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On the Grassmann Graph of Linear Codes

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Abstract

Let $\Gamma(n, k)$ be the Grassmann graph formed by the k-dimensional subspaces of a vector space of dimension n over a field $\mathbb{F}$ and, for $t \in \mathbb{N} \setminus \{0\}$, let $\Delta_t(n, k)$ be the subgraph of $\Gamma(n, k)$ formed by the set of linear $[n, k]$-codes having minimum dual distance at least $t + 1$. We show that if $|\mathbb{F}| \geq \binom{n}{t}$ then $\Delta_t(n, k)$ is connected and it is isometrically embedded in $\Gamma(n, k)$. This generalizes some results of [3] and [2].

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1. Introduction

Let $V := V(n, \mathbb{F})$ be a n-dimensional vector space over a field $\mathbb{F}$ and for $k = 1, \ldots, n-1$, denote by $\Gamma(n, k)$ the *k-Grassmann graph* of V, that is the graph whose vertices are the k-subspaces of V and where two vertices X, Y are connected by an edge if and only if $\dim(X \cap Y) = k - 1$. See [1] for more detail.

It is interesting to see what properties extend from the graph $\Gamma(n, k)$ to some of its subgraphs.

Suppose that B is a given basis of V; henceforth we will write the coordinates of the vectors in V with respect to B.

A $[n, k]$-*linear code* C is just a k-dimensional vector subspace of V. If B_C is an ordered basis of C, a *generator matrix* for C is the $k \times n$ matrix whose rows are the coordinates of the elements of B_C with respect to B. Given a $[n, k]$-linear code C, its *dual code* is the $[n, n - k]$-linear code $C^\perp$ given by

$$C^\perp := \{v \in V : \forall c \in C, v \cdot c = 0\}$$

where by $\cdot$ we mean the standard symmetric bilinear form on V given by

$$(v_1, \ldots, v_n) \cdot (c_1, \ldots, c_n) = v_1 c_1 + \cdots + v_n c_n.$$

Since the $\cdot$ is non-degenerate, $C^{\perp\perp} = C$. We say that C has *dual minimum distance* $t + 1$ if and only if the minimum Hamming distance of the dual $C^\perp$ of C is $t + 1$.

The condition for a $[n, k]$-linear C having dual minimum distance at least $t + 1$ can be easily read on any generator matrix of it. Indeed (see, e.g., Proposition 2.7), C has dual minimum

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distance at least $t + 1$ if and only if any t-columns of any generator matrix of C are linearly independent.

For $t \in \mathbb{N} \setminus \{0\}$, let $\mathcal{C}_t(n, k)$ be the set of all $[n, k]$-linear codes with dual minimum distance at least $t + 1$ and denote by $\Delta_t(n, k)$ the subgraph of $\Gamma(n, k)$ induced by the elements of $\mathcal{C}_t(n, k)$, i.e. the vertex set of $\Delta_t(n, k)$ is formed by the elements in $\mathcal{C}_t(n, k)$ and two vertices X and Y are adjacent in $\Delta_t(n, k)$ if and only if $\dim(X \cap Y) = k - 1$. We shall call $\Delta_t(n, k)$ the *Grassmann graph* of the linear $\mathcal{C}_t(n, k)$ codes.

Note that for $t = 1$, $\mathcal{C}_1(n, k)$ is the class of the *non-degenerate $[n, k]$-linear codes* and for $t = 2$, $\mathcal{C}_2(n, k)$ is the class of the *projective $[n, k]$-linear codes.*

In general, we say that a subgraph is isometrically embedded in a larger graph if there exists a distance-preserving map among them (see also Definition 2.2); see also [6].

In [2] Kwiatkowski and Pankov studied the graph $\Delta_1(n, k)$ and more recently in [3] Kwiatkowski, Pankov and Pasini, considered the graph $\Delta_2(n, k)$ in the case $\mathbb{F}$ is a finite field of order q.

In [2, Corollary 2], the authors show that $\Delta_1(n, k)$ is connected and isometrically embedded in $\Gamma(n, k)$ if and only if $n < (q + 1)^2 + k - 2$. In [3, Theorem 1] it is shown that a sufficient condition for the graph $\Delta_2(n, k)$ to be isometrically embedded in $\Gamma(n, k)$ is $q \geq \binom{n}{2}$.

In this paper we generalize the results of [2] and [3] to the graphs $\Delta_t(n, k)$ for arbitrary $t \leq k$ and arbitrary fields $\mathbb{F}$.

More in detail, our main result is the following

Theorem 1. *Let t, k, n be integers such that $1 \leq t \leq k \leq n < \infty$. Suppose that $\mathbb{F}$ is a field with $|\mathbb{F}| \geq \binom{n}{t}$. Then the graph $\Delta_t(n, k)$ is connected and isometrically embedded into the k-Grassmann graph $\Gamma(n, k)$. Furthermore, the diameter of $\Delta_t(n, k)$ and $\Gamma(n, k)$ are the same.*

Remark 1.1. The hypotheses in Theorem 1 are sufficient for the graph $\Delta_t(n, k)$ to be connected and to be isometrically embedded into $\Gamma(n, k)$ but in general they are not necessary. We leave to a future work to determine if the graph $\Delta_t(n, k)$ might be connected also under some weaker assumptions on q or t and see if there are cases where the embedding is not isometric. We leave also to a future work to generalize these results for $\mathbb{F}$ a possibly non-commutative division ring and also to the infinite dimensional cases both for n and for k.

1.1. Structure of the paper

In Section 2 we recall some basic definitions and preliminary results which shall be used in order to prove Theorem 1. Section 3 contains the proof of our main results; in particular, in Subsection 3.1 we shall prove that the graph $\Delta_t(n, k)$ is connected and isometrically embedded in $\Gamma(n, k)$ for $q \geq \binom{n}{t}$, while in Subsection 3.2 we shall show that the sets $\mathcal{C}_t(n, k)$, when not empty, always contain codes which are at maximum distance in the Grassmann graph $\Gamma(n, k)$. Finally, in Subsection 3.3 we prove Theorem 1.

2. Preliminaries

As mentioned in the Introduction, $\mathbb{F}$ is a field and $V := V(n, \mathbb{F})$ denotes a n-dimensional vector space over $\mathbb{F}$. Let $B = (e_1, \ldots, e_n)$ be a given ordered basis of V with respect to which all the vectors will be written in coordinates. For k and t integers such that $1 \leq t \leq k \leq n - 1$, $\mathcal{C}_t(n, k)$ is the class of $[n, k]$-linear codes having dual minimum distance at least $t + 1$. More explicitly,

$$\mathcal{C}_t(n, k) := \{C \subseteq V : \dim C = k, d^{\perp}(C) \geq t + 1\}$$

where $d^\perp(C) := d_{\min}(C^\perp)$ is the minimum distance of the dual code $C^\perp$ which means that the weight $wt(v)$ of any codeword $v = (v_1, \ldots, v_n) \in C^\perp$ is at least $t+1$, i.e.

$$\mathrm{wt}(v) := |\{i : v_i \neq 0\}| \geq t + 1.$$

Clearly, if $\mathcal{C}_t(n,k) \neq \emptyset$, then necessarily $t \leq k \leq n$.

If $t = k$ and $\mathbb{F} = \mathbb{F}_q$, the elements of $\mathcal{C}_k(n,k)$ are exactly the *maximum distance separable* $[n,k]$-codes (see e.g. [5]), that is codes whose minimum distance $d_{\min}$ attains the Singleton bound $d_{\min} = n - k + 1$; see Corollary 2.8.

Remark 2.1. The condition $t \leq k \leq n$ is necessary but in general not sufficient to ensure that $\mathcal{C}_t(n,k)$ is not empty. Indeed, if $\mathbb{F} = \mathbb{F}_q$, even for arbitrary values of q, it is not straightforward to determine if $\mathcal{C}_t(n,k) \neq \emptyset$ or characterize the elements of $\mathcal{C}_t(n,k)$. For instance the celebrated MDS conjecture implies $\mathcal{C}_k(n,k) = \emptyset$ for $n > q + 2$. On the other hand, if $n < q + 1$, then by Lemma 3.16, $\mathcal{C}_t(n,k) \neq \emptyset$ for all $t \leq k \leq n$.

In order to avoid trivial cases, we shall henceforth suppose that the parameters n, k, t and q if $\mathbb{F} := \mathbb{F}_q$, have been chosen so that $\mathcal{C}_t(n,k) \neq \emptyset$; in Lemma 3.16 it shall be shown that under the assumptions of Theorem 1 this is always true.

We recall from the Introduction that $\Delta_t(n,k)$ is the subgraph of $\Gamma(n,k)$ induced by the elements of $\mathcal{C}_t(n,k)$.

Definition 2.2. Let $X \in \Delta_t(n,k)$. We define the *connected component* $\Delta_t^X(n,k)$ of X in $\Delta_t(n,k)$ as the subgraph of $\Delta_t(n,k)$ whose vertices are all $Y \in \Delta_t(n,k)$ such that there is a path in $\Delta_t(n,k)$ joining X and Y. The graph $\Delta_t(n,k)$ is *connected* if $\Delta_t^X(n,k) = \Delta_t(n,k)$ for some (and, consequently for all) $X \in \Delta_t(n,k)$.

For any $X, Y \in \mathcal{C}_t(n,k)$ write $d(X,Y)$ for the distance between X and Y in the Grassmann graph $\Gamma(n,k)$ and $d_t(X,Y)$ for the distance between X and Y in $\Delta_t(n,k)$. If $\Delta_t^X(n,k) \neq \Delta_t^Y(n,k)$, that is X and Y are in different connected components of $\Delta_t(n,k)$ we put $d_t(X,Y) = \infty$. We recall that the *diameter* of a graph is the maximum of the distances among two of its vertices.

Since every edge of $\Delta_t(n,k)$ is an edge of $\Gamma(n,k)$, it is straightforward to see that $d_t(X,Y) \geq d(X,Y)$ for all $X, Y \in \Delta_t(n,k)$.

Definition 2.3. We say that $\Delta_t(n,k)$ is *isometrically embedded* in $\Gamma(n,k)$ if for any $X, Y \in \Delta_t(n,k)$ we have $d_t(X,Y) = d(X,Y) = k - \dim(X \cap Y)$.

Note that if $\Delta_t(n,k)$ is isometrically embedded in $\Gamma(n,k)$, then $\Delta_t(n,k)$ is also connected.

2.1. Some basic results

Definition 2.4. Let $(i_1, \ldots, i_t) \in \mathbb{N}^t$ be a t-uple of integers such that $1 \leq i_1 < i_2 < \cdots < i_t \leq n$. We denote by $C_{i_1 \ldots i_t} := \cap_{j:=1}^t (x_{i_j} = 0)$ the $(n-t)$-dimensional subspace of V obtained as the intersection of the coordinate hyperplanes of V of equations $x_{i_j} = 0$. We shall call $C_{i_1 \ldots i_t}$ the $(i_1, \ldots, i_t)$-*coordinate subspace* of V.

The *monomial group* $\mathcal{M}(V)$ of V consists of all linear transformations of V which map the set of subspaces $\{\langle e_1 \rangle, \ldots, \langle e_n \rangle\}$ in itself. It is straightforward to see that $\mathcal{M}(V) \cong \mathbb{F}^* \wr S_n$ where $\wr$ denotes the wreath product and S_n is the symmetric group of order n; see [5, Chapter 8,§5] for more details.

Definition 2.5. Two $[n,k]$-linear codes X and Y are *equivalent* if there exist a monomial transformation $\rho \in \mathcal{M}(V)$ such that $X = \rho(Y)$.

Suppose X is a $[n,k]$-linear code with generator matrix G_X. If $A \in \mathrm{GL}(k,\mathbb{F})$ then $G'_X = AG_X$ is also a generator matrix for X.

It follows that two $[n,k]$-linear codes X and Y with generator matrices respectively G_X and G_Y are *equivalent* if there exists $A \in \mathrm{GL}(k,\mathbb{F})$, a permutation matrix $P \in \mathrm{GL}(n,\mathbb{F})$ and a diagonal matrix $D \in \mathrm{GL}(n,\mathbb{F})$ such that

$$G_X = AG_Y(PD).$$

Equivalence between linear codes is an equivalence relation and the equivalence class of a code X corresponds to the orbit of X under the action of $\mathcal{M}(V)$ on the k-dimensional subspaces of V.

Also, it can be readily seen that two codes are equivalent if and only if any two of their generator matrices belong to the same orbit under the action of the group $\mathrm{PGL}(k,\mathbb{F}) : (\mathbb{F}^* \wr S_n)$, where $\mathrm{PGL}(k,\mathbb{F})$ acts on the right of the generator matrix.

With mostly harmless abuse of notation, in the remainder of this paper we shall not distinguish between the action of $\mathcal{M}(V)$ on the codes (regarded as subspaces of V) and that on the columns of their generator matrices.

Since equivalent codes have the same parameters (in particular they have the same minimum dual distance), we have that $\mathcal{C}_t(n,k)$ consists of unions of orbits under the action of $\mathcal{M}(V)$.

For $j = 1, \ldots, n$, let $x^j : V \to \mathbb{F}_q$ be the jth-coordinate linear functional of V which acts on the vectors e_i, $1 \le i \le n$, of B as $x^j(e_i) = \begin{cases} 0 \ if \ i \ne j \\ 1 \ if \ i = j \end{cases}$.

Observe that, for any $v \in V$ and j with $1 \le j \le n$, we have that $x^j(v)$ is exactly the j-th component of v with respect to the basis B. So, if X is a $[n,k]$-linear code and $B_X = (b_1, \ldots, b_k)$ is a given basis of X with respect to which the generator matrix G_X is written, then for any i,j with $1 \le i \le k$ and $1 \le j \le n$, then $x^j(b_i)$ is exactly the (i,j)- entry in the matrix G_X. So, the j-th column of G_X represents the restriction $x^j|_{B_X}$ of the functional x^j to the basis B_X. By linearity, we can say that the j-th column of G_X represents the restriction $x^j|_X$ of the functional x^j to X. This has the following important consequence.

Lemma 2.6. *Let $X \subseteq V$ be a $[n,k]$-linear code and G_X be a generator matrix of X. A set of coordinate functionals restricted to X is linearly independent if and only if the columns of G_X representing them are linearly independent.*

If $\mathbb{F} := \mathbb{F}_q$, we shall also use the notation

$$[m]_q := \frac{q^m - 1}{q - 1}$$

for the number of 1-dimensional subspaces of an m-dimensional vector space.

The equivalence between (1) and (2) in the following proposition is well known; however, since many results of the present work rely on it, we present a complete proof for the convenience of the reader.

Proposition 2.7. *Let X be a $[n,k]$-linear code and denote by G_X a generator matrix of X. The following are equivalent.*

(1) X has minimum dual distance at least $t + 1$.

(2) Any t columns of G_X are linearly independent.

(3) For any $1 \le i_1 < i_2 < \cdots < i_t \le n$ we have $dim(X \cap C_{i_1 \ldots i_t}) = k - t$ where $C_{i_1 \ldots i_t} := \cap_{j:=1}^{t}(x_{i_j} = 0)$ is the $(n-t)$-dimensional $(i_1, \ldots, i_t)$-coordinate subspace of V.

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Proof. The matrix G_X is a parity check matrix for the dual code $X^\perp$. Write the columns of G_X as $G_1, \ldots, G_n$ and let $y = (y_1, \ldots, y_n) \in X^\perp$. Then

$$G_X y^t = G_1 y_1 + \cdots + G_n y_n = \mathbf{0}. \tag{1}$$

Assume (1). Then, for any $y \in X^\perp$, $y \neq \mathbf{0}$, we have $\mathrm{wt}(y) \geq t+1$. Suppose by contradiction that there is a set of t-columns of G_X which are linearly dependent. To simplify the exposition, assume without much loss of generality that this set comprises the first t-columns. Then,

$$G_1 y_1 + G_2 y_2 + \cdots + G_t y_t = \mathbf{0}$$

with at least one entry y_i different from 0; so the vector $(y_1, \ldots, y_t, 0, \ldots, 0) \neq \mathbf{0}$ is in $X^\perp$ with $\mathrm{wt}(y) \leq t < t+1$. This contradicts (1).

Conversely, assume (2) and take $y \in X^\perp$ with $\mathrm{wt}(y) = d$. Then $G_X y^T = 0$. If $d = 0$, that is $y = \mathbf{0}$, then there is nothing to prove. If $d \neq 0$, suppose, again without much loss of generality, that exactly the first d entries $y_1, \ldots, y_d$ of y are non-zero. Then

$$G_1 y_1 + \cdots + G_d y_d = \mathbf{0}.$$

In particular, the first d columns of G_X must be linearly dependent; by (2) we necessarily have $d > t$ since any set of t columns of G_X is independent; this implies (1).

We now prove the equivalence between (2) and (3). Suppose that (3) holds. Then, $\dim(X \cap C_{i_1 \ldots i_t}) = k - t$ which means that the restrictions $x^{i_1}|_X, \ldots, x^{i_t}|_X$ of the t coordinate functionals $x^{i_1}, \ldots, x^{i_t}$ of V to X are linearly independent. Then (2) follows from Lemma 2.6.

Conversely, assume (2) holds and suppose by contradiction that (3) is false, that is that there exists a set of indexes $i_1, \ldots, i_t$ such that $\dim(X \cap C_{i_1 \ldots i_t}) \geq k - t + 1$. Then, for some $j \in \{1, \ldots, t\}$, we have

$$X \cap C_{i_1 \ldots i_{j-1} i_{j+1} \ldots i_t} \subseteq X \cap C_{i_j}.$$

In terms of coordinate functionals this means

$$x^{i_j}|_X \in \langle x^{i_1}|_X, \ldots x^{i_{j-1}}|_X, x^{i_{j+1}}|_X, \ldots, x^{i_t}|_X \rangle.$$

So $x^{i_j}|_X$ is a linear combination of the remaining coordinate functionals. In particular, by Lemma 2.6, this means that the column G_{i_j} of any generator matrix G_X of X is a linear combination of the columns $G_{i_1}, \ldots, G_{i_{j-1}}, G_{i_{j+1}}, \ldots, G_{i_t}$. This contradicts (2). $\square$

The following is an immediate consequence of Proposition 2.6.

Corollary 2.8. *The set $\mathcal{C}_1(n, k)$ consists of all $[n, k]$-linear non-degenerate codes; $\mathcal{C}_2(n, k)$ consists of all $[n, k]$-linear projective codes; the set $\mathcal{C}_k(n, k)$, if $\mathbb{F} = \mathbb{F}_q$, consists of all $[n, k]$-linear MDS codes.*

Proof. Only the statement about $\mathcal{C}_k(n, k)$ needs to be proved as the descriptions of $\mathcal{C}_1(n, k)$ and $\mathcal{C}_2(n, k)$ follow directly from Proposition 2.7. Suppose $C \in \mathcal{C}_k(n, k)$, i.e. C is a $[n, k]$-code having dual minimum distance at least $k + 1$. Then, by definition of $\mathcal{C}_k(n, k)$, $C^\perp$ is a $[n, n-k]$-code with minimum distance at least $k + 1 = n - (n - k) + 1$ and, as such it is a MDS-code. Since the duals of MDS codes are MDS codes, $C = C^{\perp\perp}$ is also MDS.

Conversely, suppose C to be a $[n, k]$-linear MDS code; then $C^\perp$ is also MDS and has minimum distance $k + 1$. It follows that $C \in \mathcal{C}_k(n, k)$. $\square$

3. Proof of Theorem 1

We proceed by steps; first, in Section 3.1 we provide a condition for the graph $\Delta_t(n,k)$ to be connected and isometrically embedded in $\Gamma(n,k)$; then in Section 3.2 we show that any class $\mathcal{C}_t(n,k)$ contains elements which are at maximum distance in $\Gamma(n,k)$; finally in Section 3.3 we complete the proof of Theorem 1.

3.1. The connectedness of the graph

The following are two elementary lemmas of linear algebra.

Lemma 3.1. *Let $X \in \mathcal{C}_t(n,k)$ and let H be a hyperplane of X. If $y \notin X$ then*

$$\dim(\langle H, y \rangle \cap C) \leq k - t + 1$$

for every $(n-t)$-dimensional coordinate subspace C of V.

Proof. Suppose by contradiction

$$\dim(\langle H, y \rangle \cap C) \geq k - t + 2.$$

Since $H \subseteq X$ we also have $\dim(\langle X, y \rangle \cap C) \geq k - t + 2$.

Any vector in $\langle H, y \rangle$ can be written in the form $x + \alpha y$ where $x \in H$ and $\alpha \in \mathbb{F}$. In particular, there are $k - t + 2$ linearly independent vectors v_i in $\langle X, y \rangle \cap C$ of the form

$$v_i = x_i + \alpha_i y$$

where $x_i \in H$ and $\alpha_i \in \mathbb{F}$. By Gaussian elimination (we remove y), we have at least $k - t + 1$ vectors in X which are linearly independent and contained in C; so $\dim(X \cap C) \geq k - t + 1$, which is a contradiction because $X \in \mathcal{C}_t(n,k)$ (see Proposition 2.7). $\qquad\square$

Lemma 3.2. *Let S be a vector space of dimension s, $H_1 \neq H_2$ be two distinct hyperplanes of S with fixed bases B_1 and B_2. Then, there exists a basis B of S contained in $B_1 \cup B_2$.*

Proof. Since $H_1 \neq H_2$, there exists at least one element $b \in B_2 \setminus H_1$. Consider $B = B_1 \cup \{b\}$. This is a linearly independent set consisting of s distinct elements, $B \subseteq S$ and $\dim(S) = s$. It follows that B is a basis of S with $B \subseteq B_1 \cup B_2$. $\qquad\square$

Recall that by $d(X, Y)$ we mean the distance in $\Gamma(n,k)$ while $d_t(X, Y)$ denotes the distance in $\Delta_t(n,k)$.

The following definitions are used in the proof of Lemma 3.8.

Definition 3.3. Put $\binom{\{1,\ldots,n\}}{t} := \{(i_1, \ldots, i_t) \in \mathbb{N}^t : 1 \leq i_1 < i_2 < \cdots < i_t \leq n\}$ and define as *colors* the elements of it. Endow $\binom{\{1,\ldots,n\}}{t}$ with the natural lexicographic order on the t-uples.

Take $X, Y \in \mathcal{C}_t(n,k)$ with $X \neq Y$ and let H be a hyperplane of X such that $X \cap Y \subseteq H$. The *coloration induced by H* is the map

$$\psi_H : Y/(H \cap Y) \to \binom{\{1,\ldots,n\}}{t} \cup \{\infty\}$$

sending any vector $[p] \in Y/(H \cap Y)$ to the smallest (in the lexicographic order) color $(i_1, \ldots, i_t)$ such that

$$\dim(\langle H, p \rangle \cap C_{i_1 \ldots i_t}) = k - t + 1$$

where $C_{i_1 \ldots i_t}$ is the $(i_1, \ldots, i_t)$-coordinate subspace as defined in Definition 2.4. If no such color exists we put $\psi_H([p]) = \infty$.

The function ψ_H is well defined. Indeed, if $b \in [a] = a + (H \cap Y)$, then $b = a + h$ for some $h \in H \cap Y$ and $\langle H, b \rangle = \langle H, a + h \rangle = \langle H, a \rangle$; so $\psi_H([a]) = \psi_H([b])$.

Henceforth we shall silently represent each element $[p]$ of $Y/(X \cap Y)$ by means of its representative element p.

Definition 3.4. Under the same assumptions as in Definition 3.3, we say that a set $T \subseteq Y/(H \cap Y)$ is *monochromatic* if $\forall r, s \in T$, $\psi_H(r) = \psi_H(s) \neq \infty$, i.e. all of its elements have the same color.

Definition 3.5. Under the same assumptions as in Definition 3.3, we say that a subspace S of $Y/(H \cap Y)$ with $\dim(S) = s$ is *colorable* if there exists at least one monochromatic basis of S. If S is colorable, we define the *color* $\psi_H(S)$ of S as the minimum color of a basis of S.

In symbols, let

$$\mathfrak{F}(S) := \{ f = (p_1, \ldots, p_s) \colon f \text{ is a basis of } S \text{ and } \psi_H(p_1) = \cdots = \psi_H(p_s) \neq \infty \}$$

be the set of monochromatic bases of S. If $f \in \mathfrak{F}(S)$, denote by $\psi_H(f)$ the color of any element in f. Hence

Lemma 3.6. *The subspace S is colorable if and only if $\mathfrak{F}(S) \neq \emptyset$.*

If S is colorable, the color of S is

$$\psi_H(S) := \min\{ \psi_H(f) \colon f \in \mathfrak{F}(S) \}.$$

In other words, S has color c if there are s independent vectors in S all with the same color c and any other set of s independent vectors in S either is not monochromatic or has color $c' \geq c$.

Note that a colorable subspace S with color c is not, in general, a monochromatic set.

Lemma 3.7. *Let $X, Y \in \mathcal{C}_t(n, k)$ with $\dim(X \cap Y) = k - d \geq k - t$. If $\mathbb{F}$ is a field with $|\mathbb{F}| \geq \binom{n}{t}$ then there exists a code $Z \in \mathcal{C}_t(n, k)$ such that $\dim(X \cap Z) = k - 1$ and $\dim(Z \cap Y) = k - d + 1$.*

Proof. We prove that for every hyperplane H of X containing $X \cap Y$, there exists $z \in Y \setminus (X \cap Y)$ such that $Z := \langle H, z \rangle \in \mathcal{C}_t(n, k)$.

By way of contradiction suppose the contrary. Hence there exists a hyperplane H of X with $X \cap Y \subseteq H$ such that for every $z \in Y \setminus (X \cap Y)$, we have $\langle H, z \rangle \notin \mathcal{C}_t(n, k)$.

Equivalently, by Proposition 2.7, we suppose that there exists a hyperplane H of X with $X \cap Y \subseteq H$ such that for every $z \in Y \setminus (X \cap Y)$ there exist indexes $i_1 < i_2 < \cdots < i_t$ such that $\dim(\langle H, z \rangle) \cap C_{i_1 \ldots i_t}) \geq k - t + 1$. By Lemma 3.1, we have $\dim(\langle H, z \rangle) \cap C_{i_1 \ldots i_t}) = k - t + 1$.

Under these assumptions, we will prove the following claim which leads to a contradiction.

Claim 1. *There exist indexes $i_1, \ldots, i_t$ and linearly independent vectors $p_1, \ldots, p_d \in Y \setminus (X \cap Y)$ such that $[p_1], \ldots, [p_d]$ are linearly independent in $Y/(X \cap Y)$,*

$$\dim R_i = k - t + 1 \text{ and } \dim(R_1 + \cdots + R_d) \geq k - t + d$$

where we put $H_i := \langle H, p_i \rangle$ and $R_i = H_i \cap C_{i_1 \ldots i_t}$.

Note that for every i, $R_i \subseteq \langle H, Y \rangle$. From Claim 1, we have

$$\dim(Y + R_1 + \cdots + R_d) \leq \dim(H + Y) = k + d - 1 \tag{2}$$

and

$$\dim(Y \cap (R_1 + \cdots + R_d)) = \dim(Y) + \dim(R_1 + \cdots + R_d) - \dim(R_1 + \cdots + R_d + Y) \geq$$
$$\geq k + (k - t + d) - (k + d - 1) \geq k - t + 1. \tag{3}$$

Since $R_i \subseteq C_{i_1 \ldots i_t}$ for any i, it follows

$$\dim(C_{i_1 \ldots i_t} \cap Y) \geq \dim(Y \cap (R_1 + \cdots + R_t)) \geq k - t + 1, \tag{4}$$

which is a contradiction because $\dim(C_{i_1 \ldots i_t} \cap Y) = k - t$, since $Y \in \mathcal{C}_t(n, k)$ (see Proposition 2.7).

So, in order to get the thesis, we need to prove Claim 1.

Let S be a subspace of $Y/(H \cap Y)$ with $\dim(S) = s$. We show by induction on s that S is colorable. First, by our hypotheses, for any $p \in S$ we have $\varphi_H(p) \neq \infty$.

Suppose $\dim(S) = 2$. The projective space $PG(S)$ is then a projective line of $PG(Y/H \cap Y))$ so there are $|\mathbb{F}| + 1$ points in $PG(S)$. By hypothesis we have $|\mathbb{F}| \geq \binom{n}{t}$ possible colors (see Definition 3.3). Hence there are at least 2 linearly independent vectors p_1 and p_2 in S such that $\psi_H(p_1) = \psi_H(p_2)$. Hence, $\mathfrak{F}(S) \neq \emptyset$. By Lemma 3.6, S is colorable.

Suppose now $\dim(S) > 2$. Put $s := \dim(S)$. By induction hypothesis, all subspaces S' of S with dimension $\dim(S') = \dim(S) - 1$ are colorable, that is they all admit a monochromatic basis. For any $(s-2)$-subspace S'' of S there are $|\mathbb{F}| + 1$ distinct $(s-1)$-dimensional subspaces S' of S with $S'' \leq S' \leq S$. Also $PG(S/S'')$ is a projective line. Since $|\mathbb{F}| + 1 > \binom{n}{t}$, there are at least two of such subspaces, say S_1 and S_2 with $S_1 \neq S_2$ (hence $\langle S_1, S_2 \rangle = S$) which have the same color $\psi_H(S_1) = \psi_H(S_2)$.

Let B_1 and B_2 be bases of respectively S_1 and S_2 with $\psi_H(B_1) = \psi_H(B_2)$ $(=\psi_H(S_1))$. By Lemma 3.2, there is a basis B of S contained in $B_1 \cup B_2$. So S admits at least one monochromatic basis and $\mathfrak{F}(S) \neq \emptyset$. By Lemma 3.6, S is colorable.

Hence, it is always possible to determine a monochromatic set of d independent vectors $\{p_1, p_2, \ldots, p_d\}$ of Y such that $[p_1], \ldots, [p_d]$ are independent in $Y/(X \cap Y)$. So, we have (recall that $H_i = \langle H, p_i \rangle$)

$$\dim(H_1 \cap C_{i_1 \ldots i_t}) = \dim(H_2 \cap C_{i_1 \ldots i_t}) = \cdots = \dim(H_d \cap C_{i_1 \ldots i_t}) = k - t + 1. \tag{5}$$

Since $H \subseteq X$ and $X \in \mathcal{C}_t(n, k)$, we have $\dim(H \cap C_{i_1 \ldots i_t}) \leq k - t$. Recalling that by definition $R_i := H_i \cap C_{i_1 \ldots i_t} = \langle H, p_i \rangle \cap C_{i_1 \ldots i_t}$, we have, by (5), that $\dim(R_i) = \dim(H \cap C_{i_1 \ldots i_t}) + 1 = k - t + 1$ and $\dim(H \cap C_{i_1 \ldots i_t}) = k - t$.

In particular, (note that $p_i \notin H$), it is always possible to find for $1 \leq i \leq d$ an element $h_i \in H$ and a non-null element $\alpha_i \in \mathbb{F}_q$ such that the point $\alpha_i p_i + h_i \in R_i$ is such that

$$\alpha_i p_i + h_i \in C_{i_1 \ldots i_t};$$

up to a scalar multiple we can assume $\alpha_i = 1$ for all i.

We now show that $\dim(R_1 + R_2 + \cdots + R_d) \geq k - t + d$. Suppose the contrary. Then, without loss of generality, we can assume $R_d \subseteq R_1 + R_2 + \cdots + R_{d-1}$. In particular

$$p_d + h_d \in R_1 + \cdots + R_{d-1},$$

whence

$$p_d = \beta_1 p_1 + \cdots + \beta_{d-1} p_{d-1} + h$$

with $h \in H$ a suitable element and $\beta_i \in \mathbb{F}$ for $1 \leq i \leq d - 1$. So, given that $p_1, \ldots, p_d \in Y$,

$$p_d - (\beta_1 p_1 + \cdots + \beta_{d-1} p_{d-1}) = h \in H \cap Y = X \cap Y,$$

that is

$$[p_d] + [p_1] + \cdots + [p_{d-1}] = [0]$$

in $Y/(X \cap Y)$. This contradicts the first part (already proved) of Claim 1, since $[p_1], \ldots, [p_d]$ are linearly independent vectors of $Y/(X \cap Y)$. It follows $\dim(R_1 + R_2 + \cdots + R_d) \geq k - t + d$. So Claim 1 holds. This completes the proof of the theorem. $\square$

Note that if $\mathbb{F} := \mathbb{F}_q$ is a finite field, then Lemma 3.7 gives the following

Lemma 3.8. *Let $X, Y \in \mathcal{C}_t(n,k)$ with $\dim(X \cap Y) = k - d \geq k - t$. If $\mathbb{F} := \mathbb{F}_q$ and $q \geq \binom{n}{t}$ then there exist $[d]_q$ distinct codes $Z \in \mathcal{C}_t(n,k)$ such that $\dim(X \cap Z) = k-1$ and $\dim(Z \cap Y) = k-d+1$.*

Proof. For any hyperplane H of X containing $X \cap Y$ it is possible to apply the argument in the proof of Lemma 3.7. Thus there are at least $[d]_q$ distinct codes Z with the required property. $\square$

Lemma 3.9. *Suppose $\mathbb{F}$ is a field with $|\mathbb{F}| \geq \binom{n}{t}$. Then for any $X \in \mathcal{C}_t(n,k)$ and for every $U \subset X$ with $\dim U < k - t$ there exists $X' \in \mathcal{C}_t(n, k-1)$ satisfying $U \subset X' \subset X$.*

Proof. A hyperplane H of X is an element of $\mathcal{C}_t(n, k-1)$ if and only if H does not contain $X \cap C_{i_1 \ldots i_t}$ for any $i_1 < \cdots < i_t$. Indeed, if $C_{i_1 \ldots i_t} \cap X \subseteq H$ for some $i_1 < \cdots < i_t$, then $\dim(H \cap C_{i_1 \ldots i_t}) \geq \dim(X \cap C_{i_1 \ldots i_t}) = k-t > k-t-1$ and $H \notin \mathcal{C}_t(n, k-1)$. Conversely, suppose that for any $i_1 < \cdots < i_t$, $C_{i_1 \ldots i_t} \cap X \nsubseteq H$; then $\dim(H \cap X \cap C_{i_1 \ldots i_t}) = \dim(H \cap C_{i_1 \ldots i_t}) = k-1-t$ for any choice of the indexes; so $H \in \mathcal{C}_t(n, k-1)$.

By Definition of $C_{i_1 \ldots i_t}$ (see Definition 2.4), there exist at most $\binom{n}{t}$ distinct spaces $C_{i_1 \ldots i_t}$. Now we distinguish two cases.

- If $\mathbb{F} := \mathbb{F}_q$ is a finite field, each of the spaces $C_{i_1 \ldots i_t}$ is contained in $[t]_q$ distinct hyperplanes of X. So the number of hyperplanes containing at least one $X \cap C_{i_1 \ldots i_t}$ is at most

$$\binom{n}{t}[t]_q.$$

  On the other hand U is contained in $[m]_q$ distinct hyperplanes where $m = \dim(X/U)$. Since $m > t$ we have

$$[m]_q \geq [t+1]_q = q^t + q^{t-1} + \cdots + q + 1 \geq \binom{n}{t}q^{t-1} + \binom{n}{t}q^{t-2} + \cdots + \binom{n}{t} + 1 > \binom{n}{t}[t]_q.$$

  This shows that there is at least one hyperplane X' of X containing U and none of the $C_{i_1 \ldots i_t}$.

- Suppose $\mathbb{F}$ is an infinite field and denote by X^* the dual space of X. Then, for every $U \subseteq X$ with $\dim U < k - t$ the set of hyperplanes X' of X containing U determine a subspace $\mathcal{U}$ of $\mathrm{PG}(X^*)$ of vector dimension at least $k - (k-t-1) = t+1$. Since each of the spaces $X \cap C_{i_1 \ldots i_t}$ has dimension $k - t$ (because $X \in \mathcal{C}_t(n,k)$), the set $\mathcal{H}_{i_1 \ldots i_t}$ of hyperplanes containing $C_{i_1 \ldots i_t}$ is a subspace of $\mathrm{PG}(X^*)$ of vector dimension t. In particular the set of all hyperplanes of X containing at least one $C_{i_1 \ldots i_t}$ is the union of $\binom{n}{t}$ subspaces of $\mathrm{PG}(X^*)$ each of vector dimension t.

  Since the field $\mathbb{F}$ is infinite, it is impossible for a projective space of vector dimension at least $t+1$ to be the union of a finite number of projective spaces of dimension t.

  It follows that there is at least one element

$$X' \in \mathcal{U} \setminus \bigcup_{\substack{(i_1 \ldots i_t) \in \\ \binom{\{i_1, \ldots, i_n\}}{t}}} \mathcal{H}_{i_1 \ldots i_t}$$

  This leads to the same conclusion as in the case in which $\mathbb{F}$ is finite.

$\square$

We are now ready to prove the following theorem.

Theorem 3.10. *Suppose $\mathbb{F}$ is a field with $|\mathbb{F}| \geq \binom{n}{t}$. Then $\Delta_t(n,k)$ is connected and isometrically embedded into $\Gamma(n,k)$.*

Proof. Take $X, Y \in \mathcal{C}_t(n,k)$ with $X \neq Y$. Put $\dim(X \cap Y) := k - d$. If $k - d \geq k - t$, then the thesis follows from Lemma 3.7. Suppose now $k - d < k - t$. By Lemma 3.9 there exists $X' \subseteq X$ with $X \cap Y \subseteq X'$ and $X' \in \mathcal{C}_t(n, k-1)$. Let Y' a $(k-d)+1$- dimensional subspace of Y containing $X \cap Y$. Put $T = \langle X', Y' \rangle$. Then $\dim(T) = k$; also

$$\dim(T \cap C_{i_1 \dots i_t}) \leq \dim(X' \cap C_{i_1 \dots i_t}) + 1 = k - 1 - t + 1 = k - t$$

for all $1 \leq i_1 < i_2 < \cdots < i_t \leq n$. In particular $T \in \mathcal{C}_t(n,k)$ and $\dim(X \cap T) = k - 1$, $\dim(Y \cap T) = k - d + 1$. By recursively applying this argument we get that $\Delta_t(n,k)$ is connected.

By construction, the length $d_t(X,Y)$ of the path joining X and Y in $\Delta_t(n,k)$ is at most $k - \dim(X \cap Y) := d(X,Y)$, i.e. $d_t(X,Y) \leq d(X,Y)$. Since $d_t(X,Y) \geq d(X,Y)$ in general, we get the thesis. $\qquad\square$

Remark 3.11. As a consequence of Theorem 3.10,

$$\mathrm{diam}(\Delta_t(n,k)) \leq \mathrm{diam}(\Gamma(n,k))$$

when $|\mathbb{F}| \geq \binom{n}{t}$. In the following section we shall show that these two diameters are actually the same.

3.2. Codes at maximum distance

In this section we do not assume any hypothesis on the parameters, apart that they have been chosen so that there exists at least one $[n,k]$-linear code with dual minimum distance at least t, i.e. $\mathcal{C}_t(n,k) \neq \emptyset$. Hence, the graph $\Delta_t(n,k)$ is not assumed to connected. This observation justifies the following.

Definition 3.12. We say that two codes in $X, Y \in \mathcal{C}_t(n,k)$ are *opposite* in $\Delta_t(n,k)$ if they belong to the same connected component $\Delta_t^X(n,k) = \Delta_t^Y(n,k)$ of $\Delta_t(n,k)$ and $d_t(X,Y) = \mathrm{diam}(\Delta_t^X(n,k))$.

Definition 3.13. We say that two k-dimensional subspaces $X, Y \subseteq V$ are *opposite* in $\Gamma(n,k)$ if $\dim(X \cap Y) = \max\{2k - n, 0\}$.

Observe that if $2k \leq n$, being opposite in $\Gamma(n,k)$ means $\dim(X + Y) = 2k$ (equivalently, $X \cap Y = \{0\}$), while if $2k > n$, it means $\dim(X + Y) = n$.

Lemma 3.14. *Suppose $t \leq k \leq n$, $\mathcal{C}_t(n,k) \neq \emptyset$ and $|\mathbb{F}| > \max\{k, n-k\}$. Then, for any code $C \in \mathcal{C}_t(n,k)$ there exists a code $D \in \mathcal{C}_t(n,k)$ which is equivalent and opposite to C in $\Gamma(n,k)$.*

Proof. Suppose $2k \leq n$ and let $C \in \mathcal{C}_t(n,k)$ with G as generator matrix. By elementary row operations on G, which leave C invariant, and column operations by means of $\rho \in \mathcal{M}(V)$ (see Definition 2.5), we can obtain a generator matrix

$$G' := \begin{pmatrix} I & A & B \end{pmatrix}$$

for an equivalent code $\rho(C) =: C' \in \mathcal{C}_t(n,k)$. Here I is the $k \times k$ identity matrix, A is a $k \times k$ matrix of rank t (since any t columns of a generator matrix of a code in $\mathcal{C}_t(n,k)$ are linearly independent) and B is a $k \times (n - 2k)$. Take $\lambda \in \mathbb{F} \setminus \{0\}$ and consider the matrix

$$G''_\lambda := \begin{pmatrix} \lambda A & I & B \end{pmatrix}.$$

Since G''_λ is obtained from G' by applying transformations induced by the monomial group $\mathcal{M}(V)$, the code C''_λ having G''_λ as generator matrix, is equivalent to C'. In particular $C''_\lambda \in \mathcal{C}_t(n, k)$.

We want to show that it is always possible to choose λ so that C''_λ and C' are in direct sum as subspaces of V, that is the matrix

$$\begin{pmatrix} I & A & B \\ \lambda A & I & B \end{pmatrix}$$

has rank $2k$. By elementary row operations, subtracting from the second block $\lambda \begin{pmatrix} I & A & B \end{pmatrix}$, we see that the rank of the matrix above is the same as the rank of

$$\begin{pmatrix} I & A & B \\ 0 & I - \lambda A^2 & (I - \lambda A)B \end{pmatrix}$$

In particular, this rank is definitely $2k$ if $\det(I - \lambda A^2) \neq 0$.

On the other hand $\det(I - \lambda A^2) = 0$ if and only if λ^{-1} is an eigenvalue of A^2. Since A^2 is a $k \times k$ matrix, of rank at most t, the number of its eigenvalues is at most $t \leq k < |\mathbb{F}|$. So, there is at least one $\lambda \in \mathbb{F}^*$ which is a non-null eigenvalue of A. For such a λ, the matrix G''_λ represents a code $C''_\lambda \in \mathcal{C}_t(n, k)$ such that $\dim(C' \cap C''_\lambda) = 0$, that is C' and C''_λ are opposite in $\Gamma(n, k)$.

Suppose now $2k > n$. Let $C \in \mathcal{C}_t(n, k)$ be a code with generator matrix G. Using elementary row operations on G and a monomial transformation $\rho \in \mathcal{M}(V)$, we can obtain a code $C' := \rho(C)$ equivalent to C whose generator matrix G' is in systematic form, i.e

$$G' = \begin{pmatrix} I_{n-k} & 0 & A_1 \\ 0 & I_{2k-n} & A_2 \end{pmatrix},$$

where A_1 and A_2 are suitable matrices of dimensions respectively $(n-k) \times (n-k)$ and $(2k-n) \times (n-k)$. For $\lambda \in \mathbb{F} \setminus \{0\}$, let now $G''_\lambda = \begin{pmatrix} \lambda A_1 & 0 & I_{n-k} \\ \lambda A_2 & I_{2k-n} & 0 \end{pmatrix}$ and C''_λ be the code with generator matrix G''_λ. The matrix G''_λ is obtained by permuting and multiplying some of the columns of the matrix G by a non-zero scalar λ; as such the code C''_λ generated by G''_λ is equivalent to C and C'; thus, $C''_\lambda \in \mathcal{C}_t(n, k)$ for all $\lambda \neq 0$.

Since $2k > n$, the codes C' and C''_λ are opposite if and only if $\dim(C' + C''_\lambda) = n$, that is to say the rank of the matrix $\bar{G}_\lambda = \begin{pmatrix} G' \\ G''_\lambda \end{pmatrix}$ is maximum and equal to n.

Explicitly, the structure of the matrix $\bar{G}_\lambda$ is

$$\bar{G}_\lambda = \begin{pmatrix} I_{n-k} & 0 & A_1 \\ 0 & I_{2k-n} & A_2 \\ \lambda A_1 & 0 & I_{n-k} \\ \lambda A_2 & I_{2k-n} & 0 \end{pmatrix}.$$

By using column operations we see that

$$\text{rank} \begin{pmatrix} I_{n-k} & 0 & A_1 \\ 0 & I_{2k-n} & A_2 \\ \lambda A_1 & 0 & I_{n-k} \\ \lambda A_2 & I_{2k-n} & 0 \end{pmatrix} \geq \text{rank} \begin{pmatrix} I_{n-k} & 0 & A_1 \\ 0 & I_{2k-n} & A_2 \\ \lambda A_1 & 0 & I_{n-k} \end{pmatrix} = \text{rank} \begin{pmatrix} I_{n-k} & 0 & A_1 \\ 0 & I_{2k-n} & 0 \\ \lambda A_1 & 0 & I_{n-k} \end{pmatrix} =$$

$$\text{rank} \begin{pmatrix} I_{n-k} & 0 & 0 \\ 0 & I_{2k-n} & 0 \\ \lambda A_1 & 0 & I_{n-k} - \lambda A_1^2 \end{pmatrix}.$$

So, if λ^{-1} is not an eigenvalue of A_1^2, we have that rank $(\bar{G}_\lambda) = n$. As the matrix A_1^2 has dimension $(n-k) \times (n-k)$, we have that A_1^2 has at most $n-k$ eigenvalues; as $|\mathbb{F}| > n - k$ there are some values of λ such that this rank is maximum; for these values of λ we get that $C', C_\lambda'' \in \mathcal{C}_t(n,k)$ are opposite.

Observe now that for any two codes $X, Y \in \mathcal{C}_t(n,k)$ and any $\eta \in \mathcal{M}(V)$, we have $d(X,Y) = d(\eta(X), \eta(Y))$ in $\Gamma(n,k)$. Since $C' = \rho(C)$ for $\rho \in \mathcal{M}(V)$, put $D = \rho^{-1}(C_\lambda'')$, where C_λ'' is the code constructed above. Then,

$$d(C, D) = d(\rho^{-1}(C'), \rho^{-1}(C_\lambda'')) = d(C', C_\lambda'').$$

It follows that C and D are codes in $\mathcal{C}_t(n,k)$ which are equivalent and opposite in $\Gamma(n,k)$. $\square$

Corollary 3.15. *Suppose $\mathcal{C}_t(n,k) \neq \emptyset$ and $\Delta_t(n,k)$ to be isometrically embedded into $\Gamma(n,k)$. If $|\mathbb{F}| > \max\{k, n-k\}$, then*

$$\mathrm{diam}(\Delta_t(n,k)) = \mathrm{diam}(\Gamma(n,k)).$$

Proof. By Lemma 3.14, there are at least two codes $X, Y \in \mathcal{C}_t(n,k)$ with $d(X,Y) = \mathrm{diam}(\Gamma(n,k))$. Since $\Delta_t(n,k)$ is isometrically embedded in $\Gamma(n,k)$ we also have $d_t(X,Y) = \mathrm{diam}(\Gamma(n,k))$. On the other hand, for any $X, Y \in \mathcal{C}_t(n,k)$,

$$d_t(X, Y) = d(X, Y) \leq \mathrm{diam}(\Gamma(n,k)).$$

It follows $\mathrm{diam}(\Delta_t(n,k)) = \mathrm{diam}(\Gamma(n,k))$. $\square$

Note that for $q \geq \binom{n}{t}$, the assumptions of Corollary 3.15 hold.

3.3. Proof of Theorem 1

Lemma 3.16. *If $\mathbb{F}$ is a field with $|\mathbb{F}| + 1 \geq n$ then $\mathcal{C}_t(n,k) \neq \emptyset$ for all t and k with $1 \leq t \leq k \leq n$.*

Proof. Note that for all t we have $\mathcal{C}_t(n,k) \subseteq \mathcal{C}_{t-1}(n,k)$. So, in order to get the lemma we just need to show that $\mathcal{C}_k(n,k) \neq \emptyset$ under our assumptions. It is well known that if $n \leq q+1$ for $\mathbb{F} := \mathbb{F}_q$ a finite field or order q, there exist $[n,k]$–linear MDS codes. Since the dual of an MDS code is MDS, it is immediate to see that any k-columns of the generator matrix of a $[n,k]$–MDS code C are independent. It follows that $C \in \mathcal{C}_k(n,k) \neq \emptyset$.

Suppose now $\mathbb{F}$ is an arbirary infinite field. There exist at least n distinct elements $a_1, a_2, \ldots, a_n \in \mathbb{F}$. Consider the matrix

$$G := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ a_1^2 & a_2^2 & \ldots & a_n^2 \\ \vdots & & & \vdots \\ a_1^{k-1} & a_2^{k-1} & \ldots & a_n^{k-1} \end{pmatrix}$$

Any $k \times k$ minor $M_{i_1 \ldots i_k}$ of G, comprising the columns $i_1, \ldots, i_k$ is a Vandermonde matrix with determinant

$$\det(M_{i_1 \ldots i_k}) = \prod_{1 \leq r < s \leq k} (a_{i_s} - a_{i_r}) \neq 0.$$

In particular, the code C with generator matrix G belongs to $\mathcal{C}_k(n,k)$ which is consequently non-empty. $\square$

Theorem 1 follows from Theorem 3.10, Corollary 3.15 and Lemma 3.16.

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