigwedge_{i=1}^{n-1} (x_{a_i} \to x_{a_{i+1}})$$

encodes the fact that $\mathbf{S}$ is totally ordered. In other words finite integral residuated semilattices are able to *speak about themselves*: any term encoding information about $\mathbf{S}$ is called a **diagram**. A **Jankov formula** for $\mathbf{S}$ is simply any equation in the variables X_S involving diagrams of $\mathbf{S}$.

Now suppose that $\mathbf{S}$ is a finite subdirectly irreducible residuated semilattice with minimal nontrivial congruence μ; since $1/\mu$ is a finite subalgebra, it has a minimum element that will be denoted by $a_\star$. If $S = \{a_1, \dots, a_n, a_\star, 1\}$ let's define a diagram (called the **t-diagram**) for $\mathbf{S}$ in the following way

$$T_{\mathbf{S}}(X_S) = \bigwedge \{x_{u *^{\mathbf{S}} v} \leftrightarrow x_u * x_v : u, v \in S, * \in \{\vee, \wedge, \to, \leftarrow, \cdot, 1\}\}.$$

Observe that $T_{\mathbf{S}}(X_S)$ encodes all the operation tables of $\mathbf{S}$ and so, by design, $T^{\mathbf{S}}_{\mathbf{S}}(a_1, \dots, a_n, a_\star, 1) = 1$.

Lemma 3.6. *Let* $\mathbf{S}$ *be a finite subdirectly irreducible integral residuated semilattice* $S = \{a_1, \dots, a_n, a_\star, 1\}$ *and let* $\mathbf{B}$ *any integral residuated semilattice. Then* $\mathbf{S} \in \mathbf{IS}(\mathbf{B})$ *if and only if there are* $b_1, \dots, b_n, b_\star \in B$ *with* $b_\star \neq 1$ *and*

$$T^{\mathbf{B}}_{\mathbf{S}}(b_1, \dots, b_n, b_\star, 1) = 1.$$

Proof. Suppose $\mathbf{S} \in \mathbf{IS}(\mathbf{B})$, let h be an embedding witnessing this fact and let's define $b_1 = h(a_1), \dots, b_n = h(a_n), b_\star = h(a_\star)$. Since h is an embedding $b_\star \neq 1$ and, since $T^{\mathbf{S}}_{\mathbf{S}}(a_1, \dots, a_n, a_\star, 1) = 1$, by applying h we get the desired conclusion.

Suppose then that $b_{a_1}, \dots, b_{a_n}, b_* \in B$ are such that $T^{\mathbf{B}}_{\mathbf{S}}(b_{a_1}, \dots, b_{a_n}, b_\star, 1) = 1$. Let's define a function $h : S \longrightarrow B$ by setting

$$h(a_1) = b_{a_1}, \dots, h(a_n) = b_{a_n}, h(a_\star) = b_\star, h(1) = 1.$$

Since 1 is the top element, each conjunct in $T^{\mathbf{B}}_{\mathbf{S}}(b_{a_1}, \dots, b_{a_n}, b_\star, 1)$ must be equal to 1; hence if $* \in \{\vee, \wedge, \rightarrow, \cdot\}$

$$\begin{aligned} h(a_i *^{\mathbf{S}} a_j) &= h(a_{a_i *^{\mathbf{S}} a_j}) = b_{a_i *^{\mathbf{S}} a_j} \\ &= b_{a_i} *^{\mathbf{B}} b_{a_j} && \text{since} \quad b_{a_i *^{\mathbf{S}} a_j} \leftrightarrow b_{a_i} *^{\mathbf{B}} b_{a_j} = 1 \\ &= h(a_i) *^{\mathbf{B}} h(a_j). \end{aligned}$$

This is enough to conclude the h is a homomorphism. Now by definition $h(1) \neq h(a_\star)$ so $(1, a_\star) \notin \ker(h)$; but the pair $(1, a_\star)$ generates the minimal congruence of $\mathbf{A}$, so the only possibility is that $\ker(h)$ is the zero congruence. Hence h is an embedding and the proof is finished. ☐

Note that the lemma above holds trivially for integral finite subdirectly irreducible residuated lattices or FL-algebras (and was proved in [30] in the commutative case), by redefining the t-diagram accordingly. Let $\mathbf{S}$ be a finite subdirectly irreducible integral residuated semilattice and let $J_{\mathbf{S}}(X_S) = T_{\mathbf{S}}(X_S) \rightarrow x_\star$; then $J_{\mathbf{S}}(X_S) \approx 1$ is a Jankov formula. If $\mathbf{S}$ is a finite subdirectly irreducible algebra, then $\mathbf{S} \not\models J_{\mathbf{S}}(X_S) \approx 1$ by design; hence if $\mathbf{S} \in \mathsf{U}$ and W is the subvariety of U axiomatized by $J_{\mathbf{S}}(X_S) \approx 1$, then for any $\mathbf{B} \in \mathsf{W}$, $\mathbf{A} \notin \mathbf{V}(\mathbf{B})$. It follows that $\mathsf{W} \subseteq \mathsf{W}^{\mathsf{U}}_{\mathbf{S}}$ and if equality holds $\mathbf{S}$ is splitting for U. This is exactly the way V. Jankov [23] showed that any finite subdirectly irreducible Heyting algebra is splitting for any subvariety of Heyting algebras to which it belongs, but there are many cases in which $\mathsf{W} \neq \mathsf{W}^{\mathsf{U}}_{\mathbf{S}}$. As a matter of fact in [30] the authors showed that if $\mathsf{U} = \mathsf{FL}_{ew}$, then $\mathsf{W} = \mathsf{W}^{\mathsf{U}}_{\mathbf{S}}$ if and only if $\mathbf{S} = \mathbf{2}$.

4. Divisible residuated semilattices

An integral residuated semilattice $\mathbf{S}$ is **divisible** if the ordering is the inverse divisibility ordering i.e. for all $a, b \in S$

$$a \leq b \qquad \text{if and only if} \qquad \exists c, d \text{ with } a = cb = bd.$$

Lemma 4.1. [21] *For an integral residuated semilattice* $\mathbf{S}$ *the following are equivalent*

(1) $\mathbf{S}$ *is divisible;*
(2) *if* $a, b \in A$ *and* $a \leq b$, *then* $(b \to a)b = a = b(b \leftarrow a)$;
(3) *if* $a, b \in A$, *then* $(b \to a)b = a \wedge b = b(b \leftarrow a)$;
(4) *if* $a, b, c \in A$, *then*

$$(a \to b) \to (a \to c) = (b \to a) \to (b \to c)$$
$$(a \leftarrow b) \leftarrow (a \leftarrow c) = (b \leftarrow a) \leftarrow (b \leftarrow c).$$

Divisible FL-algebras are called **GBL-algebras**; they form a variety, denoted by GBL and also integral GBL-algebras form a variety, denoted by GBL_w. The variety of commutative GBL-algebras is denoted by GBL_e and that of commutative and integral GBL-algebras is denoted by GBL_{ew}. It is a normal variety, so by Theorem 2.6, $\mathbf{S}^{rs}(\mathsf{GBL}_{ew})$ is a variety as well. Its members are called **hoops** [12] and the variety is denoted by H. Integral and divisible residuated semilattices are called **pseudohoops** [22] and the variety is denoted by PH. It is evident that any subreduct of an algebra in GBL_w is a pseudohoop but, since GBL_w is not normal, we cannot claim directly as above that the variety of pseudohoops coincides with the class of subreducts of algebras in GBL_w[2].

Observe however that both in integral GBL-algebras and in pseudohoops, congruences are completely determined by the congruence filters, and a congruence filter of an algebra in GBL_w is a filter of its pseudohoop subreduct. This implies that any result we prove for pseudohoops holds also for integral GBL-algebras, as long as it depends only on the behavior of equations and congruences. From now on we will state our main results for pseudohoops, leaving to the reader the easy translation to integral GBL-algebras.

Lemma 4.2. *If* $\mathbf{S}$ *is pseudohoop then, for all* $a, b \in S$ *and* $n \in \mathbb{N}$

$$(a \to b)^n \leftarrow (b \to a) = b \to a$$
$$(a \leftarrow b)^n \to (b \leftarrow a) = b \leftarrow a.$$

Proof. We prove the first one; the reader can check that a similar argument works for the second as well. The proof is by induction starting with $n = 1$ ($n = 0$ is trivial). Observe that $a \leq (a \to b) \leftarrow a$ and $a \leq (a \leftarrow b) \to b$; so since $\to$ is antimonotonic in the first argument, we get

$$((a \to b) \leftarrow a) \to b \leq a \to b \leq ((a \to b) \leftarrow a) \to a.$$

This in turn implies

$$[((a \to b) \leftarrow a) \to b] \to [((a \to b) \leftarrow a) \to a] = 1$$

[2]it is indeed true that pseudohoops coincide with subreducts of algebras in GBL_w, but explaining why would force us to take a detour that we do not believe necessary at this point; however we can point the reader to [33] and [35], where all the necessary informations can be found.

which by Lemma 4.1(4) is equivalent to

$$(b \to ((a \to b) \leftarrow a)) \to (b \to a) = 1$$

and hence (by Lemma 2.1) to

$$((a \to b) \leftarrow (b \to a)) \to (b \to a) = 1$$

and finally to

$$(a \to b) \leftarrow (b \to a) \leq b \to a.$$

On the other hand $(a \to b)(b \to a) \leq (b \to a)$, so $b \to a \leq (a \to b) \leftarrow (b \to a)$; thus antisimmetricity of the ordering proves our claim for $n = 1$.

Assume now that the conclusion holds for n; then

$$\begin{aligned}(a \to b)^{n+1} \leftarrow (b \to a) &= (a \to b) \leftarrow ((a \to b)^n \leftarrow (b \to a)) \\ &= (a \to b) \leftarrow (b \to a) = b \to a.\end{aligned}$$

□

A **Wajsberg pseudohoop** is a pseudohoop satisfying the equations

$$(x \to y) \leftarrow y \approx (y \to x) \leftarrow x$$
$$(x \leftarrow y) \to y \approx (y \leftarrow x) \to x.$$

A **Wajsberg hoop** is a commutative Wajsberg pseudohoop. Note that in a totally ordered Wajsberg pseudohoop

$$a \vee b = (a \leftarrow b) \to b = (a \to b) \leftarrow b.$$

The next Lemma has been proved in [12] for hoops and extended to the noncommutative case in [16].

Lemma 4.3. [16] *For a totally ordered pseudohoop* $\mathbf{S}$ *the following are equivalent:*

1. $\mathbf{S}$ *is a Wajsberg pseudohoop;*
2. *for all* $a, b \in S$, *if* $b \to a = a$, *either* $a = 1$ *or* $b = 1$*;*
3. *for all* $a, b \in S$, *if* $b \leftarrow a = a$, *either* $a = 1$ *or* $b = 1$.

Theorem 4.4. *Suppose* $\mathbf{A}$ *is a pseudohoop with a unique minimal filter* F*; then* $\mathbf{F}$ *is a totally ordered Wajsberg pseudohoop. Hence any normal and simple pseudohoop is a totally ordered Wajsberg pseudohoop.*

Proof. That $\mathbf{F}$ is a subpseudohoop of $\mathbf{A}$ is clear; moreover since F is minimal as filter it cannot have proper filter, since they would be also filters of $\mathbf{A}$, violating minimality of F. So let $a \not\leq b$, so that $a \to b \neq 1$; then, since $\mathbf{F}$ has no nontrivial filters, there is an m with $(a \to b)^m \leq b \to a$, i.e. $(a \to b)^m \leftarrow (b \to a) = 1$. But by Lemma 4.2, $(a \to b)^n \leftarrow (b \to a) = b \to a$ for all n, so it must be $b \to a = 1$, i.e. $b \leq a$ and $\mathbf{F}$ is totally ordered. Suppose now that $b \to a = a$; if $b \neq 1$ then the filter generated by b is F, so there is an m with $b^m \to a = 1$. On the other hand from $b \to a = a$ we can infer that $b^n \to a = a$ for all n. So we must have $a = 1$ and $\mathbf{F}$ is Wajsberg from Lemma 4.3.

If $\mathbf{S}$ is a normal and simple pseudohoop then S is the unique minimal filter of $\mathbf{S}$ (since every filter is a congruence filter); thus the conclusion follows from the first part. $\square$

Since commutative residuated semilattices are normal we get as a corollary the well-known fact that a simple hoop is a totally ordered Wajsberg hoop [12]. A pseudohoop $\mathbf{S}$ is **cancellative** if it is cancellative as a partially ordered monoid; it is well-known (and easily checked) that cancellative pseudohoops form a variety axiomatized by $y \to xy \approx y \leftarrow yx \approx x$.

Lemma 4.5. *Let $\mathbf{S}$ be a totally ordered Wajsberg pseudohoop. If $a, b \in S$ and there is a $c \in S$ such that $c < ab, ba$, then $b \to ab = b \leftarrow ba = a$; hence any totally ordered Wajsberg pseudohoop is either bounded or cancellative.*

Proof. Since $c < ab$, then $ab \not\leq c$ and hence $a \not\leq b \to c$ so, since $\mathbf{S}$ is totally ordered, $b \to c \leq a$. Then

$$\begin{aligned} a &= a \vee (b \to c) \\ &= (a \to (b \to c)) \leftarrow (b \to c) \\ &= (ab \to c) \leftarrow (b \to c) \\ &= b \to ((ab \to c) \leftarrow c) \\ &= b \to (ab \vee c) = b \to ab. \end{aligned}$$

That $b \leftarrow ba = a$ is proven similarly. $\square$

Suppose that $\mathbf{A}$ is a pseudohoop with a unique minimal nontrivial filter U (this is the case if $\mathbf{A}$ is a subdirectly irreducible and normal) and let $u \in U \setminus \{1\}$ such that $u \to a = a$ for some $a \in A \setminus \{1\}$ (if no such pair of elements exists, then by Lemma 3.2 no non trivial cut is possible) and set

$$F' = \{a \in A \setminus \{1\} : u \to a = a = u \leftarrow a\} \qquad F = F' \cup \{1\}.$$

Lemma 4.6. *Let $\mathbf{A}$, U, u and F' as above; then*

(1) *for all $a \in A \setminus \{1\}$ there is an $u \in U \setminus \{1\}$ with $a \leq u$;*
(2) *$U \cap F' = \emptyset$;*
(3) *$a \in F'$ if and only if $a \neq 1$ and for all $u \in U \setminus \{1\}$, $u \to a = u \leftarrow a = a$;*
(4) *$a \in A \setminus F'$ if and only if for all $u \in U$ and $u \to a = u \leftarrow a = a$ implies $u = 1$ or $a = 1$.*

Proof. (1) Let $a \in A \setminus \{1\}$; since U is minimal $U \subseteq \mathrm{Fil}_{\mathbf{A}}(a)$ and if $v \in U \setminus \{1\}$ then $a^n \leq v$ for some n. Choose n to be minimal and let $u = a^{n-1} \to v$; then $v \leq u$ and $u \neq 1$, so $u \in U \setminus \{1\}$. Moreover $a \to u = a \to (a^{n-1} \to v) = a^n \to v = 1$, so $a \leq u$.

(2) Note that $\mathbf{U}$ is a subpseudohoop of $\mathbf{A}$ and it cannot have any proper filter, since any proper filter of $\mathbf{U}$ would be a filter of $\mathbf{A}$ smaller than U, contradicting minimality of $\mathbf{U}$. Hence $\mathbf{U}$ is a normal and simple pseudohoop and hence a Wajsberg pseudohoop (Theorem 4.4); if $a \in U \cap F'$, we can find an $u \in U \setminus \{1\}$ with $u \to a = a$; by Lemma 4.3 we must have $a = 1$. But $1 \notin U \cap F'$ so $a \notin U \cap F'$ and $U \cap F' = \emptyset$.

(3) Suppose $a \neq 1$ and let $v \in U \setminus \{1\}$ such that $v \to a = v \leftarrow a = a$; it is easily seen that $v^n \to a = v^n \leftarrow a = a$ for any n. Now since $u \leq (u \to a) \leftarrow a$ (Lemma 2.1) we have $(u \to a) \leftarrow a \in U$ and hence there exists an n with $v^n \leq (u \to a) \leftarrow a$. Then

$$\begin{aligned} 1 &= v^n \to ((u \to a) \leftarrow a) \\ &= (u \to a) \leftarrow (v^n \to a) \qquad \text{by Lemma 2.1} \\ &= (u \to a) \leftarrow a. \end{aligned}$$

So $u \to a \leq a$ and thus the equality holds. We can show that $u \leftarrow a = a$ by repeating the argument starting from $u \leq (u \leftarrow a) \to a$. Note also that (4) is the counterpositive of (3). □

Lemma 4.7. *Let* $\mathbf{A}$, F', u *and* U *as above and let* $a \in F'$. *Then:*

(1) *for all* $u \in U$, $a \leq u$;
(2) *for all* $u \in U$, $ua = au = a$;
(3) *for all* $b \in F'$, $b \leq a$ *implies* $b \in F'$;
(4) $F = F' \cup \{1\}$ *is the universe of a subpseudohoop of* $\mathbf{A}$.

Proof. (1) Pick $u \in U$; then

$$\begin{aligned} a \to u = (u \to a) \leftarrow (a \to u) \qquad \text{by Lemma 4.2} \\ a \leftarrow (a \to u) = a^2 \to u. \end{aligned}$$

This is the first step of an obvious induction showing that $a \to u = a^n \to u$ for all n. On the other hand, minimality of U implies that $U \subseteq \mathrm{Fil}_{\mathbf{A}}(a)$, so for some m, $a^m \leq u$. But this implies $1 = a^m \to u = a \to u$, so $a \leq u$.

(2) Compute $ua = u(u \leftarrow a) = u \wedge a = a$ and $au = (u \to a)u = u \wedge a = a$ (by (1)).

(3) Suppose $b \leq a$; then $u \to b \leq u \to a = a$. Now

$$\begin{aligned} (u \to b) \to b &= 1 \to ((u \to b) \to b) \\ &= ((u \to b) \to a) \to ((u \to b) \to b) \\ &= (a \to (u \to b)) \to (a \to b) \qquad \text{by Lemma 4.2} \\ &= ((u \to a) \to (u \to b)) \to (a \to b) \\ &= ((a \to u) \to (a \to b)) \to (a \to b) \qquad \text{by Lemma 4.2} \\ &= (1 \to (a \to b)) \to (a \to b) = 1 \qquad \text{by (1);} \end{aligned}$$

thus $u \to b \leq b$ and the equality holds. That $u \leftarrow b = b$ is proved similarly and so $b \in F'$ by Lemma 4.6(3).

(4) Let $b \in F'$; then

$$a \to b = ua \to b = u \to (a \to b),$$

and

$$a \to b = a \to (u \leftarrow b) = u \leftarrow (a \to b).$$

So by Lemma 4.6(3) either $a \to b = 1$ or $a \to b \in F'$ and $F' \cup \{1\}$ is closed under $\to$. The proof that it is closed under $\leftarrow$ is the same and it is closed under product and $\wedge$ since it is closed downward by (3). □

Let now $S = A \setminus F'$; how far are we from proving that (F, S) is a cut? Conditions (1), (2) of the definition are satisfied clearly and conditions (3),(4) and (5) are satisfied relatively to F and U. Hence if $U = S$ we are done and $\mathbf{A} = \mathbf{F} \oplus \mathbf{U}$; if $U \neq S$ we need an additional property. A filter U of an integral residuated semilattice **A satisfies condition (C)** for some $u \in U$ if

$$\text{for all } a \in A,\ u \to a = a \text{ if and only if } u \leftarrow a = a.$$

Theorem 4.8. *Let* $\mathbf{A}$ *be a pseudohoop having a unique minimal filter* U *satisfying condition (C) and let* $u \in U$ *such that* $F' = \{a \in A \setminus \{1\} : u \to a = a\}$ *is nonempty. Then, if* $S = A \setminus F'$ *and* $F = F' \cup \{1\}$*,* (F, S) *is a cut of* $\mathbf{A}$ *and* $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$*. Moreover* $\mathbf{S}$ *is a totally ordered Wajsberg pseudohoop.*

Proof. Since U satisfies condition (C), U, F' and u satisfy the hypotheses of Lemma 4.6 and 4.7. It follows that to prove the first part of the conclusion we need only show that (F, S) satisfies condition (3), (4) and (5) of the definition of cut. Let now $b \in S$ and $a \in F'$; note that $a \to b = ua \to b = u \to (a \to b)$. Hence $u \leftarrow (a \to b) = a \to b$ as well and by Lemma 4.6(3), either $a \to b \in F'$ or $a \to b = 1$; but $b \notin F'$, $b \leq a \to b$ and F' is closed downward by Lemma 4.7(3). So $a \to b \notin F'$, hence $a \to b = 1$ and $a \leq b$. This fulfils condition (3) of the definition of cut.

Next we show that if $a \in F'$ and $b \in S$, then $b \to a, b \leftarrow a \in F'$; in fact $b \to a = b \to (u \leftarrow a) = u \leftarrow (b \to a)$; by condition (C), $u \to (b \to a) = b \to a$ and by Lemma 4.6(3), either $b \to a \in F$ or $b \to a = 1$. But the latter is impossible since $b \not\leq a$, so $b \to a \in F'$. The proof that $b \leftarrow a \in F$ is similar.

Now suppose that $a, b \in S$ and $ab \in F$; then either $b \to ab \in F'$ or $b \to ab = 1$. But $ab \leq b \to ab$ and $ab \in S$, so $b \to ab \notin F'$ and hence $b \to ab = 1$; so $b \leq ab$ and thus $b = ab$. This is a contradiction which implies $ab \in S$ and fulfills condition (4) of the definition of cut.

Let now $a \in F$ and $b \in S$; then by the above $a \leq b$, $b \to a \in F$ and hence $(b \to a) \leftarrow a \in F$. But $b \leq (b \to a) \leftarrow a$ and $b \notin F'$, so $(b \to a) \leftarrow a \notin F'$; hence $(b \to a) \leftarrow a = 1$ and so $b \to a = a$. Similarly we can show that $b \leftarrow a = a$ and moreover

$$ab = (b \to a)b = a \wedge b = a \qquad ba = b(b \leftarrow a) = b \wedge a = a.$$

This fulfills condition (5) of the definition of cut, so $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$.

Now let $a, b \in S$ and suppose $b \not\leq a$; then $b \to a < 1$. Pick $u \in U$; since U is minimal there is a n such that $(b \to a)^n \leq u$. So $u \to (a \to b) \leq (b \to a)^n \to (a \to b) = a \to b$ by Lemma 4.2; again using condition (C) and Lemma 4.6(3) we get $a \to b \in F$, but $a \to b \notin F'$ since $b \notin F'$. So $a \to b = 1$, $a \leq b$ and $\mathbf{S}$ is totally ordered. Finally suppose that for $a, b \in S$, $b \to a = a$; then $b^n \to a = a$ for all $n \in \mathbb{N}$. If $b \neq 1$ and $u \in U$, then $b^n \leq u$ for some n; so $a \leq u \to a \leq b^n \to a = a$, which implies $u \to a = a$. Again as above, $a \in F$, but $a \notin F'$; so $a = 1$ and $\mathbf{S}$ satisfies Lemma 4.3. Hence $\mathbf{S}$ is a totally ordered Wajsberg pseudohoop. □

Now we observe:

- in a commutative pseudohoop every filter satisfies condition (C), hence any subdirectly irreducible hoop is of the form $\mathbf{F} \oplus \mathbf{S}$ where $\mathbf{S}$ is a totally ordered Wajsberg hoop [12];
- in a normal pseudohoop every filter satisfies condition (C) ([26], Lemma 3.5), hence any normal subdirectly irreducible pseudohoop is of the form $\mathbf{F} \oplus \mathbf{S}$, where $\mathbf{S}$ is a totally ordered Wajsberg pseudohoop.

All these results above can be transferred to normal and integral GBL-algebras, provided we can take care of the join in the decomposition. But P. Jipsen and F. Montagna [26] showed that for normal and integral GBL-algebras the above decomposition always gives rise to an ordinal sum of type (1) or (2), thus solving any problem. In the same paper, using their decomposition, the authors were able to show that every finite GBL-algebra is integral and commutative.

The analogous result for pseudohoops has been proved only very recently by the author [4], also using the decomposition given by Theorem 4.8:

Theorem 4.9. [4] *Every finite pseudohoop is a hoop (i.e. it is integral and commutative).*

If we want to characterize the splitting algebras in subvarieties of GBL_w or PH we would like to have the finite model property, so that we can reduce the problem to finite subdirectly irreducible algebras. Unfortunately there is a substantial limitation:

Corollary 4.10. *If a variety $\mathsf{V} \subseteq \mathsf{PH}$ has the finite model property then it is contained in H; if a variety $\mathsf{V} \subseteq \mathsf{GBL}_w$ has the finite model property, then it is contained in GBL_{ew}.*

Proof. Suppose then that V is not contained in PH; then there must be a non commutative $\mathbf{S} \in \mathsf{V}$, that must be infinite by Theorem 4.9. However any finite algebra in V is commutative (again by Theorem 4.9) so V cannot be generated by its finite algebras. This is equivalent to saying that V does not have the finite model property. The statement about GBL_{ew}-algebras follows similarly from the quoted result in [26]. □

So even in normal varieties of pseudohoops (GBL_w-algebras) there might be infinite, albeit finitely generated, splitting algebras. However both the variety H of hoops [5] and GBL_{ew} [26] have the finite model property; we shall investigate their splitting algebras in the next section.

5. Splittings in GBL_{ew}

By the end of this section we will obtain a complete characterization of the splitting algebras in GBL_{ew}; since hoops are subreducts of algebras in GBL_{ew} we will get almost for free a similar characterization for hoops. However in the process we will obtain some partial results also for normal varieties of integral GBL-algebras.

Let's introduce some totally ordered Wajsberg algebras:

- $[0,1]$ is the real interval with operations induced by the *Wajsberg norm.* i.e $xy = \max(x+y-1,0)$, $x \to y = \min(1+x-y,1)$;
- $\mathbf{Wa}_n$ is the subalgebra (or the subhoop) of $[0,1]$ generated by $\frac{1}{n}$; it is well-known that they are all simple and that they are the only finite totally ordered Wajsberg hoops [29].

Lemma 5.1. *Let* $\mathbf{A}$ *be any algebra in* GBL_w*; then*

$$\mathbf{SH}(\mathbf{A} \oplus \mathbf{Wa}_m) = \mathbf{SH}(\mathbf{A}) \cup \mathbf{S}(\mathbf{A} \oplus \mathbf{Wa}_m).$$

Proof. We compute:

$$\begin{aligned}\mathbf{SH}(\mathbf{A} \oplus Wa_m) &= \mathbf{S}(\mathbf{H}(\mathbf{A}) \cup \mathbf{A} \oplus \mathbf{H}(\mathbf{Wa}_m)) \quad \text{by Lemma 3.1}\\ &= \mathbf{S}(\mathbf{H}(\mathbf{A}) \cup \mathbf{A} \oplus \mathbf{Wa}_m) \quad \text{since } \mathbf{Wa}_m \text{ is also simple}\\ &= \mathbf{SH}(\mathbf{A}) \cup \mathbf{S}(\mathbf{A} \oplus \mathbf{Wa}_m).\end{aligned}$$

□

Lemma 5.2. *Let* $\mathbf{A}$ *be a finite algebra in* GBL_w *and let* J *be a set of positive integers. Then*

(1) $\mathbf{P}_u(\{\mathbf{A} \oplus \mathbf{Wa}_j : j \in J\}) = \{\mathbf{A} \oplus \mathbf{B} : \mathbf{B} \in \mathbf{P}_u(\{\mathbf{Wa}_i : j \in J\})\}$*;*
(2) *if* J *is an infinite set, then* $\mathbf{A} \oplus [0,1] \in \mathbf{HSP}_u(\{\mathbf{A} \oplus \mathbf{Wa}_j : j \in J\})$*.*

Proof. (1) is an easy consequence of Lemma 3.1. For (2) let $\mathsf{K} = \{\mathbf{Wa}_j : j \in J\}$; since J is infinite, it generates the entire variety of Wajsberg hoops; moreover, by Lemma 2.7, we may conclude that any finitely subdirectly irreducible Wajsberg hoop lies in $\mathbf{HSP}_u(\{\mathbf{Wa}_j : j \in J\})$. But $[0,1]$ is totally ordered so it is a finitely subdirectly irreducible Wajsberg hoop; hence $[0,1] \in \mathbf{HSP}_u(\{\mathbf{Wa}_j : j \in J\})$. The conclusion follows now by applying (1) and Lemma 3.1. □

Lemma 5.3. *Let* $\mathbf{A} = \mathbf{F} \oplus \mathbf{Wa}_n$ *where* $n \neq 1$ *and* $\mathbf{F}$ *is a finite and integral GBL-algebra. Suppose that* V *is a variety of normal and integral GBL-algebras such that* $\mathbf{F} \oplus \mathbf{Wa}_k \in \mathsf{V}$ *for infinitely many* k*, with* $n \not| k$*. Then* $\mathbf{A}$ *is not splitting for* V*.*

Proof. If $\mathbf{A} \notin \mathsf{V}$ then $\mathbf{A}$ is not splitting for V by definition. So we may assume that $\mathbf{F} \oplus \mathbf{Wa}_n \in \mathsf{V}$. Let $K = \{k \geq n : n \not| k\}$; since $n \neq 1$, K is infinite. First we show that $\mathbf{F} \oplus \mathbf{Wa}_n \notin \mathbf{V}(\mathbf{F} \oplus \mathbf{Wa}_k)$ for any $k \in K$. Now clearly $\mathbf{F} \oplus \mathbf{Wa}_n \notin \mathbf{SH}(\mathbf{F})$ for cardinality reasons. On the other hand Lemma 3.1 says that any proper ordinal sum in $\mathbf{S}(\mathbf{F} \oplus \mathbf{Wa}_k)$ must have a subalgebra of $\mathbf{Wa}_k$ as second component; but $\mathbf{Wa}_n \notin \mathbf{S}(\mathbf{Wa}_k)$, since $n \not| k$, so $\mathbf{F} \oplus \mathbf{Wa}_n \notin \mathbf{S}(\mathbf{F} \oplus \mathbf{Wa}_k)$. By Lemma 5.1, $\mathbf{F} \oplus \mathbf{Wa}_n \notin \mathbf{SH}(\mathbf{F} \oplus \mathbf{Wa}_k)$. But now we can apply Jónsson Lemma to $\mathbf{V}(\mathbf{F} \oplus \mathbf{Wa}_k)$, to conclude that any finite subdirectly irreducible algebra in $\mathbf{V}(\mathbf{F} \oplus \mathbf{Wa}_k)$ is in $\mathbf{HS}(\mathbf{F} \oplus \mathbf{Wa}_k) = \mathbf{SH}(\mathbf{F} \oplus \mathbf{Wa}_k)$ (where the equality holds since V has the CEP). But $\mathbf{F} \oplus \mathbf{Wa}_n$ is a finite subdirectly irreducible algebra that is not there, so $\mathbf{F} \oplus \mathbf{Wa}_n \notin \mathbf{V}(\mathbf{F} \oplus \mathbf{Wa}_k)$.

It follows that the variety $\mathsf{W}^{\mathsf{V}}_{\mathbf{A}}$ must contain the set $\{\mathbf{F} \oplus \mathbf{Wa}_k : k \in K'\}$ for some infinite set $K' \subseteq K$. Now apply Lemma 5.2 to conclude that $\mathbf{F} \oplus [0,1] \in \mathbf{HSP}_u(\{\mathbf{F} \oplus \mathbf{Wa}_k : k \in K'\})$; but by Lemma 5.1 and 5.2,

$\mathbf{F} \oplus [0,1] \in \mathsf{W}_{\mathbf{A}}^{\mathsf{V}}$ and hence $\mathbf{A} \in \mathsf{W}_{\mathbf{A}}^{\mathsf{V}}$, since $\mathbf{Wa}_m$ is a subalgebra of $[0,1]$. So $\mathbf{A}$ cannot be a splitting algebra for V. □

Corollary 5.4. *Let $\mathbf{A} = \mathbf{F} \oplus \mathbf{Wa}_n$ where $n \neq 1$ and $\mathbf{F}$ is a finite and integral GBL-algebra. Then V is not splitting for GBL_{ew} or GBL_w^k for any $k \geq 2$.*

Next note that $\mathbf{Wa}_1 = \mathbf{2}$ is contained in any GBL-algebra, so it is splitting for any variety and the conjugate variety is the trivial variety. We will show that, if $\mathbf{F}$ is a normal finite integral GBL-algebra, then $\mathbf{A} = \mathbf{F} \oplus \mathbf{2}$ is splitting for any normal variety of GBL-algebras to which it belongs. We will proceed in a way similar to the one we have explained in Section 3 for Heyting algebras, i.e. using a suitable Jankov formula; if $A = \{0, a_1, \dots, a_n, a_\star, 1\}$ we define the following diagram

$$D_{\mathbf{A}}(X_A) = \bigwedge_{i,j=1}^{n} ((x_{a_i} \leftrightarrow x_{a_j}) \to x_\star) \wedge \bigwedge_{i=1}^{n} (x_{a_i} \to x_\star) \wedge (x_\star \to x_\star^2).$$

In this diagram we encode the fact all the a_i are distinct, that $a_\star$ is a coatom, so that $2 = \{a_\star, 1\}$, and that it is idempotent. Let's define

$$\widehat{J}_{\mathbf{A}}(X_A) = D_{\mathbf{A}}(X_A) \to (T_{\mathbf{A}}(X_A) \to x_\star);$$

the Jankov formula we are going to use is $\widehat{J}_{\mathbf{A}}(X_A) \approx 1$. The rationale for this choice is the following: the request that $T_{\mathbf{A}}(X_A) \to x_\star \approx 1$ appears to be too strong, so we have to settle for less. Hence we ask that $T_{\mathbf{A}}(X_A) \to x_\star$ be large enough to be above $D_{\mathbf{A}}(X_A)$. Roughly speaking we consider the algebras in GBL_{ew} that have the following property: if they are generated by the same number of elements as $\mathbf{A}$ and they have a unique maximal idempotent coatom, then not all the operation tables are encoded by the t-diagram of $\mathbf{A}$. Note that $\mathbf{A} \not\models \widehat{J}_{\mathbf{A}}(X_A) \approx 1$ by design.

Lemma 5.5. *Let $\mathbf{A} = \mathbf{F} \oplus \mathbf{2}$ where $\mathbf{F}$ is a finite integral and normal GBL-algebra with $F = \{0, a_1, \dots, a_n, 1\}$ and $2 = \{a_\star, 1\}$; let $\mathbf{B}$ be any integral and normal subdirectly irreducible GBL-algebra generated by elements $b_1, \dots, b_n, c$. If $\widehat{J}_{\mathbf{A}}^{\mathbf{B}}(0, b_1, \dots, b_n, c, 1) \neq 1$, then $\mathbf{A} \in \mathbf{IS}(\mathbf{B})$.*

Proof. Since $\mathbf{B}$ is integral and normal, by Theorem 4.8 (and the results in [26]) $\mathbf{B} \cong \mathbf{F}_1 \oplus \mathbf{S}_1$ for some totally ordered Wajsberg pseudohoop $\mathbf{S}_1$ (and the sum is always of type (1) or (2)).

Observe that the hypotheses imply

$$D_{\mathbf{A}}^{\mathbf{B}}(0, b_1, \dots, b_n, c, 1) \not\leq T_{\mathbf{A}}^{\mathbf{B}}(0, b_1, \dots, b_n, c, 1) \to^{\mathbf{B}} c.$$

We will prove all our statement by counterpositive; first observe that all the b_i must be distinct; for, if $b_i = b_j$, then

$$D_{\mathbf{A}}^{\mathbf{B}}(0, b_1, \dots, b_n, c, 1) \leq (b_i \leftrightarrow b_j) \to^{\mathbf{B}} c = c \leq T_{\mathbf{A}}^{\mathbf{B}}(0, b_1, \dots, b_n, c, 1) \to^{\mathbf{B}} c.$$

Next $c \in S_1$; otherwise for some i, $b_i \in S_1$ and

$$D_{\mathbf{A}}^{\mathbf{B}}(0, b_1, \dots, b_n, c, 1) \leq b_i \to^{\mathbf{B}} c = c \leq T_{\mathbf{A}}^{\mathbf{B}}(0, b_1, \dots, b_n, c, 1) \to^{\mathbf{B}} c.$$

Now suppose that $b_i \in S_1$ for some i; then

$$\begin{aligned} D^{\mathbf{B}}_{\mathbf{A}}(0, b_1, \dots, b_n, c, 1) &\le b_i \to^{\mathbf{B}} c = (c \to^{\mathbf{B}} c) \wedge (b_i \to^{\mathbf{B}} c) \\ &= (b_i \vee^{\mathbf{B}} c) \to^{\mathbf{B}} c = ((c \to^{\mathbf{B}} b_i) \to^{\mathbf{B}} b_i) \to^{\mathbf{B}} c \\ &= ((c \to^{\mathbf{B}} b_i) \leftrightarrow^{\mathbf{B}} b_i) \to c \le T^{\mathbf{B}}_{\mathbf{A}}(0, b_1, \dots, b_n, c, 1) \to^{\mathbf{B}} c. \end{aligned}$$

Hence $b_i \in F_1$ for all i. Moreover if $c = c \to^{\mathbf{B}} c^2$, then

$$D^{\mathbf{B}}_{\mathbf{A}}(0, b_1, \dots, b_n, c, 1) \le c \to^{\mathbf{B}} c^2 = c \le T^{\mathbf{B}}_{\mathbf{A}}(0, b_1, \dots, b_n, c, 1) \to^{\mathbf{B}} c$$

so $c \neq c \to^{\mathbf{B}} c^2$ and hence $\mathbf{S}_1$ is not a cancellative Wajsberg pseudohoop. It follows by Lemma 4.5 that $\mathbf{S}_1$ is a bounded Wajsberg pseudohoop and hence has a minimal element that we call $b_\star$.

Next we claim that the map $h : \mathbf{F} \longrightarrow \mathbf{F}_1$ defined by $h(a_i) = b_i$, $i = 1, \dots, n$ and $h(a_\star) = b_\star$ is a homomorphism. In fact suppose that for some $* \in \{\vee, \wedge, \to, 0, 1\}$, $a_i *^{\mathbf{A}} a_j = a_k$ but $b_i *^{\mathbf{B}} b_j \leftrightarrow^{\mathbf{B}} b_k \neq 1$, i.e. $b_i *^{\mathbf{B}} b_j \leftrightarrow^{\mathbf{B}} b_k \in F_1 \setminus \{1\}$. Then

$$1 = (b_i *^{\mathbf{B}} b_j \leftrightarrow^{\mathbf{B}} b_k) \to c \le T^{\mathbf{B}}_{\mathbf{A}}(0, b_1, \dots, b_n, c, 1) \to^{\mathbf{B}} c;$$

hence $T^{\mathbf{B}}_{\mathbf{A}}(0, b_1, \dots, b_n, c, 1) \to c = 1$, that is impossible. Now, since all the b_i are distinct, h is really an embedding of $\mathbf{F}$ into $\mathbf{F}_1$. It follows that $\mathbf{A} = \mathbf{F} \oplus \mathbf{2}$ embeds into $\mathbf{B} = \mathbf{F}_1 \oplus \mathbf{S}_1$ as desired. $\square$

Now we are ready to prove the main theorem.

Theorem 5.6. *Let* V *be a normal variety of integral GBL-algebras, and let* $\mathbf{F}$ *be a finite integral GBL-algebra such that* $\mathbf{A} = \mathbf{F} \oplus \mathbf{2} \in \mathsf{V}$*. Then* $\mathbf{A}$ *is splitting for* V *with splitting equation* $\widehat{J}_{\mathbf{A}}(X_A) \approx 1$*. Moreover the conjugate variety is* $\mathsf{W}^{\mathsf{V}}_{\mathbf{A}} = \{\mathbf{B} \in \mathsf{V} : \mathbf{A} \notin \mathbf{HS}(\mathbf{B})\}$*.*

Proof. Since $\mathbf{A} \not\models \widehat{J}_{\mathbf{A}}(X_A) \approx 1$ by design to prove the first part it is enough to show that if $\mathbf{B} \in \mathsf{V}$ and $\mathbf{A} \notin \mathbf{V}(\mathbf{B})$, then $\mathbf{B} \models \widehat{J}_{\mathbf{A}}(X_A) \approx 1$ or, equivalently, that if $\mathbf{B} \not\models \widehat{J}_{\mathbf{A}}(X_A) \approx 1$, then $\mathbf{A} \in \mathbf{V}(\mathbf{B})$.

Assume then that $\mathbf{B} \not\models \widehat{J}_{\mathbf{A}}(X_A) \approx 1$; then there are $b_1, \dots, b_n, c \in B$ with $d = \widehat{J}^{\mathbf{B}}_{\mathbf{A}}(0, b_1, \dots, b_n, c, 1) \neq 1$. Let $\mathbf{B}_1$ the subalgebra of $\mathbf{B}$ generated by $\{b_1, \dots, b_n, c\}$ (so that $d \in B_1$) and let θ be maximal in the set $\{\alpha \in \mathrm{Con}(\mathbf{B}_1) : (d, 1) \notin \alpha\}$; then $\mathbf{B}_1/\theta$ is subdirectly irreducible, $\mathbf{B}_1$ is normal (since it belongs to V), $b_1/\theta, \dots, b_n/\theta, c/\theta$ generate $\mathbf{B}_1/\theta$ and moreover $\widehat{J}^{\mathbf{B}\theta}_{\mathbf{A}}(0/\theta, b_1/\theta, \dots, b_n/\theta, c/\theta) \neq 1$. By Lemma 5.5

$$\mathbf{A} \in \mathbf{IS}(\mathbf{B}_1/\theta) \subseteq \mathbf{SHS}(\mathbf{B}) = \mathbf{HS}(\mathbf{B}) \subseteq \mathbf{V}(\mathbf{B}).$$

This also shows that $\{\mathbf{B} \in \mathsf{V} : \mathbf{A} \notin \mathbf{HS}(\mathbf{B})\} \subseteq \mathsf{W}^{\mathsf{V}}_{\mathbf{A}}$; for the converse if $\mathbf{B} \models \widehat{J}_{\mathbf{A}}(X_A) \approx 1$, then $\mathbf{HS}(\mathbf{B}) \models \widehat{J}_{\mathbf{A}}(X_A) \approx 1$. Since $\mathbf{A} \not\models \widehat{J}_{\mathbf{A}}(X_A) \approx 1$ it must be $\mathbf{A} \notin \mathbf{HS}(\mathbf{B})$. $\square$

Corollary 5.7. *Suppose that* V *is a normal variety of integral GBL-algebras containing* GBL_{ew}*; then a finite subdirectly irreducible GBL-algebra* $\mathbf{A}$ *is splitting for* V *if and only if* $\mathbf{A} = \mathbf{F} \oplus \mathbf{2}$*.*

Proof. If $\mathbf{A}$ is a finite integral GBL-algebra, then it is commutative; if it is subdirectly irreducible then $\mathbf{A} = \mathbf{F} \oplus \mathbf{S}$ for some finite subdirectly irreducible Wajsberg hoop $\mathbf{S}$. Therefore $\mathbf{A} = \mathbf{F} \oplus \mathbf{Wa}_k$ for some k. Now $\mathbf{F}$ (or $\mathbf{F} \cup \{m\}$ if the sum is of type (2)) is a finite GBL-algebra and hence it is commutative as well. In any case $\mathbf{F} \oplus \mathbf{Wa}_n$ is commutative for all n, so it belongs to GBL_{ew} and hence to V. So V satisfies the hypotheses of Lemma 5.3, hence if $\mathbf{A}$ is splitting then $\mathbf{A} = \mathbf{F} \oplus \mathbf{2}$.

For the converse we just invoke Theorem 5.6. □

Since GBL_{ew} has the finite model property we get:

Corollary 5.8. *A subdirectly irreducible algebra in* GBL_{ew} *is splitting for* GBL_{ew} *if and only if it is* $\mathbf{F} \oplus \mathbf{2}$ *for some finite* $\mathbf{F} \in \mathsf{GBL}$.

Lemma 5.3 holds for variety of pseudohoops almost without modifications; this is not true for Theorem 5.6 since we cannot affirm that a finite pseudhoop is commutative. However hoops do have the finite model property [5] so:

Theorem 5.9. *A subdirectly irreducible hoop* $\mathbf{A}$ *is splitting for the variety* H *of hoops if and only if* $A = \mathbf{F} \oplus \mathbf{2}$ *for some finite hoop* $\mathbf{F}$.

Proof. The "only if" part is the same as in Theorem 5.6 and most of the "if" part is as in Lemma 5.3. We really have to show that $\mathbf{2}$ is splitting for H, since we can no longer state that $\mathbf{V}(\mathbf{2})$ is the only atom in the lattice of subvarieties (and in fact it is not, see [8]). However let V be the variety of hoops axiomatized by the equation $x \to x^2 \approx x$; we claim that $\mathbf{2}$ is splitting in H with conjugate variety V. It obvious that $\mathbf{2} \notin \mathsf{V}$. Conversely assume that W is a variety of hoops in which $x \to x^2 \approx x$ fails; then there is a subdirectly irreducible hoop $\mathbf{A} \in \mathsf{W}$ that is generated by an element a such that $a \to a^2 \neq a$. Since $\mathbf{A}$ is subdirectly irreducible, then it is isomorphic with $\mathbf{F} \oplus \mathbf{S}$ for some totally ordered Wajsberg hoop $\mathbf{S}$ (see the remarks below Theorem 4.8); but since it is monogenerated, $\mathbf{F}$ must be trivial, so $\mathbf{A} = \mathbf{S}$, a Wajsberg hoop. Clearly $\mathbf{A}$ cannot be cancellative, so it must be bounded by Lemma 4.5; hence $\mathbf{2}$ is a subalgebra of $\mathbf{A}$ and $\mathbf{2} \in \mathsf{W}$. This proves the theorem. □

6. Conclusions

Let's summarize what we have done in the previous section:

- we have characterized all the (necessarily finite) splitting algebras in GBL_{ew};
- we have described all the *finite* splitting algebras in sufficiently large normal varieties of integral GBL-algebras.

A logical step forward is to examine the following problem: given an infinite but finitely generated integral GBL-algebra $\mathbf{F}$ and a finitely generated totally ordered Wajsberg pseudohoop $\mathbf{S}$, find necessary and sufficient conditions for $\mathbf{F} \oplus \mathbf{S}$ to be splitting for some normal variety of GBL-algebras. We have no idea so far on how to deal with the general problem; however if we assume that **F** itself is totally ordered, then we can prove more. To be more precise we can investigate splittings in varieties generated by totally ordered integral residuated semilattices; such varieties are called *representable* and they have several properties that can be used to facilitate our investigation. Representable GBL_{ew} algebras are called *BL-algebras* and representable hoops are called *basic hoops* [5]. They both have the finite model property [5] and their splitting algebras have been completely characterized [3]. We will continue investigating representable varieties in a separate paper [2].

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