The mathematical possibilities of the music concept of mode from Ptolemy to Messiaen

The “mathematical possibilities” of the music concept of mode from Ptolemy to Messiaen

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Abstract

In the 5th century A.D. Boethius translates the Greek terms *tonos* and *tropos* into the Latin *modus*. The theory of Greek *tonoi* is thus connected to the practice of church modes, producing what has been defined “l’imbroglio des modes”. The main difficulties connected to this concept arise from the fact that both Greek and Medieval theorists presented the modes sometimes as different sequences obtained by linearizing the same cycle of music intervals, sometimes as pitch translations of the same sequence. After the “tonal” period, when the modes were reduced to two (our Major and Minor), in the 20th century the search for new modal possibilities was resumed, and led to the encounter with what Messiaen called “mathematical impossibilities”. In this paper, we try to get out of the “imbroglio” by using an extensional method, which consists (a) of building a model that contains all the mathematical possibilities related to the concept of mode and (b) identifying the subsets of the model that correspond to the various forms that this concept has historically taken on.

Introduction

> One point will attract our attention at the outset: the *charme* of impossibilities. [...] This *charme*, at once voluptuous and contemplative, resides particularly in certain mathematical impossibilities in the modal and rhythmic domains. Modes, which cannot be transposed beyond a certain number of transpositions, because one always falls again into the same notes, rhythms which cannot be use in retrograde, because in such a case one finds the same order of values again - these are two striking impossibilities. (Messiaen, 1944, Ch.1).

This explicit reference to mathematics, dealing with musical modes, is unusual (and, for this reason, famous). The concept of mode was marginalized during the long relationship between music and mathematics, which was predominantly focused on the concept of scale (“division of the canon”). The separation between scales and modes dates back to the beginning of Music theory. In the *Republic* (III, 398), Plato reports Damon's theory, which attributes to the various *harmoniai* (a term that will converge into the Latin *modus*) the power to influence human behaviour: the Lydian is soft and convivial, the Mixolydian mournful and lamenting, the Dorian and the Phrygian are suitable for warriors.[1] At the same time, Plato, in the *Timaeus* (34C-36D),

[1] Similar considerations were repeated until the Renaissance. Maurolicus, around the middle of the 16th century, states that: [Modus] Dorius somnolentia expellit; Hypodorius somnium inducit; Phrygius incitat, asperat;

tunes the "soul of the world" according to the numerical intervals of the Pythagorean diatonic scale, thus starting the conception of a mathematically harmonious cosmos. So, because of the enormous importance that these two Platonic dialogues had in the transmission of Greek music theory to later ages, the Theory of Scales took the path of mathematics and metaphysics, the Theory of Modes that of ethics and morals.

According to a traditional definition, "a *mode* is essentially a question of internal relationship of notes within a scale especially of the predominance of one of them over the others as a tonic, its predominance being established in any of all of a number of ways: e.g., frequent recurrence, its appearance in a prominent position as the first note or the last, the delaying of its expected occurrence by some kind of embellishment." (Winnington-Ingram, 1936, 2). In other words, a mode is defined by a note of the scale that is more important than the others (the *fundamental note* or *tonic*) which determines, among these, a hierarchy based on their relationship with it. In this definition, the hardest point to transfer in quantitative terms is the method for determining the fundamental note in a given melody. It is not simply a matter of identifying the note that appears most often, because the quality of such occurrences needs to be assessed. In general, the fundamental note is the final note of the melody (in the Middle Ages it was called *finalis*), but this cannot be considered an absolute rule. However, in order to make an analysis of the concept of mode, it is not necessary to determine a procedure to identify the fundamental note. It is sufficient to assume that such a note exists, and start the theory from this assumption.

Even with this simplification, the development of the concept of mode remains full of contradictions. The problem is that, from the Greeks to the present day, the same term has often indicated different objects, and the same object has been indicated by different terms. The eminent French composer and musicologist Jacques Chailley, in his *L'imbroglio des modes* (an Italian word borrowed from French), gives these methodological indications: to refer directly to the texts that are contemporary with the facts studied, and clearly define the terms.[2]

In the present paper, devoted to analyzing the forms that the musical concept of mode has historically taken on, we have tried to follow the above mentioned indications: we have restricted ourselves to the texts that we believe to be fundamental, and have devoted all of Section 1 to defining the terms. Two aspects of these definitions seem to be new to us. The first one is the systematic use of the mathematical concept of cycle, and the clear distinction between cycle and sequence. The second is the employment of a method that in Logic is called *extensional*. In somewhat rough terms, an extensional method is employed when, facing problematic concepts, rather than directly defining them, one first constructs a model that contains all the "mathematical possibilities" related to these concepts, and then finds, among these possibilities, the ones that satisfy the concepts themselves. The set of the objects of the model that satisfies a given concept is called *extension of the concept*. Thus, even if its deepest essence is not captured, the concept is delimited and "defined" in a non-ambiguous way. In a context where terms are much more numerous than possible concepts, the employment of this method seemed to us to be quite appropriate.

Hypophrygius blanditur, lascivit; Lydius laudat, consolatur; Hypolidius compatitur, laetificat; Mixolidius varius, querulus, audax; Hypomisulidius excitat. (*Musica*, 122).

[2] These two points express a desire which is common to anyone who faces the study of modes: make *tabula rasa* and start over again. "*Après avoir usé beaucoup d'aspirine, j'ai fini par envoyer le commentateurs au diable*". (*L'imbroglio*, 1960, Avant-Propos).

The paper is divided into three parts. In the first part we define the concepts of scale and mode, trying to describe the *standard form* that they have taken in the modern era and in the ambit of *Tonal music* (i.e., the kind of music that has characterized the so-called classical period of western music and that characterizes every type of contemporary popular music). This standard form has been the point of arrival of a process that had started at the time of the Greeks and continued, with intense interest, throughout the Middle Ages and the Renaissance. The mathematically more salient points of this process will be analyzed in Section 2. But the *standard form* was also the starting point from which new scalar and new modal forms came about from the end of the 19th century. We will study them in Section 3, mainly referring to Olivier Messiaen's work.

1 Standard definitions

The nomenclature of harmonic theory is, probably, the least "rational" among those of the theories with mathematical content, perhaps because no other theory came from the contribution of people whose interests were so different. It is therefore necessary to start by giving some definitions, in particular those of scale and mode. We will do this by remaining faithful to the meaning these concepts have today, so that the reader can identify them without difficulty. In the next section we will see if and how these definitions are consistent with the use that has been made of these concepts in the past.

Essential elements of our definitions are the use of the mathematical concept of cycle and the strict distinction between cycles and sequences. From now on, we will denote cycles (both of intervals and notes) between round brackets, and sequences between angle brackets. Therefore, given three distinct elements a, b, c, we will have $(a,b,c) = (b,c,a) = (c,a,b)$, but $<a,b,c> \neq <b,c,a> \neq <c,a,b>$.

The concept of mode presupposes that of scale, which in turn is based on that of *consonance*, and in particular on that of *consonance of octave*. Thanks to a discovery that tradition attributes to Pythagoras, we can quantitatively define this concept: two sounds are in the relation of octave if the ratio between the lengths of the strings that produce them is 2:1. Of course, the formation of the concept of octave, which is at the basis of music as we conceive it, precedes its quantification and derives from the feeling of homogeneity - of "consonance" - that two sounds at the interval of an octave produce. This feeling is so strong that the two sounds, albeit radically different in pitch, are perceived as a single sound. This equivalence is confirmed by the notation that, from the Middle Ages onwards, use the same letters A, B, C, ... (in Italian and French, La, Si, Do (Ut), ...) to denote sounds that are one or more octaves apart. In summary: the consonance of octave produces an equivalence relation between sounds and A, B, C ... are the names given to its equivalence classes.

Let us now consider the scales. Although, in nature, the variation of the pitch of a sound takes place mainly in a continuous manner, in music - or at least in western music - this variation has been treated mainly as a discrete phenomenon. Musical instruments, with the significant exception of strings, produce a finite set of pre-defined notes. The "scale problem" consists of choosing, within the *continuum* of the sounds, an appropriate selection of them. Thanks to the equivalence *modulo* octave, this selection can be circumscribed within a single octave, i.e., within the set of sounds whose boundary notes (or, more precisely, the length of their strings) are in the ratio of 2:1. In this way, a pattern is generated, which, repeating itself cyclically octave after octave, generates the sound space. Considering that:

(a) a *music interval* is defined by a ratio (between the lengths of the two strings that produce the sounds of the interval);

(b) the music operation of adding two musical intervals corresponds to the mathematical operation of multiplication of the corresponding ratios,[3]

we can give the following definition of scale:

•(1.1) A *scale* is a sequence of k numeric values $d_1,...,d_k$ which express k music intervals and whose product $d_1 \cdot ... \cdot d_k$ is 2 (the value corresponding to the octave). This sequence of intervals is used as a cycle ($d_1,...,d_k$), because the musical space we want to get can start from anywhere in the sequence and be generated by repeating the sequence itself an appropriate number, not necessarily integer, of times.

The emphasis on the concept of cycle is ours. This mathematical concept is not explicitly used in music theory. Sets of intervals, or notes, are always presented as a sequence, both when they actually have to represent a sequence, with a beginning and an end, and when it is understood that the sequence can repeat itself cyclically. On the other hand, we think that it is important, as we shall see later, to point out that, for the scales, the repeatability is essential while the starting point is not.

Historically, the most important scales were the *Pythagorean diatonic scale* (also known as *Timaeus's Scale*) characterized by the following cycle of 7 intervals:

$$(9/8,\ 9/8,\ 256/243,\ 9/8,\ 9/8,\ 9/8,\ 256/243)$$

and the *Natural Scale* (also known as *Zarlino's scale*, or *Just intonation*), whose corresponding cycle is

$$(9/8,\ 10/9\ ,16/15\ ,9/8\ ,10/9\ ,9/8\ ,16/15).$$

From the evolution of these two scales came the *equal-tempered diatonic scale*. This time, values $d_1 \cdot ... \cdot d_k$ cannot be expressed in fractional terms, since they are irrational numbers. The cycle is

$$(\sqrt[6]{2},\ \sqrt[6]{2},\ \sqrt[12]{2},\ \sqrt[6]{2},\ \sqrt[6]{2},\ \sqrt[6]{2},\ \sqrt[12]{2}).$$

If we set $\sqrt[6]{2} = t$ (*tempered tone*) and $\sqrt[12]{2} = s$ (*tempered semitone*) we obtain the cycle (t,t,s,t,t,t,s), which expresses the familiar succession of tones and semitones of our scale. From a mathematical point of view, the difference between Pythagorean, natural and equal-tempered scales is very profound: the first, but not the other two, only employs powers of 2 and 3, and was therefore the scale preferred by the Pythagorean school and the reference scale during the Middle Ages. The second one, but not the other two, has only ratios of the type $(n+1)/n$, the so called *epimoric ratios*, the only ones worthy of employment within a scale according to Ptolemy. The third one, but not the other two, uses irrational numbers, which is why it has been

[3] See for instance Bellissima (2011).

ostracized for a long time, but thanks to which the semitone is the exact half of the tone and hence sub-multiple of any interval of the scale.[4]

With regard to the sound, however, the differences between the three scales are rather small, and what we have termed as three scales are often defined as three *tunings* of the same scale: the *diatonic scale*. In this case, the quantitative concept is that of "tuning", while the term "scale" assumes a qualitative value, which in the diatonic case expresses the cycle (*t,t,s,t,t,t,s*), where the five tones are more or less equal to each other and the two semitones are more or less equal to half a tone. In this sense, the concept of scale is close to the concept of *genus* of the Greeks, a qualitative concept used to group the various tunings by similarity. On the other hand, the concept of tuning corresponds to the Greek concept of *division of the canon*. Nowadays, the term "tuning" can be misleading. It leads us to think about the pitch of the notes ("what is the frequency of A?"), while the divisions of the canon - from the one described in Euclid's *Sectio Canonis* to the innumerable ones described by Ptolemy - consisted of a series of ratios and did not give any indication of the pitch of the notes, i.e. on the initial state of the monochord (*canon*). Therefore, we will not use the term "tuning" and, according to the above-given definition, will attribute a quantitative value to the term "scale".

By dividing each tone of the equal-tempered diatonic scale into two equal parts, we obtain the *equal-temperate chromatic scale* (black and white keys of our pianos), which divides the octave into 12 equal intervals:

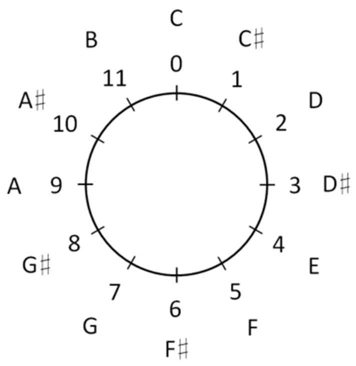
$$(\sqrt[12]{2}, \sqrt[12]{2}, \sqrt[12]{2}, \sqrt[12]{2}, \sqrt[12]{2}, \sqrt[12]{2}, \sqrt[12]{2}, \sqrt[12]{2}, \sqrt[12]{2}, \sqrt[12]{2}, \sqrt[12]{2}, \sqrt[12]{2}).$$

Thanks to the equality of all semitones, it is possible to obtain the equal-tempered diatonic scale starting from any point of the equal-tempered chromatic scale. This is the characteristic that has allowed this diatonic scale to obscure the other diatonic scales, until they disappeared.

In addition to the meaning of the term "scale" which has been described above (see (1.1)), this term has another meaning, a more familiar one, which is used when we speak of "scale of F major", or "scale of D minor". Leaving aside the adjective major and minor for a moment, the element of novelty is that in these cases there is a reference to the pitch of the sounds, expressed by the names F and D. So we will call *interval scales* those treated so far, and *scales of notes* (also named as *transposition scales* or *transposition keys*) these new scales, which we now define.

First of all, let us take a certain sound *x* and give it the name C (for our purposes, it is not important what the pitch of *x* actually is). This name will denote not only the sounds of the same pitch of *x* but, in agreement with what we have seen above, also the sounds that are separated from *x* by one or more octaves. Once the pitch of a note is fixed, the pitch of the other notes of the chromatic scale is obtained as a result. In virtue of the equality of its intervals, the 12 notes can be arranged on the "dodecaphonic circle" (also called "musical clock") as in Figure 1.

[4] Of course, $\sqrt[12]{2}$ is the square root of $\sqrt[6]{2}$, and not its half. As a result of the previous point (b), the terminology concerning the composition of intervals is antilogarithmic with respect to the algebra of their ratios.



Dodecaphonic Circle

Figure 1.

Let us now consider the *C major scale*. It consists of the sequence of notes C-D-E-F-G-A-B-C. We can denote this scale by the sequence of numbers

<0,2,4,5,7,9,11>

not indicating the highest note, that doubles the lowest one to the upper octave. Another example: the *E major scale*, whose notes are E-F#-G#-A-B-C#-D#-E, can be denoted by the sequence <4,6,8,9,11,1,3>, of *E minor scale*, whose notes are E-F#-G-A-B-C-D-E, by (**4**,6,7,9,11,0,2) (we consider minor scales in their *natural form*). Clearly, the notes are meant to grow in pitch (the numbers do not indicate individual sounds but classes of sounds *modulo* octaves).

Each scale of notes can be interpreted as a concrete representation of the more abstract concept of *tonality*. The scale of C major represents the *tonality of C major* in the sense that music written in this key uses, with due exceptions, the notes of scale of C major, and it has C as its fundamental note. As we anticipated in the Introduction, we do not need to define the concept of fundamental note. For us it is simply a highlighted point within a cycle or within a sequence of notes. We represent the tonality of C major by the cycle of notes

(**0**,2,4,5,7,9,11)

in which the fundamental note C is in bold. We use again the cyclic representation to mark the fact that we are considering a set of notes whose boundaries are not explicitly limited and which is not required to start or end with the fundamental note. So, the two expressions (**0**,2,4,5,7,9,11) and, for instance, (7,9,11,**0**,2,4,5) are equivalent.

We can repeat what we did for the tonality of C major for every other tonality. We will therefore represent the tonality of E major by (**4**,6,8,9,11,1,3), the tonality of C minor by (**0**,2,3,5,7,8,10), the tonality of E minor by (**4**,6,7,9,11,0,2).

The two previous examples of major scales, like all the major scales, have in common the fact that, starting with the fundamental note and performing an entire cycle, the intervals between consecutive notes form the sequence <*t,t,s,t,t,t,s*>. In the same way, the two previous examples of minor scales, like all the minor scales, have in common the fact that, starting with the fundamental note and performing an entire cycle, the intervals between consecutive notes form the sequence <*t,s,t,t,s,t,t*>. The two sequences are different, but they are *linearizations* of the same cycle of intervals, which is that of the diatonic scale ((*t,t,s,t,t,t,s*) = (*t,s,t,t,s,t,t*)). We prefer to express these linearizations, rather than through sequences, by means of cycles in which the interval on which the linearization begins, that is *the interval immediately after the* *fundamental note*, is in bold. In this way it becomes clearer that the concept of mode, in its modern conception, is not bound to a single octave form. So, we prefer to represent the Major mode not by <*t*,*t*,*s*,*t*,*t*,*t*,*s*>, but by

$$(\boldsymbol{t},t,s,t,t,t,s),$$

the Minor mode not by <*t*,*s*,*t*,*t*,*s*,*t*,*t*>, but by

$$(\boldsymbol{t},s,t,t,s,t,t),$$

or by the mathematically equivalent expression

$$(t,t,s,t,t,\boldsymbol{t},s).$$

The previous definitions show how the concept of mode is intermediate between the concept of interval scale and that of tonality. A mode is a cycle of intervals with a significant interval: it becomes an interval scale by deleting the reference to a particular interval, and becomes a tonality by passing from intervals to notes which form these intervals.

Switching from an interval scale to a mode by highlighting an interval of the cycle correspond to a linearization of the cycle itself, and the diatonic scale has the property that all the seven possible linearizations of its cycle, obtained by beginning on one of its intervals, are different from each other:[5]

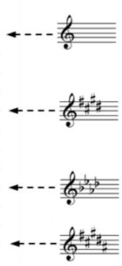
<*t*,*t*,*s*,*t*,*t*,*t*,*s*>, <*t*,*s*,*t*,*t*,*t*,*s*,*t*>, <*s*,*t*,*t*,*t*,*s*,*t*,*t*>, <*t*,*t*,*t*,*s*,*t*,*t*,*s*>, <*t*,*t*,*s*,*t*,*t*,*s*,*t*>, <*t*,*s*,*t*,*t*,*s*,*t*,*t*>, <*s*,*t*,*t*,*s*,*t*,*t*,*t*>.

Therefore, the diatonic scale has 7 different modes. Except the last one, all the others have a medieval correspondent (se Section 2) and can be denoted by using the ancient names. In order: *Ionian* (our *Major*), *Dorian*, *Phrygian*, *Lydian*, *Mixolydian*, *Aeolian* (our *Minor*). From the end of the Middle Ages, many of the previous modes progressively disappeared and, apart from occasional exceptions, only two survived: Major and Minor.

1.1 The model

We define a model to represent in an extensional way the concepts of diatonic scale, mode, tonality and scale of notes. The last concept, besides describing the more concrete object (a precise sequence of notes that can be performed on an instrument) is also the most particular, in the sense that all the other concepts are obtained from it by means of successive generalizations. The scales of notes are therefore the most suitable objects to become the elements of the model. Since each of the 7 possible modes of the diatonic scale can be used to produce 12 different scales of notes, the scales of notes that follow the diatonic pattern are 84.

Major	(Dorian)	(Phrygian)	(Lydian)	(Mixolydian)	Minor	---
<0,2,4,5,7,9,11>	<2,4,5,7,9,11,0>	<4,5,7,9,11,0,2>	<5,7,9,11,0,2,4>	<7,9,11,0,2,4,5>	<9,11,0,2,4,5,7>	<11,0,2,4,5,7,9>
<1,3,5,6,8,10,0>	<3,5,6,8,10,0,1>	<5,6,8,10,0,1,3>	<6,8,10,0,1,3,5>	<8,10,0,1,3,5,6>	<10,0,1,3,5,6,8>	<0,1,3,5,6,8,10>
<2,4,6,7,9,11,1>	<4,6,7,9,11,1,2>	<6,7,9,11,1,2,4>	<7,9,11,1,2,4,6>	<9,11,1,2,4,6,7>	<11,1,2,4,6,7,9>	<1,2,4,6,7,9,11>
<3,5,7,8,10,0,2>	<5,7,8,10,0,2,3>	<7,8,10,0,2,3,5>	<8,10,0,2,3,5,7>	<10,0,2,3,5,7,8>	<0,2,3,5,7,8,10>	<2,3,5,7,8,10,0>
<4,6,8,9,11,1,3>	<6,8,9,11,1,3,4>	<8,9,11,1,3,4,6>	<9,11,1,3,4,6,8>	<11,1,3,4,6,8,9>	<1,3,4,6,8,9,11>	<3,4,6,8,9,11,1>
<5,7,9,10,0,2,4>	<7,9,10,0,2,4,5>	<9,10,0,2,4,5,7>	<10,0,2,4,5,7,9>	<0,2,4,5,7,9,10>	<2,4,5,7,9,10,0>	<4,5,7,9,10,0,2>
<6,8,10,11,1,3,5>	<8,10,11,1,3,5,6>	<10,11,1,3,5,6,8>	<11,1,3,5,6,8,10>	<1,3,5,6,8,10,11>	<3,5,6,8,10,11,1>	<5,6,8,10,11,1,3>
<7,9,11,0,2,4,6>	<9,11,0,2,4,6,7>	<11,0,2,4,6,7,9>	<0,2,4,6,7,9,11>	<2,4,6,7,9,11,0>	<4,6,7,9,11,0,2>	<6,7,9,11,0,2,4>
<8,10,0,1,3,5,7>	<10,0,1,3,5,7,8>	<0,1,3,5,7,8,10>	<1,3,5,7,8,10,0>	<3,5,7,8,10,0,1>	<5,7,8,10,0,1,3>	<7,8,10,0,1,3,5>
<9,11,1,2,4,6,8>	<11,1,2,4,6,8,9>	<1,2,4,6,8,9,11>	<2,4,6,8,9,11,1>	<4,6,8,9,11,1,2>	<6,8,9,11,1,2,4>	<8,9,11,1,2,4,6>
<10,0,2,3,5,7,9>	<0,2,3,5,7,9,10>	<2,3,5,7,9,10,0>	<3,5,7,9,10,0,2>	<5,7,9,10,0,2,3>	<7,9,10,0,2,3,5>	<9,10,0,2,3,5,7>
<11,1,3,4,6,8,10>	<1,3,4,6,8,10,11>	<3,4,6,8,10,11,1>	<4,6,8,10,11,1,3>	<6,8,10,11,1,3,4>	<8,10,11,1,3,4,6>	<10,11,1,3,4,6,8>



Diatonic matrix

Figure 2.

[5] As we shall see in Section 3, in some scales of the 20th century this property will no longer be valid.

We arrange them in the form of a matrix 12×7 (the *Diatonic matrix*) as in Figure 2, following this criterion. The element $s_{1,1}$ is the C major scale <0,2,4,5,7,9,11> . Given an element $s_{i,j}$ and an integer k, the element $s_{i,j+k \text{ (mod 7)}}$ can be obtained by employing the same cycle of notes of $s_{i,j}$, rotating forwards each note of k positions if k is positive, backwards of $-k$ positions if k is negative. On the other hand, the element $s_{i+k \text{ (mod 12)},j}$ is obtained by rotating forwards every note of the sequence $s_{i,j}$ of k semitones if k is positive, backwards of $-k$ semitones if k is negative. Beginning on the C major scale is not essential because, having to do with sums *modulo* 7 or *modulo* 12, the matrix may be understood as a torus.

Now we determine on this model the extensions of the concepts defined above.

•(1,2) The extension of a tonality is a single element of the model, consisting of the scale whose notes belong to the cycle that expresses the tonality and which begins with the fundamental note of the tonality itself.

Therefore, the concept of scale of notes and that of tonality, though qualitatively different, have the same extension. As we will see in Sections 3, this coincidence can end if we consider interval scales different from the diatonic one.

•(1.3) The extension of a mode is a single column.

Indeed, all the elements of a given column produce the same succession of intervals, starting from each of the 12 possible notes (see Figure 2).

•(1.4) The extension of the diatonic scale is the whole matrix.

The model suggests the possible importance of a concept whose extension is a single row of the model. The elements that belong to the same row are the possible linearizations of the same *cycle of notes*. This concept is well present in musical theory, even if it is simply called *set of notes*. In practice, it corresponds to a certain disposition of the key alterations (in Figure 2 only a few of them have been represented). So:

•(1.5) The extension of a cycle of notes is a single row.

On a cycle of notes it is possible to obtain 7 different tonalities, one for each mode. However, the modern reduction of the number of modes has consequently reduced this number to 2. For example, from the cycle of notes (0,2,4,5,7,9,11) it is possible to obtain the tonality of C major (**0**,2,4,5,7,9,11) and its *relative minor*, the tonality of A minor (0,2,4,5,7,**9**,11). The model highlights the following fact:

•(1.6) The concept of tonality is the conjunction of the concepts of mode and cycle of notes.

Indeed, the extension of the concept of tonality is the intersection of the extensions of the concepts of mode and cycle of notes. The diagram in Figure 3 shows the relationships between the five concepts we have encountered.

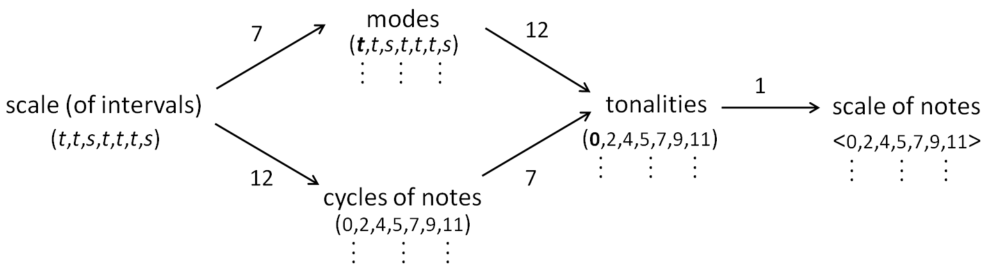


Figure 3.

2. Ancient modes

The most profound difference between the ancient and the modern forms of the concept of mode derives from the fact that the modern form refers to a potentially unlimited musical space, while the ancient forms refer to an explicitly limited space. A melody - which in the medieval period almost always came from the liturgical repertoire - had to develop (apart from sporadic exceptions) within the one-octave limit, and the mode of the melody depended on the type of this octave.

The explicit limitation of the span of the music space changes the structural aspects of the concept of mode. It is no longer enough to indicate the point from which the diatonic cycle of interval is linearized, but it is also necessary to know how the fundamental note is placed within the octave that represents the musical space of the melody. The fundamental note is thus characterized by a double parameter: its position within the cycle of intervals and that within the musical space.

The two-dimensional matrix defined in the previous section is no longer a suitable model for extensionally representing this kind of mode. Indeed:

i) a sequence of notes can no longer be understood as a pattern that, being repeated an indefinite number of times, generates the music space. This sequence (after having doubled the first note to the upper octave) must represent the musical space of the melody.

ii) the fundamental note does not necessarily coincide with the first note of the sequence and must therefore be indicated (for example, putting it in bold).

An element of the new model must therefore have a form of this kind:

<3,5,**6**,8,10,0,1>.

In the specific case, this sequence indicates that:

(i) The music space within which the melody has to develop is D#, F, F#, G#, A#, C, C#, D#.

(ii) The fundamental note is F#.

In each sequence, the position of the fundamental note may vary in 7 ways. Therefore, the new model can be built by placing vertically side by side 7 copies of the Diatonic matrix of Figure 2, which vary only for the notes in bold. In the first matrix, the notes in bold are the first of each sequence, in the second one the second notes, and so on (see Figure 4). Since the model is formed by matrices, it is convenient to orient the axes as in Figure 4 and begin the indexes from 1.

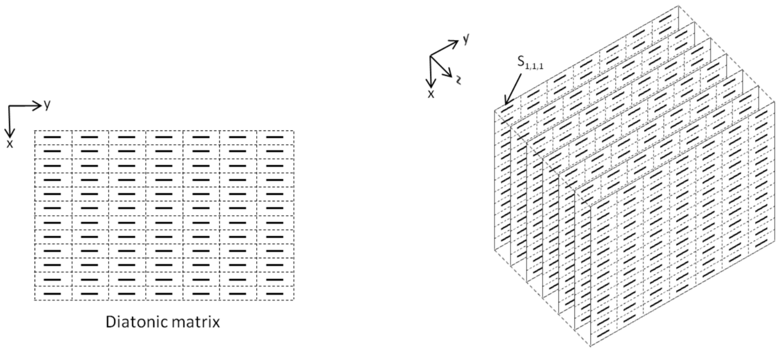


Figure 4.

From this construction it turns out that:

i) The sequences of notes belonging to the same plane *XY* have in common the same position of the fundamental note within the octave (see Figure 5a).

ii) The sequences of notes belonging to the same plane *YZ* have in common the same cycle of notes.

Therefore:

•(2.1) The extension of a cycle of notes, which in the two-dimensional model was a row, is now a horizontal plane (see Figure 5a, where only a few of them have been represented).

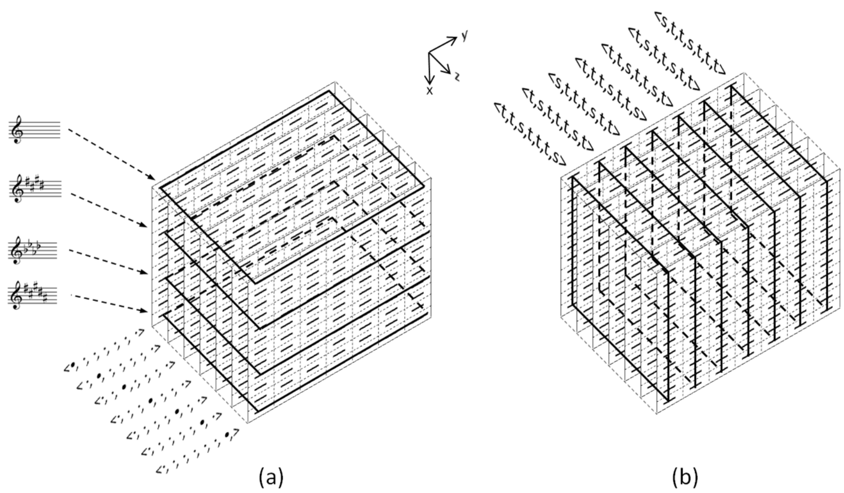


Figure 5.

iii) The sequences of notes belonging to the same plane *XZ* have in common the same sequence of intervals. In the two-dimensional model, in which it was understood that the fundamental note was the first of each sequence of notes, each sequence of intervals corresponded to a mode. If, on the other hand, the fundamental note is not necessarily the first one, then a sequence of intervals loses its modal value. A pure sequence is what Boethius calls *octave species*. So:

•(2.2) Each plane *XZ* is the extension of an octave species (see Figure 5b).

To facilitate the comparison with ancient modes, we determine the extension on this three-dimensional model of the standard concept of mode considered in Section 1. The concept has been defined as a cycle of intervals with one of them in bold, which corresponds to a linearization of the cycle itself. For example, the Major mode can be represented by any of the 7 expressions $(\boldsymbol{t},t,s,t,t,t,s)$, $(s,\boldsymbol{t},t,s,t,t,t)$, $(t,s,\boldsymbol{t},t,s,t,t)$, $(t,t,s,\boldsymbol{t},t,s,t)$, $(t,t,t,s,\boldsymbol{t},t,s)$, $(s,t,t,t,s,\boldsymbol{t},t,)$, $(t,s,t,t,t,s,\boldsymbol{t})$ which, being cycles, are *equivalent* to each other. However, if the music space is no longer potentially unlimited, then the reference to the cycles (meant as generative schemes) disappears, and the previous 7 expressions must be converted into the 7 sequences of intervals $<\boldsymbol{t},t,s,t,t,t,s>$, $<s,\boldsymbol{t},t,s,t,t,t>$,.$<t,s,\boldsymbol{t},t,s,t,t>$, $<t,t,s,\boldsymbol{t},t,s,t>$, $<t,t,t,s,\boldsymbol{t},t,s>$, $<s,t,t,t,s,\boldsymbol{t},t>$, $<t,s,t,t,t,s,\boldsymbol{t}>$, which are *different* from each other and represent the 7 different ways in which, in the Major mode, the fundamental note can be placed within the music space of an octave.

The extension of the Major mode on the three-dimensional model is therefore constituted by the seven columns represented in Figure 6 (note that, since the columns of the matrix are obtained by rotating the cycle forward, the position of each note proceeds backwards). Figure 6 also shows the extension of the Minor mode. The extension of the remaining modes has the same form: a set of 7 columns going backwards on different *XY* plans.

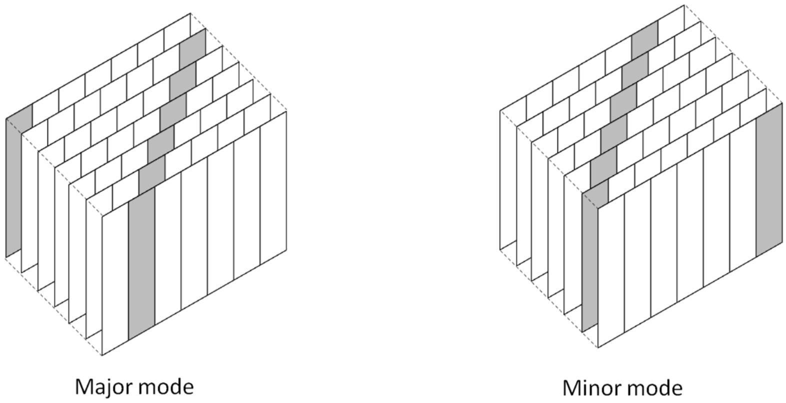


Figure 6.

2.1 Authentic and plagal modes

We start by considering the medieval modes as they were defined by Renaissance theorists Glareanus (*Dodecachordon*, 1547) and Zarlino (*Le Istituzioni Harmoniche*, 1561). This form is the last one that the medieval modes took on before taking the modern form described in Section 1. In History of Music texts they are usually presented as linearizations of the cycle (C, D, E, F, G, A, B), where each linearization occurs twice: one with the fundamental note coinciding with the first one, the other with the fundamental note coinciding with the fourth one.

Dorian:	from D to D	Hypodorian:	from A to A	fundamental note: D
Prygian:	from E to E	Hypoprygian:	from B to B	fundamental note: E
Lydian:	from F to F	Hypolydian:	from C to C	fundamental note: F
Mixolydian:	from G to G	Hypomixolydian:	from D to D	fundamental note: G
Minor (Aeolian):	from A to A	Hypoaeolian:	from E to E	fundamental note: A
Major (Ionian):	from C to C	Hypoionian:	from G to G	fundamental note: C

Table 1

The number of these modes is 12 (6+6), and not 14, because there are no modes whose fundamental note is B. The reason for this, as well as the fact that, for each linearization, only two fundamental notes are considered, can be explained in the following terms.

The medieval definition of the modes was not directly related to the linearizations of the octave. This interval was considered as the conjunction of a fifth (understood as a *pentachord*, i.e., five notes spanning an interval of fifth) and a fourth (understood as a *tetrachord*, that is, four notes spanning an interval of fourth). The fundamental note of the resulting octave had to be, in any case, the lowest note of the fifth. So, if the fourth was placed over the fifth, we had an *authentic* mode, i.e, an octave whose fundamental note was the lowest one. If the fourth was placed under the fifth, we had a *plagal* (or *hypo*) mode, i.e, an octave whose fundamental note was the fourth one. The reason why B could not be a fundamental note was that the interval between B and the fifth note over B, i.e. F, is not , as well as in all the other cases, an interval of fifth (i.e., three tones plus one semitone), but a *tritone*, a very dissonant interval that in the Middle Ages was called “diabulus in musica”.

In more ancient times, not only the linearization beginning on B, but also those beginning on A and C were not considered (which, paradoxically, correspond to the only modes that survived, our Minor and Major). Thus, the so-called *ecclesiastic modes* were 4+4: two of D, two of E, two of F and two of G. Each pair was denoted, in order, by the Greek-Latin names *protus*, *deuterus*, *tritus* and *tetrardus*.

We consider now the extension of the 12 medieval modes on our three-dimensional model. By taking literally the previous definitions, each mode corresponds to a sequence of seven notes with one of them being highlighted (see Table 2).

	authentic (*non-hypo*)	plagal (*hypo*)
Dorian:	<**2**,4,5,7,9,11,0>	<9,11,0,**2**,4,5,7>
Prygian:	<**4**,5,7,9,11,0,2>	<11,0,2,**4**,5,7,9>
Lydian:	<**5**,7,9,11,0,2,4>	<0,2,4,**5**,7,9,11>
Mixolydian:	<**7**,9,11,0,2,4,5>	<2,4,5,**7**,9,11,0>
Minor (Aeolian):	<**9**,11,0,2,4,5,7>	<4,5,7,**9**,11,0,2>
Major (Ionian):	<**0**,2,4,5,7,9,11>	<7,9,11,**0**,2,4,5>

Table 2

Each extension is therefore a single element of the model. Since white keys only have been considered, all these elements stand on the highest horizontal plane, i.e. the plane YZ $\{x = 1\}$. The 6 elements corresponding to authentic modes are on the line $\{x = 1, z = 1\}$; the 6 elements corresponding to plagal modes are on the line $\{x = 1, z = 4\}$ (see Figure 7).

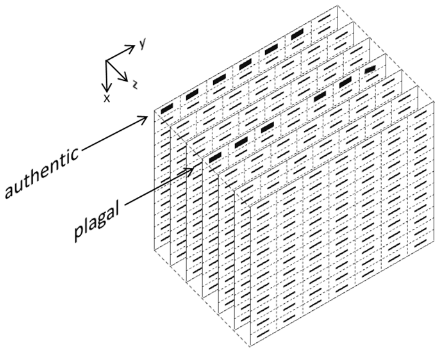


Figure 7.

Identifying a mode with a single sequence of notes is very restrictive, because it constrains each mode to a fixed pitch-range. However, the above definitions can be understood in a less rigid form. As we said, the musical space within which a melody had to be performed - and according to which its mode was determined - was the octave. However, this octave was inserted into a larger space, originally constituted by the *Greater Perfect System* of the Greeks, consisting of a double octave. During the Middle Ages this System had been progressively altered: the original fifteen notes had increased until they reached the number of twenty - the *Icosichordum* of Guido d'Arezzo - and the combinations of Greek letters by which the notes were indicated had been replaced by Latin letters, finally cyclically arranged *modulo* 7: A, B, ..., G , a, b, ..., g, aa, bb ... (this notation is still in use today, outside Italy and France).

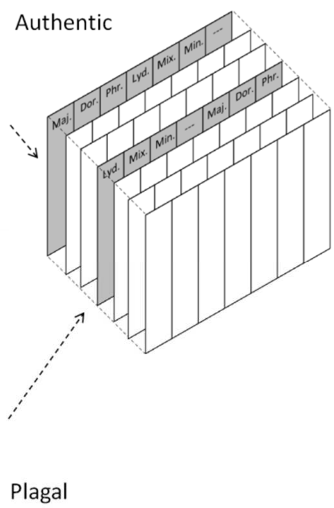
Having only white keys, each linearization of the diatonic cycle (*t,t,s,t,t,t,s*) could be inserted into the System only by starting with a particular note. For example, the authentic Dorian mode, whose corresponding sequence is <*t,s,t,t,t,s,t*>, could only begin on D, because, on the white keys, only by beginning on D is it possible to obtain this sequence. The fact that a given mode had to begin on a specific note was therefore a consequence of the objects that were available.

As already mentioned, medieval theorists, when defining the modes, did not directly refer to the notes as in our previous Table 1. The definition of the modes was an opportunity to give vent to the passion for combinatorial enumerations. Both the fourth, i.e., the tetrachord characterized by the cycle (*t,t,s*), and the fifth, i.e., the pentachord characterized by the cycle (*t,t,t,s*), admit as many distinct linearizations (*species*) as there are intervals, and thus one less than the number of the notes. (*Semper una minus species erit quam fuerint voces*. Boethius, *Inst. Mus*. IIII. 14). In defining the modes, theorists always referred to the three species of the fourth (<*t,t,s*>, <*t,s,t*>, <*s,t,t*>) and the four species of the fifth (<*t,t,t,s*>, <*t,t,s,t*>, <*t,s,t,t*>, <*s,t,t,t*>). For example, the authentic Dorian mode was presented as the conjunction of the third species of fifth with the second species of fourth. In this way we would have 12 (4×3) authentic modes and 12 (3×4) plagal modes. However, the only conjunctions that were taken into account were those that produced one of the possible linearizations of the octave. For example, the conjunction of <*t,s,t*> with <*t,t,t,s*> was not considered, because <*t,s,t,t,t,t,s*> is not a linearization of the diatonic cycle (*t,t,s,t,t,t,s*). Therefore, it would have been much easier to refer directly to the octave species; but the reference to the fifth and fourth, in addition to being more “combinatorial”, provided much more information about the structure of the octave.

This way of proceeding shows that the pitch of the notes, as well as their position within the Greater Perfect System, was not a primary interest but only a secondary consequence. There is further evidence in this regard. The Greater Perfect System was sometimes enriched with another kind of note corresponding to B flat. By adding this black key, each octave species allowed not one but two possible starting notes. Referring again to the authentic Dorian mode, the sequence <*t,s,t,t,t,s,t*> could begin not only on D (thus obtaining the scale D-E-F-G-A-B-C-D) but also from A (thus obtaining the scale A-B flat-C-D-E-F-G-A). These possibilities were exploited. Further evidence also comes from the hexachords of Guido d'Arezzo. A *hexachord* corresponds to a scale in the Major mode without its seventh note, i.e., to the sequence of intervals <*t,t,s,t,t*>. In this case, if we have both B and B flat, there are three possible starting notes: C, from which neither B nor B flat are employed, F, from which B flat is employed, and G, from which B is employed. All these possibilities were considered, and respectively named as *hexachordum naturalis*, *mollis* and *durus*. Moreover, the names Ut (Do), Re, Mi, Fa, Sol, La, that nowadays are used in Italy and France to denote pitch-classes, in the Middle Ages denoted the first, the second,..., the sixth note of the hexachord, which could change in pitch. Increasing the number of altered notes (black keys), also increased the possibilities to obtain a given sequence of intervals starting from different notes of the System, and two important theorists like Jaques de Liège (*Speculum Musice*) and Marchetto da Padova (*Lucidarium*) explicitly assert that these scales of notes were different applications of the same mode.

What has been said so far shows that it is a limitation to reduce the extension of a medieval mode to a single point of the model (that is, a single sequence of *notes* with one of them in bold). On the other hand, it is possible to consider this sequence as an example of one of *intervals* with one of them in bold. In this way, the extension of a mode on the model is not a single element but a column (see Figure 8).

Major	Dorian	Phrygian	Lydian	Mixolydian	Minor	---
<**0**,2,4,5,7,9,11>	<**2**,4,5,7,9,11,0>	<**4**,5,7,9,11,0,2>	<**5**,7,9,11,0,2,4>	<**7**,9,11,0,2,4,5>	<**9**,11,0,2,4,5,7>	<**11**,0,2,4,5,7,9>
<**1**,3,5,6,8,10,0>	<**3**,5,6,8,10,0,1>	<**5**,6,8,10,0,1,3>	<**6**,8,10,0,1,3,5>	<**8**,10,0,1,3,5,6>	<**10**,0,1,3,5,6,8>	<**0**,1,3,5,6,8,10>
<**2**,4,6,7,9,11,1>	<**4**,6,7,9,11,1,2>	<**6**,7,9,11,1,2,4>	<**7**,9,11,1,2,4,6>	<**9**,11,1,2,4,6,7>	<**11**,1,2,4,6,7,9>	<**1**,2,4,6,7,9,11>
<**3**,5,7,8,10,0,2>	<**5**,7,8,10,0,2,3>	<**7**,8,10,0,2,3,5>	<**8**,10,0,2,3,5,7>	<**10**,0,2,3,5,7,8>	<**0**,2,3,5,7,8,10>	<**2**,3,5,7,8,10,0>
<**4**,6,8,9,11,1,3>	<**6**,8,9,11,1,3,4>	<**8**,9,11,1,3,4,6>	<**9**,11,1,3,4,6,8>	<**11**,1,3,4,6,8,9>	<**1**,3,4,6,8,9,11>	<**3**,4,6,8,9,11,1>
<**5**,7,9,10,0,2,4>	<**7**,9,10,0,2,4,5>	<**9**,10,0,2,4,5,7>	<**10**,0,2,4,5,7,9>	<**0**,2,4,5,7,9,10>	<**2**,4,5,7,9,10,0>	<**4**,5,7,9,10,0,2>
<**6**,8,10,11,1,3,5>	<**8**,10,11,1,3,5,6>	<**10**,11,1,3,5,6,8>	<**11**,1,3,5,6,8,10>	<**1**,3,5,6,8,10,11>	<**3**,5,6,8,10,11,1>	<**5**,6,8,10,11,1,3>
<**7**,9,11,0,2,4,6>	<**9**,11,0,2,4,6,7>	<**11**,0,2,4,6,7,9>	<**0**,2,4,6,7,9,11>	<**2**,4,6,7,9,11,0>	<**4**,6,7,9,11,0,2>	<**6**,7,9,11,0,2,4>
<**8**,10,0,1,3,5,7>	<**10**,0,1,3,5,7,8>	<**0**,1,3,5,7,8,10>	<**1**,3,5,7,8,10,0>	<**3**,5,7,8,10,0,1>	<**5**,7,8,10,0,1,3>	<**7**,8,10,0,1,3,5>
<**9**,11,1,2,4,6,8>	<**11**,1,2,4,6,8,9>	<**1**,2,4,6,8,9,11>	<**2**,4,6,8,9,11,1>	<**4**,6,8,9,11,1,2>	<**6**,8,9,11,1,2,4>	<**8**,9,11,1,2,4,6>
<**10**,0,2,3,5,7,9>	<**0**,2,3,5,7,9,10>	<**2**,3,5,7,9,10,0>	<**3**,5,7,9,10,0,2>	<**5**,7,9,10,0,2,3>	<**7**,9,10,0,2,3,5>	<**9**,10,0,2,3,5,7>
<**11**,1,3,4,6,8,10>	<**1**,3,4,6,8,10,11>	<**3**,4,6,8,10,11,1>	<**4**,6,8,10,11,1,3>	<**6**,8,10,11,1,3,4>	<**8**,10,11,1,3,4,6>	<**10**,11,1,3,4,6,8>



Lydian	Mixolydian	Minor	---	Major	Dorian	Phrygian
<0,2,4,**5**,7,9,11>	<2,4,5,**7**,9,11,0>	<4,5,7,**9**,11,0,2>	<5,7,9,**11**,0,2,4>	<7,9,11,**0**,2,4,5>	<9,11,0,**2**,4,5,7>	<11,0,2,**4**,5,7,9>
<1,3,5,**6**,8,10,0>	<3,5,6,**8**,10,0,1>	<5,6,8,**10**,0,1,3>	<6,8,10,**0**,1,3,5>	<8,10,0,**1**,3,5,6>	<10,0,1,**3**,5,6,8>	<0,1,3,**5**,6,8,10>
<2,4,6,**7**,9,11,1>	<4,6,7,**9**,11,1,2>	<6,7,9,**11**,1,2,4>	<7,9,11,**1**,2,4,6>	<9,11,1,**2**,4,6,7>	<11,1,2,**4**,6,7,9>	<1,2,4,**6**,7,9,11>
<3,5,7,**8**,10,0,2>	<5,7,8,**10**,0,2,3>	<7,8,10,**0**,2,3,5>	<8,10,0,**2**,3,5,7>	<10,0,2,**3**,5,7,8>	<0,2,3,**5**,7,8,10>	<2,3,5,**7**,8,10,0>
<4,6,8,**9**,11,1,3>	<6,8,9,**11**,1,3,4>	<8,9,11,**1**,3,4,6>	<9,11,1,**3**,4,6,8>	<11,1,3,**4**,6,8,9>	<1,3,4,**6**,8,9,11>	<3,4,6,**8**,9,11,1>
<5,7,9,**10**,0,2,4>	<7,9,10,**0**,2,4,5>	<9,10,0,**2**,4,5,7>	<10,0,2,**4**,5,7,9>	<0,2,4,**5**,7,9,10>	<2,4,5,**7**,9,10,0>	<4,5,7,**9**,10,0,2>
<6,8,10,**11**,1,3,5>	<8,10,11,**1**,3,5,6>	<10,11,1,**3**,5,6,8>	<11,1,3,**5**,6,8,10>	<1,3,5,**6**,8,10,11>	<3,5,6,**8**,10,11,1>	<5,6,8,**10**,11,1,3>
<7,9,11,**0**,2,4,6>	<9,11,0,**2**,4,6,7>	<11,0,2,**4**,6,7,9>	<0,2,4,**6**,7,9,11>	<2,4,6,**7**,9,11,0>	<4,6,7,**9**,11,0,2>	<6,7,9,**11**,0,2,4>
<8,10,0,**1**,3,5,7>	<10,0,1,**3**,5,7,8>	<0,1,3,**5**,7,8,10>	<1,3,5,**7**,8,10,0>	<3,5,7,**8**,10,0,1>	<5,7,8,**10**,0,1,3>	<7,8,10,**0**,1,3,5>
<9,11,1,**2**,4,6,8>	<11,1,2,**4**,6,8,9>	<1,2,4,**6**,8,9,11>	<2,4,6,**8**,9,11,1>	<4,6,8,**9**,11,1,2>	<6,8,9,**11**,1,2,4>	<8,9,11,**1**,2,4,6>
<10,0,2,**3**,5,7,9>	<0,2,3,**5**,7,9,10>	<2,3,5,**7**,9,10,0>	<3,5,7,**9**,10,0,2>	<5,7,9,**10**,0,2,3>	<7,9,10,**0**,2,3,5>	<9,10,0,**2**,3,5,7>
<11,1,3,**4**,6,8,10>	<1,3,4,**6**,8,10,11>	<3,4,6,**8**,10,11,1>	<4,6,8,**10**,11,1,3>	<6,8,10,**11**,1,3,4>	<8,10,11,**1**,3,4,6>	<10,11,1,**3**,4,6,8>

Figure 8.

So, another form of the concept of mode has been identified. Though independent of pitch, it is however more restrictive than the modern standard form, as can be inferred from their different extensions on the model (compare Figure 6 and Figure 8). While the modern form corresponds to a *cycle* of intervals with one of them in bold, this new form corresponds to a sequence of intervals with one of them in bold. With respect to the modern form, the new form, which we call *I-mode* (where *I* stands for “intermediate”), has an extra constraint: it is bound to a single species of octave. While modern modes are 7, I-modes are 49, even if medieval theorists actually considered only 12 of them.

	authentic (non-*hypo*)	plagal (*hypo*)
Dorian:	<***t***,*s*,*t*,*t*,*t*,*s*,*t*>	<*t*,*s*,*t*,***t***,*s*,*t*,*t*>
Prygian:	<***s***,*t*,*t*,*t*,*s*,*t*,*t*>	<*s*,*t*,*t*,***s***,*t*,*t*,*t*>
Lydian:	<***t***,*t*,*t*,*s*,*t*,*t*,*s*>	<*t*,*t*,*s*,***t***,*t*,*t*,*s*>
Mixolydian:	<***t***,*t*,*s*,*t*,*t*,*s*,*t*>	<*t*,*s*,*t*,***t***,*t*,*s*,*t*>
Minor (Aeolian):	<***t***,*s*,*t*,*t*,*s*,*t*,*t*>	<*s*,*t*,*t*,***t***,*s*,*t*,*t*>
Major (Ionian):	<***t***,*t*,*s*,*t*,*t*,*t*,*s*>	<*t*,*t*,*s*,***t***,*t*,*s*,*t*>

Table 3

The extensions of the medieval modes involve only two *XY* planes of the model: the plane $\{z = 1\}$ for the authentic modes and the plane $\{z = 4\}$ for the plagal ones. The three-dimensional model in its entirety will be employed in the next section, dealing with the Greek modes.

2.2 The *tonoi* of Ptolemy

The names of the ecclesiastic modes refer to Greek or Asian populations. The medieval theorists believed they were reproducing the Greek *tonoi* or *tropoi* (terms that Boethius translated with the Latin term *modus*) but, as is well-known, this process is spoilt by many errors, due to the lack of clear definitions and the contradictory nature of the sources. Not only. As we shall see, no correspondence is possible, as it is not a simple problem of changing names, but one of changing properties.

The only author in which the discussion on *tonoi* involves numerical data is Ptolemy. In his *Harmonics*, II.15, he presents 35 (5×7) sequences of ratios that describe the same number of sequences of intervals spanning an octave. They concern 5 different diatonic scales. For each scale there are 7 tables containing sequences of ratios which describe the 7 possible linearizations of scale itself. Ptolemy denotes these linearizations as *tonoi*. The sequences are in descending order, in the Greek style. To avoid confusion, we transcribed them in reverse order, leaving everything else unchanged.

As an example, we consider the first of the 5 Ptolemaic scales, which is characterized by the cycle (22/21, 12/11, 7/6, 9/8, 28/27, 8/7, 9/8).[6] The choice of the scale is not important here; so we can identify the previous cycle with the usual diatonic cycle (*s*,*t*,*t*,*t*,*s*,*t*,*t*). If anything, the fact that this scale of Ptolemy uses four different types of tone (7/6, 8/7, 9/8 and 12/11) and two different types of semitones (22/21 and 28/27) facilitates our work: even if, as we know, each interval of the usual diatonic cycle can be identified by its position within the cycle itself, it is true that, if the intervals are different from each other, then it is easier to distinguish them.

The 7 tables related to this scale therefore represent the 7 possible linearizations of the cycle (22/21, 12/11, 7/6, 9/8, 28/27, 8/7, 9/8), and each of them is named in two different ways. For instance, the sequence <28/27, 8/7, 9/8, 22/21, 12/11, 7/6, 9/8>, i.e. <*s*,*t*,*t*,*s*,*t*,*t*,*t*>, is named as *Mixolydian tonos from nete*, but also *Lydian tonos from mese*; the sequence <22/21, 12/11, 7/6,

[6] It has been obtained from the “mixture” of the tetrachords <22/21, 12/11, 7/6> and <28/27, 8/7, 9/8> separated by an interval of a tone of 9/8.

9/8, 28/27, 8/7, 9/8>, i.e. <*s,t,t,t,s,t,t*>, is the *Dorian from nete* but also the *Mixolydian from mese*.

To explain the meaning of these names we have to return to the Greater Perfect System. Although it had the span of two octaves, it was not considered as the simple conjunction of two octaves. The module generating the System was not the octave but the tetrachord. The central octave of the System was obtained by two tetrachords "disjoined" by an interval of tone. The notes bounding these two tetrachords were named as *hypate*, *mese*, *paramese* and *nete*. Above this octave, and sharing its highest note ("in conjunction"), there was a further tetrachord; below it was another, also in conjunction with the lowest note of the central octave. The double octave range was completed by the addition of one more note at the bottom (it was named as *proslambanomenos*, which means "added"). Since the sequences of intervals were read downward, the lower octave of the System started "from *mese*", the central one "from *nete*" (see Figure 9).

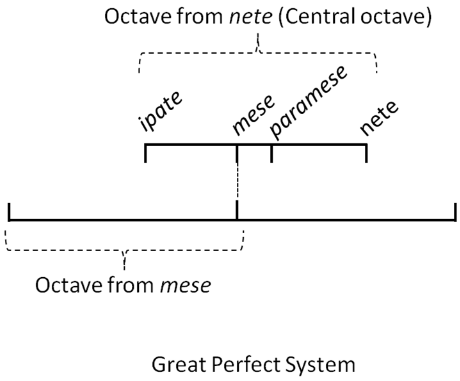


Figure 9.

To obtain the various *tonoi*, Ptolemy rotates the System. We can imagine it as a *tapis roulant*, dissected according to the intervals of a given diatonic scale. During rotation, intervals disappear from the top and the same intervals reappear at the bottom. The possible configurations of the system are seven (they correspond to the linearizations of a cycle of two diatonic octaves)[7]. For each of these, Ptolemy first describes the sequence of intervals of the lower octave ("from *mese*"). This sequence is equal to the sequence of the higher octave, and hence this description would be sufficient to characterize the whole System. However, in honour of the central octave around which the system was built, Ptolemy also describes the sequence of intervals of that octave ("from *nete*"), which is obviously different from the previous two (see Table 4).

Octave from *mese*	Octave from *nete*	intervals
	(central octave)	
Dorian	Hypodorian	<*t,s,t,t,s,t,t*>
Phrygian	Hypophrygian	<*t,t,s,t,t,s,t*>
Lydian	Hypolydian	<*t,t,t,s,t,t,s*>
Mixolydian	Dorian	<*s,t,t,t,s,t,t*>
Hypodorian	Phrygian	<*t,s,t,t,t,s,t*>

[7] In general, if an *n*-cycle has *m* distinct linearizations, the doubling of this cycle still has *m* linearizations, because after *n* steps the double cycle returns to the same configuration. Thus, the double octave cycle (*t,t,s,t,t,t,s,t,t,s,t,t,t,s*) has so many linearizations as the octave cycle (*t,t,s,t,t,t,s*).

Hypophrygian	Lydian	<*t*,*t*,*s*,*t*,*t*,*t*,*s*>
Hypolydian	Mixolydian	<*s*,*t*,*t*,*s*,*t*,*t*,*t* >

Table 4

In our model, the extensions of Ptolemaic sequences are the 7 planes *XZ* which, as we know, are the extensions of the *octave species* (see Figure 5b). So far, the *tonoi* have no modal value, since it is not specified what the fundamental note is. This value can be obtained from the distinction that Ptolemy, as already Aristoxenus before him, made between *tetic* names and *dynamic* names of the notes of the System (*Harm.* II, 6-11). Referring to the previous example, we can imagine the *tapis roulant* stopping on a point at which the central octave has the Dorian form (see Table 4). At this point, the names of the notes are written on the *tapis* and also on the floor, in correspondence. The names on the *tapis* are the dynamic ones (they characterize the notes one relatively to each other), the names on the floor are the *tetic* ones (they characterize the notes with respect to the boundaries of the System). Tetic and the dynamic names only coincide in the Dorian *tonos*. When the *tapis roulant* begins to rotate, other octave species are obtained, but tetic and the dynamic names do not fit anymore.

The distinction between the two classes of names seems to have the purpose of giving dynamic names a role that we could call “modal”. According to Ptolemy, the dynamic name of a note expresses in fact “the way in which it is related to something else” (*Harm.* II.5). At this point, the most natural candidate to begin the fundamental note is the dynamic *mese*. The term *mese* means “central”, and this centrality is not just a metric fact: as we have seen, it is the whole System that expands from its centre. (However, choosing another note, as long as it is dynamic, does not make a significant difference). The dotted line of Figure 11a describes the movement of the *dynamic mese*, which is in the center (*mese*) only in the Dorian *tonos*.

Of course, putting in bold the dynamic *mese* does not mean identifying its role with the modern one of the fundamental note. We can prove that this is false. In all Ptolemaic modes, dynamic *mese* is the note that precedes the tone of disjunction between the two tetrachords of type <*s*,*t*,*t*> that form the central octave. This tone corresponds to the third of the three consecutive tones of the diatonic cycle (in the scale we are considering, it corresponds to the fraction 9/8 between 7/6 and 28/27). The fact that this interval remains unchanged in all *tonoi* has an immediate, important consequence: all these *tonoi* belong to the same mode, in the modern sense. More precisely, Ptolemaic *tonoi* are the linearizations of the cycle (***t***,*s*,*t*,*t*,*s*,*t*,*t*) which define our Minor mode (see Table 5 and Figure 10).

VIII from *mese*	VIII from *nete*	intervals	line *X*
	(central octave)		
Dorian	Hypodorian	<***t***,*s*,*t*,*t*,*s*,*t*,*t*>	$y = 6$, $z = 6$
Phrygian	Hypophrygian	<*t*,***t***,*s*,*t*,*t*,*s*,*t*>	$y = 5$, $z = 6$
Lydian	Hypolydian	<*t*,*t*,***t***,*s*,*t*,*t*,*s*>	$y = 4$, $z = 6$
Mixolydian	Dorian	<*s*,*t*,*t*,***t***,*s*,*t*,*t*>	$y = 3$, $z = 6$
Hypodorian	Phrygian	<*t*,*s*,*t*,*t*,***t***,*s*,*t*>	$y = 2$, $z = 6$
Hypophrygian	Lydian	<*t*,*t*,*s*,*t*,*t*,***t***,*s*>	$y = 1$, $z = 6$
Hypolydian	Mixolydian	<*s*,*t*,*t*,*s*,*t*,*t*,***t*** >	$y = 7$, $z = 6$

Table 5

Now, the extension on the model of each *tonos* of Table 5 is a column: the *tonoi* are therefore I-modes. However, there is no correspondence between Ptolemaic *tonoi* of Table 5 and medieval modes of Table 3, and no correspondence is possible. While each pair of medieval modes (authentic and plagal) belongs to a different modern mode and has a modal value comparable to the modern one (compare Figures 8 and 2), all Ptolemaic *tonoi* belong to the same modern mode. The modal value of a *tonos* is therefore different from ours, and only concerns the position of the melodic centre within the octave (compare Figure 10, in which the names refer to the characteristic octave, and Figure 6).

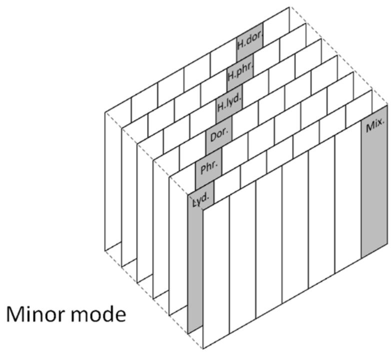


Figure 10.

Up to this point, no reference has been made to the pitch of the *tonoi*. If, referring to the *tapis roulant*, we assume that the floor is stationary, then all the *tonoi* are in the same pitch-range. But the presence of absolute movements is not clear, even because of lack of proper terminology. In *Harm.* II.10 Ptolemy, after criticizing the classifications of the *tonoi* present at his time, proposes his proper classification, in which the names of the *tonoi* are associated with the notes of a diatonic scale. Of course, if we consider this scale stationary, then the *tonoi* vary in pitch. But this point is not clear. There is, however, a well-decipherable numerical table from which it is possible to deduce that the pitch of the *tonoi* is not important. For each *tonos* of each scale Ptolemy also provides, in addition to a sequence of ratios, a sequence of rational numbers in those ratios. These numbers are always between 120 and 60, except for a deviation of a semitone (the maximum value that appears in the table is 124+(4/9), the minimum is 56+(136/729)). For example, the numbers corresponding to the characteristic octave of the Dorian *tonos* for the scale we have taken as a reference are 120, 114+(6/11), 105, 90, 80, 77+(1/7), 67+(1/2), 60. The values of all *tonoi* of all scales are grouped in ascending order in a single table (in Greek *canon*): the *canon canonion*, the *tabella tabellarum*. This table of tables makes sense only if all the *tonoi* can be put together in the same pitch range, and this presupposes that they have no pitch-constraints. In this case, the table of tables allows one to get all the notes on the same *canon* (this time in the sense of monochord).

2.3 From *tonoi* to *modi*: Boethius

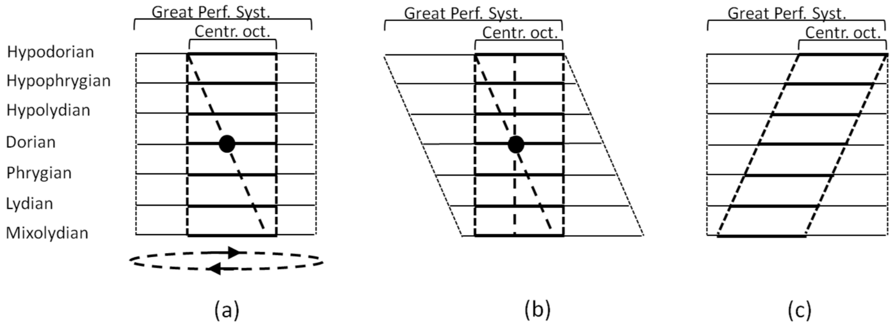


Figure 11.

Boethian *De Institutione Musica* and *De Institutione Arithmetica* are the texts through which the Greek harmonic theory passed to the Middle Ages. As we have already mentioned, he is the author who translated the Greek terms *tonos* and *tropos* with Latin *modus*:

> Ex diapason igitur consonantiae specibus existunt, qui appellantur modi, quos eosdem tropos vel tonos nominant. Sunt autem tropi constitutiones in totis vocum ordinibus vel gravitate vel acumine differentes. (*Inst. Mus.*, IV.15).

In the Boethian definition there is a reference to the octave species (*diapason consonantiae species*), which is well-present in Ptolemy, but there is also an unequivocal reference to the pitch of the sounds (*vel gravitate vel acumine differentes*), that we have not found in Ptolemy. However, an explicit reference to the pitch of the *tonoi* was well present in other Greek theorists. Aristides Quintilianus, Gaudentius and Alypius, three relatively unknown authors who lived between the time of Ptolemy and that of Boethius, describe the *tonoi* as simple transpositions in pitch of the Great Perfect System, whose sequence of intervals remains unchanged. Until only the octave species are considered, the modes cannot be more than 7; but, if the pitch is also considered, this limitation falls. Gaudentius lists 14 modes, Aristides Quintilianus and Alypius (and Cassiodorus, who follows Alypius) 15.

From the previous quote we can see that Boethius attempted to coordinate octave species and pitch. At first (*Inst. Mus.* IV.14) he described the octave species as simple rotations of the diatonic cycle (named as *prima*, *secunda*,..., *septima species*), but he did not associate them with modes, nor gave them the names of ancient populations. Modes and names of populations appear in the next chapter (*Inst. Mus.*, IV.15 *De modorum exordiis, in quo dispositio natarum per singulos modos ac voces*). A clear incongruence shows that he perhaps tries to put together the Ptolemaic cyclical motion of the double octave with its translation, in the manner of Alypius. The result is incoherent. In a cyclical motion each note of the System must ascend a tone (or a semitone) except the last one, which must take the place of the first one. Instead, Boethius raises the first note (*proslambanomenos*), while the second note (*hypate hypaton*) - and not the last one - takes the place of the first one; finally, all the other notes are raised. (*Si quis igitur proslambanomenon in acumen intendat tono hypatenque hypaton eodem tono adtenuet ceterasque omnes tono faciat acutiores, acutior toto ordo proveniet*). In other words, Boethius exchanges the first two notes and translates the others, leaving a hole where the third note was.

Apart from this incident, in *Inst. Mus.*,IV.15-16 Boethius associates the names of the modes with the rigid transpositions of the two octaves of the Greater Prefect System. With the addition of an eighth mode (the Hypermixolydian, addition that Boethius erroneously attribute to Ptolemy)[8], the musical space reaches the exact span of three octaves. A two-dimensional diagram (*Descriptio* II) describes the arrangement of the modes.

Up to this point the modes are simply changes in pitch of the same pattern, in the manner of Aristides Quintilianus, Gaudentius and Alypius (even if Boethian modes are only 8, instead of 14 or 15).[9] Nevertheless, in his definition, Boethius links modes and octave species to each other. The nature of this link is not explained, but *Descriptio* II - in which the modes are represented as in Figure 11b and in which only *tetic* notations appear - suggests a non-artificial connection that allows us to obtain the Ptolemaic *tonoi*. It is enough to take as a reference octave not an octave that moves with the double octave system, but a "*tetic* octave", i.e an octave whose boundaries are fixed (in metaphor, the boundaries are written on the floor, while the *tapis roulant*, without its rotating movement, is translated as a carpet). The choice of this "*tetic* octave" is forced: it must be the second of the three octaves: it is the only one which remains within the double octave system while it translates (see Figure 11b). But this octave is also the central octave of the Dorian *tonos*, and therefore the most significant one. If the System translates, the forms taken from the octave between these fixed boundaries are exactly those listed in Table 3, which correspond to Ptolemaic octave species. To reach a modal value, it is necessary to highlight a note. If we put in bold the *mese* of the moving System, i.e. the *dynamic mese*, then the correspondence with the Ptolemaic modes of Table 5 is complete (see Figure 10b, where the motion of the bold note is represented by the diagonal dotted line).

But if, on the other hand, we put in bold the *tetic mese* (the one written on the floor and represented by the vertical dotted line in Figure 10b) and, as usual, associate it with the note A (i.e. 9), then we leave the Ptolemaic *tonoi* and approach the medieval modes. What we obtain is Table 6, which is partly new, partly known.

	Central VIII	
Hypodorian	<4,6,7,**9**,11,0,2>	<*t*,*s*,*t*,***t***,*s*,*t*,*t*>
Hypophrygian	<4,6,8,**9**,11,1,2>	<*t*,*t*,*s*,***t***,*t*,*s*,*t*>
Hypolydian	<3,5,7,**9**,10,0,2>	<*t*,*t*,*t*,***s***,*t*,*t*,*s*>
Dorian	<4,5,7,**9**,11,0,2>	<*s*,*t*,*t*,***t***,*s*,*t*,*t*>
Phrygian	<4,6,7,**9**,11,1,2>	<*t*,*s*,*t*,***t***,*t*,*s*,*t*>
Lydian	<4,6,8,**9**,11,1,3>	<*t*,*t*,*s*,***t***,*t*,*t*,*s*>
Mixolydian	<4,5,7,**9**,10,0,2>	<*s*,*t*,*t*,***s***,*t*,*t*,*t*>

Table 6

[8] In *Harm.*, II, at the beginning of Ch. 10, Ptolemy says that the eighth *tonos* is "included superfluously". On the double octave of the System, an octave can be shifted eight times; however, its last position has the same sequence of intervals as the first one.

[9] Gaffurius, in his *Theorica Musice*, further simplifies Boethius and presents the modes as rigid translations of the octave over the double octave. He also lists eight modes and, like Boethius, attributes the eighth mode to Ptolemy.

The third column of the table, in which the octave species appear with an interval in bold, contains I-modes and coincides, apart from the names, with the column of the medieval plagal modes (see Table 3).[10] However, the sequences of notes in the first column are new, especially with regard to this fact: they all are included, with a small exception in the Hypolydian, between the same boundaries. The extension on our model of an I-mode is a column. For each I-mode, Table 6 and Table 2 choose two different sequences among those that belong to its extension. All the sequences in Table 6 have the same fundamental note (and therefore the same boundaries), while those in Table 2 have the same cycle of notes (the white keys) and, to achieve this, they must vary in pitch.

The transition from Ptolemaic *tonoi* to Boethian modes and then to medieval modes appears therefore as a series of simplifications on a set of two systems in relative motion. In the Ptolemaic case, the double octave rotates on itself dragging the fundamental note but not the boundaries of the characteristic octave, which remain fixed (Figure 11a). In the case of Boethius, the double octave simply translates (Figure 11b). Finally, in the medieval case, the double octave remains motionless: only the extremes of the characteristic octave move and, together with them, the fundamental note (see Figure 11c, not taking into account the names of the modes).

3. The *Scales of limited transpositions*

The process of reducing the number of modes, from the 12 of the Middle Ages to the Major-Minor dichotomy, coincides with the formation of the tonal conception of music, which is still alive today in every form of popular music. On the other hand, towards the end of the 19th century, the so-called "classical music" took other ways. The change was partly achieved by abandoning the diatonic scale. The composers of this period drew more freely from the set of notes of the chromatic scale, and some of them, such as Debussy, Scriabin, Schönberg and Messiaen, came to formalize the use of scales which were alternative to the diatonic one. These scales do not presuppose a different tuning of the instruments (on the same piano we can play Mozart and Schönberg), since they can be obtained as subsets of the equal-tempered scale.

The most famous of these new scales is the whole-tone scale, usually attributed to Debussy. It corresponds to the cycle of intervals (t,t,t,t,t,t), i.e. ($\sqrt[6]{2}$, $\sqrt[6]{2}$, $\sqrt[6]{2}$, $\sqrt[6]{2}$, $\sqrt[6]{2}$, $\sqrt[6]{2}$).[11] Of course, it makes no sense to talk about modes of this scale: a mode is a linearization of a cycle of intervals, and in this case all the linearizations are equal to each other.

The composer who studied more systematically the relationship between modes and new scales was Olivier Messiaen. This musician, in addition to the whole-tone scale, considers six other scales that he calls - in the next section we will try to understand why - "modes of limited transpositions", but that, in line with the literature in this regard, we will call *scales of limited transpositions*.

[10] The only match between names regards the Lydian mode. The Hypolydian has no medieval correspondent, since it would have B as a fundamental note.

[11] Schönberg, who devotes to this scale Chapter XX of his *Harmonielehre*, does not accept its attribution to Debussy. He asserts that "the whole-tone scale has occurred to all contemporary musicians quite of its own accord, as a natural consequence of the most recent events in music".

To study the differences between these scales and the diatonic one, we consider the *Second scale of Messiaen*, whose cycle of intervals is (*s*,*t*,*s*,*t*,*s*,*t*,*s*,*t*). Messiaen presents it as a scale of notes beginning on C (see Figure 12).

Figure 12.

It corresponds to the numerical sequence <0,1,3,4,6,7,9,10>. We take this sequence as the starting point for constructing the model. We can follow the procedure already employed for the diatonic case and described in Section 1.1. In fact, these new scales can be in a modern way understood as cycles, and thus we can come back again to a two-dimensional matrix. The matrix type, of course, changes. Since the number of the notes of the Second scale of Messiaen is not 7 but 8, its model is a 12×8 matrix (see Figure 13).

	m							
	<0,1,3,4,6,7,9,10>	<1,3,4,6,7,9,10,0>	<3,4,6,7,9,10,0,1>	<4,6,7,9,10,0,1,3>	<6,7,9,10,0,1,3,4>	<7,9,10,0,1,3,4,6>	<9,10,0,1,3,4,6,7>	<10,0,1,3,4,6,7,9>
c	<1,2,4,5,7,8,10,11>	<2,4,5,7,8,10,11,1>	<4,5,7,8,10,11,1,2>	<5,7,8,10,11,1,2,4>	<7,8,10,11,1,2,4,5>	<8,10,11,1,2,4,5,7>	<10,11,1,2,4,5,7,8>	<11,1,2,4,5,7,8,10>
	<2,3,5,6,8,9,11,0>	<3,5,6,8,9,11,0,2>	<5,6,8,9,11,0,2,3>	<6,8,9,11,0,2,3,5>	<8,9,11,0,2,3,5,6>	<9,11,0,2,3,5,6,8>	<11,0,2,3,5,6,8,9>	<0,2,3,5,6,8,9,11>
	<3,4,6,7,9,10,0,1>	<4,6,7,9,10,0,1,3>	<6,7,9,10,0,1,3,4>	<7,9,10,0,1,3,4,6>	<9,10,0,1,3,4,6,7>	<10,0,1,3,4,6,7,9>	<0,1,3,4,6,7,9,10>	<1,3,4,6,7,9,10,0>
	<4,5,7,8,10,11,1,2>	<5,7,8,10,11,1,2,4>	<7,8,10,11,1,2,4,5>	<8,10,11,1,2,4,5,7>	<10,11,1,2,4,5,7,8>	<11,1,2,4,5,7,8,10>	<1,2,4,5,7,8,10,11>	<2,4,5,7,8,10,11,1>
	<5,6,8,9,11,0,2,3>	<6,8,9,11,0,2,3,5>	<8,9,11,0,2,3,5,6>	<9,11,0,2,3,5,6,8>	<11,0,2,3,5,6,8,9>	<0,2,3,5,6,8,9,11>	<2,3,5,6,8,9,11,0>	<3,5,6,8,9,11,0,2>
	<6,7,9,10,0,1,3,4>	<7,9,10,0,1,3,4,6>	<9,10,0,1,3,4,6,7>	<10,0,1,3,4,6,7,9>	<0,1,3,4,6,7,9,10>	<1,3,4,6,7,9,10,0>	<3,4,6,7,9,10,0,1>	<4,6,7,9,10,0,1,3>
	<7,8,10,11,1,2,4,5>	<8,10,11,1,2,4,5,7>	<10,11,1,2,4,5,7,8>	<11,1,2,4,5,7,8,10>	<1,2,4,5,7,8,10,11>	<2,4,5,7,8,10,11,1>	<4,5,7,8,10,11,1,2>	<5,7,8,10,11,1,2,4>
	<8,9,11,0,2,3,5,6>	<9,11,0,2,3,5,6,8>	<11,0,2,3,5,6,8,9>	<0,2,3,5,6,8,9,11>	<2,3,5,6,8,9,11,0>	<3,5,6,8,9,11,0,2>	<5,6,8,9,11,0,2,3>	<6,8,9,11,0,2,3,5>
	<9,10,0,1,3,4,6,7>	<10,0,1,3,4,6,7,9>	<0,1,3,4,6,7,9,10>	<1,3,4,6,7,9,10,0>	<3,4,6,7,9,10,0,1>	<4,6,7,9,10,0,1,3>	<6,7,9,10,0,1,3,4>	<7,9,10,0,1,3,4,6>
	<10,11,1,2,4,5,7,8>	<11,1,2,4,5,7,8,10>	<1,2,4,5,7,8,10,11>	<2,4,5,7,8,10,11,1>	<4,5,7,8,10,11,1,2>	<5,7,8,10,11,1,2,4>	<7,8,10,11,1,2,4,5>	<8,10,11,1,2,4,5,7>
	<11,0,2,3,5,6,8,9>	<0,2,3,5,6,8,9,11>	<2,3,5,6,8,9,11,0>	<3,5,6,8,9,11,0,2>	<5,6,8,9,11,0,2,3>	<6,8,9,11,0,2,3,5>	<8,9,11,0,2,3,5,6>	<9,11,0,2,3,5,6,8>

Matrix of the Second scale of Messiaen

Figure 13.

Clearly, the same procedure can be used to make the model of each scale *S* obtained from the chromatic scale, i.e. whose intervals are multiples of the tempered semitone. If the intervals of the scale *S* are *n*, then the model will be a 12×*n* matrix. As we know, the choice of the scale of notes from which to start is not important, since the matrix is understood as a torus.

The elements belonging either to the same row or the same column are, by construction, always different from each other. In the diatonic case, all the elements of the matrix were also different from each other. On the other hand, in case of scales different from the diatonic one it may occur that some elements of the matrix are the same. The matrices for which this occurs (such as the one for the Second scale of Messiaen) are characterized by the following property:

•(3.1) The matrix for a scale *S* contains equal elements if and only if one (and therefore each) element of the matrix is the union of some cosets of a subgroup *H* of the group Z/12 different from {0}. (Clearly, *H* itself is considered as a coset).

In this case we say that *H generates S*. There holds that:

•(3.2) The interval scales whose matrix contains equal elements are precisely the scales of limited transpositions.

Indeed, the name derives from the fact that if the matrix for S has some equal elements then the "transposition keys" (i.e., the scales of notes) that can be obtained from S are less than $12\times n$, and thus in a "limited number". Let us denote the proper subgroups of $Z/12$ different from $\{0\}$ in this way:

$$H_2 = \{0,6\};\ H_3 = \{0,4,8\};\ H_4 = \{0,3,6,9\};\ H_6 = \{0,2,4,6,8,10\}.$$

If we consider the Second scale of Messiaen, we can see that the set of notes of the scale <0,1,3,4,6,7,9,10>, from which the matrix has been generated, is equal to $H_4 \cup (H_4 + 1)$. Each other scale of notes of the matrix has either the same set of notes, or the set $(H_4 +1) \cup (H_4 + 2)$, or $(H_4 +2) \cup (H_4 + 3)$, or $(H_4 +3) \cup H_4$.

Obviously, if a scale S is generated by a subgroup H of $Z/12$, then it can also be generated by every subgroup K of H. Let h be the cardinal of the maximum subgroup H that generates S (in the previous example, $h = |H_4| = 4$). This number plays a fundamental role in understanding the characteristics of S. First of all, h determines how equal elements appear in the matrix. If we set

$$m = n/h \quad \text{and} \quad c = 12/h,$$

where n is the number of the intervals of S, then the vector $(-c, m)$ connects the elements of the matrix which are equal to each other (see Figure 13). More precisely, given two elements $s_{x,y}$ and $s_{x',y'}$

•(3.3) $s_{x,y} = s_{x',y'}$ iff there exists a number k s.t. $x' = x - kc \pmod{12}$ and $y' = y + km \pmod{n}$.

From (3.3) it follows that two columns that are at a distance of m columns (or multiples of m) contain the same elements and are the rotation one of the other. The same happens for two rows that are at a distance of c rows (or multiples of c). This implies that:

•(3.4) The extension of a mode of a scale S consists of h (= n/m) columns of the matrix for S. Therefore, the number of modes of S is m.

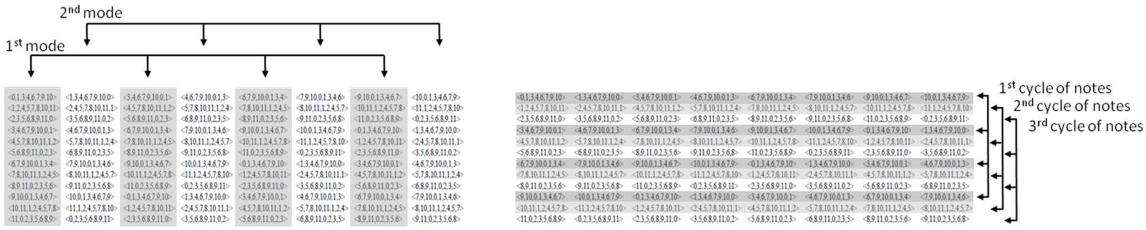
Then, the number of modes of the Second scale of Messiaen is $n/h = 8/4 = 2$ (See Figure 14a). The cycle (s,t,s,t,s,t,s,t) has in fact only 2 distinct linearizations: <s,t,s,t,s,t,s,t> and <t,s,t,s,t,s,t,s>. In Section 1 we expressed these linearizations by means of cycles in which the starting interval was in bold. In the case of scales of limited transpositions, however, there is not a single interval on which a given linearization can begin, but there are h. Therefore, the two modes of the Second scale of Messiaen can be expressed as ($\boldsymbol{s},t,\boldsymbol{s},t,\boldsymbol{s},t,\boldsymbol{s},t$) and ($\boldsymbol{t},s,\boldsymbol{t},s,\boldsymbol{t},s,\boldsymbol{t},s$).

•(3.5) The extension of a cycle of notes of a scale S consists of h (= $12/c$) rows of the matrix for S. Therefore, the number of cycles of notes of S is c.

Then, the number of cycles of notes of the Second scale of Messiaen is $12/h = 12/4 = 3$. They are (0,1,3,4,6,7,9,10), (1,2,4,5,7,8,10,11) and (2,3,5,6,8,9,11,0) (see Figure 14b).

Moreover, from (3.3) it follows also that :

•(3.6) Each scale of notes appears h times in the matrix.



(a) (b)

Figure 14.

In non-diatonic contexts, it does not make much sense to speak of tonality, at least in a classical sense. But it may be of some interest to see how its extensional correspondent evolves. In the diatonic case the concept of tonality is equivalent to the conjunction of the concepts of mode and cycle of notes (see (1.6)). We can generalize this fact and define *tonality* a concept whose extension is the intersection between the extension of a mode and that of a cycle of notes, two concepts that also retain a clear meaning in a non-diatonic context.

In the Diatonic matrix the extension of a tonality is reduced to a single scale of notes, since the extension of a mode is a single column and the extension of a cycle of notes is a single row. In a matrix for a scale of limited transpositions, this uniqueness is lost. Since the extension of a mode is composed of h columns (see 3.4) and that of a cycle of notes of h rows (see 3.5), their intersection has h^2 elements. Thus:

•(3.7) The extension of a tonality of a scale S consists of h^2 elements. Therefore, the number of tonalities of S is $12n/h^2$, i.e. $c{\cdot}m$ (the number of cycles of notes multiplied by the number of modes).

Finally, from (3.7) and (3.6) we obtain that

•(3.8) The number of scales of notes that belongs to a tonality is h.

Apart from the term “tonality”, which may be inappropriate, models for non-diatonic scales therefore reveal the significance of a concept that is intermediate between that of a scale of notes, on the one hand, and those of a mode and cycle of notes on the other (see Figure 15, which is the non-diatonic correspondent of Figure 3).

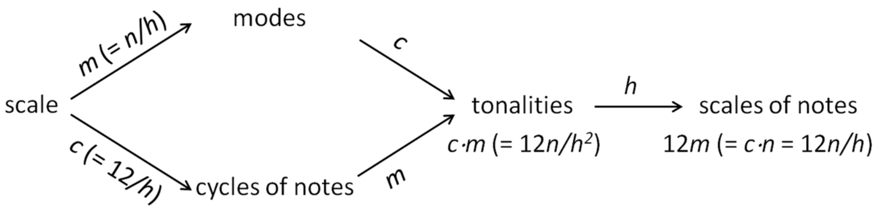


Figure 15.

From (3.7) it follows that the number of “tonalities” of the Second scale of Messiaen is $c{\cdot}m$ = $3{\cdot}2 = 6$. As we did in the case of the modes, we put in bold all the notes from which a given mode can be obtained. So the tonalities of the 1th mode are (**0**,1,**3**,4,**6**,7,**9**,10), (**1**,2,**4**,5,**7**,8,**10**,11), (**2**,3,**5**,6,**8**,9,**11**,0), and those of the 2th mode are (**1**,3,**4**,6,**7**,9,**10**,0), (**2**,4,**5**,7,**8**,10,**11**,1) and (**3**,5,**6**,8,**9**,11,**0**,2).

We think that the presence of several highlighted notes in the same "tonality" is the graphic demonstration of these words of Messiaen:

> [Modes of limited transpositions] are at once in the atmosphere of several tonalities, without polytonality, the composer being free to give predominance to one of the tonalities or to leave the tonal impression unsettled. (*Techn.*, Ch.XVI)

By (*) we can list all the scales of limited transpositions.[12]

		n		m	c	
1)	$(s,s,s,s,s,s,s,s,s,s,s,s)$	12	$Z/12$	1	1	
2)	(t,t,t,t,t,t)	6	H_6	1	2	M1
3)	$(3s,3s,3s,3s)$	4	H_4	1	3	
4)	(s,t,s,t,s,t,s,t)	8	$H_4 \cup H_4+1$	2	3	M2
5)	$(2t,2t,2t)$	3	H_3	1	4	
6)	$(s,3s,s,3s,s,3s)$	6	$H_3 \cup H_3+1$	2	4	
7)	(s,s,t,s,s,t,s,s,t)	9	$H_3 \cup H_3+1 \cup H_3+2$	3	4	M3
8)	$(3t,3t)$	2	H_2	1	6	
9)	$(s,5s,s,5s)$	4	$H_2 \cup H_2+1$	2	6	
10)	$(t,2t,t,2t)$	4	$H_2 \cup H_2+2$	2	6	
11)	$(s,s,2t,s,s,2t)$	6	$H_2 \cup H_2+1 \cup H_2+2$	3	6	M5
12)	$(s,t,3s,s,t,3s)$	6	$H_2 \cup H_2+1 \cup H_2+3$	3	6	
13)	$(s,3s,t,s,3s,t)$	6	$H_2 \cup H_2+1 \cup H_2+4$	3	6	
14)	$(s,s,s,3s,s,s,s,3s)$	8	$H_2 \cup H_2+1 \cup H_2+2 \cup H_2+3$	4	6	M4
15)	(s,s,t,t,s,s,t,t)	8	$H_2 \cup H_2+1 \cup H_2+2 \cup H_2+4$	4	6	M6
16)	(s,s,s,s,t,s,s,s,s,t)	10	$H_2 \cup H_2+1 \cup H_2+2 \cup H_2+3 \cup H_2+4$	5	6	M7

Table 7

In the third column there is the number of notes of S; in the fourth the subgroup of $Z/12$ and the cosets whose union forms a scale of notes of S beginning on C; in the fifth the number of the modes; in the sixth the number of the cycles of notes. Seven of these scale have been studied and used by Messiaen (in the last column of the table there is the number with which he denoted them).

3.1 Descriptions by two composers

The descriptions that some composers have made of the scales of limited transpositions represent a modern example of the difficulties related to a language whose terms are not clearly defined. In the diatonic field - within which musical terminology was formed - this lack of clarity (primarily, in our opinion, the lack of distinction between cycles and sequences) is

[12] This list may also be obtained by means of more general theorems. They are part of a field which has been developed since Polya's work in the forties of the last century, and goes under the name of Musical combinatorics (see for instance Polya-Read (1987), Read (1997) and Broué (2001)).

counterbalanced by the familiarity with the context. But away from this context, this compensation weakens. We consider two composers: Schönberg and Messiaen.

Schönberg presents the whole-tone scale, the second of Table 7, in this way.

> I have never overestimated the value of the whole-tone chords and the whole-tone scale. As enticing as it seemed, that two such scales (there can only be two, because the third would be a repetition of the first) could displace the twelve major and twelve minor scales in a manner similar to the displacement in their time of the eighty-four church modes, I nevertheless sensed immediately that the exclusive use of this scale would bring about an emasculation of expression, erasing all individuality. (*Harm.*, Ch.20).

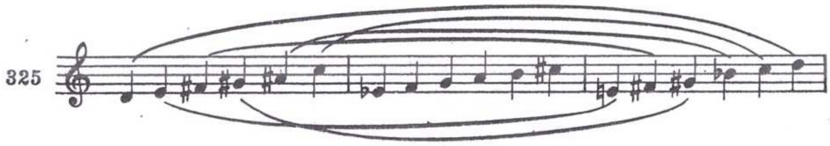


Figure 16.

We will comment on Schönberg's judgment later. Now let us just look at his description. The whole-tone scale is one. When Schönberg talks about "two of such scales (there can only be two, because the third would be a repetition of the first)" he is counting the cycles of notes. Figure 16, that he uses to explain this fact, shows a cyclic permutation, a one-one correspondence between the first and the third scale of notes on the pentagram. These 2 cycles correspond to the 12 cycles of the diatonic scale, and not to 12+12 scale of notes (major and minor). In turn, these 24 scales (12 for each of the two surviving modes) correspond to the 84 possible scales of notes of the diatonic scale (12 for each of the its 7 possible modes). 84 is in fact the cardinality of the Diatonic matrix (See Figure 2). But for no reason can these 84 objects be called "modes", and less than ever "church modes".

Messiaen's case is more significant, as he is the only composer who studies the scales of limited transpositions in a systematic way. Messiaen is a musician, but the subject he faces is in many respects purely mathematical. As we have seen in the introduction, he himself recognizes this nature, speaking of "mathematical impossibilities". What he presents us with, calling them "modes" and representing them "in the first transposition", are seven sequences of notes beginning on C. Let us see how he introduces his Second scale (the fourth of our list), which we analyzed in the previous section:

> One already finds traces of it in Sadko by Rimsky-Korsakov. Alexander Scriabin uses it in a more conscious fashion; Ravel and Strawinsky have used it transiently. But all that remains in the state of timid sketch. Mode 2 is transposable 3 times, as is the chord of diminished seventh. It is divided into 4 symmetrical groups of 3 notes each. (*Techn.*, Ch. XVI).

We begin with a marginal observation. Above and below the pentagrams with which Messiaen presents its scales appear horizontal brackets that delimit what he calls "symmetrical groups of notes" (see Figure 12). These clusters are indicated - and counted - considering both extremes. That of counting both extremes of a sequence of notes is an ancient custom in music theory, linked to the denomination of the intervals. An interval is called a *fourth*, a *fifth*, an *octave* if, on the diatonic scale, it spans 4, 5 or 8 notes, counting both extremes. This way of naming goes back to the Greeks, who called *diatessaron* (through 4) and *diapente* (through 5) the intervals of fourth and fifth. From a strictly musical point of view, this custom has a valid motivation: to be counted are not k abstract intervals but k+1 concrete notes. However, this way of counting creates difficulties when the intervals are composed between them: a fourth and a fifth form an octave and not a ninth, three fourths form a tenth and not a twelfth, and so on. The mathematical unsuitability of this way of counting is also clear in our case. Speaking of "4 symmetrical groups of 3 notes each" Messiaen hides the real structure of the scale. In his description number 12 (4 groups of 3 notes each) appears, while the number in question is 8 (4 groups of 2 notes each), that is number n of the intervals of the scale. In general, the number of "symmetrical groups of notes" of a scale of limited transpositions is h, i.e. the cardinal of the subgroup H of $Z/12$ generating the scale itself, and, only counting one boundary, each group is formed by n/h notes, as many as the cosets used in the scale construction. (Clearly, H itself is considered as a coset).

Coming back to Messiaen's terminology, we first analyze the meaning of the terms that he uses, and then we will try to justify this use. Regarding the meaning of the terms, two inconsistencies can be observed:

1) what he calls “modes” are actually interval scales;

2) what he calls “transpositions” are actually cycles of notes.

Let us start with the second inconsistency, which is the easiest to deal with. In the same way as Schönberg, Messiaen, in order to show the limited number of “transpositions” of his Second scale, observes that the fourth transposition has the same notes of the first, the fifth of the second, and so on. It is not the transposition scales that are also to be counted this time: they are *sequences* of notes, and the fourth sequence is different from the first for the simple reason that it begins on a different note. It is the cycles of notes that are counted: the first and the fourth transpositions use the same set, they are two linearizations of the same cycle. Therefore, it is not the transpositions, whose number is always 12, to be “limited”, but the cycles of notes.

Regarding the first inconsistency, i.e, the exchange between “scale” and “mode”, we first quote a passage in which, the subject “mode” is twice shifted to “scale” (the italics are ours).

> The *modes* of limited transpositions have nothing in common with the three great modal system of India, China and ancient Greece, no more than with the modes of plainchants (relatives of the Greek modes), all these *scales* being transposable twelve times. […]
>
> The first *mode* is divided into 6 groups of 2 notes each. It is transposable twice. It is the whole-tone *scale*.(*Techn.*, Ch. XVI)

Messiaen treats the terms “scale” and “mode” as synonyms, but they have always denoted distinct objects. A proof that the objects that he calls “modes” are actually scales is in the numbers. It is not an absolute proof, because the number of the objects counted by Messiaen is wrong in any case, but it is highly indicative: only if these objects are scales the error becomes “reasonable”. Messiaen affirms that the “modes” having a limited number of transpositions are 7. This number can rise to 11, as he finds 4 other “modes” that, oddly, he does not count because they are subsets (“truncations”) of “modes” already counted. From Table 6 it turns out that the modes (in our sense) having this property are 38: too far from 11. The scales are instead 16 and, excluding those that have two or three notes and are devoid of musical interest, their number comes close to 11. But, beyond the numbers, Messiaen's entire treatment makes sense only if what he denotes through a sequence of notes and calls “mode” is intended as a cycle and not as a sequence of intervals.

The reasons for this terminological choice (which, unlike the case of Schönberg, continues steadily) are difficult to understand. Messiaen seems to proceed in this way, not so infrequent in music theory: in order to indicate an object *x* which is abstract and without a clear definition, he indicates another object *y*, as concrete as possible, which is a particularization of *x*, and he leaves to the context the task of clarifying what specific properties of *y* should be considered and what not. In our case, *x* is the concept of interval scale. This concept is devoid of a general definition in music theory, because it is always accompanied by adjectives such as *diatonic*, *chromatic*, *pythagoric*, etc.. In order to present a new interval scale, a mathematician such as Ptolemy indicates a sequence of numeric ratios that denote intervals (that is, he defines *x*), while a musician such as Messiaen indicates a sequence of notes (this is his *y*). In this case, the artifice of using a concrete object instead of a more abstract one is successful. The reason for this success can be shown by Figure 15. If we read the diagram in the direction of the arrows, from left to right, from abstract to concrete, then each single object is in correspondence with many (a scale has many modes, a mode has many tonalities, and so on). But if we go backward, from right to left, from concrete to abstract, then all passages are univocal. A scale of notes comes from a single tonality, which in turn comes from only one mode and only one cycle of notes, which in turn come from a single interval scale. This is why Messiaen can denote in a non-ambiguous way the cycle of intervals (*s*,*t*,*s*,*t*,*s*,*t*,*s*,*t*) by the sequence of notes <0,1,3,4,6,7,9,10>: the reader easily understands this object, “plays it” and then, depending on the context, does not consider the pitch of the notes, or the interval from which the sequence begins. But why call the object *x* “mode”? The explanation cannot be in the fact that the concept of scale is devoid of a good definition in music theory: the concept of mode is also in the same situation, if not worse. The reason for this choice may be the following. In the previous section we showed that, given a scale, the numbers of its modes, its cycles of notes and its tonalities are interconnected, because all these numbers depend on cardinal *h* of the generating subgroup *H*. In particular:

•(3.9) A scale of *n* intervals has less than *n* modes iff it has less than 12 cycles of notes iff each mode has less than 12 cycles of notes.

For a mathematician, the most important element that distinguishes the scales of limited transpositions from the diatonic scale is the fact that they have less than *n* modes. He conceives these scales as cycles of *n* elements in which not all the *n* possible linearizations are different from each other. The name he would suggest would be “scales with a limited number of modes”. Not so a 20th century musician. For him, the most salient feature of the diatonic scale is not the possibility to have 7 distinct modes: from the 18th century onwards, the modes actually employed were only two, Major and Minor. The most salient feature is that each of these two modes develops on 12 different cycles of notes to which correspond 12 transposition keys (the experience of a modern music learner starts from the 12 Major scales and the 12 Minor scales). So, when this fact fails, the musician associates this limitation not with the interval scale, but the mode.

Conclusion

To the question "What is the sound of music employing scales with few modes like?" Messiaen's answer refers to the "*charme*, at once voluptuous and contemplative, that resides particularly in certain mathematical impossibilities" (see the passage quoted in the Introduction). Schönberg's response changed over time. In the *Harmonielehre*, written around 1910, he argued that "the exclusive use of this scale [the whole-tone scale] would bring about an emasculation of expression, erasing all individuality" (see the passage quoted in the previous section). But he had to radically change opinion, at least in appearance. His previous judgment concerns Scale Number 2 of our Table 7. A few years later he was to exclusively employ Scale Number 1 of the same table. In the twenties of the 20th century he began to develop the "*Method of Composing with Twelve Tones Which are Related Only with One Another*". This method consists primarily of the constant use of a set of 12 different notes (the "series"). All 12 notes must have the same importance: in order to ensure this equality, the method imposes that no note is repeated before the eleven other notes of the chromatic scale have been used. Clearly, such a method necessarily implies the total disappearance of the concept of mode. A mode is based on the predominance of one note over the others, predominance obtained thanks to the fact that the diatonic scale is a proper and "irregular" subset of the chromatic scale. But Schönberg's scale is the chromatic scale itself. Its matrix is a 12×12 matrix, ad it is symmetric. On it, the extensions of scale, mode, cycle of notes and tonality coincide with each other, all of them being equal to the whole matrix. Using for the last time the expression of Messiaen, any modal distinction is "mathematically impossible".

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