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# On the Grassmann Graph of Linear Codes

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## Abstract

Let  $\Gamma(n, k)$  be the Grassmann graph formed by the  $k$ -dimensional subspaces of a vector space of dimension  $n$  over a field  $\mathbb{F}$  and, for  $t \in \mathbb{N} \setminus \{0\}$ , let  $\Delta_t(n, k)$  be the subgraph of  $\Gamma(n, k)$  formed by the set of linear  $[n, k]$ -codes having minimum dual distance at least  $t + 1$ . We show that if  $|\mathbb{F}| \geq \binom{n}{t}$  then  $\Delta_t(n, k)$  is connected and it is isometrically embedded in  $\Gamma(n, k)$ . This generalizes some results of [3] and [2].

*Keywords:* Grassmann graph, Linear Codes, Diameter

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## 1. Introduction

Let  $V := V(n, \mathbb{F})$  be a  $n$ -dimensional vector space over a field  $\mathbb{F}$  and for  $k = 1, \dots, n-1$ , denote by  $\Gamma(n, k)$  the  $k$ -Grassmann graph of  $V$ , that is the graph whose vertices are the  $k$ -subspaces of  $V$  and where two vertices  $X, Y$  are connected by an edge if and only if  $\dim(X \cap Y) = k - 1$ . See [1] for more detail.

It is interesting to see what properties extend from the graph  $\Gamma(n, k)$  to some of its subgraphs.

Suppose that  $B$  is a given basis of  $V$ ; henceforth we will write the coordinates of the vectors in  $V$  with respect to  $B$ .

A  $[n, k]$ -linear code  $C$  is just a  $k$ -dimensional vector subspace of  $V$ . If  $B_C$  is an ordered basis of  $C$ , a generator matrix for  $C$  is the  $k \times n$  matrix whose rows are the coordinates of the elements of  $B_C$  with respect to  $B$ . Given a  $[n, k]$ -linear code  $C$ , its dual code is the  $[n, n - k]$ -linear code  $C^\perp$  given by

$$C^\perp := \{v \in V : \forall c \in C, v \cdot c = 0\}$$

where by  $\cdot$  we mean the standard symmetric bilinear form on  $V$  given by

$$(v_1, \dots, v_n) \cdot (c_1, \dots, c_n) = v_1 c_1 + \dots + v_n c_n.$$

Since the  $\cdot$  is non-degenerate,  $C^{\perp\perp} = C$ . We say that  $C$  has dual minimum distance  $t + 1$  if and only if the minimum Hamming distance of the dual  $C^\perp$  of  $C$  is  $t + 1$ .

The condition for a  $[n, k]$ -linear  $C$  having dual minimum distance at least  $t + 1$  can be easily read on any generator matrix of it. Indeed (see, e.g., Proposition 2.7),  $C$  has dual minimum

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distance at least  $t + 1$  if and only if any  $t$ -columns of any generator matrix of  $C$  are linearly independent.

For  $t \in \mathbb{N} \setminus \{0\}$ , let  $\mathcal{C}_t(n, k)$  be the set of all  $[n, k]$ -linear codes with dual minimum distance at least  $t + 1$  and denote by  $\Delta_t(n, k)$  the subgraph of  $\Gamma(n, k)$  induced by the elements of  $\mathcal{C}_t(n, k)$ , i.e. the vertex set of  $\Delta_t(n, k)$  is formed by the elements in  $\mathcal{C}_t(n, k)$  and two vertices  $X$  and  $Y$  are adjacent in  $\Delta_t(n, k)$  if and only if  $\dim(X \cap Y) = k - 1$ . We shall call  $\Delta_t(n, k)$  the *Grassmann graph* of the linear  $\mathcal{C}_t(n, k)$  codes.

Note that for  $t = 1$ ,  $\mathcal{C}_1(n, k)$  is the class of the *non-degenerate*  $[n, k]$ -linear codes and for  $t = 2$ ,  $\mathcal{C}_2(n, k)$  is the class of the *projective*  $[n, k]$ -linear codes.

In general, we say that a subgraph is isometrically embedded in a larger graph if there exists a distance-preserving map among them (see also Definition 2.2); see also [6].

In [2] Kwiatkowski and Pankov studied the graph  $\Delta_1(n, k)$  and more recently in [3] Kwiatkowski, Pankov and Pasini, considered the graph  $\Delta_2(n, k)$  in the case  $\mathbb{F}$  is a finite field of order  $q$ .

In [2, Corollary 2], the authors show that  $\Delta_1(n, k)$  is connected and isometrically embedded in  $\Gamma(n, k)$  if and only if  $n < (q + 1)^2 + k - 2$ . In [3, Theorem 1] it is shown that a sufficient condition for the graph  $\Delta_2(n, k)$  to be isometrically embedded in  $\Gamma(n, k)$  is  $q \geq \binom{n}{2}$ .

In this paper we generalize the results of [2] and [3] to the graphs  $\Delta_t(n, k)$  for arbitrary  $t \leq k$  and arbitrary fields  $\mathbb{F}$ .

More in detail, our main result is the following

**Theorem 1.** *Let  $t, k, n$  be integers such that  $1 \leq t \leq k \leq n < \infty$ . Suppose that  $\mathbb{F}$  is a field with  $|\mathbb{F}| \geq \binom{n}{t}$ . Then the graph  $\Delta_t(n, k)$  is connected and isometrically embedded into the  $k$ -Grassmann graph  $\Gamma(n, k)$ . Furthermore, the diameter of  $\Delta_t(n, k)$  and  $\Gamma(n, k)$  are the same.*

**Remark 1.1.** The hypotheses in Theorem 1 are sufficient for the graph  $\Delta_t(n, k)$  to be connected and to be isometrically embedded into  $\Gamma(n, k)$  but in general they are not necessary. We leave to a future work to determine if the graph  $\Delta_t(n, k)$  might be connected also under some weaker assumptions on  $q$  or  $t$  and see if there are cases where the embedding is not isometric. We leave also to a future work to generalize these results for  $\mathbb{F}$  a possibly non-commutative division ring and also to the infinite dimensional cases both for  $n$  and for  $k$ .

### 1.1. Structure of the paper

In Section 2 we recall some basic definitions and preliminary results which shall be used in order to prove Theorem 1. Section 3 contains the proof of our main results; in particular, in Subsection 3.1 we shall prove that the graph  $\Delta_t(n, k)$  is connected and isometrically embedded in  $\Gamma(n, k)$  for  $q \geq \binom{n}{t}$ , while in Subsection 3.2 we shall show that the sets  $\mathcal{C}_t(n, k)$ , when not empty, always contain codes which are at maximum distance in the Grassmann graph  $\Gamma(n, k)$ . Finally, in Subsection 3.3 we prove Theorem 1.

## 2. Preliminaries

As mentioned in the Introduction,  $\mathbb{F}$  is a field and  $V := V(n, \mathbb{F})$  denotes a  $n$ -dimensional vector space over  $\mathbb{F}$ . Let  $B = (e_1, \dots, e_n)$  be a given ordered basis of  $V$  with respect to which all the vectors will be written in coordinates. For  $k$  and  $t$  integers such that  $1 \leq t \leq k \leq n - 1$ ,  $\mathcal{C}_t(n, k)$  is the class of  $[n, k]$ -linear codes having dual minimum distance at least  $t + 1$ . More explicitly,

$$\mathcal{C}_t(n, k) := \{C \subseteq V : \dim C = k, d^\perp(C) \geq t + 1\}$$

where  $d^\perp(C) := d_{\min}(C^\perp)$  is the minimum distance of the dual code  $C^\perp$  which means that the weight  $\text{wt}(v)$  of any codeword  $v = (v_1, \dots, v_n) \in C^\perp$  is at least  $t + 1$ , i.e.

$$\text{wt}(v) := |\{i : v_i \neq 0\}| \geq t + 1.$$

Clearly, if  $\mathcal{C}_t(n, k) \neq \emptyset$ , then necessarily  $t \leq k \leq n$ .

If  $t = k$  and  $\mathbb{F} = \mathbb{F}_q$ , the elements of  $\mathcal{C}_k(n, k)$  are exactly the *maximum distance separable*  $[n, k]$ -codes (see e.g. [5]), that is codes whose minimum distance  $d_{\min}$  attains the Singleton bound  $d_{\min} = n - k + 1$ ; see Corollary 2.8.

**Remark 2.1.** The condition  $t \leq k \leq n$  is necessary but in general not sufficient to ensure that  $\mathcal{C}_t(n, k)$  is not empty. Indeed, if  $\mathbb{F} = \mathbb{F}_q$ , even for arbitrary values of  $q$ , it is not straightforward to determine if  $\mathcal{C}_t(n, k) \neq \emptyset$  or characterize the elements of  $\mathcal{C}_t(n, k)$ . For instance the celebrated MDS conjecture implies  $\mathcal{C}_k(n, k) = \emptyset$  for  $n > q + 2$ . On the other hand, if  $n < q + 1$ , then by Lemma 3.16,  $\mathcal{C}_t(n, k) \neq \emptyset$  for all  $t \leq k \leq n$ .

In order to avoid trivial cases, we shall henceforth suppose that the parameters  $n, k, t$  and  $q$  if  $\mathbb{F} := \mathbb{F}_q$ , have been chosen so that  $\mathcal{C}_t(n, k) \neq \emptyset$ ; in Lemma 3.16 it shall be shown that under the assumptions of Theorem 1 this is always true.

We recall from the Introduction that  $\Delta_t(n, k)$  is the subgraph of  $\Gamma(n, k)$  induced by the elements of  $\mathcal{C}_t(n, k)$ .

**Definition 2.2.** Let  $X \in \Delta_t(n, k)$ . We define the *connected component*  $\Delta_t^X(n, k)$  of  $X$  in  $\Delta_t(n, k)$  as the subgraph of  $\Delta_t(n, k)$  whose vertices are all  $Y \in \Delta_t(n, k)$  such that there is a path in  $\Delta_t(n, k)$  joining  $X$  and  $Y$ . The graph  $\Delta_t(n, k)$  is *connected* if  $\Delta_t^X(n, k) = \Delta_t(n, k)$  for some (and, consequently for all)  $X \in \Delta_t(n, k)$ .

For any  $X, Y \in \mathcal{C}_t(n, k)$  write  $d(X, Y)$  for the distance between  $X$  and  $Y$  in the Grassmann graph  $\Gamma(n, k)$  and  $d_t(X, Y)$  for the distance between  $X$  and  $Y$  in  $\Delta_t(n, k)$ . If  $\Delta_t^X(n, k) \neq \Delta_t^Y(n, k)$ , that is  $X$  and  $Y$  are in different connected components of  $\Delta_t(n, k)$  we put  $d_t(X, Y) = \infty$ . We recall that the *diameter* of a graph is the maximum of the distances among two of its vertices.

Since every edge of  $\Delta_t(n, k)$  is an edge of  $\Gamma(n, k)$ , it is straightforward to see that  $d_t(X, Y) \geq d(X, Y)$  for all  $X, Y \in \Delta_t(n, k)$ .

**Definition 2.3.** We say that  $\Delta_t(n, k)$  is *isometrically embedded* in  $\Gamma(n, k)$  if for any  $X, Y \in \Delta_t(n, k)$  we have  $d_t(X, Y) = d(X, Y) = k - \dim(X \cap Y)$ .

Note that if  $\Delta_t(n, k)$  is isometrically embedded in  $\Gamma(n, k)$ , then  $\Delta_t(n, k)$  is also connected.

### 2.1. Some basic results

**Definition 2.4.** Let  $(i_1, \dots, i_t) \in \mathbb{N}^t$  be a  $t$ -uple of integers such that  $1 \leq i_1 < i_2 < \dots < i_t \leq n$ . We denote by  $C_{i_1 \dots i_t} := \bigcap_{j=1}^t (x_{i_j} = 0)$  the  $(n - t)$ -dimensional subspace of  $V$  obtained as the intersection of the coordinate hyperplanes of  $V$  of equations  $x_{i_j} = 0$ . We shall call  $C_{i_1 \dots i_t}$  the  $(i_1, \dots, i_t)$ -*coordinate subspace* of  $V$ .

The *monomial group*  $\mathcal{M}(V)$  of  $V$  consists of all linear transformations of  $V$  which map the set of subspaces  $\{\langle e_1 \rangle, \dots, \langle e_n \rangle\}$  in itself. It is straightforward to see that  $\mathcal{M}(V) \cong \mathbb{F}^* \wr S_n$  where  $\wr$  denotes the wreath product and  $S_n$  is the symmetric group of order  $n$ ; see [5, Chapter 8, §5] for more details.

**Definition 2.5.** Two  $[n, k]$ -linear codes  $X$  and  $Y$  are *equivalent* if there exist a monomial transformation  $\rho \in \mathcal{M}(V)$  such that  $X = \rho(Y)$ .

Suppose  $X$  is a  $[n, k]$ -linear code with generator matrix  $G_X$ . If  $A \in \text{GL}(k, \mathbb{F})$  then  $G'_X = AG_X$  is also a generator matrix for  $X$ .

It follows that two  $[n, k]$ -linear codes  $X$  and  $Y$  with generator matrices respectively  $G_X$  and  $G_Y$  are *equivalent* if there exists  $A \in \text{GL}(k, \mathbb{F})$ , a permutation matrix  $P \in \text{GL}(n, \mathbb{F})$  and a diagonal matrix  $D \in \text{GL}(n, \mathbb{F})$  such that

$$G_X = AG_Y(PD).$$

Equivalence between linear codes is an equivalence relation and the equivalence class of a code  $X$  corresponds to the orbit of  $X$  under the action of  $\mathcal{M}(V)$  on the  $k$ -dimensional subspaces of  $V$ .

Also, it can be readily seen that two codes are equivalent if and only if any two of their generator matrices belong to the same orbit under the action of the group  $\text{PGL}(k, \mathbb{F}) : (\mathbb{F}^* \wr S_n)$ , where  $\text{PGL}(k, \mathbb{F})$  acts on the right of the generator matrix.

With mostly harmless abuse of notation, in the remainder of this paper we shall not distinguish between the action of  $\mathcal{M}(V)$  on the codes (regarded as subspaces of  $V$ ) and that on the columns of their generator matrices.

Since equivalent codes have the same parameters (in particular they have the same minimum dual distance), we have that  $\mathcal{C}_t(n, k)$  consists of unions of orbits under the action of  $\mathcal{M}(V)$ .

For  $j = 1, \dots, n$ , let  $x^j : V \rightarrow \mathbb{F}_q$  be the  $j^{\text{th}}$ -coordinate linear functional of  $V$  which acts on the vectors  $e_i$ ,  $1 \leq i \leq n$ , of  $B$  as  $x^j(e_i) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ .

Observe that, for any  $v \in V$  and  $j$  with  $1 \leq j \leq n$ , we have that  $x^j(v)$  is exactly the  $j$ -th component of  $v$  with respect to the basis  $B$ . So, if  $X$  is a  $[n, k]$ -linear code and  $B_X = (b_1, \dots, b_k)$  is a given basis of  $X$  with respect to which the generator matrix  $G_X$  is written, then for any  $i, j$  with  $1 \leq i \leq k$  and  $1 \leq j \leq n$ , then  $x^j(b_i)$  is exactly the  $(i, j)$ -entry in the matrix  $G_X$ . So, the  $j$ -th column of  $G_X$  represents the restriction  $x^j|_{B_X}$  of the functional  $x^j$  to the basis  $B_X$ . By linearity, we can say that the  $j$ -th column of  $G_X$  represents the restriction  $x^j|_X$  of the functional  $x^j$  to  $X$ . This has the following important consequence.

**Lemma 2.6.** *Let  $X \subseteq V$  be a  $[n, k]$ -linear code and  $G_X$  be a generator matrix of  $X$ . A set of coordinate functionals restricted to  $X$  is linearly independent if and only if the columns of  $G_X$  representing them are linearly independent.*

If  $\mathbb{F} := \mathbb{F}_q$ , we shall also use the notation

$$[m]_q := \frac{q^m - 1}{q - 1}$$

for the number of 1-dimensional subspaces of an  $m$ -dimensional vector space.

The equivalence between (1) and (2) in the following proposition is well known; however, since many results of the present work rely on it, we present a complete proof for the convenience of the reader.

**Proposition 2.7.** *Let  $X$  be a  $[n, k]$ -linear code and denote by  $G_X$  a generator matrix of  $X$ . The following are equivalent.*

- (1)  $X$  has minimum dual distance at least  $t + 1$ .
- (2) Any  $t$  columns of  $G_X$  are linearly independent.
- (3) For any  $1 \leq i_1 < i_2 < \dots < i_t \leq n$  we have  $\dim(X \cap C_{i_1 \dots i_t}) = k - t$  where  $C_{i_1 \dots i_t} := \bigcap_{j=1}^t (x_{i_j} = 0)$  is the  $(n - t)$ -dimensional  $(i_1, \dots, i_t)$ -coordinate subspace of  $V$ .

*Proof.* The matrix  $G_X$  is a parity check matrix for the dual code  $X^\perp$ . Write the columns of  $G_X$  as  $G_1, \dots, G_n$  and let  $y = (y_1, \dots, y_n) \in X^\perp$ . Then

$$G_X y^t = G_1 y_1 + \dots + G_n y_n = \mathbf{0}. \quad (1)$$

Assume (1). Then, for any  $y \in X^\perp$ ,  $y \neq \mathbf{0}$ , we have  $\text{wt}(y) \geq t + 1$ . Suppose by contradiction that there is a set of  $t$ -columns of  $G_X$  which are linearly dependent. To simplify the exposition, assume without much loss of generality that this set comprises the first  $t$ -columns. Then,

$$G_1 y_1 + G_2 y_2 + \dots + G_t y_t = \mathbf{0}$$

with at least one entry  $y_i$  different from 0; so the vector  $(y_1, \dots, y_t, 0, \dots, 0) \neq \mathbf{0}$  is in  $X^\perp$  with  $\text{wt}(y) \leq t < t + 1$ . This contradicts (1).

Conversely, assume (2) and take  $y \in X^\perp$  with  $\text{wt}(y) = d$ . Then  $G_X y^T = 0$ . If  $d = 0$ , that is  $y = \mathbf{0}$ , then there is nothing to prove. If  $d \neq 0$ , suppose, again without much loss of generality, that exactly the first  $d$  entries  $y_1, \dots, y_d$  of  $y$  are non-zero. Then

$$G_1 y_1 + \dots + G_d y_d = \mathbf{0}.$$

In particular, the first  $d$  columns of  $G_X$  must be linearly dependent; by (2) we necessarily have  $d > t$  since any set of  $t$  columns of  $G_X$  is independent; this implies (1).

We now prove the equivalence between (2) and (3). Suppose that (3) holds. Then,  $\dim(X \cap C_{i_1 \dots i_t}) = k - t$  which means that the restrictions  $x^{i_1}|_X, \dots, x^{i_t}|_X$  of the  $t$  coordinate functionals  $x^{i_1}, \dots, x^{i_t}$  of  $V$  to  $X$  are linearly independent. Then (2) follows from Lemma 2.6.

Conversely, assume (2) holds and suppose by contradiction that (3) is false, that is that there exists a set of indexes  $i_1, \dots, i_t$  such that  $\dim(X \cap C_{i_1 \dots i_t}) \geq k - t + 1$ . Then, for some  $j \in \{1, \dots, t\}$ , we have

$$X \cap C_{i_1 \dots i_{j-1} i_{j+1} \dots i_t} \subseteq X \cap C_{i_j}.$$

In terms of coordinate functionals this means

$$x^{i_j}|_X \in \langle x^{i_1}|_X, \dots, x^{i_{j-1}}|_X, x^{i_{j+1}}|_X, \dots, x^{i_t}|_X \rangle.$$

So  $x^{i_j}|_X$  is a linear combination of the remaining coordinate functionals. In particular, by Lemma 2.6, this means that the column  $G_{i_j}$  of any generator matrix  $G_X$  of  $X$  is a linear combination of the columns  $G_{i_1}, \dots, G_{i_{j-1}}, G_{i_{j+1}}, \dots, G_{i_t}$ . This contradicts (2).  $\square$

The following is an immediate consequence of Proposition 2.6.

**Corollary 2.8.** *The set  $\mathcal{C}_1(n, k)$  consists of all  $[n, k]$ -linear non-degenerate codes;  $\mathcal{C}_2(n, k)$  consists of all  $[n, k]$ -linear projective codes; the set  $\mathcal{C}_k(n, k)$ , if  $\mathbb{F} = \mathbb{F}_q$ , consists of all  $[n, k]$ -linear MDS codes.*

*Proof.* Only the statement about  $\mathcal{C}_k(n, k)$  needs to be proved as the descriptions of  $\mathcal{C}_1(n, k)$  and  $\mathcal{C}_2(n, k)$  follow directly from Proposition 2.7. Suppose  $C \in \mathcal{C}_k(n, k)$ , i.e.  $C$  is a  $[n, k]$ -code having dual minimum distance at least  $k + 1$ . Then, by definition of  $\mathcal{C}_k(n, k)$ ,  $C^\perp$  is a  $[n, n - k]$ -code with minimum distance at least  $k + 1 = n - (n - k) + 1$  and, as such it is a MDS-code. Since the duals of MDS codes are MDS codes,  $C = C^{\perp\perp}$  is also MDS.

Conversely, suppose  $C$  to be a  $[n, k]$ -linear MDS code; then  $C^\perp$  is also MDS and has minimum distance  $k + 1$ . It follows that  $C \in \mathcal{C}_k(n, k)$ .  $\square$

### 3. Proof of Theorem 1

We proceed by steps; first, in Section 3.1 we provide a condition for the graph  $\Delta_t(n, k)$  to be connected and isometrically embedded in  $\Gamma(n, k)$ ; then in Section 3.2 we show that any class  $\mathcal{C}_t(n, k)$  contains elements which are at maximum distance in  $\Gamma(n, k)$ ; finally in Section 3.3 we complete the proof of Theorem 1.

#### 3.1. The connectedness of the graph

The following are two elementary lemmas of linear algebra.

**Lemma 3.1.** *Let  $X \in \mathcal{C}_t(n, k)$  and let  $H$  be a hyperplane of  $X$ . If  $y \notin X$  then*

$$\dim(\langle H, y \rangle \cap C) \leq k - t + 1$$

for every  $(n - t)$ -dimensional coordinate subspace  $C$  of  $V$ .

*Proof.* Suppose by contradiction

$$\dim(\langle H, y \rangle \cap C) \geq k - t + 2.$$

Since  $H \subseteq X$  we also have  $\dim(\langle X, y \rangle \cap C) \geq k - t + 2$ .

Any vector in  $\langle H, y \rangle$  can be written in the form  $x + \alpha y$  where  $x \in H$  and  $\alpha \in \mathbb{F}$ . In particular, there are  $k - t + 2$  linearly independent vectors  $v_i$  in  $\langle X, y \rangle \cap C$  of the form

$$v_i = x_i + \alpha_i y$$

where  $x_i \in H$  and  $\alpha_i \in \mathbb{F}$ . By Gaussian elimination (we remove  $y$ ), we have at least  $k - t + 1$  vectors in  $X$  which are linearly independent and contained in  $C$ ; so  $\dim(X \cap C) \geq k - t + 1$ , which is a contradiction because  $X \in \mathcal{C}_t(n, k)$  (see Proposition 2.7).  $\square$

**Lemma 3.2.** *Let  $S$  be a vector space of dimension  $s$ ,  $H_1 \neq H_2$  be two distinct hyperplanes of  $S$  with fixed bases  $B_1$  and  $B_2$ . Then, there exists a basis  $B$  of  $S$  contained in  $B_1 \cup B_2$ .*

*Proof.* Since  $H_1 \neq H_2$ , there exists at least one element  $b \in B_2 \setminus H_1$ . Consider  $B = B_1 \cup \{b\}$ . This is a linearly independent set consisting of  $s$  distinct elements,  $B \subseteq S$  and  $\dim(S) = s$ . It follows that  $B$  is a basis of  $S$  with  $B \subseteq B_1 \cup B_2$ .  $\square$

Recall that by  $d(X, Y)$  we mean the distance in  $\Gamma(n, k)$  while  $d_t(X, Y)$  denotes the distance in  $\Delta_t(n, k)$ .

The following definitions are used in the proof of Lemma 3.8.

**Definition 3.3.** Put  $\binom{\{1, \dots, n\}}{t} := \{(i_1, \dots, i_t) \in \mathbb{N}^t : 1 \leq i_1 < i_2 < \dots < i_t \leq n\}$  and define as colors the elements of it. Endow  $\binom{\{1, \dots, n\}}{t}$  with the natural lexicographic order on the  $t$ -uples.

Take  $X, Y \in \mathcal{C}_t(n, k)$  with  $X \neq Y$  and let  $H$  be a hyperplane of  $X$  such that  $X \cap Y \subseteq H$ . The coloration induced by  $H$  is the map

$$\psi_H : Y/(H \cap Y) \rightarrow \binom{\{1, \dots, n\}}{t} \cup \{\infty\}$$

sending any vector  $[p] \in Y/(H \cap Y)$  to the smallest (in the lexicographic order) color  $(i_1, \dots, i_t)$  such that

$$\dim(\langle H, p \rangle \cap C_{i_1 \dots i_t}) = k - t + 1$$

where  $C_{i_1 \dots i_t}$  is the  $(i_1, \dots, i_t)$ -coordinate subspace as defined in Definition 2.4. If no such color exists we put  $\psi_H([p]) = \infty$ .

The function  $\psi_H$  is well defined. Indeed, if  $b \in [a] = a + (H \cap Y)$ , then  $b = a + h$  for some  $h \in H \cap Y$  and  $\langle H, b \rangle = \langle H, a + h \rangle = \langle H, a \rangle$ ; so  $\psi_H([a]) = \psi_H([b])$ .

Henceforth we shall silently represent each element  $[p]$  of  $Y/(X \cap Y)$  by means of its representative element  $p$ .

**Definition 3.4.** Under the same assumptions as in Definition 3.3, we say that a set  $T \subseteq Y/(H \cap Y)$  is *monochromatic* if  $\forall r, s \in T, \psi_H(r) = \psi_H(s) \neq \infty$ , i.e. all of its elements have the same color.

**Definition 3.5.** Under the same assumptions as in Definition 3.3, we say that a subspace  $S$  of  $Y/(H \cap Y)$  with  $\dim(S) = s$  is *colorable* if there exists at least one monochromatic basis of  $S$ . If  $S$  is colorable, we define the *color*  $\psi_H(S)$  of  $S$  as the minimum color of a basis of  $S$ .

In symbols, let

$$\mathfrak{F}(S) := \{f = (p_1, \dots, p_s) : f \text{ is a basis of } S \text{ and } \psi_H(p_1) = \dots = \psi_H(p_s) \neq \infty\}$$

be the set of monochromatic bases of  $S$ . If  $f \in \mathfrak{F}(S)$ , denote by  $\psi_H(f)$  the color of any element in  $f$ . Hence

**Lemma 3.6.** *The subspace  $S$  is colorable if and only if  $\mathfrak{F}(S) \neq \emptyset$ .*

If  $S$  is colorable, the color of  $S$  is

$$\psi_H(S) := \min\{\psi_H(f) : f \in \mathfrak{F}(S)\}.$$

In other words,  $S$  has color  $c$  if there are  $s$  independent vectors in  $S$  all with the same color  $c$  and any other set of  $s$  independent vectors in  $S$  either is not monochromatic or has color  $c' \geq c$ .

Note that a colorable subspace  $S$  with color  $c$  is not, in general, a monochromatic set.

**Lemma 3.7.** *Let  $X, Y \in \mathcal{C}_t(n, k)$  with  $\dim(X \cap Y) = k - d \geq k - t$ . If  $\mathbb{F}$  is a field with  $|\mathbb{F}| \geq \binom{n}{t}$  then there exists a code  $Z \in \mathcal{C}_t(n, k)$  such that  $\dim(X \cap Z) = k - 1$  and  $\dim(Z \cap Y) = k - d + 1$ .*

*Proof.* We prove that for every hyperplane  $H$  of  $X$  containing  $X \cap Y$ , there exists  $z \in Y \setminus (X \cap Y)$  such that  $Z := \langle H, z \rangle \in \mathcal{C}_t(n, k)$ .

By way of contradiction suppose the contrary. Hence there exists a hyperplane  $H$  of  $X$  with  $X \cap Y \subseteq H$  such that for every  $z \in Y \setminus (X \cap Y)$ , we have  $\langle H, z \rangle \notin \mathcal{C}_t(n, k)$ .

Equivalently, by Proposition 2.7, we suppose that there exists a hyperplane  $H$  of  $X$  with  $X \cap Y \subseteq H$  such that for every  $z \in Y \setminus (X \cap Y)$  there exist indexes  $i_1 < i_2 < \dots < i_t$  such that  $\dim(\langle H, z \rangle \cap C_{i_1 \dots i_t}) \geq k - t + 1$ . By Lemma 3.1, we have  $\dim(\langle H, z \rangle \cap C_{i_1 \dots i_t}) = k - t + 1$ .

Under these assumptions, we will prove the following claim which leads to a contradiction.

**Claim 1.** *There exist indexes  $i_1, \dots, i_t$  and linearly independent vectors  $p_1, \dots, p_d \in Y \setminus (X \cap Y)$  such that  $[p_1], \dots, [p_d]$  are linearly independent in  $Y/(X \cap Y)$ ,*

$$\dim R_i = k - t + 1 \text{ and } \dim(R_1 + \dots + R_d) \geq k - t + d$$

where we put  $H_i := \langle H, p_i \rangle$  and  $R_i = H_i \cap C_{i_1 \dots i_t}$ .

Note that for every  $i$ ,  $R_i \subseteq \langle H, Y \rangle$ . From Claim 1, we have

$$\dim(Y + R_1 + \dots + R_d) \leq \dim(H + Y) = k + d - 1 \quad (2)$$

and

$$\begin{aligned} \dim(Y \cap (R_1 + \dots + R_d)) &= \dim(Y) + \dim(R_1 + \dots + R_d) - \dim(R_1 + \dots + R_d + Y) \geq \\ &\geq k + (k - t + d) - (k + d - 1) \geq k - t + 1. \end{aligned} \quad (3)$$



Since  $R_i \subseteq C_{i_1 \dots i_t}$  for any  $i$ , it follows

$$\dim(C_{i_1 \dots i_t} \cap Y) \geq \dim(Y \cap (R_1 + \dots + R_t)) \geq k - t + 1, \quad (4)$$

which is a contradiction because  $\dim(C_{i_1 \dots i_t} \cap Y) = k - t$ , since  $Y \in \mathcal{C}_t(n, k)$  (see Proposition 2.7).

So, in order to get the thesis, we need to prove Claim 1.

Let  $S$  be a subspace of  $Y/(H \cap Y)$  with  $\dim(S) = s$ . We show by induction on  $s$  that  $S$  is colorable. First, by our hypotheses, for any  $p \in S$  we have  $\varphi_H(p) \neq \infty$ .

Suppose  $\dim(S) = 2$ . The projective space  $\text{PG}(S)$  is then a projective line of  $\text{PG}(Y/H \cap Y)$  so there are  $|\mathbb{F}| + 1$  points in  $\text{PG}(S)$ . By hypothesis we have  $|\mathbb{F}| \geq \binom{n}{t}$  possible colors (see Definition 3.3). Hence there are at least 2 linearly independent vectors  $p_1$  and  $p_2$  in  $S$  such that  $\psi_H(p_1) = \psi_H(p_2)$ . Hence,  $\mathfrak{F}(S) \neq \emptyset$ . By Lemma 3.6,  $S$  is colorable.

Suppose now  $\dim(S) > 2$ . Put  $s := \dim(S)$ . By induction hypothesis, all subspaces  $S'$  of  $S$  with dimension  $\dim(S') = \dim(S) - 1$  are colorable, that is they all admit a monochromatic basis. For any  $(s - 2)$ -subspace  $S''$  of  $S$  there are  $|\mathbb{F}| + 1$  distinct  $(s - 1)$ -dimensional subspaces  $S'$  of  $S$  with  $S'' \leq S' \leq S$ . Also  $\text{PG}(S/S'')$  is a projective line. Since  $|\mathbb{F}| + 1 > \binom{n}{t}$ , there are at least two of such subspaces, say  $S_1$  and  $S_2$  with  $S_1 \neq S_2$  (hence  $\langle S_1, S_2 \rangle = S$ ) which have the same color  $\psi_H(S_1) = \psi_H(S_2)$ .

Let  $B_1$  and  $B_2$  be bases of respectively  $S_1$  and  $S_2$  with  $\psi_H(B_1) = \psi_H(B_2)$  ( $= \psi_H(S_1)$ ). By Lemma 3.2, there is a basis  $B$  of  $S$  contained in  $B_1 \cup B_2$ . So  $S$  admits at least one monochromatic basis and  $\mathfrak{F}(S) \neq \emptyset$ . By Lemma 3.6,  $S$  is colorable.

Hence, it is always possible to determine a monochromatic set of  $d$  independent vectors  $\{p_1, p_2, \dots, p_d\}$  of  $Y$  such that  $[p_1], \dots, [p_d]$  are independent in  $Y/(X \cap Y)$ . So, we have (recall that  $H_i = \langle H, p_i \rangle$ )

$$\dim(H_1 \cap C_{i_1 \dots i_t}) = \dim(H_2 \cap C_{i_1 \dots i_t}) = \dots = \dim(H_d \cap C_{i_1 \dots i_t}) = k - t + 1. \quad (5)$$

Since  $H \subseteq X$  and  $X \in \mathcal{C}_t(n, k)$ , we have  $\dim(H \cap C_{i_1 \dots i_t}) \leq k - t$ . Recalling that by definition  $R_i := H_i \cap C_{i_1 \dots i_t} = \langle H, p_i \rangle \cap C_{i_1 \dots i_t}$ , we have, by (5), that  $\dim(R_i) = \dim(H \cap C_{i_1 \dots i_t}) + 1 = k - t + 1$  and  $\dim(H \cap C_{i_1 \dots i_t}) = k - t$ .

In particular, (note that  $p_i \notin H$ ), it is always possible to find for  $1 \leq i \leq d$  an element  $h_i \in H$  and a non-null element  $\alpha_i \in \mathbb{F}_q$  such that the point  $\alpha_i p_i + h_i \in R_i$  is such that

$$\alpha_i p_i + h_i \in C_{i_1 \dots i_t};$$

up to a scalar multiple we can assume  $\alpha_i = 1$  for all  $i$ .

We now show that  $\dim(R_1 + R_2 + \dots + R_d) \geq k - t + d$ . Suppose the contrary. Then, without loss of generality, we can assume  $R_d \subseteq R_1 + R_2 + \dots + R_{d-1}$ . In particular

$$p_d + h_d \in R_1 + \dots + R_{d-1},$$

whence

$$p_d = \beta_1 p_1 + \dots + \beta_{d-1} p_{d-1} + h$$

with  $h \in H$  a suitable element and  $\beta_i \in \mathbb{F}$  for  $1 \leq i \leq d - 1$ . So, given that  $p_1, \dots, p_d \in Y$ ,

$$p_d - (\beta_1 p_1 + \dots + \beta_{d-1} p_{d-1}) = h \in H \cap Y = X \cap Y,$$

that is

$$[p_d] + [p_1] + \dots + [p_{d-1}] = [0]$$

in  $Y/(X \cap Y)$ . This contradicts the first part (already proved) of Claim 1, since  $[p_1], \dots, [p_d]$  are linearly independent vectors of  $Y/(X \cap Y)$ . It follows  $\dim(R_1 + R_2 + \dots + R_d) \geq k - t + d$ . So Claim 1 holds. This completes the proof of the theorem.  $\square$

Note that if  $\mathbb{F} := \mathbb{F}_q$  is a finite field, then Lemma 3.7 gives the following

**Lemma 3.8.** *Let  $X, Y \in \mathcal{C}_t(n, k)$  with  $\dim(X \cap Y) = k - d \geq k - t$ . If  $\mathbb{F} := \mathbb{F}_q$  and  $q \geq \binom{n}{t}$  then there exist  $[d]_q$  distinct codes  $Z \in \mathcal{C}_t(n, k)$  such that  $\dim(X \cap Z) = k - 1$  and  $\dim(Z \cap Y) = k - d + 1$ .*

*Proof.* For any hyperplane  $H$  of  $X$  containing  $X \cap Y$  it is possible to apply the argument in the proof of Lemma 3.7. Thus there are at least  $[d]_q$  distinct codes  $Z$  with the required property.  $\square$

**Lemma 3.9.** *Suppose  $\mathbb{F}$  is a field with  $|\mathbb{F}| \geq \binom{n}{t}$ . Then for any  $X \in \mathcal{C}_t(n, k)$  and for every  $U \subset X$  with  $\dim U < k - t$  there exists  $X' \in \mathcal{C}_t(n, k - 1)$  satisfying  $U \subset X' \subset X$ .*

*Proof.* A hyperplane  $H$  of  $X$  is an element of  $\mathcal{C}_t(n, k - 1)$  if and only if  $H$  does not contain  $X \cap C_{i_1 \dots i_t}$  for any  $i_1 < \dots < i_t$ . Indeed, if  $C_{i_1 \dots i_t} \cap X \subseteq H$  for some  $i_1 < \dots < i_t$ , then  $\dim(H \cap C_{i_1 \dots i_t}) \geq \dim(X \cap C_{i_1 \dots i_t}) = k - t > k - t - 1$  and  $H \notin \mathcal{C}_t(n, k - 1)$ . Conversely, suppose that for any  $i_1 < \dots < i_t$ ,  $C_{i_1 \dots i_t} \cap X \not\subseteq H$ ; then  $\dim(H \cap X \cap C_{i_1 \dots i_t}) = \dim(H \cap C_{i_1 \dots i_t}) = k - 1 - t$  for any choice of the indexes; so  $H \in \mathcal{C}_t(n, k - 1)$ .

By Definition of  $C_{i_1 \dots i_t}$  (see Definition 2.4), there exist at most  $\binom{n}{t}$  distinct spaces  $C_{i_1 \dots i_t}$ . Now we distinguish two cases.

- If  $\mathbb{F} := \mathbb{F}_q$  is a finite field, each of the spaces  $C_{i_1 \dots i_t}$  is contained in  $[t]_q$  distinct hyperplanes of  $X$ . So the number of hyperplanes containing at least one  $X \cap C_{i_1 \dots i_t}$  is at most

$$\binom{n}{t} [t]_q.$$

On the other hand  $U$  is contained in  $[m]_q$  distinct hyperplanes where  $m = \dim(X/U)$ . Since  $m > t$  we have

$$[m]_q \geq [t+1]_q = q^t + q^{t-1} + \dots + q + 1 \geq \binom{n}{t} q^{t-1} + \binom{n}{t} q^{t-2} + \dots + \binom{n}{t} + 1 > \binom{n}{t} [t]_q.$$

This shows that there is at least one hyperplane  $X'$  of  $X$  containing  $U$  and none of the  $C_{i_1 \dots i_t}$ .

- Suppose  $\mathbb{F}$  is an infinite field and denote by  $X^*$  the dual space of  $X$ . Then, for every  $U \subseteq X$  with  $\dim U < k - t$  the set of hyperplanes  $X'$  of  $X$  containing  $U$  determine a subspace  $\mathcal{U}$  of  $\text{PG}(X^*)$  of vector dimension at least  $k - (k - t - 1) = t + 1$ . Since each of the spaces  $X \cap C_{i_1 \dots i_t}$  has dimension  $k - t$  (because  $X \in \mathcal{C}_t(n, k)$ ), the set  $\mathcal{H}_{i_1 \dots i_t}$  of hyperplanes containing  $C_{i_1 \dots i_t}$  is a subspace of  $\text{PG}(X^*)$  of vector dimension  $t$ . In particular the set of all hyperplanes of  $X$  containing at least one  $C_{i_1 \dots i_t}$  is the union of  $\binom{n}{t}$  subspaces of  $\text{PG}(X^*)$  each of vector dimension  $t$ .

Since the field  $\mathbb{F}$  is infinite, it is impossible for a projective space of vector dimension at least  $t + 1$  to be the union of a finite number of projective spaces of dimension  $t$ .

It follows that there is at least one element

$$X' \in \mathcal{U} \setminus \bigcup_{\substack{(i_1 \dots i_t) \in \\ \{\{i_1, \dots, i_n\}\}_t}} \mathcal{H}_{i_1 \dots i_t}$$

This leads to the same conclusion as in the case in which  $\mathbb{F}$  is finite.

$\square$

We are now ready to prove the following theorem.

**Theorem 3.10.** *Suppose  $\mathbb{F}$  is a field with  $|\mathbb{F}| \geq \binom{n}{t}$ . Then  $\Delta_t(n, k)$  is connected and isometrically embedded into  $\Gamma(n, k)$ .*

*Proof.* Take  $X, Y \in \mathcal{C}_t(n, k)$  with  $X \neq Y$ . Put  $\dim(X \cap Y) := k - d$ . If  $k - d \geq k - t$ , then the thesis follows from Lemma 3.7. Suppose now  $k - d < k - t$ . By Lemma 3.9 there exists  $X' \subseteq X$  with  $X \cap Y \subseteq X'$  and  $X' \in \mathcal{C}_t(n, k - 1)$ . Let  $Y'$  a  $(k - d) + 1$ -dimensional subspace of  $Y$  containing  $X \cap Y$ . Put  $T = \langle X', Y' \rangle$ . Then  $\dim(T) = k$ ; also

$$\dim(T \cap C_{i_1 \dots i_t}) \leq \dim(X' \cap C_{i_1 \dots i_t}) + 1 = k - 1 - t + 1 = k - t$$

for all  $1 \leq i_1 < i_2 < \dots < i_t \leq n$ . In particular  $T \in \mathcal{C}_t(n, k)$  and  $\dim(X \cap T) = k - 1$ ,  $\dim(Y \cap T) = k - d + 1$ . By recursively applying this argument we get that  $\Delta_t(n, k)$  is connected.

By construction, the length  $d_t(X, Y)$  of the path joining  $X$  and  $Y$  in  $\Delta_t(n, k)$  is at most  $k - \dim(X \cap Y) := d(X, Y)$ , i.e.  $d_t(X, Y) \leq d(X, Y)$ . Since  $d_t(X, Y) \geq d(X, Y)$  in general, we get the thesis.  $\square$

**Remark 3.11.** As a consequence of Theorem 3.10,

$$\text{diam}(\Delta_t(n, k)) \leq \text{diam}(\Gamma(n, k))$$

when  $|\mathbb{F}| \geq \binom{n}{t}$ . In the following section we shall show that these two diameters are actually the same.

### 3.2. Codes at maximum distance

In this section we do not assume any hypothesis on the parameters, apart that they have been chosen so that there exists at least one  $[n, k]$ -linear code with dual minimum distance at least  $t$ , i.e.  $\mathcal{C}_t(n, k) \neq \emptyset$ . Hence, the graph  $\Delta_t(n, k)$  is not assumed to be connected. This observation justifies the following.

**Definition 3.12.** We say that two codes in  $X, Y \in \mathcal{C}_t(n, k)$  are *opposite* in  $\Delta_t(n, k)$  if they belong to the same connected component  $\Delta_t^X(n, k) = \Delta_t^Y(n, k)$  of  $\Delta_t(n, k)$  and  $d_t(X, Y) = \text{diam}(\Delta_t^X(n, k))$ .

**Definition 3.13.** We say that two  $k$ -dimensional subspaces  $X, Y \subseteq V$  are *opposite* in  $\Gamma(n, k)$  if  $\dim(X \cap Y) = \max\{2k - n, 0\}$ .

Observe that if  $2k \leq n$ , being opposite in  $\Gamma(n, k)$  means  $\dim(X + Y) = 2k$  (equivalently,  $X \cap Y = \{0\}$ ), while if  $2k > n$ , it means  $\dim(X + Y) = n$ .

**Lemma 3.14.** *Suppose  $t \leq k \leq n$ ,  $\mathcal{C}_t(n, k) \neq \emptyset$  and  $|\mathbb{F}| > \max\{k, n - k\}$ . Then, for any code  $C \in \mathcal{C}_t(n, k)$  there exists a code  $D \in \mathcal{C}_t(n, k)$  which is equivalent and opposite to  $C$  in  $\Gamma(n, k)$ .*

*Proof.* Suppose  $2k \leq n$  and let  $C \in \mathcal{C}_t(n, k)$  with  $G$  as generator matrix. By elementary row operations on  $G$ , which leave  $C$  invariant, and column operations by means of  $\rho \in \mathcal{M}(V)$  (see Definition 2.5), we can obtain a generator matrix

$$G' := (I \quad A \quad B)$$

for an equivalent code  $\rho(C) =: C' \in \mathcal{C}_t(n, k)$ . Here  $I$  is the  $k \times k$  identity matrix,  $A$  is a  $k \times k$  matrix of rank  $t$  (since any  $t$  columns of a generator matrix of a code in  $\mathcal{C}_t(n, k)$  are linearly independent) and  $B$  is a  $k \times (n - 2k)$ . Take  $\lambda \in \mathbb{F} \setminus \{0\}$  and consider the matrix

$$G''_\lambda := (\lambda A \quad I \quad B).$$

Since  $G''_\lambda$  is obtained from  $G'$  by applying transformations induced by the monomial group  $\mathcal{M}(V)$ , the code  $C''_\lambda$  having  $G''_\lambda$  as generator matrix, is equivalent to  $C'$ . In particular  $C''_\lambda \in \mathcal{C}_t(n, k)$ .

We want to show that it is always possible to choose  $\lambda$  so that  $C''_\lambda$  and  $C'$  are in direct sum as subspaces of  $V$ , that is the matrix

$$\begin{pmatrix} I & A & B \\ \lambda A & I & B \end{pmatrix}$$

has rank  $2k$ . By elementary row operations, subtracting from the second block  $\lambda(I \ A \ B)$ , we see that the rank of the matrix above is the same as the rank of

$$\begin{pmatrix} I & A & B \\ 0 & I - \lambda A^2 & (I - \lambda A)B \end{pmatrix}$$

In particular, this rank is definitely  $2k$  if  $\det(I - \lambda A^2) \neq 0$ .

On the other hand  $\det(I - \lambda A^2) = 0$  if and only if  $\lambda^{-1}$  is an eigenvalue of  $A^2$ . Since  $A^2$  is a  $k \times k$  matrix, of rank at most  $t$ , the number of its eigenvalues is at most  $t \leq k < |\mathbb{F}|$ . So, there is at least one  $\lambda \in \mathbb{F}^*$  which is a non-null eigenvalue of  $A$ . For such a  $\lambda$ , the matrix  $G''_\lambda$  represents a code  $C''_\lambda \in \mathcal{C}_t(n, k)$  such that  $\dim(C' \cap C''_\lambda) = 0$ , that is  $C'$  and  $C''_\lambda$  are opposite in  $\Gamma(n, k)$ .

Suppose now  $2k > n$ . Let  $C \in \mathcal{C}_t(n, k)$  be a code with generator matrix  $G$ . Using elementary row operations on  $G$  and a monomial transformation  $\rho \in \mathcal{M}(V)$ , we can obtain a code  $C' := \rho(C)$  equivalent to  $C$  whose generator matrix  $G'$  is in systematic form, i.e

$$G' = \begin{pmatrix} I_{n-k} & 0 & A_1 \\ 0 & I_{2k-n} & A_2 \end{pmatrix},$$

where  $A_1$  and  $A_2$  are suitable matrices of dimensions respectively  $(n-k) \times (n-k)$  and  $(2k-n) \times (n-k)$ . For  $\lambda \in \mathbb{F} \setminus \{0\}$ , let now  $G''_\lambda = \begin{pmatrix} \lambda A_1 & 0 & I_{n-k} \\ \lambda A_2 & I_{2k-n} & 0 \end{pmatrix}$  and  $C''_\lambda$  be the code with generator matrix  $G''_\lambda$ . The matrix  $G''_\lambda$  is obtained by permuting and multiplying some of the columns of the matrix  $G'$  by a non-zero scalar  $\lambda$ ; as such the code  $C''_\lambda$  generated by  $G''_\lambda$  is equivalent to  $C$  and  $C'$ ; thus,  $C''_\lambda \in \mathcal{C}_t(n, k)$  for all  $\lambda \neq 0$ .

Since  $2k > n$ , the codes  $C'$  and  $C''_\lambda$  are opposite if and only if  $\dim(C' + C''_\lambda) = n$ , that is to say the rank of the matrix  $\bar{G}_\lambda = \begin{pmatrix} G' \\ G''_\lambda \end{pmatrix}$  is maximum and equal to  $n$ .

Explicitly, the structure of the matrix  $\bar{G}_\lambda$  is

$$\bar{G}_\lambda = \begin{pmatrix} I_{n-k} & 0 & A_1 \\ 0 & I_{2k-n} & A_2 \\ \lambda A_1 & 0 & I_{n-k} \\ \lambda A_2 & I_{2k-n} & 0 \end{pmatrix}.$$

By using column operations we see that

$$\begin{aligned} \text{rank} \begin{pmatrix} I_{n-k} & 0 & A_1 \\ 0 & I_{2k-n} & A_2 \\ \lambda A_1 & 0 & I_{n-k} \\ \lambda A_2 & I_{2k-n} & 0 \end{pmatrix} &\geq \text{rank} \begin{pmatrix} I_{n-k} & 0 & A_1 \\ 0 & I_{2k-n} & A_2 \\ \lambda A_1 & 0 & I_{n-k} \end{pmatrix} = \text{rank} \begin{pmatrix} I_{n-k} & 0 & A_1 \\ 0 & I_{2k-n} & 0 \\ \lambda A_1 & 0 & I_{n-k} \end{pmatrix} = \\ &\text{rank} \begin{pmatrix} I_{n-k} & 0 & 0 \\ 0 & I_{2k-n} & 0 \\ \lambda A_1 & 0 & I_{n-k} - \lambda A_1^2 \end{pmatrix}. \end{aligned}$$

So, if  $\lambda^{-1}$  is not an eigenvalue of  $A_1^2$ , we have that  $\text{rank}(\bar{G}_\lambda) = n$ . As the matrix  $A_1^2$  has dimension  $(n-k) \times (n-k)$ , we have that  $A_1^2$  has at most  $n-k$  eigenvalues; as  $|\mathbb{F}| > n-k$  there are some values of  $\lambda$  such that this rank is maximum; for these values of  $\lambda$  we get that  $C', C''_\lambda \in \mathcal{C}_t(n, k)$  are opposite.

Observe now that for any two codes  $X, Y \in \mathcal{C}_t(n, k)$  and any  $\eta \in \mathcal{M}(V)$ , we have  $d(X, Y) = d(\eta(X), \eta(Y))$  in  $\Gamma(n, k)$ . Since  $C' = \rho(C)$  for  $\rho \in \mathcal{M}(V)$ , put  $D = \rho^{-1}(C''_\lambda)$ , where  $C''_\lambda$  is the code constructed above. Then,

$$d(C, D) = d(\rho^{-1}(C'), \rho^{-1}(C''_\lambda)) = d(C', C''_\lambda).$$

It follows that  $C$  and  $D$  are codes in  $\mathcal{C}_t(n, k)$  which are equivalent and opposite in  $\Gamma(n, k)$ .  $\square$

**Corollary 3.15.** *Suppose  $\mathcal{C}_t(n, k) \neq \emptyset$  and  $\Delta_t(n, k)$  to be isometrically embedded into  $\Gamma(n, k)$ . If  $|\mathbb{F}| > \max\{k, n-k\}$ , then*

$$\text{diam}(\Delta_t(n, k)) = \text{diam}(\Gamma(n, k)).$$

*Proof.* By Lemma 3.14, there are at least two codes  $X, Y \in \mathcal{C}_t(n, k)$  with  $d(X, Y) = \text{diam}(\Gamma(n, k))$ . Since  $\Delta_t(n, k)$  is isometrically embedded in  $\Gamma(n, k)$  we also have  $d_t(X, Y) = \text{diam}(\Gamma(n, k))$ . On the other hand, for any  $X, Y \in \mathcal{C}_t(n, k)$ ,

$$d_t(X, Y) = d(X, Y) \leq \text{diam}(\Gamma(n, k)).$$

It follows  $\text{diam}(\Delta_t(n, k)) = \text{diam}(\Gamma(n, k))$ .  $\square$

Note that for  $q \geq \binom{n}{t}$ , the assumptions of Corollary 3.15 hold.

### 3.3. Proof of Theorem 1

**Lemma 3.16.** *If  $\mathbb{F}$  is a field with  $|\mathbb{F}| + 1 \geq n$  then  $\mathcal{C}_t(n, k) \neq \emptyset$  for all  $t$  and  $k$  with  $1 \leq t \leq k \leq n$ .*

*Proof.* Note that for all  $t$  we have  $\mathcal{C}_t(n, k) \subseteq \mathcal{C}_{t-1}(n, k)$ . So, in order to get the lemma we just need to show that  $\mathcal{C}_k(n, k) \neq \emptyset$  under our assumptions. It is well known that if  $n \leq q+1$  for  $\mathbb{F} := \mathbb{F}_q$  a finite field of order  $q$ , there exist  $[n, k]$ -linear MDS codes. Since the dual of an MDS code is MDS, it is immediate to see that any  $k$ -columns of the generator matrix of a  $[n, k]$ -MDS code  $C$  are independent. It follows that  $C \in \mathcal{C}_k(n, k) \neq \emptyset$ .

Suppose now  $\mathbb{F}$  is an arbitrary infinite field. There exist at least  $n$  distinct elements  $a_1, a_2, \dots, a_n \in \mathbb{F}$ . Consider the matrix

$$G := \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{k-1} & a_2^{k-1} & \dots & a_n^{k-1} \end{pmatrix}$$

Any  $k \times k$  minor  $M_{i_1 \dots i_k}$  of  $G$ , comprising the columns  $i_1, \dots, i_k$  is a Vandermonde matrix with determinant

$$\det(M_{i_1 \dots i_k}) = \prod_{1 \leq r < s \leq k} (a_{i_s} - a_{i_r}) \neq 0.$$

In particular, the code  $C$  with generator matrix  $G$  belongs to  $\mathcal{C}_k(n, k)$  which is consequently non-empty.  $\square$

Theorem 1 follows from Theorem 3.10, Corollary 3.15 and Lemma 3.16.

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