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Abstract

We study the mean curvature motion of a droplet flowing by mean curvature on a horizontal hyperplane with a possibly nonconstant prescribed contact angle. Using the solutions constructed as a limit of an approximation algorithm of Almgren-Taylor-Wang and Luckhaus-Sturzenhecker, we show the existence of a weak evolution, and its compatibility with a distributional solution. We also prove various comparison results.

Résumé

Nous étudions le mouvement par courbure moyenne d'une goutte qui glisse par courbure moyenne sur un hyperplan horizontal avec un angle de contact prescrit éventuellement non constant. En utilisant les solutions construites comme limites d'un algorithme d'approximation dû à Almgren, Taylor et Wang et Luckhaus et Sturzenhecker, nous montrons l'existence d'une évolution faible, et sa compatibilité avec une solution au sens des distribution. Nous démontrons également plusieurs résultats de comparaison.

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1. Introduction

Historically, capillarity problems attracted attention because of their applications in physics, for instance in the study of wetting phenomena [18, 22], energy minimizing drops and their

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adhesion properties [1, 17, 20, 48], as well as because of their connections with minimal surfaces, see e.g. [14, 29] and references therein.

In this paper we are interested in the study of the evolution of a droplet flowing on a horizontal hyperplane under curvature driven forces with a prescribed (possibly nonconstant) contact angle. Although there are results in the literature describing the static and dynamic behaviours of droplets [2, 12, 49], not too much seems to be known concerning their mean curvature motion. Various results have been obtained for mean curvature flow of hypersurfaces with Dirichlet boundary conditions [35, 46, 47, 52] and zero-Neumann boundary condition [3, 34, 38, 51]. It is also worthwhile to recall that, when the contact angle is constant, the evolution is related to the so-called mean curvature flow of surface clusters, also called space partitions (networks, in the plane): in two dimensions local well-posedness has been shown in [16], and authors of [39] derived global existence of the motion of grain boundaries close to an equilibrium configuration. See also [43] for related results. In higher space dimensions short time existence for symmetric partitions of space into three phases with graph-type interfaces has been derived in [30, 31]. Very recently, authors of [26] have shown short time existence of the mean curvature flow of three surface clusters.

If we describe the evolving droplet by a set $E(t) \subset \Omega$, $t \ge 0$ the time, where $\Omega = \mathbb{R}^n \times (0, +\infty)$ is the upper half-space in $\mathbb{R}^{n+1}$, the evolution problem we are interested in reads as

$$V = H_{E(t)} \qquad \text{on } \Omega \cap \partial E(t), \tag{1.1}$$

where V is the normal velocity and $H_{E(t)}$ is the mean curvature of $\partial E(t)$, supplied with the contact angle condition on the contact set (the boundary of the wetted area):

$$\nu_{E(t)} \cdot e_{n+1} = \beta \qquad \text{on } \overline{\Omega \cap \partial E(t)} \cap \partial\Omega, \tag{1.2}$$

where $\nu_{E(t)}$ is the outer unit normal to $\overline{\Omega \cap \partial E(t)}$ at $\partial\Omega$, and $\beta : \partial\Omega \to [-1, 1]$ is the cosine of the prescribed contact angle. We do not allow $\partial E(t)$ to be tangent to $\partial\Omega$, i.e. we suppose $|\beta| \le 1 - 2\kappa$ on $\partial\Omega$ for some $\kappa \in (0, \frac{1}{2}]$. Following [38], in Appendix B we show local well-posedness of (1.1)-(1.2).

Short time existence describes the motion only up to the first singularity time. In order to continue the flow through singularities one needs a notion of weak solution. Concerning the case without boundary, there are various notions of generalized solutions, such as Brakke's varifold-solution [15], the viscosity solution (see [32] and references therein), the Almgren-Taylor-Wang [4] and Luckhaus-Sturzenhecker [41] solution, the minimal barrier solution (see [10] and references therein); see also [27, 37] for other different approaches.

In the present paper we want to adapt the scheme proposed in [4, 41], and later extended to the notions of *minimizing movement* and *generalized minimizing movement* by De Giorgi [25] (see also [6, 8]) to solve (1.1)-(1.2). Let us recall the definition.

Definition 1.1. *Let S be a topological space, $F : S \times S \times [1, +\infty) \times \mathbb{Z} \to [-\infty, +\infty]$ be a functional and $u : [0, +\infty) \to S$. We say that u is a generalized minimizing movement associated to F, S (shortly GMM) starting from $a \in S$ and we write $u \in GMM(F, S, \mathbb{Z}, a)$, if there exist $w : [1, +\infty) \times \mathbb{Z} \to S$ and a diverging sequence $\{\lambda_j\}$ such that*

$$\lim_{j \to +\infty} w(\lambda_j, [\lambda_j t]) = u(t) \quad \textit{for any } t \ge 0,$$

and the functions $w(\lambda, k)$, $\lambda \ge 1$, $k \in \mathbb{Z}$, are defined inductively as $w(\lambda, k) = a$ for $k \le 0$ and

$$F(w(\lambda, k+1), w(\lambda, k), \lambda, k) = \min_{s \in S} F(s, w(\lambda, k), \lambda, k) \qquad \forall k \ge 0.$$

If $GMM(F, S, \mathbb{Z}, a)$ consists of a unique element it is called a minimizing movement starting from a.

In the sequel, we take $S = BV(\Omega, \{0,1\})$, $F = \mathcal{A}_\beta : BV(\Omega, \{0,1\}) \times BV(\Omega, \{0,1\}) \times [1, +\infty) \times \mathbb{Z} \to (-\infty, +\infty]$ defined by

$$\mathcal{A}_\beta(E, E_0, \lambda) = C_\beta(E, \Omega) + \lambda \int_{E \Delta E_0} d_{E_0}\, dx,$$

where $E_0 \in BV(\Omega, \{0,1\})$ is the initial set, d_{E_0} is the distance to $\Omega \cap \partial E_0$ and

$$C_\beta(E, \Omega) = P(E, \Omega) - \int_{\partial\Omega} \beta \chi_E \, d\mathcal{H}^n$$

is the capillary functional. If $\Omega = \mathbb{R}^{n+1}$ (hence when the term $\int_{\partial\Omega} \beta\chi_E \, d\mathcal{H}^n$ is not present), the weak evolution (GMM) has been studied in [4] and [41], see also [44] for the Dirichlet case. Further, when no ambiguity appears we use $GMM(E_0)$ to denote a GMM starting from $E_0 \in BV(\Omega, \{0,1\})$.

After setting in Section 2 the notation, and some properties of finite perimeter sets, in Section 3 we study the functional $C_\beta(\cdot, \Omega)$ and its level-set counterpart $\mathsf{C}_\beta(\cdot, \Omega)$, including lower semicontinuity and coercivity, which will be useful in Section 6. In particular, the map $E \mapsto \mathcal{A}_\beta(E, E_0, \lambda)$ is $L^1(\Omega)$-lower semicontinuous if and only if $\|\beta\|_\infty \le 1$ (Lemma 3.5). Although we can also establish the coercivity of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ (Proposition 3.2), compactness theorems in BV cannot be applied because of the unboundedness of Ω. However, in Theorem 4.1 we prove that if $E_0 \in BV(\Omega, \{0,1\})$ is bounded and $\|\beta\|_\infty < 1$, then $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ has a minimizer in $BV(\Omega, \{0,1\})$, and any minimizer is bounded. In Lemma 4.6 we study the behaviour of minimizers as $\lambda \to +\infty$. In Proposition 4.4 we show existence of constrained minimizers of $C_\beta(\cdot, \Omega)$, which will be used in the proof of existence of GMMs and in comparison principles. In Appendix A we need to generalize such existence and uniform boundedness results to minimizers of functionals of type $C_\beta(\cdot, \Omega) + \mathcal{V}$ under suitable hypotheses on $\mathcal{V}$.

In Section 5 we study the regularity of minimizers of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ (Theorem 5.3). We point out the uniform density estimates for minimizers of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ and constrained minimizers of $C_\beta(\cdot, \Omega)$ (Theorem 5.1 and Proposition 5.8), which are the main ingredients in the existence proof of GMMs (Section 7), and in the proof of coincidence with distributional solutions (Section 8).

In Section 6 we prove the following comparison principle for minimizers of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ (Theorem 6.1): *if E_0, F_0 are bounded, $E_0 \subseteq F_0$, $\|\beta_1\|_\infty, \|\beta_2\|_\infty < 1$ and $\beta_1 \le \beta_2$, then*

a) there exists a minimizer F_λ^* of $\mathcal{A}_{\beta_2}(\cdot, F_0, \lambda)$ containing any minimizer of $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$;

b) there exists a minimizer $E_{\lambda *}$ of $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$ contained in any minimizer of $\mathcal{A}_{\beta_2}(\cdot, F_0, \lambda)$.

If in addition $\mathrm{dist}(\Omega \cap \partial E_0, \Omega \cap \partial F_0) > 0$, *then all minimizers E_λ and F_λ of $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$ and $\mathcal{A}_{\beta_2}(\cdot, F_0, \lambda)$ respectively, satisfy $E_\lambda \subseteq F_\lambda$.* As a corollary, we show that if E^+ is a bounded minimizer of $C_\beta(\cdot, \Omega)$ in the collection $\mathcal{E}(E^+)$ of all finite perimeter sets containing E^+, and if $\|\beta\|_\infty < 1$, then for any $E_0 \subseteq E^+$, any minimizer E_λ of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ satisfies $E_\lambda \subseteq \overline{E^+}$ (Proposition 6.11 b)).

In Section 7 we apply the scheme in Definition 1.1 to the functional $\mathcal{A}_\beta(\cdot, E_0, \lambda)$: as in [41, 45] we build a locally $\frac{1}{2}$-Hölder continuous generalized minimizing movement $t \in [0, +\infty) \mapsto E(t) \in BV(\Omega, \{0,1\})$ starting from a bounded set $E_0 \in BV(\Omega, \{0,1\})$ (Theorem 7.1). Moreover, using the results of Section 6, we prove that any GMM starting from a bounded set stays bounded. In general, for two GMMs one cannot expect a comparison principle (for example in the presence of fattening). However, the notions of *maximal* and *minimal* GMM (Definition 7.2) are always comparable if the initial sets are comparable (Theorem 7.3). This requires regularity of minimizers of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ and $C_\beta(\cdot, \Omega)$, see Sections 4 and 5. Finally, in Section 8 we prove that, under a suitable conditional convergence assumption and if $1 \le n \le 6$, our GMM solution is, in fact, a *distributional solution* to (1.1)-(1.2).

2. Some preliminaries

2.1. Notation

χ_F stands for the characteristic function of the Lebesgue measurable set $F \subseteq \mathbb{R}^{n+1}$ and $|F|$ denotes its Lebesgue measure. The set of $L^1(\Omega)$-functions having bounded total variation in an open set $\Omega \subseteq \mathbb{R}^{n+1}$ is denoted by $BV(\Omega)$, and

$$BV(\Omega, \{0,1\}) := \{E \subseteq \Omega : \ \chi_E \in BV(\Omega)\}.$$

Given $E \subseteq BV(\Omega, \{0,1\})$ we denote by $P(E,\Omega)$ the *perimeter* of E in Ω, i.e. $P(E,\Omega) := \int_\Omega |D\chi_E|$, by $\partial^* E$ the essential boundary of E, and by $\nu_E(x)$ the measure-theoretical exterior normal to E at $x \in \partial^* E$. Since Lebesgue equivalent sets in Ω have the same perimeter in Ω, we assume that any set $E \subset \Omega$ we consider coincides with the set

$$\left\{x \in \mathbb{R}^{n+1} : \ \lim_{r\to 0+} \frac{|B_r(x) \cap E|}{|B_r(x)|} = 1\right\}$$

of points of density one, where $B_r(x)$ is the ball of radius $r > 0$ centered at x. Recall that $\overline{\partial^* E} = \partial E$. For simplicity, set $P(E, \mathbb{R}^{n+1}) = P(E)$. We say that $E \subset \mathbb{R}^{n+1}$ has locally finite perimeter in $\mathbb{R}^{n+1}$, if $P(E, \Omega') < +\infty$ for every bounded open set $\Omega' \subset \mathbb{R}^{n+1}$. The collection of all sets of locally finite perimeter is denoted by $BV_{\mathrm{loc}}(\Omega, \{0,1\})$. We refer to [7, 33] for a complete information about BV-functions and sets of finite perimeter.

For a fixed nonempty $E_0 \in BV(\Omega, \{0,1\})$ set

$$\mathcal{E}(E_0) := \{E \in BV(\Omega, \{0,1\}) : \ E_0 \subseteq E\}, \tag{2.1}$$

which is $L^1(\Omega)$-closed.

Given $\rho > 0$ and $l > 0$ let $C_\rho^l = \hat{B}_\rho \times (0,l)$ stand for the truncated cylinder in $\mathbb{R}^{n+1}$ of height l, whose basis is an open ball $\hat{B}_\rho \subset \mathbb{R}^n$ centered at the origin of radius $\rho > 0$; also set $\Omega_l := \mathbb{R}^n \times (0,l)$.

2.2. Some properties of sets of finite perimeter

By [23, Theorem II], for every $E \in BV_{\mathrm{loc}}(\Omega, \{0,1\})$ the additive set function $O \mapsto \int_O |D\chi_E|$ defined on the open sets $O \subseteq \Omega$ extends to a measure $B \mapsto \int_B |D\chi_E|$ defined on the Borel σ-algebra of Ω. Moreover, $P(\cdot, \Omega)$ is strongly subadditive, i.e.

$$P(E \cap F, \Omega) + P(E \cup F, \Omega) \le P(E,\Omega) + P(F,\Omega) \quad \text{for any } E, F \in BV(\Omega, \{0,1\}). \tag{2.2}$$

Let Ω be an open set with Lipschitz boundary and $E \in BV_{\mathrm{loc}}(\mathbb{R}^{n+1}, \{0,1\})$. We denote the interior and exterior traces of the set E on $\partial\Omega$ respectively by χ_E^+ and χ_E^- and we recall that $\chi_E^\pm \in L^1_{\mathrm{loc}}(\partial\Omega)$. Moreover, the integration by parts formula holds [23]:

$$\int_\Omega \chi_E \operatorname{div} g\, dx = -\int_\Omega g \cdot D\chi_E + \int_{\partial\Omega} (\chi_E^+ - \chi_E^-) g \cdot \nu_\Omega \, d\mathcal{H}^n \qquad \forall g \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}), \tag{2.3}$$

where ν_Ω is the outer unit normal to $\partial\Omega$.

If $V \subseteq \Omega$ is an open set with Lipschitz boundary, then

$$P(E,\Omega) = P(E,V) + P(E,\Omega\setminus\overline{V}) + \int_{\Omega\cap\partial V} |\chi_E^+ - \chi_E^-| \, d\mathcal{H}^n.$$

The trace set of $E \subseteq \Omega$ on $\partial\Omega$ is denoted by $\mathrm{Tr}(E)$. With a slight abuse of notation we set $\chi_{\mathrm{Tr}(E)} = \chi_E$. Note that

$$P(E,\overline{\Omega}) := P(E,\Omega) + \int_{\partial\Omega} \chi_E \, d\mathcal{H}^n = P(E).$$

In general, even if $E \in BV(\Omega,\{0,1\})$, the traces $\chi_E^\pm$ are in $L^1_{\mathrm{loc}}(\partial\Omega)$, but not in $L^1(\partial\Omega)$. For instance, if $\Omega = \big(\mathbb{R}\times(0,+\infty)\big) \cup A \subset \mathbb{R}^2$ and $A = \bigcup_{m=2}^{+\infty} (m - \frac{1}{m^2}, m + \frac{1}{m^2}) \times (-1,0]$, then $E = A \in BV(\Omega,\{0,1\})$, whereas $\mathcal{H}^1(\mathrm{Tr}(E)) = +\infty$. In Lemma 2.1 we show that $\chi_E \in L^1(\partial\Omega)$ for any $E \in BV(\Omega,\{0,1\})$, provided that Ω is a half-space.

From now on we fix $\Omega := \mathbb{R}^n \times (0,+\infty)$; we often identify $\partial\Omega = \mathbb{R}^n \times \{0\}$ with $\mathbb{R}^n$, so that $E \subset \partial\Omega$ means $E \subset \mathbb{R}^n$, and $\pi : \Omega \to \partial\Omega$ denotes the projection

$$\pi(\hat{x}, x_{n+1}) := \hat{x}, \quad x = (\hat{x}, x_{n+1}) \in \Omega.$$

2.3. Controlling the trace of a set by its perimeter

The following lemma shows that the $L^1(\partial\Omega)$-norm of the trace of $E \in BV(\Omega,\{0,1\})$ is controlled by $P(E,\Omega)$.

Lemma 2.1. *For any $E \in BV(\Omega,\{0,1\})$ and for any $\beta \in L^\infty(\partial\Omega)$ the inequalities*

$$\left|\int_{\partial\Omega} \beta\chi_E \, d\mathcal{H}^n\right| \le \int_\Omega |\beta\circ\pi| \, |D\chi_E| \le \|\beta\|_\infty \, P(E,\Omega) \tag{2.4}$$

hold. In particular, $P(E) < +\infty$.

Proof. The last inequality of (2.4) is immediate. The first inequality is enough to be shown for $\beta \ge 0$.

If β is locally Lipschitz, then (2.4) follows from the divergence theorem. Indeed, suppose that $\mathrm{supp}(\beta)$ is compact. Since $\mathrm{div}((\beta\circ\pi)e_{n+1}) = 0$, we have

$$0 = \int_E \mathrm{div}((\beta\circ\pi)e_{n+1}) \, dx = \int_{\Omega\cap\partial^* E} (\beta\circ\pi)\, \nu_E \cdot e_{n+1} \, d\mathcal{H}^n - \int_{\partial\Omega} \beta\chi_E \, d\mathcal{H}^n.$$

Hence nonnegativity of β implies that

$$\int_{\partial\Omega} \beta\chi_E \, d\mathcal{H}^n \le \int_{\Omega\cap\partial^* E} \beta\circ\pi \, d\mathcal{H}^n = \int_\Omega \beta\circ\pi \, |D\chi_E|. \tag{2.5}$$

If $\mathrm{supp}(\beta)$ is not compact, we use $\eta_k(|x|)\beta(x)$ in (2.5) instead of $\beta(x)$, where $\eta_k : [0,+\infty) \to [0,+\infty)$ is Lipschitz, linear in $[k,k+1]$, $\eta_k = 1$ in $[0,k]$ and $\eta_k = 0$ in $[k+1,+\infty)$. Now (2.4) follows from the monotone convergence theorem. In particular, when $\beta \equiv 1$ we have

$$P(E) = P(E,\Omega) + \int_{\partial\Omega} \chi_E d\mathcal{H}^n \le 2P(E,\Omega).$$

Assume that $\beta = \chi_{\hat{O}}$ for some open set $\hat{O} \subseteq \partial\Omega$. Consider a sequence $\{\beta_k\}$ of nonnegative locally Lipschitz functions converging $\mathcal{H}^n$ -almost everywhere to β on $\partial\Omega$ such that $\beta_k \leq \beta$ and $\mathrm{supp}\beta_k \subseteq \overline{\hat{O}}$. By Fatou's lemma we get

$$\int_{\partial\Omega} \beta\chi_E d\mathcal{H}^n \leq \liminf_{k\to+\infty} \int_{\partial\Omega} \beta_k\chi_E d\mathcal{H}^n \leq \liminf_{k\to+\infty} \int_{\Omega} \beta_k \circ \pi\, |D\chi_E| \leq \int_{\Omega} \beta \circ \pi\, |D\chi_E|.$$

Finally, if $\beta \in L^\infty(\partial\Omega)$ is any nonnegative function, then the statement of the lemma follows by an approximation argument. □

From Lemma 2.1 it follows that $E \in BV(\Omega, \{0,1\})$ if and only if $E \in BV(\mathbb{R}^{n+1}, \{0,1\})$.

Remark 2.2. If $u \in BV(\Omega)$, then its trace belongs to $L^1(\partial\Omega)$. Indeed, it is well-known that

$$\int_\Omega |u| dx = \int_{-\infty}^0 \int_\Omega \chi_{\{u<t\}}(x)\, dxdt + \int_0^{+\infty} \int_\Omega \chi_{\{u>t\}}(x)\, dxdt, \tag{2.6}$$

$$\int_\Omega |Du| = \int_{-\infty}^0 P(\{u<t\}, \Omega)\, dt + \int_0^{+\infty} P(\{u>t\}, \Omega)\, dt, \tag{2.7}$$

in particular, $\{u > t\}, \{u < s\} \in BV(\Omega)$ for a.e. $t > 0$ and $s < 0$. Using (2.4) with $\beta \equiv 1$, for a.e. $t > 0$ and $s < 0$ we get

$$\int_{\partial\Omega} \chi_{\{u>t\}} d\mathcal{H}^n \leq P(\{u>t\}, \Omega), \qquad \int_{\partial\Omega} \chi_{\{u<s\}} d\mathcal{H}^n \leq P(\{u<s\}, \Omega)$$

and whence

$$\int_{\partial\Omega} |u|\, d\mathcal{H}^n \leq \int_\Omega |Du|.$$

Notice that for every $\beta \in L^\infty(\partial\Omega)$ one has also

$$\int_{\partial\Omega} \beta u\, d\mathcal{H}^n = -\int_{-\infty}^0 \int_{\partial\Omega} \beta\chi_{\{u<t\}}\, d\mathcal{H}^n dt + \int_0^{+\infty} \int_{\partial\Omega} \beta\chi_{\{u>t\}}\, d\mathcal{H}^n dt. \tag{2.8}$$

The following lemma is the analog to the comparison theorem in [6, page 216][1].

Lemma 2.3. *Let E_0 be a closed convex set such that $\nu_{E_0} \cdot e_{n+1} \geq 0$ $\mathcal{H}^n$ -a.e. on $\Omega \cap \partial E_0$. Then $P(E_0, \Omega) \leq P(E, \Omega)$ for every $E \in \mathcal{E}(E_0)$.*

3. Capillary functionals

Let $\beta \in L^\infty(\partial\Omega)$. The capillary functional $C_\beta(\cdot, \Omega) : BV(\Omega, \{0,1\}) \to \mathbb{R}$ and its "level set" version $\mathsf{C}_\beta(\cdot, \Omega) : BV(\Omega) \to \mathbb{R}$ are defined as

$$C_\beta(E, \Omega) := P(E, \Omega) - \int_{\partial\Omega} \beta\chi_E\, d\mathcal{H}^n, \tag{3.1}$$

[1] For any $E \in BV(\mathbb{R}^{n+1}, \{0,1\})$ and any closed convex set $C \subseteq \mathbb{R}^{n+1}$ the inequality $P(E \cap C) \leq P(E)$ holds; equality occurs if and only if $|E \setminus C| = 0$.

and

$$\mathsf{C}_\beta(u,\Omega) := \int_\Omega |Du| - \int_{\partial\Omega} \beta u d\mathcal{H}^n,$$

respectively. Note that $\mathsf{C}_\beta(\cdot,\Omega)$ is convex, $\mathsf{C}_\beta(u,\Omega) = \mathsf{C}_{-\beta}(-u,\Omega)$ for any $u \in BV(\Omega)$, and $C_\beta(E,\Omega) = \mathsf{C}_\beta(\chi_E,\Omega)$ for any $E \in BV(\Omega,\{0,1\})$. Moreover, when $\|\beta\|_\infty \leq 1$, by (2.4) the functional $C_\beta(\cdot,\Omega)$ is nonnegative, and the same holds for $\mathsf{C}_\beta(\cdot,\Omega)$ as by (2.6)-(2.8) one has

$$\mathsf{C}_\beta(u,\Omega) = \int_{-\infty}^0 C_{-\beta}(\{u<t\},\Omega)\,dt + \int_0^{+\infty} C_\beta(\{u>t\},\Omega)\,dt. \tag{3.2}$$

The functional $\mathsf{C}_\beta(\cdot,\Omega)$ will be useful for the comparison principles (Section 6).

3.1. Coercivity and lower semicontinuity

The next lemma is a localized version of [17, Lemma 4], which is needed to prove coercivity of $C_\beta(\cdot,\Omega)$ and $\mathsf{C}_\beta(\cdot,\Omega)$ and will be frequently used (see for example the proofs of Theorem A.3 and Theorem 5.1).

Lemma 3.1. *Assume that $\|\beta\|_\infty \leq 1$ and $E \in BV(\Omega,\{0,1\})$. Then for any open set $A \subseteq \Omega$ with $A \in BV_{\rm loc}(\mathbb{R}^{n+1},\{0,1\})$ and*

$$\mathcal{H}^n\Big([\pi^{-1}(\pi(A)) \setminus A] \cap \Omega \cap \partial^* E\Big) = 0 \tag{3.3}$$

the inequality

$$P(E,A) - \int_{\partial\Omega} \beta\chi_{E\cap A}\,d\mathcal{H}^n \geq \frac{1-\operatorname{ess\,sup}\beta}{2}\left[P(E,A) + \int_{\partial\Omega}\chi_{E\cap A}\,d\mathcal{H}^n\right] \tag{3.4}$$

holds.

Proof. Let us first show that if $F \subset \Omega$ has locally finite perimeter in $\mathbb{R}^{n+1}$, then

$$\chi_F \leq \chi_{\pi(F)} \quad \mathcal{H}^n\text{-a.e. on } \partial\Omega. \tag{3.5}$$

Set $\hat G := \{\hat x \in \mathrm{Tr}(F) : \chi_{\pi(F)}(\hat x) = 0\}$. For any $\varepsilon > 0$ take an open set $\hat O \subseteq \partial\Omega$ such that $\hat G \subseteq \hat O$ and $\mathcal{H}^n(\hat O \setminus \hat G) < \varepsilon$. Since $\mathcal{H}^n(\pi(F) \cap \hat G) = 0$, one has

$$\begin{aligned}|F \cap \pi^{-1}(\hat G)| &= \int_{\pi^{-1}(\hat G)} \chi_F\,dx = \int_0^{+\infty} dx_{n+1}\int_{\hat G}\chi_F(\hat x, x_{n+1})d\mathcal{H}^n(\hat x)\\ &= \int_0^{+\infty}\mathcal{H}^n(\hat G \cap \{(\hat x,0):\ (\hat x,x_{n+1}) \in F\})dx_{n+1} = \int_0^{+\infty}\mathcal{H}^n(\hat G \cap \pi(F))dx_{n+1} = 0.\end{aligned}$$

Let $\hat B_\rho \subset \mathbb{R}^n$ denote the ball of radius $\rho > 0$ centered at the origin. Recall that for any $\gamma > 0$ the following estimate [33, page 35] holds:

$$\int_{\hat O \cap \hat B_\rho}\chi_F d\mathcal{H}^n \leq P(F,(\hat O \cap \hat B_\rho)\times(0,\gamma)) + \frac{1}{\gamma}\int_{(\hat O\cap\hat B_\rho)\times(0,\gamma)}\chi_F\,dx.$$

Then using $\hat{G} \subseteq \mathrm{Tr}(F)$, we establish

$$\begin{aligned}\mathcal{H}^n(\hat{G} \cap \hat{B}_\rho) \leq & \int_{\hat{O}\cap\hat{B}_\rho} \chi_F d\mathcal{H}^n \leq P(F, (\hat{O} \cap \hat{B}_\rho) \times (0, \gamma)) \\ & + \frac{1}{\gamma} \int_{(\hat{G}\cap\hat{B}_\rho)\times(0,\gamma)} \chi_F \, dx + \frac{1}{\gamma} \int_{((\hat{O}\setminus\hat{G})\cap\hat{B}_\rho)\times(0,\gamma)} \chi_F \, dx \\ \leq & P(F, \hat{O} \times (0, \gamma)) + \frac{1}{\gamma} |F \cap \pi^{-1}(\hat{G})| + \mathcal{H}^n(\hat{O} \setminus \hat{G}) < P(F, \hat{O} \times (0, \gamma)) + \varepsilon.\end{aligned}$$

Now letting $\varepsilon, \gamma \to 0^+$ we get $\mathcal{H}^n(\hat{G} \cap \hat{B}_\rho) = 0$ and (3.5) follows from letting $\rho \to +\infty$.

We have

$$\int_\Omega \chi_{\pi(A)} \circ \pi \, \frac{1 + \beta \circ \pi}{2} \, |D\chi_E| = \int_{\pi^{-1}(\pi(A))} \frac{1 + \beta \circ \pi}{2} \, |D\chi_E| = \int_A \frac{1 + \beta \circ \pi}{2} \, |D\chi_E|, \tag{3.6}$$

where in the second equality we used (3.3). Moreover, from (3.5) with $F = A$ we get

$$\int_{\partial\Omega} \frac{1 + \beta}{2} \chi_{E\cap A} \, d\mathcal{H}^n = \int_{\partial\Omega} \chi_A \, \frac{1 + \beta}{2} \chi_E \, d\mathcal{H}^n \leq \int_{\partial\Omega} \chi_{\pi(A)} \, \frac{1 + \beta}{2} \chi_E \, d\mathcal{H}^n. \tag{3.7}$$

Now, using Lemma 2.1 with β replaced with $(1 + \beta)\chi_{\pi(A)}/2$, from (3.6) and (3.7) we obtain

$$\int_{\partial\Omega} \frac{1 + \beta}{2} \chi_{E\cap A} \, d\mathcal{H}^n \leq \int_A \frac{1 + \beta \circ \pi}{2} \, |D\chi_E|. \tag{3.8}$$

Finally, adding the identities

$$\begin{aligned} P(E, A) = \int_A |D\chi_E| &= \int_A \frac{1 - \beta \circ \pi}{2} \, |D\chi_E| + \int_A \frac{1 + \beta \circ \pi}{2} \, |D\chi_E|, \\ -\int_{\partial\Omega} \beta \chi_{E\cap A} \, d\mathcal{H}^n &= \int_{\partial\Omega} \frac{1 - \beta}{2} \chi_{E\cap A} \, d\mathcal{H}^n - \int_{\partial\Omega} \frac{1 + \beta}{2} \chi_{E\cap A} \, d\mathcal{H}^n,\end{aligned}$$

and using (3.8) we deduce

$$P(E, A) - \int_{\partial\Omega} \beta \chi_{E\cap A} \, d\mathcal{H}^n \geq \int_A \frac{1 - \beta \circ \pi}{2} \, |D\chi_E| + \int_{\partial\Omega} \frac{1 - \beta}{2} \chi_{E\cap A} \, d\mathcal{H}^n.$$

This relation yields (3.4). □

Proposition 3.2 (Coercivity of the capillary functionals). *If* $-1 \leq \beta \leq 1 - 2\kappa$ $\mathcal{H}^n$*-a.e. on* $\partial\Omega$ *for some* $\kappa \in [0, \frac{1}{2}]$, *then*

$$\kappa P(E) \leq C_\beta(E, \Omega) \leq P(E) \qquad \forall E \in BV(\Omega, \{0, 1\}). \tag{3.9}$$

Moreover, if $\|\beta\|_\infty \leq 1 - 2\kappa$ *for some* $\kappa \in [0, \frac{1}{2}]$, *then*

$$\kappa \int_{\overline{\Omega}} |Du| \leq \mathsf{C}_\beta(u, \Omega) \leq \int_{\overline{\Omega}} |Du| \qquad \forall u \in BV(\Omega). \tag{3.10}$$

Proof. The inequality $\kappa P(E) \le C_\beta(E,\Omega)$ follows from Lemma 3.1 with $A=\Omega$. Moreover, by virtue of Lemma 2.1,

$$\|\beta\|_\infty \le 1 \quad \Longrightarrow \quad C_\beta(E,\Omega) \le P(E) \qquad \forall E \in BV(\Omega,\{0,1\}). \tag{3.11}$$

Now (3.10) follows from the inequalities

$$\kappa P(\{u<t\},\Omega) + \kappa \int_{\partial\Omega} \chi_{\{u<t\}}\, d\mathcal{H}^n \le C_{-\beta}(\{u<t\},\Omega) \le P(\{u<t\},\Omega) + \int_{\partial\Omega} \chi_{\{u<t\}}\, d\mathcal{H}^n$$

for a.e. $t<0$ and

$$\kappa P(\{u>t\},\Omega) + \kappa \int_{\partial\Omega} \chi_{\{u>t\}}\, d\mathcal{H}^n \le C_{\beta}(\{u>t\},\Omega) \le P(\{u>t\},\Omega) + \int_{\partial\Omega} \chi_{\{u>t\}}\, d\mathcal{H}^n$$

for a.e. $t>0$, from (2.6)-(2.8), (3.2) and by [33, Remark 2.14], possibly after extending u to 0 outside Ω. □

Remark 3.3. From the proof of Proposition 3.2 it follows that if $u \ge 0$, then (3.10) holds for any $\beta \in L^\infty(\partial\Omega)$ with $-1 \le \beta \le 1-2\kappa$; if $u \le 0$, (3.10) is valid whenever $-1+2\kappa \le \beta \le 1$.

Remark 3.4. If $\beta > 1$ on a set of infinite $\mathcal{H}^n$-measure, then $C_\beta(\cdot,\Omega)$ is unbounded from below. Note also that if $\|\beta\|_\infty \le 1$, then $\emptyset$ is the unique minimizer of $C_\beta(\cdot,\Omega)$ in $BV(\Omega,\{0,1\})$. Indeed, clearly,

$$0 = C_\beta(\emptyset,\Omega) = \min_{E\in BV(\Omega,\{0,1\})} C_\beta(E,\Omega).$$

If there were a minimizer $E \ne \emptyset$ of $C_\beta(\cdot,\Omega)$, there would exist $l>0$ such that $|E \setminus \Omega_l| > 0$. Now since $\mathrm{Tr}(E) = \mathrm{Tr}(E\cap\overline{\Omega_l})$, by [6, page 216] we have

$$0 = C_\beta(E,\Omega) > C_\beta(E\cap\overline{\Omega_l},\Omega) \ge 0,$$

a contradiction.

Lemma 3.5 (Lower semicontinuity). *Assume that $\beta \in L^\infty(\partial\Omega)$. Then the functionals $C_\beta(\cdot,\Omega)$ and $\mathsf{C}_\beta(\cdot,\Omega)$ are $L^1(\Omega)$-lower semicontinuous if and only if $\|\beta\|_\infty \le 1$.*

Proof. Assume that $\|\beta\|_\infty \le 1$. In this case the lower semicontinuity of $C_\beta(\cdot,\Omega)$ is proven in [17, Lemma 2]. Let us prove the lower semicontinuity of $\mathsf{C}_\beta(\cdot,\Omega)$. Take $u_k, u \in BV(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$. By (2.6) we may assume that $\int_\Omega |\{u_k<t\}\Delta\{u<t\}|\,dx \to 0$ as $k\to+\infty$ for a.e. $t\in\mathbb{R}$. Then using the nonnegativity of summands, the lower semicontinuity of $C_\beta(\cdot,\Omega)$ and Fatou's Lemma in (3.2) we establish

$$\begin{aligned}
\liminf_{k\to+\infty} \mathsf{C}_\beta(u_k,\Omega) &\ge \liminf_{k\to+\infty} \int_{-\infty}^0 C_{-\beta}(\{u_k<t\},\Omega)\,dt + \liminf_{k\to+\infty} \int_0^{+\infty} C_\beta(\{u_k>t\},\Omega)\,dt \\
&\ge \int_{-\infty}^0 \liminf_{k\to+\infty} C_{-\beta}(\{u_k<t\},\Omega)\,dt + \int_0^{+\infty} \liminf_{k\to+\infty} C_\beta(\{u_k>t\},\Omega)\,dt \\
&\ge \int_{-\infty}^0 C_{-\beta}(\{u<t\},\Omega)\,dt + \int_0^{+\infty} C_\beta(\{u>t\},\Omega)\,dt = \mathsf{C}_\beta(u,\Omega).
\end{aligned}$$

Now assume that $\|\beta\|_\infty > 1$, i.e. the set $\{\hat x \in \partial\Omega : |\beta(\hat x)| > 1\}$ has positive $\mathcal{H}^n$-measure. Let for some $\varepsilon, \delta_0 > 0$ the set $\hat A := \{\beta > 1+\varepsilon\}$ satisfy $|\hat A| \ge \delta_0$. By Lusin's theorem, for any

$k > \frac{4\|\beta\|_\infty}{\varepsilon\delta_0}$ there exists $\beta_k \in C(\partial\Omega)$ such that $\mathcal{H}^n(\{\beta \neq \beta_k\}) < \frac{1}{k}$ and $\|\beta_k\|_\infty \leq \|\beta\|_\infty$. Let k be so large that $\mathcal{H}^n(\{\beta_k > 1+\varepsilon\}) \geq \delta_0/2$ and choose an open set $\hat{O} \subset \{\beta_k > 1+\varepsilon\}$ of finite perimeter such that $\delta_0/4 \leq \mathcal{H}^n(\hat{O}) < +\infty$. Define the sequence of sets $E_m := \hat{O} \times (0, \frac{1}{m}) \subset \Omega$. Clearly, $E_m \to \emptyset$ in $L^1(\Omega)$ as $m \to +\infty$. Then, indicating by $P(\hat{O})$ the perimeter of $\hat{O}$ in $\mathbb{R}^n$, from the relations

$$\begin{aligned} C_\beta(E_m,\Omega) =& \frac{1}{m}\, P(\hat{O}) + \mathcal{H}^n(\hat{O}) - \int_{\hat{O}} \beta d\mathcal{H}^n \\ \leq& \frac{1}{m}\, P(\hat{O}) + \mathcal{H}^n(\hat{O}) - \int_{\hat{O}} \beta_k d\mathcal{H}^n + \int_{\hat{O}} |\beta - \beta_k| d\mathcal{H}^n \\ \leq& \frac{1}{m}\, P(\hat{O}) - \varepsilon\mathcal{H}^n(\hat{O}) + 2\|\beta\|_\infty \mathcal{H}^n(\hat{O} \cap \{\beta \neq \beta_k\}) \leq \frac{1}{m}\, P(\hat{O}) - \frac{\varepsilon\delta_0}{4}, \end{aligned}$$

we establish

$$\liminf_{m\to+\infty} C_\beta(E_m,\Omega) \leq -\frac{\varepsilon\delta_0}{4} < 0 = C_\beta(\emptyset,\Omega).$$

Since $\mathsf{C}_\beta(\chi_E,\Omega) = C_\beta(E,\Omega)$, one has also $\liminf_{m\to+\infty} \mathsf{C}_\beta(\chi_{E_m},\Omega) < 0 = \mathsf{C}_\beta(0,\Omega)$. Hence $C_\beta(\cdot,\Omega)$ and $\mathsf{C}_\beta(\cdot,\Omega)$ are not $L^1(\Omega)$-lower semicontinuous.

Finally, the case $|\{\beta < -1-\varepsilon\}| > 0$ can be treated in a similar way. $\square$

Remark 3.6. If Ω is an arbitrary bounded open set with Lipschitz boundary and $\|\beta\|_\infty \leq 1$, then the lower semicontinuity of $C_\beta(\cdot,\Omega)$ is a consequence of [5, Theorem 3.4]. In this case $C_\beta(\cdot,\Omega)$ is bounded from below by $-\mathcal{H}^n(\partial\Omega)$. Hence again Fatou's lemma and (3.2) yield lower semicontinuity of $\mathsf{C}_\beta(\cdot,\Omega)$.

4. Capillary Almgren-Taylor-Wang-type functional

In the sequel, for a given nonempty set $F \subseteq \Omega$, d_F stands for the distance function from the boundary of ∂F in Ω:

$$d_F(x) := \mathrm{dist}(x, \Omega \cap \partial F).$$

The function

$$\tilde{d}_F(x) := \begin{cases} -d_F(x) & \text{if } x \in F, \\ d_F(x) & \text{if } x \in \Omega \setminus F, \end{cases}$$

is called the *signed distance function* from ∂F in Ω negative inside F. The distance from the empty set is assumed to be equal to $+\infty$.

Notice that for $E, F \subseteq \Omega$, $F \neq \emptyset$,

$$\int_{E\Delta F} d_F\, dx = \int_{E\setminus F} \tilde{d}_F\, dx - \int_{F\setminus E} \tilde{d}_F\, dx = \int_E \tilde{d}_F\, dx - \int_F \tilde{d}_F\, dx,$$

provided $\int_{E\cap F} d_F dx < +\infty$. Moreover, we assume $\int_{E\Delta F} d_F dx := 0$ whenever $|E\Delta F| = 0$.

Given $\beta \in L^\infty(\partial\Omega)$, $E_0 \in BV(\Omega,\{0,1\})$ and $\lambda \geq 1$, recalling the definition of $C_\beta(\cdot,\Omega)$ in (3.1), we define the *capillary Almgren-Taylor-Wang-type* functional $\mathcal{A}_\beta(\cdot, E_0, \lambda) : BV(\Omega,\{0,1\}) \to [-\infty, +\infty]$ with contact angle β, as

$$\mathcal{A}_\beta(E, E_0, \lambda) := C_\beta(E,\Omega) + \lambda \int_{E\Delta E_0} d_{E_0}\, dx, \tag{4.1}$$

so that

$$\mathcal{A}_\beta(E,E_0,\lambda) = P(E,\Omega) + \lambda\int_E \tilde d_{E_0}\,dx - \int_{\partial\Omega}\beta\chi_E\,d\mathcal{H}^n - \lambda\int_{E_0}\tilde d_{E_0}\,dx \tag{4.2}$$

whenever $\displaystyle\int_{E\cap E_0} d_{E_0}dx < +\infty$.

4.1. Existence of minimizers of the functional $\mathcal{A}_\beta(\cdot,E_0,\lambda)$

We always suppose that $\lambda \ge 1$ and in this section we assume that

$$\begin{cases} E_0 \in BV(\Omega,\{0,1\}) \text{ is nonempty and bounded },\\ \beta\in L^\infty(\partial\Omega) \text{ and } \exists\kappa\in(0,\frac12] : -1\le\beta\le 1-2\kappa\ \mathcal{H}^n\text{-a.e on } \partial\Omega. \end{cases} \tag{4.3}$$

Hence, there exists a cylinder $C_D^H = \hat B_D\times(0,H)$ containing E_0 whose basis is an open ball $\hat B_D\subset\mathbb{R}^n$ of radius $D>0$ and height

$$H = 1+\max\Big\{x_{n+1} :\ x=(x',x_{n+1})\in\overline{E_0}\Big\}.$$

Define

$$R_0 := R_0(n,\kappa,E_0) = D+1+\max\Big\{8^{n^2+n+1}\left(\frac{P(E_0)}{\kappa}\right)^{\frac{n+1}{n}},\ 4\mu(\kappa,n)\Big\}, \tag{4.4}$$

where $\mu(\kappa,n) = (1/\kappa+2)^{\frac{n+1}{n}}$. The proof of the next result is essentially postponed to Appendix A, since the main idea does not differ too much from [17].

Theorem 4.1 (**Existence of minimizers and uniform bound**). *Suppose that* (4.3) *holds. Then the minimum problem*

$$\inf_{E\in BV(\Omega,\{0,1\})}\mathcal{A}_\beta(E,E_0,\lambda) \tag{4.5}$$

has a solution E_λ. *Moreover, any minimizer is contained in* $C^H_{R_0}$.

Proof. Let $f=\lambda\tilde d_{E_0}$ and

$$\mathcal{V} : BV(\Omega,\{0,1\})\to(-\infty,+\infty],\quad \mathcal{V}(E) := \int_E f dx.$$

Then $\mathcal{V}$ satisfies Hypothesis A.1 and by Remark A.4, $\mathcal{R}_0\le R_0$. Now the proof directly follows from Theorem A.3. □

Remark 4.2. If $E_0=\emptyset$, then (4.5) has a unique solution $E_\lambda=\emptyset$. Moreover, for some choices of $\lambda\ge1$ and $\emptyset\ne E_0\in BV(\Omega,\{0,1\})$, the empty set solves (4.5). For example, let B_ρ be the ball centered at x such that $x_{n+1}\ge 4\rho+4$. If $\lambda\rho\le n$, then as in [11, 19], one can show that $E_\lambda=\emptyset$ is the unique minimizer of $\mathcal{A}_\beta(\cdot,B_\rho,\lambda)$.

Remark 4.3. Let F minimize $\mathcal{A}_\beta(\cdot,E_0,\lambda)$ in $BV(C^H_{R_0},\{0,1\})$. Then F is an unconstrained minimizer, i.e.

$$\mathcal{A}_\beta(F,E_0,\lambda) = \min_{E\in BV(\Omega,\{0,1\})}\mathcal{A}_\beta(E,E_0,\lambda). \tag{4.6}$$

Indeed, let E_λ be any minimizer of $\mathcal{A}_\beta(\cdot,E_0,\lambda)$. Clearly, $\mathcal{A}_\beta(F,E_0,\lambda)\ge\mathcal{A}_\beta(E_\lambda,E_0,\lambda)$. On the other hand, by Theorem 4.1 $E_\lambda\subseteq C^H_{R_0}$ and by minimality of F in $C^H_{R_0}$ we have $\mathcal{A}_\beta(F,E_0,\lambda)\le\mathcal{A}_\beta(E_\lambda,E_0,\lambda)$, which implies (4.6).

Recalling Remark 3.4 and definition (2.1) of $\mathcal{E}(E_0)$ we have also the following result.

Proposition 4.4 (Existence of constrained minimizers of C_β). *Under assumptions* (4.3) *the constrained minimum problem*

$$\inf_{E \in BV(\Omega,\{0,1\}),\, E \in \mathcal{E}(E_0)} C_\beta(E, \Omega) \tag{4.7}$$

has a solution. In addition, any minimizer E^+ satisfies $E^+ \subseteq C^H_{R_0}$, where R_0 is given by (4.4)*, and E^+ is also a solution of*

$$\inf_{E \in BV(\Omega,\{0,1\}),\, E \in \mathcal{E}(E^+)} C_\beta(E, \Omega).$$

Proof. Set

$$\mathcal{V} : BV(\Omega, \{0, 1\}) \to [0, +\infty], \quad \mathcal{V}(E) := \begin{cases} 0 & \text{if } E \in \mathcal{E}(E_0), \\ +\infty & \text{if } E \in BV(\Omega, \{0, 1\}) \setminus \mathcal{E}(E_0). \end{cases} \tag{4.8}$$

Then $\mathcal{V}$ satisfies Hypothesis A.1 and $\mathcal{R}_0 \le R_0$. Now existence of a minimizer E^+ of $C_\beta(\cdot, \Omega)$ in $\mathcal{E}(E_0)$ and the inclusion $E^+ \subseteq C^H_{R_0}$ follow from Theorem A.3. To show the last statement we observe that the inclusion $E_0 \subseteq E^+$ implies $\mathcal{E}(E^+) \subseteq \mathcal{E}(E_0)$. Hence the minimality of E^+ yields the inequality $C_\beta(E^+, \Omega) \le C_\beta(E, \Omega)$ for any $E \in \mathcal{E}(E^+)$. □

Solutions of (4.7) will be called constrained minimizers of $C_\beta(\cdot, \Omega)$ in $\mathcal{E}(E_0)$.

Example 4.5. Suppose that $E_0 \subset \Omega$ is a closed convex set so that $\nu_{E_0} \cdot e_{n+1} \ge 0$ $\mathcal{H}^n$ -a.e. on $\Omega \cap \partial E_0$. Then for every $\beta \in L^\infty(\partial\Omega, [-1, 0])$ the set E_0 is a constrained minimizer of $C_\beta(\cdot, \Omega)$ in $\mathcal{E}(E_0)$. Indeed, by Lemma 2.3 $P(E_0, \Omega) \le P(E, \Omega)$ for all $E \in \mathcal{E}(E_0)$, therefore

$$C_\beta(E, \Omega) - C_\beta(E_0, \Omega) = P(E, \Omega) - P(E_0, \Omega) + \int_{\partial\Omega} (-\beta)\chi_{E\setminus E_0} d\mathcal{H}^n \ge 0.$$

The following lemma shows the behaviour of E_λ as $\lambda \to +\infty$.

Lemma 4.6 (Asymptotics of E_λ as time goes to 0^+). *Assume* (4.3) *and* $|\overline{E_0} \setminus E_0| = 0$. *Then any minimizer E_λ satisfies:*

a) $\lim_{\lambda\to+\infty} |E_\lambda \Delta E_0| = 0$,

b) $\lim_{\lambda\to+\infty} C_\beta(E_\lambda, \Omega) = C_\beta(E_0, \Omega)$,

c) $\lim_{\lambda\to+\infty} \lambda \int_{E_\lambda \Delta E_0} d_{E_0}\, dx = 0$.

Proof. a) We have

$$\kappa P(E_\lambda) \le \mathcal{A}_\beta(E_\lambda, E_0, \lambda) \le \mathcal{A}_\beta(E_0, E_0, \lambda) = C_\beta(E_0, \Omega) \le P(E_0).$$

Moreover, from $\mathcal{A}_\beta(E_\lambda, E_0, \lambda) \le P(E_0)$ and (2.4) we get $\lambda \int_{E_\lambda \Delta E_0} d_{E_0}\, dx \le P(E_0)$, hence

$$\lim_{\lambda\to+\infty} \int_{E_\lambda \Delta E_0} d_{E_0}\, dx = 0. \tag{4.9}$$

Recall from Theorem 4.1 that $E_\lambda \subseteq C^H_{R_0}$ for all $\lambda \geq 1$. Hence, by compactness, from every diverging sequence $\{\lambda_i\}$ we can select a subsequence $\{\lambda_{i_k}\}$ such that

$$E_{\lambda_{i_k}} \to E_\infty \quad \text{in } L^1(\Omega)$$

for some $E_\infty \in BV(C^H_{R_0}, \{0,1\})$. From (4.9) we deduce that $\int_{E_\infty \Delta E_0} d_{E_0}\, dx = 0$, and thus, since $d_{E_0} \geq 0$ and by assumption $|\overline{E_0} \setminus E_0| = 0$, we get $|E_\infty \Delta E_0| = 0$. Now the arbitrariness of $\{\lambda_j\}$ implies a).

b) Clearly, $C_\beta(E_\lambda, \Omega) \leq \mathcal{A}_\beta(E_\lambda, E_0, \lambda) \leq C_\beta(E_0, \Omega)$ for all $\lambda \geq 1$. Then by a) and by the $L^1(\Omega)$ -lower semicontinuity of $C_\beta(\cdot, \Omega)$ (Lemma 3.5) we establish

$$C_\beta(E_0, \Omega) \leq \liminf_{\lambda \to +\infty} C_\beta(E_\lambda, \Omega) \leq \limsup_{\lambda \to +\infty} C_\beta(E_\lambda, \Omega) \leq C_\beta(E_0, \Omega),$$

and b) follows.

c) follows from b) and nonnegativity of $\lambda \int_{E_\lambda \Delta E_0} d_{E_0}\, dx$, since

$$\limsup_{\lambda \to +\infty} \lambda \int_{E_\lambda \Delta E_0} d_{E_0}\, dx \leq \lim_{\lambda \to +\infty} [C_\beta(E_0, \Omega) - C_\beta(E_\lambda, \Omega)] = 0.$$

$\square$

5. Density estimates and regularity of minimizers

In this section we assume that

$$\begin{cases} E_0 \in BV(\Omega, \{0,1\}) \text{ is nonempty and bounded }, \\ \beta \in L^\infty(\partial\Omega) \text{ and } \exists \kappa \in (0, \tfrac{1}{2}] : \ \|\beta\|_\infty \leq 1 - 2\kappa. \end{cases} \tag{5.1}$$

Define

$$R(n,\kappa) := \left(2^{n+3}\, \frac{\omega_n + (n+1)\omega_{n+1}}{\omega_{n+1}\kappa^{n+1}}\right)^{\frac{1}{2}}, \ \gamma(n,\kappa) := \frac{\kappa(n+1)}{\sqrt{R(n,\kappa)^2 + 4\kappa(n+1)} + R(n,\kappa)}, \tag{5.2}$$

and

$$C(n,\kappa) := (n+1)\omega_{n+1} + 2\omega_n + \frac{\kappa(n+1)}{2}\,\omega_{n+1}, \quad c(n,\kappa) := c_{n+1}\left(\frac{\kappa}{4}\right)^n, \tag{5.3}$$

where c_{n+1} is the relative isoperimetric constant for the ball, i.e.

$$c_{n+1}\ \min\{|B_r \cap F|, |B_r \setminus F|\}^{\frac{n}{n+1}} \leq P(F, B_r), \qquad r > 0,\ F \in BV(B_r, \{0,1\}).$$

The aim of this section is to prove the following uniform density estimates for minimizers of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$, needed to prove regularity of minimizers (Theorem 5.3) and Proposition 5.7.

Theorem 5.1. *Assume that* E_0 *and* β *are as in* (5.1) *and* $E_\lambda \in BV(\Omega, \{0,1\})$ *is a minimizer of* $\mathcal{A}_\beta(\cdot, E_0, \lambda)$. *Then either* $E_\lambda = \emptyset$ *or*

$$\left(\frac{\kappa}{4}\right)^{n+1} \leq \frac{|E_\lambda \cap B_r(x)|}{\omega_{n+1} r^{n+1}} \leq 1 - \left(\frac{\kappa}{4}\right)^{n+1}, \tag{5.4}$$

$$c(n,\kappa) \le \frac{P(E_\lambda, B_r(x))}{r^n} \le C(n,\kappa) \tag{5.5}$$

for every $x \in \partial E_\lambda$ *and* $r \in (0, \frac{\gamma(n,\kappa)}{\lambda^{1/2}})$. *In particular,*

$$\mathcal{H}^n(\partial E_\lambda \setminus \partial^* E_\lambda) = 0. \tag{5.6}$$

We postpone the proof after several auxiliary results. First we show a weaker version of Theorem 5.1; the difference stands in that Proposition 5.2 holds for $r \le O(\frac{1}{\lambda})$ and $O(\frac{1}{\lambda})$ depends on E_0, whereas Theorem 5.1 is valid for $r \le O(\frac{1}{\lambda^{1/2}})$ and $O(\frac{1}{\lambda^{1/2}})$ is independent of E_0.

Proposition 5.2. *Under the assumptions of Theorem 5.1, setting*

$$\Lambda := \Lambda(\lambda, n, \kappa, P(E_0)) = \lambda\, \mathrm{diam}(\hat{B}_{D+R_0+1} \times (-1, H+1)),$$

for any nonempty E_λ, $x \in \partial E_\lambda$ *and* $r \in (0, \min\{1, \frac{\kappa(n+1)}{2\Lambda}\})$, *the density estimates* (5.4)-(5.5) *hold.*

Proof. For completeness we give the full proof of the proposition using the methods of [41, 45]. We recall that one could also employ the density estimates for almost minimizers of the capillary functional (see for instance [21, Lemma 2.8]).

Set $r_0 := \min\{1, \frac{\kappa(n+1)}{2\Lambda}\}$, and fix $x \in \partial^* E_\lambda$. Let $B_r := B_r(x)$ be the ball of radius $r \in (0, r_0)$ centered at x, we can choose r such that

$$\mathcal{H}^n(\partial B_r \cap \partial E_\lambda) = 0.$$

First we show that E_λ satisfies

$$\kappa P(E_\lambda \cap B_r) \le 2\mathcal{H}^n(E_\lambda \cap \partial B_r) + \Lambda |E_\lambda \cap B_r|. \tag{5.7}$$

Comparing $\mathcal{A}_\beta(E_\lambda, E_0, \lambda)$ with $\mathcal{A}_\beta(E_\lambda \setminus B_r, E_0, \lambda)$, for a.e. $s \in (r, r_0)$ we establish

$$\begin{aligned} P(E_\lambda, B_s \cap \Omega) - &\int_{B_r \cap \partial\Omega} \beta \chi_{E_\lambda \cap B_r}\, d\mathcal{H}^n + \lambda \int_{E_\lambda \cap B_r} \tilde{d}_{E_0} dy \\ &\le P(E_\lambda, (B_s \setminus \overline{B}_r) \cap \Omega) + \mathcal{H}^n(E_\lambda \cap \partial B_r). \end{aligned}$$

Sending $s \to r^+$ we get

$$P(E_\lambda, B_r \cap \Omega) - \int_{B_r \cap \partial\Omega} \beta \chi_{E_\lambda}\, d\mathcal{H}^n + \lambda \int_{E_\lambda \cap B_r} \tilde{d}_{E_0} dy \le \mathcal{H}^n(E_\lambda \cap \partial B_r). \tag{5.8}$$

By Theorem 4.1 $E_\lambda \subseteq C^H_{R_0}$ and thus, since $r_0 \le 1$, for any $y \in B_r$

$$\lambda |\tilde{d}_{E_0}(y)| \le \lambda\, \mathrm{diam}(\hat{B}_{D+R_0+1} \times (-1, H+1)) = \Lambda. \tag{5.9}$$

Moreover, using (3.9) for $E_\lambda \cap B_r$ we get (5.7):

$$\begin{aligned} \kappa P(E_\lambda \cap B_r) \le& P(E_\lambda, B_r \cap \Omega) + \mathcal{H}^n(E_\lambda \cap \partial B_r) - \int_{B_r \cap \partial\Omega} \beta \chi_{E_\lambda}\, d\mathcal{H}^n \\ \le& 2\mathcal{H}^n(E_\lambda \cap \partial B_r) + \Lambda |E_\lambda \cap B_r|. \end{aligned}$$

Now by the isoperimetric inequality,

$$P(E_\lambda \cap B_r) \geq (n+1)\omega_{n+1}^{\frac{1}{n+1}} |E_\lambda \cap B_r|^{\frac{n}{n+1}}. \tag{5.10}$$

Set $m(r) := |E_\lambda \cap B_r|$. Then m is absolutely continuous, $m(0) = 0$, $m(r) > 0$ for all $r > 0$ and $m'(r) = \mathcal{H}^n(E_\lambda \cap \partial B_r)$ for a.e. $r \in (0, r_0)$. Consequently, (5.7) and (5.10) give

$$\kappa(n+1)\omega_{n+1}^{\frac{1}{n+1}} m(r)^{\frac{n}{n+1}} \leq 2m'(r) + \Lambda m(r) = 2m'(r) + \Lambda m(r)^{\frac{n}{n+1}} m(r)^{\frac{1}{n+1}}. \tag{5.11}$$

Since $m(r) \leq \omega_{n+1} r^{n+1}$ and $r \leq \frac{\kappa(n+1)}{2\Lambda}$, from the last inequality we obtain

$$\frac{\kappa}{4}(n+1)\omega_{n+1}^{\frac{1}{n+1}} m(r)^{\frac{n}{n+1}} \leq m'(r).$$

Integrating we get the lower volume density estimate

$$m(r) \geq \left(\frac{\kappa}{4}\right)^{n+1} \omega_{n+1} r^{n+1}, \quad \forall r \in (0, r_0).$$

Let us prove the upper volume density estimate in (5.4). Since $E_\lambda \subseteq \Omega$ if $x \in \partial\Omega \cap \partial^* E_\lambda$, the inequality

$$\frac{|B_r \setminus E_\lambda|}{\omega_{n+1} r^{n+1}} \geq \frac{1}{2} > \left(\frac{\kappa}{4}\right)^{n+1} \qquad \forall r > 0 \tag{5.12}$$

is trivial. So assume that $x \in \Omega \cap \partial^* E_\lambda$. Since $\mathcal{A}_\beta(E_\lambda, E_0, \lambda) \leq \mathcal{A}_\beta((E_\lambda \cup B_r) \cap \Omega, E_0, \lambda)$, arguing as in the proof of (5.8) we get

$$P(E_\lambda, B_r \cap \Omega) + \int_{\partial\Omega} \beta\chi_{(B_r\cap\Omega)\setminus E_\lambda} \, d\mathcal{H}^n \leq \mathcal{H}^n((\Omega \setminus E_\lambda) \cap \partial B_r) + \lambda \int_{(B_r\cap\Omega)\setminus E_\lambda} \tilde{d}_{E_0} dy. \tag{5.13}$$

From the isoperimetric inequality, (3.9), (5.13) and also (5.9), it follows that

$$\begin{aligned} \kappa(n+1)\omega_{n+1}^{\frac{1}{n+1}} |(B_r \setminus E_\lambda) \cap \Omega|^{\frac{n}{n+1}} &\leq \kappa P((B_r \setminus E_\lambda) \cap \Omega) \leq C_{-\beta}((B_r \setminus E_\lambda) \cap \Omega, \Omega) \\ &\leq P(E_\lambda, B_r \cap \Omega) + \int_{\partial\Omega} \beta\chi_{(B_r\cap\Omega)\setminus E_\lambda} \, d\mathcal{H}^n + \mathcal{H}^n((\Omega \setminus E_\lambda) \cap \partial B_r) \\ &\leq 2\mathcal{H}^n((\Omega \setminus E_\lambda) \cap \partial B_r) + \Lambda |(B_r \setminus E_\lambda) \cap \Omega|. \end{aligned} \tag{5.14}$$

Repeating the same arguments as before we establish

$$\frac{|B_r \setminus E_\lambda|}{\omega_{n+1} r^{n+1}} \geq \frac{|(B_r \setminus E_\lambda) \cap \Omega|}{\omega_{n+1} r^{n+1}} \geq \left(\frac{\kappa}{4}\right)^{n+1} \qquad \forall r \in (0, r_0).$$

Let us now show (5.5). From (5.8) we get

$$\begin{aligned} P(E_\lambda, B_r) &= P(E_\lambda, B_r \cap \Omega) + \int_{B_r \cap \partial\Omega} \chi_{E_\lambda} \, d\mathcal{H}^n \\ &\leq \mathcal{H}^n(E_\lambda \cap \partial B_r) + \int_{B_r \cap \partial\Omega} (1+\beta)\chi_{E_\lambda} \, d\mathcal{H}^n + \Lambda |E_\lambda \cap B_r| \\ &\leq (n+1)\omega_{n+1} r^n + 2\omega_n r^n + \omega_{n+1} r^n (\Lambda r) \\ &\leq \left[(n+1)\omega_{n+1} + 2\omega_n + \omega_{n+1}\frac{\kappa(n+1)}{2}\right] r^n \end{aligned}$$

for a.e $r \in (0, r_0)$. Since $P(E_\lambda, \cdot)$ is a nonnegative measure, this inequality holds for all $r \in (0, r_0)$. This proves the upper perimeter estimate in (5.5).

The lower perimeter density estimate in (5.5) follows from (5.4) and the relative isoperimetric inequality (see for example [7, page 152]). $\square$

Theorem 5.3 (**Regularity of minimizers up to the boundary).** *Assume that* E_0 *and* β *satisfy* (5.1). *Then any nonempty minimizer* E_λ *is open in* $\mathbb{R}^{n+1}$ *and* $\Omega \cap \partial^* E_\lambda$ *is an* n*-dimensional manifold of class* $C^{2,\alpha}$ *for a suitable* $\alpha \in (0,1)$*, and* $\mathcal{H}^s(\Omega \cap (\partial E_\lambda \setminus \partial^* E_\lambda)) = 0$ *for all* $s > n-7$*. Moreover, if* $\beta \in \mathrm{Lip}(\partial\Omega)$*, then*

a) $\mathcal{H}^n((\partial E_\lambda \cap \partial\Omega)\Delta(\mathrm{Tr}(E_\lambda))) = 0$;

b) $\partial E_\lambda \cap \partial\Omega$ *is a set of finite perimeter in* $\partial\Omega$ *and*

$$\mathcal{H}^{n-1}(\partial(\partial E_\lambda \cap \partial\Omega) \setminus \partial^*(\partial E_\lambda \cap \partial\Omega)) = 0,$$

where $\partial(\partial E_\lambda \cap \partial\Omega)$ *denotes the boundary of* $\partial E_\lambda \cap \partial\Omega$ *in* $\partial\Omega$*. Moreover, if* $M_\lambda = \overline{\Omega \cap \partial E_\lambda}$*, then*

$$\partial(\partial E_\lambda \cap \partial\Omega) = M_\lambda \cap \partial\Omega.$$

c) There exists a relatively closed set $\Sigma \subset M_\lambda$ *with* $\mathcal{H}^{n-1}(\Sigma \cap \partial\Omega) = 0$ *such that in a neighborhood of any* $x \in (M_\lambda \cap \partial\Omega) \setminus \Sigma$ *the set* M_λ *is a* $C^{1,1/2}$*-manifold with boundary, and*

$$\nu_{E_\lambda} \cdot e_{n+1} = \beta \quad on\ (M_\lambda \cap \partial\Omega) \setminus \Sigma.$$

Proof. Since E_λ is a minimizer of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ in every ball $B \subset \Omega$, we can apply [44, Corollary 3.1] to prove that E_λ is open and $\Omega \cap \partial^* E_\lambda$ is $C^{2,\alpha}$ with $\Omega \cap \partial E_\lambda = \Omega \cap \partial^* E_\lambda$ for $n = 2, \ldots, 6$, and $\mathcal{H}^s(\Omega \cap (\partial E_\lambda \setminus \partial^* E_\lambda)) = 0$ for all $s > n-7$. Moreover, if $\beta \in \mathrm{Lip}(\partial\Omega)$, by (5.9) the remaning assertions follow from [21, Lemma 2.16, Theorem 1.10]. $\square$

Remark 5.4. (Compare with [41, Remark 1.4] and [45].)

a) Assume that $x \in \overline{E_\lambda}$ and $r > 0$ are such that $B_r(x) \cap E_0 = \emptyset$. Then $d_{E_0} \geq 0$ in $E_\lambda \cap B_r(x)$ and from (5.8) we get

$$P(E_\lambda, B_r \cap \Omega) - \int_{B_r \cap \partial\Omega} \beta \chi_{E_\lambda}\, d\mathcal{H}^n \leq \mathcal{H}^n(E_\lambda \cap \partial B_r). \tag{5.15}$$

Then proceeding as in the proof of Proposition 5.2 we get $|E_\lambda \cap B_r| \geq (\kappa/2)^{n+1}\, \omega_{n+1} r^{n+1}$. Moreover, from (5.15) it follows that

$$P(E_\lambda, B_r \cap \Omega) \leq \mathcal{H}^n(E_\lambda \cap \partial B_r) + \int_{B_r \cap \partial\Omega} \chi_{E_\lambda}\, d\mathcal{H}^n \leq [(n+1)\omega_{n+1} + \omega_n] r^n.$$

b) Similarly, if $x \in \overline{E_\lambda}$ and $B_r(x) \cap (\Omega \setminus E_0) = \emptyset$, then $|B_r \setminus E_\lambda| \geq (\kappa/2)^{n+1}\, \omega_{n+1} r^{n+1}$.

Observe that in both cases r *need not be* in $(0, \min\{1, \frac{\kappa(n+1)}{2\Lambda}\})$ and the assumption $x \in \partial E_\lambda$ is not necessary.

The following proposition is the analog of [41, Lemma 2.1] and [45, Proposition 3.2.1].

Proposition 5.5 (L^∞ **-bound for the distance function).** *Assume that* E_0 *and* β *are as in* (5.1) *and* $E_\lambda \in BV(\Omega, \{0,1\})$ *is a minimizer of* $\mathcal{A}_\beta(\cdot, E_0, \lambda)$. *Then*

$$\sqrt{\lambda}\|d_{E_0}\|_{L^\infty(E_\lambda \Delta E_0)} \le R(n, \kappa). \tag{5.16}$$

Proof. Let $R := R(n,\kappa)$. Suppose by contradiction that there exist $\varepsilon > 0$, $\lambda \ge 1$ and $x \in E_\lambda \Delta E_0$ such that $d_{E_0}(x) > (R+\varepsilon)\lambda^{-1/2}$. Consider first the case $x \in E_\lambda \setminus E_0$. By regularity of E_λ (Theorem 5.3) we may assume that $x \in \partial E_\lambda \setminus E_0$. Note that $B_\rho \cap E_0 = \emptyset$, where $B_\rho := B_\rho(x)$, $\rho = (R+\varepsilon)\lambda^{-1/2}/2$. Since $\mathcal{A}_\beta(E_\lambda, E_0, \lambda) \le \mathcal{A}_\beta(E_\lambda \setminus B_\rho, E_0, \lambda)$, and $\tilde{d}_{E_0}(y) = d_{E_0}(y) \ge \rho$ for any $y \in B_\rho \cap E_\lambda$, from (5.8) we establish

$$\frac{(R+\varepsilon)\lambda^{1/2}}{2}|E_\lambda \cap B_\rho| \le \lambda \int_{E_\lambda \cap B_\rho} \tilde{d}_{E_0} dy \le \mathcal{H}^n(E_\lambda \cap \partial B_\rho) + \int_{B_\rho \cap \partial\Omega} \beta\chi_{E_\lambda}\, d\mathcal{H}^n \le [\omega_{n+1}(n+1) + \omega_n]\rho^n.$$

This and Remark 5.4 (a) yield[2]

$$\omega_{n+1}\frac{(R+\varepsilon)\kappa^{n+1}}{2^{n+2}}\lambda^{1/2}\rho^{n+1} \le [\omega_{n+1}(n+1) + \omega_n]\rho^n,$$

or equivalently, recalling the definition of ρ

$$(R+\varepsilon)^2 \le 2^{n+3}\,\frac{\omega_n + (n+1)\omega_{n+1}}{\omega_{n+1}\kappa^{n+1}} = R^2,$$

which is a contradiction. A similar contradiction is obtained when $x \in E_0 \setminus E_\lambda$. □

Corollary 5.6. *Assume* (4.3) *and* $|\overline{E_0} \setminus E_0| = 0$. *If* $\|\beta\|_\infty < 1$, *then* $\overline{\Omega \cap \partial E_\lambda} \xrightarrow{K} \overline{\Omega \cap \partial E_0}$ *as* $\lambda \to +\infty$, *where* $\xrightarrow{K}$ *denotes Kuratowski convergence [40].*

Proof. It suffices to show that every diverging sequence $\{\lambda_j\}$ has a subsequence $\{\lambda'_j\}$ such that

$$K - \lim_{j\to+\infty} \overline{\Omega \cap \partial E_{\lambda'_j}} = \overline{\Omega \cap \partial E_0}.$$

Choose any sequence $\lambda_j \to +\infty$. By compactness of closed sets in Kuratowski convergence [40, page 340], there exists a closed set $C \subset \overline{\Omega}$ such that up to a not relabelled subsequence $\overline{\Omega \cap \partial E_{\lambda_j}} \xrightarrow{K} C$ as $j \to +\infty$. Let us show first that $\overline{\Omega \cap \partial E_0} \subseteq C$. Take any $x \in \mathbb{R}^{n+1} \setminus C$; we may suppose that $x \in \Omega$. Since C is closed, there exists a ball $B_\rho(x)$ such that $B_\rho(x) \cap C = \emptyset$. Since $\overline{\Omega \cap \partial E_{\lambda_j}} \xrightarrow{K} C$ as $j \to +\infty$, we have $B_\rho(x) \cap \overline{\Omega \cap \partial E_{\lambda_j}} = \emptyset$ for $j \ge 1$ large enough. Therefore, $P(E_{\lambda_j}, B_\rho(x) \cap \Omega) = 0$, and by a) and lower semicontinuity, $P(E_0, B_\rho(x) \cap \Omega) = 0$. This yields $B_{\rho/2}(x) \cap \overline{\Omega \cap \partial E_0} = \emptyset$ and thus $\mathbb{R}^{n+1} \setminus C \subseteq \mathbb{R}^{n+1} \setminus \overline{\Omega \cap \partial E_0}$.

Now suppose that there exists $x \in C \setminus \overline{\Omega \cap \partial E_0}$. Then there exists $\rho > 0$ such that $B_\rho(x) \cap \overline{\Omega \cap \partial E_0} = \emptyset$. Since $x \in C$, there exists $x_j \in \overline{\Omega \cap \partial E_{\lambda_j}}$ such that $x_j \to x$. Choose $j \in \mathbb{N}$ so large that $x_j \in B_{\rho/4}(x)$ and $R(n,\kappa)\lambda_j^{-1/2} < \rho/4$, where $R(n,\kappa)$ is defined in (5.2). By Proposition 5.5, we have

$$d_{E_0}(x_j) \le R(n,\kappa)\lambda_j^{-1/2} < \frac{\rho}{4}.$$

On the other hand, by construction, $d_{E_0}(x) \ge \frac{3\rho}{4}$, which leads to a contradiction. This yields $C \subseteq \overline{\Omega \cap \partial E_0}$. □

[2] Since the upper bound for the radii in Proposition 5.2 is of order $O(\frac{1}{\lambda})$, in general, we cannot apply it with ρ.

Proof of Theorem 5.1. We repeat the same procedures of the proof of Proposition 5.2 with improved estimates for the volume term of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$. Let $R := R(n,\kappa)$, $\gamma := \gamma(n,\kappa)$. Fix $x \in \partial^* E_\lambda$, and choose $r \in (0, \gamma\lambda^{-1/2})$ such that $\mathcal{H}^n(\partial B_r \cap \partial E_\lambda) = 0$. From (5.16) it follows

$$\sup_{(E_\lambda \setminus E_0)\cap B_r} d_{E_0} \le R\lambda^{-1/2}.$$

Therefore, using the obvious inequality

$$\sup_{(E_\lambda \cap E_0)\cap B_r} d_{E_0} \le 2r + \sup_{(E_0 \setminus E_\lambda)\cap B_r} d_{E_0} \le (2\gamma + R)\lambda^{-1/2},$$

from (5.8) we establish that

$$P(E_\lambda, B_r \cap \Omega) - \int_{B_r \cap \partial\Omega} \beta\chi_{E_\lambda}\, d\mathcal{H}^n \le \mathcal{H}^n(E_\lambda \cap \partial B_r) + (R + 2\gamma)\lambda^{1/2}|E_\lambda \cap B_r|. \tag{5.17}$$

Since $m(r) := |E_\lambda \cap B_r| \le \omega_{n+1} r^{n+1}$ and $r \le \frac{\gamma}{\lambda^{1/2}}$, similarly to (5.11) from (5.17) we deduce

$$\kappa(n+1)\omega_{n+1}^{\frac{1}{n+1}} m(r)^{\frac{n}{n+1}} \le 2m'(r) + (R + 2\gamma)\lambda^{1/2} r\omega_{n+1}^{\frac{1}{n+1}} m(r)^{\frac{n}{n+1}}, \text{ for a.e. } r \in (0, \gamma\lambda^{1/2}).$$

By the definition of γ one has

$$(R + 2\gamma)\lambda^{1/2} r \le (R + 2\gamma)\gamma = \frac{1}{2}\kappa(n+1).$$

Thus,

$$\frac{\kappa}{4}(n+1)\omega_{n+1}^{\frac{1}{n+1}} m(r)^{\frac{n}{n+1}} \le m'(r) \quad \text{for a.e. } r \in (0, \gamma\lambda^{-1/2}).$$

Integrating this differential inequality we get the lower volume density estimate in (5.4).

Let us prove the upper volume density estimate in (5.4). Due to (5.12) we may suppose that $x \in \Omega \cap \partial^* E_\lambda$. As above one can estimate d_{E_0} in $(B_r \setminus E_\lambda) \cap \Omega$ as follows:

$$\sup_{\Omega \cap ((B_r \setminus E_\lambda)\setminus E_0)} d_{E_0} \le 2r + \sup_{E_\lambda \Delta E_0} d_{E_0} \le (2\gamma + R)\lambda^{-1/2}. \tag{5.18}$$

Since $\tilde d_{E_0} \le 0$ in $\Omega \cap ((B_r \setminus E_\lambda) \cap E_0)$, plugging (5.18) in (5.13) and proceeding as above we establish

$$\frac{\kappa}{4}(n+1)\omega_{n+1}^{\frac{1}{n+1}} |(B_r \setminus E_\lambda) \cap \Omega|^{\frac{n}{n+1}} \le \mathcal{H}^n((\Omega \setminus E_\lambda) \cap B_r),$$

from which the upper volume density estimates in (5.4) follows.

The proof of (5.5) is exactly the same as the proof of perimeter density estimates in Proposition 5.2. Finally, (5.6) is a standard consequence of a covering argument. □

Let us prove the following L^1 -estimate for the minimizers of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$, the analog of [41, Lemma 1.5] and [45, Proposition 3.2.3]. Notice carefully the exponent $-1/2$ of λ in (5.19).

Proposition 5.7 (L^1 **-estimate).** *Assume that E_0 and β satisfy* (5.1) *and the uniform volume density estimates* (5.4) *holds for E_0. Then for any minimizer E_λ of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ the estimate*

$$|E_\lambda \Delta E_0| \le C_{n,\kappa} P(E_0)\,\ell + \frac{1}{\ell}\int_{E_\lambda \Delta E_0} d_{E_0} dx, \quad \ell \in \left(0, \frac{\gamma(n,\kappa)}{\lambda^{1/2}}\right) \tag{5.19}$$

holds, where

$$C_{n,\kappa} := \left(\frac{8}{\kappa}\right)^{n+1} \omega_{n+1}^{\frac{1}{n+1}}\, \mathfrak{b}(n)\, c_{n+1} \tag{5.20}$$

and $\mathfrak{b}(n)$ *is the constant in Besicovitch covering theorem.*

Proof. Set

$$A := \{x \in E_\lambda \Delta E_0 : d_{E_0}(x) \geq \ell\}, \quad B := \{x \in E_\lambda \Delta E_0 : d_{E_0}(x) < \ell\}.$$

By Chebyshev inequality

$$|A| \leq \frac{1}{\ell} \int_{E_\lambda \Delta E_0} d_{E_0} dx.$$

Let us estimate $|B|$. Since E_0 is bounded, by Besicovitch covering theorem there exist at most countably many balls $\{B_\ell(x_i)\}$, $x_i \in \partial E_0$ such that any point of ∂E_0 belongs to at most $\mathfrak{b}(n)$ balls, $\partial E_0 \subset \bigcup_i B_\ell(x_i)$ and $B \subset \bigcup_i B_{2\ell}(x_i)$. Since the balls $\{B_{2\ell}(x_i)\}$ cover B, by the density estimates (5.4) and the relative isoperimetric inequality we get

$$\begin{aligned}
|B_{2\ell}(x_i)| =& 2^{n+1}\omega_{n+1}\ell^{n+1} \leq 2^{n+1}\left(\frac{4}{\kappa}\right)^{n+1} \min\{|B_\ell(x_i) \cap E_0|, |B_\ell(x_i) \setminus E_0|\} \\
\leq& \left(\frac{8}{\kappa}\right)^{n+1} \omega_{n+1}^{\frac{1}{n+1}}\, \ell\, \min\{|B_\ell(x_i) \cap E_0|, |B_\ell(x_i) \setminus E_0|\}^{\frac{n}{n+1}} \\
\leq& \left(\frac{8}{\kappa}\right)^{n+1} \omega_{n+1}^{\frac{1}{n+1}}\, \ell\, c_{n+1}\, P(E_0, B_\ell(x_i)).
\end{aligned}$$

Therefore

$$|B| \leq \left(\frac{8}{\kappa}\right)^{n+1} \omega_{n+1}^{\frac{1}{n+1}}\, c_{n+1}\, \ell \sum_i P(E_0, B_\ell(x_i)) \leq \left(\frac{8}{\kappa}\right)^{n+1} \omega_{n+1}^{\frac{1}{n+1}}\, \mathfrak{b}(n)\, c_{n+1}\, P(E_0)\, \ell.$$

Now (5.19) follows from the estimates for $|A|, |B|$ and from $|E_\lambda \Delta E_0| \leq |A| + |B|$. □

A specific choice of ℓ will be made in the proof of Theorem 7.1. We conclude this section with a proposition about the regularity of minimizers of $C_\beta(\cdot, \Omega)$.

Proposition 5.8 (Density estimates for constrained minimizers of C_β). *Assume that* E_0 *and* β *satisfy* (5.1) *and there exist* $c_1, c_2, \varepsilon \in (0,1)$ *such that for every* $x \in \partial E_0$ *and* $r \in (0, \varepsilon)$ *the inequalities*

$$e_1 \leq \frac{|B_r(x) \cap E_0|}{|B_r(x)|} \leq e_2$$

hold. Let E^+ *be a constrained minimizer of* $C_\beta(\cdot, \Omega)$ *in* $\mathcal{E}(E_0)$*. Then for every* $x \in \partial E^+$ *and* $r \in (0, \varepsilon)$

$$\begin{aligned}
e_1 \left(\frac{\kappa}{8}\right)^{n+1} \leq& \frac{|B_r(x) \cap E^+|}{|B_r(x)|} \leq 1 - \left(\frac{\kappa}{4}\right)^{n+1}, \\
c_{n+1} e_1^{\frac{n}{n+1}} (\kappa/8)^n \leq& \frac{P(E^+, B_r(x))}{r^n} \leq (n+1)\omega_{n+1} + \omega_n.
\end{aligned} \tag{5.21}$$

In particular, $\mathcal{H}^n(\partial E^+ \setminus \partial^* E^+) = 0$.

Proof. Let $x \in \partial E^+$, and $r \in (0, \varepsilon)$ be such that $\mathcal{H}^n(\partial B_r \cap \partial^* E^+) = 0$, where $B_r := B_r(x)$.

We start with the upper volume density estimate in (5.21). We may suppose $x \in \Omega \cap \partial^* E^+$, since the case $x \in \partial\Omega \cap \partial^* E^+$ is trivial. Using $C_\beta(E^+, \Omega) \le C_\beta((E^+ \cup B_r) \cap \Omega, \Omega)$, as in (5.13) we establish

$$P(E^+, B_r) + \int_{\partial\Omega} \beta \chi_{(B_r \setminus E^+) \cap \Omega} \, d\mathcal{H}^n \le \mathcal{H}^n((\Omega \setminus E^+) \cap \partial B_r). \tag{5.22}$$

Adding $\mathcal{H}^n(\partial B_r \cap (\Omega \setminus E^+))$ to both sides and proceeding as in (5.14) we get

$$\kappa(n+1)\omega_{n+1}^{\frac{1}{n+1}} |(B_r \setminus E^+) \cap \Omega|^{\frac{n}{n+1}} \le 2\mathcal{H}^n((\Omega \setminus E^+) \cap \partial B_r)$$

and hence as in the proof of Theorem 5.1

$$|B_r \setminus E^+| \ge \left(\frac{\kappa}{4}\right)^{n+1} \omega_{n+1} r^{n+1}.$$

This implies the upper volume density estimate in (5.21).

The lower volume density estimate is a little delicate, since in general we cannot use the set $E = E^+ \setminus B_r$ as a competitor since it need not belong to $\mathcal{E}(E_0)$. If $d := d_{E_0}(x) = 0$, then $x \in \partial E_0$ and, hence, using $E_0 \cap B_r \subset E^+ \cap B_r$ and the lower volume density estimate for E_0 we establish

$$\frac{|E^+ \cap B_r|}{|B_r|} \ge \frac{|E_0 \cap B_r|}{|B_r|} \ge e_1 \ge e_1 \left(\frac{\kappa}{8}\right)^{n+1}.$$

If $d > 0$ and $r \in (0, \min\{\varepsilon, d\})$, we may use comparison set $E^+ \setminus B_r$ and as in the proof of (5.4) we obtain

$$\frac{|E^+ \cap B_r|}{|B_r|} \ge \left(\frac{\kappa}{4}\right)^{n+1} \ge e_1 \left(\frac{\kappa}{8}\right)^{n+1}. \tag{5.23}$$

Suppose $d < \varepsilon$. Since one can extend (5.23) to $(0, d]$ by continuity, if $r \in (d, \min\{2d, \varepsilon\})$, then

$$\frac{|E^+ \cap B_r|}{|B_r|} \ge \frac{|E^+ \cap B_d|}{|B_d|} \cdot \left(\frac{d}{r}\right)^{n+1} \ge \left(\frac{\kappa}{8}\right)^{n+1} \ge e_1 \left(\frac{\kappa}{8}\right)^{n+1}.$$

Let $r \in [2d, \varepsilon)$ and $x_0 \in \overline{\Omega \cap \partial E_0}$ be such that $d = |x - x_0|$. Then using $B(x, r) \supset B(x_0, r - d)$, the lower density estimate for E_0 and $r - d \ge r/2$, we obtain

$$\frac{|E^+ \cap B_r|}{|B_r|} \ge \frac{|E_0 \cap B_{r-d}(x_0)|}{|B_{r-d}(x_0)|} \cdot \left(\frac{r-d}{r}\right)^{n+1} \ge e_1 \left(\frac{1}{2}\right)^{n+1} \ge e_1 \left(\frac{\kappa}{8}\right)^{n+1}.$$

Now the lower perimeter estimate follows from the volume density estimates and the relative isoperimetric inequality. The upper perimeter estimate is obtained from (5.22):

$$P(E^+, B_r) \le \mathcal{H}^n((\Omega \setminus E^+) \cap \partial B_r) - \int_{\partial\Omega} \beta \chi_{(B_r \setminus E^+) \cap \Omega} \, d\mathcal{H}^n \le ((n+1)\omega_{n+1} + \omega_n) r^n.$$

Finally, the relation $\mathcal{H}^n(\partial E^+ \setminus \partial^* E^+) = 0$ is a consequence of the density estimates together with a covering argument. $\square$

6. Comparison principles

The main result of this section is the following comparison between minimizers of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$.

Theorem 6.1 (Comparison for minimizers of $\mathcal{A}_\beta$). *Assume that $E_0, F_0,\ \beta_1,\beta_2$ satisfy (4.3). Suppose that $E_0 \subseteq F_0$ and $\beta_1 \le \beta_2$. Then*

a) there exists a minimizer F_λ^* of $\mathcal{A}_{\beta_2}(\cdot, F_0, \lambda)$ containing any minimizer of $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$;

b) there exists a minimizer $E_{\lambda *}$ of $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$ contained in any minimizer of $\mathcal{A}_{\beta_2}(\cdot, F_0, \lambda)$.

If in addition

$$\operatorname{dist}(\Omega \cap \partial E_0, \Omega \cap \partial F_0) > 0, \tag{6.1}$$

then all minimizers E_λ and F_λ of $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$ and $\mathcal{A}_{\beta_2}(\cdot, F_0, \lambda)$ respectively, satisfy

$$E_\lambda \subseteq F_\lambda.$$

Remark 6.2. We do not exclude the case that either E_λ or F_λ is empty.

Remark 6.3. For any $E_0,\ \beta$ satisfying (4.3), using Theorem 6.1 with $\beta_1 = \beta_2 = \beta$ and $F_0 = E_0,$ we establish the existence of unique minimizers $E_{\lambda *}$ and E_λ^* of $\mathcal{A}_\beta(\cdot, E_0, \lambda),$ such that any other minimizer E_λ satisfies $E_{\lambda *} \subseteq E_\lambda \subseteq E_\lambda^*$.

Definition 6.4 (Maximal and minimal minimizers). *We call E_λ^* and $E_{\lambda *}$ the maximal and minimal minimizer of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ respectively.*

Before proving Theorem 6.1 we need the following observations. Given β satisfying (4.3), $C = C_r^h,\ h, r > 0$ and $v \in L^\infty_{\mathrm{loc}}(\Omega),\ v \ge 0$ a.e. in $\Omega \setminus C,$ define the convex functional $\mathcal{B}_\beta(\cdot, v, C) : BV(\Omega, [0,1]) \to (-\infty, +\infty],$ a sort of level-set capillary Almgren-Taylor-Wang-type functional, as

$$\mathcal{B}_\beta(u, v, C) = \mathsf{C}_\beta(u, \Omega) + \int_\Omega uv\, dx.$$

Set

$$\mathcal{R}_1(C, v) := r + 1 + \max\Big\{8^{n^2+n+1} \left(\frac{C_\beta(C, \Omega) + \|v\|_{L^\infty(C)}|C|}{\kappa}\right)^{\frac{n+1}{n}},\ 4\mu(\kappa, n)\Big\}, \tag{6.2}$$

where $\mu(\kappa, n) = (1/\kappa + 2)^{\frac{n+1}{n}}$. By Example A.2 the functional

$$\mathcal{V} : BV(\Omega, \{0,1\}) \to (-\infty, +\infty], \qquad \mathcal{V}(E) := \int_E v dx$$

satisfies Hypothesis A.1. Thus, by Theorem A.3 the functional $E \in BV(\Omega, \{0, 1\}) \mapsto \mathcal{B}_\beta(\chi_E, v, C) \in \mathbb{R}$ has a minimizer, and every minimizer E_v satisfies

$$E_v \subseteq C^h_{\mathcal{R}_1(C,v)}. \tag{6.3}$$

Notice that by (2.8) and (3.2),

$$\mathcal{B}_\beta(u, v, C) = \int_0^1 \mathcal{B}_\beta(\chi_{\{u>t\}}, v, C)\, dt \quad \forall u \in BV(\Omega, [0,1]), \tag{6.4}$$

which yields that χ_{E_v} is a minimizer of $\mathcal{B}_\beta(\cdot, v, C)$ in $BV(\Omega, [0, 1])$.

The following remark is in the spirit of [13, Section 1].

Remark 6.5 (**Minimality of level sets).** From (6.4) it follows that $u \in BV(\Omega,[0,1])$ is a minimizer of $\mathcal{B}_\beta(\cdot,v,C)$ in $BV(\Omega,[0,1])$ if and only if $\chi_{\{u>t\}}$ is a minimizer of $\mathcal{B}_\beta(\cdot,v,C)$ for a.e. $t\in[0,1]$. Indeed, let for some $u\in BV(\Omega,[0,1])$ the function $\chi_{\{u>t\}}$ be a minimizer of $\mathcal{B}_\beta(\cdot,v,C)$ for a.e. $t\in[0,1]$. Then for any $w\in BV(\Omega,[0,1])$ and for a.e. $t\in[0,1]$ one has $\mathcal{B}_\beta(w,v,C)\ge\mathcal{B}_\beta(\chi_{\{u>t\}},v,C)$, therefore,

$$\mathcal{B}_\beta(u,v,C)=\int_0^1\mathcal{B}_\beta(\chi_{\{u>t\}},v,C)dt\le\mathcal{B}_\beta(w,v,C).$$

Conversely, if $u\in BV(\Omega,[0,1])$ is a minimizer of $\mathcal{B}_\beta(\cdot,v,C)$, then for a.e. $t\in[0,1]$ one has $\mathcal{B}_\beta(u,v,C)\le\mathcal{B}_\beta(\chi_{\{u>t\}},v,C)$. Hence, from (6.4) it follows that $\mathcal{B}_\beta(u,v,C)=\mathcal{B}_\beta(\chi_{\{u>t\}},v,C)$ for a.e. $t\in[0,1]$. In particular, if $u\in BV(\Omega,[0,1])$ is a minimizer of $\mathcal{B}_\beta(\cdot,v,C)$, then by (6.3) $\{u>t\}\subseteq C^h_{\mathcal{R}_1(C,v)}$ for a.e. $t\in[0,1]$, i.e. $u=0$ a.e. in $\Omega\setminus C^h_{\mathcal{R}_1(C,v)}$. Hence,

$$\min_{u\in BV(\Omega,[0,1])}\mathcal{B}_\beta(u,v,C)=\min_{u\in BV(\Omega,[0,1]),\ u=0\ \text{ a.e. in }\ \Omega\setminus C^h_{\mathcal{R}_1(C,v)}}\mathcal{B}_\beta(u,v,C). \tag{6.5}$$

Lemma 6.6. *Let E_0, β satisfy* (4.3)*, and R_0 be defined as in* (4.4)*. Then E_λ is a minimizer of $\mathcal{A}_\beta(\cdot,E_0,\lambda)$ if and only if χ_{E_λ} is a minimizer of $\mathcal{B}_\beta(\cdot,v^\lambda_{E_0},C^H_{R_0})$, where $v^\lambda_{E_0}=\lambda\chi_{C^H_{R_0}}\tilde d_{E_0}$.*

Proof. By (4.2) we have

$$\mathcal{A}_\beta(E,E_0,\lambda)=\mathcal{B}_\beta(\chi_E,v^\lambda_{E_0},C^H_{R_0})-\lambda\int_{E_0}\tilde d_{E_0}\,dx\qquad\forall E\in BV(C^H_{R_0},\{0,1\}). \tag{6.6}$$

Now if E_λ minimizes $\mathcal{A}_\beta(\cdot,E_0,\lambda)$, we have $E_\lambda\subseteq C^H_{R_0}$ (Theorem 4.1) and thus, for any $u\in BV(\Omega,[0,1])$ with $u=0$ a.e. in $\Omega\setminus C^H_{R_0}$ from (6.4)-(6.6) we deduce

$$\begin{aligned}\mathcal{B}_\beta(u,v^\lambda_{E_0},C^H_{R_0})&=\int_0^1\mathcal{B}_\beta(\chi_{\{u>t\}},v^\lambda_{E_0},C^H_{R_0})\,dt=\int_0^1\mathcal{A}_\beta(\{u>t\},E_0,\lambda)\,dt+\lambda\int_{E_0}\tilde d_{E_0}\,dx\\&\ge\int_0^1\mathcal{A}_\beta(E_\lambda,E_0,\lambda)\,dt+\lambda\int_{E_0}\tilde d_{E_0}\,dx=\mathcal{B}_\beta(\chi_{E_\lambda},v^\lambda_{E_0},C^H_{R_0}).\end{aligned}$$

By (6.5) χ_{E_λ} is a minimizer of $\mathcal{B}_\beta(\cdot,v^\lambda_{E_0},C^H_{R_0})$.

Conversely, assume that χ_{E_λ} is a minimizer of $\mathcal{B}_\beta(\cdot,v^\lambda_{E_0},C^H_{R_0})$, then by (6.6) $E_\lambda\subseteq C^H_{R_0}$ is a minimizer of $\mathcal{A}_\beta(\cdot,E_0,\lambda)$ in $BV(C^H_{R_0},\{0,1\})$. Hence, by Remark 4.3 E_λ is a minimizer of $\mathcal{A}_\beta(\cdot,E_0,\lambda)$. □

Proposition 6.7 (**Strong comparison for minimizers of $\mathcal{B}_\beta$**). *Assume that $v_1,v_2\in L^\infty_{\rm loc}(\Omega)$, $v_1>v_2$ a.e. in Ω and $v_2\ge0$ a.e. in $\Omega\setminus C$. Suppose also that $\beta_1\le\beta_2$ satisfy* (4.3)*. Let $u_1,u_2\in BV(\Omega,[0,1])$ be minimizers of $\mathcal{B}_{\beta_1}(\cdot,v_1,C)$ and $\mathcal{B}_{\beta_2}(\cdot,v_2,C)$ respectively. Then $u_1\le u_2$ a.e. in Ω.*

Proof. Adding the inequalities $\mathcal{B}_{\beta_1}(u_1,v_1,C)\le\mathcal{B}_{\beta_1}(u_1\wedge u_2,v_1,C)$ and $\mathcal{B}_{\beta_2}(u_2,v_2,C)\le\mathcal{B}_{\beta_2}(u_1\vee u_2,v_2,C)$ and using

$$\int_\Omega|D(u_1\wedge u_2)|+\int_\Omega|D(u_1\vee u_2)|\le\int_\Omega|Du_1|+\int_\Omega|Du_2|,$$

we establish

$$\int_{\partial\Omega\cap\{u_1>u_2\}} (\beta_2-\beta_1)(u_1-u_2)\,d\mathcal{H}^n \le \int_{\{u_1>u_2\}} (v_2-v_1)(u_1-u_2)\,dx.$$

Since $v_1 > v_2$ and $\beta_1 \le \beta_2$, this inequality holds if and only if $|\{u_1 > u_2\}| = 0$, i.e. $u_1 \le u_2$ a.e. in Ω. □

Proposition 6.8 (Comparison for minimizers of $\mathcal{B}_\beta$). *Assume that $v_1, v_2 \in L^\infty_{\mathrm{loc}}(\Omega)$, $v_1 \ge v_2$ a.e. in Ω and $v_2 \ge 0$ a.e. in $\Omega \setminus C$. Suppose also that $\beta_1 \le \beta_2$ satisfy* (4.3). *Then:*

a) there exists a minimizer u_{1*} of $\mathcal{B}_{\beta_1}(\cdot, v_1, C)$ such that $u_{1*} \le u_2$ for any minimizer u_2 of $\mathcal{B}_{\beta_2}(\cdot, v_2, C)$;

b) there exists a minimizer u_2^* of $\mathcal{B}_{\beta_2}(\cdot, v_2, C)$ such that $u_1 \le u_2^*$ for any minimizer u_1 of $\mathcal{B}_{\beta_1}(\cdot, v_1, C)$.

Proof. a) Take $\varepsilon \in (0,1)$. Since $v_1 + \varepsilon > v_2$ a.e. in Ω, by Proposition 6.7 any minimizer $u_1^\varepsilon, u_2 \in BV(\Omega, [0,1])$ of $\mathcal{B}_{\beta_1}(\cdot, v_1+\varepsilon, C)$ and $\mathcal{B}_{\beta_2}(\cdot, v_2, C)$ respectively, satisfies $u_1^\varepsilon \le u_2$. Let $\mathcal{R}_1 := \max\{\mathcal{R}_1(C, v_1), \mathcal{R}_1(C, v_2)\}$. By minimality, $\mathcal{B}_{\beta_1}(u_1^\varepsilon, v_1+\varepsilon, C) \le \mathcal{B}_{\beta_1}(0, v_1+\varepsilon, C) = 0$, and since by Remark 6.5 $u_1^\varepsilon = 0$ a.e. in $\Omega \setminus C^h_{\mathcal{R}_1}$, recalling (3.10) we get

$$\kappa \int_\Omega |Du_1^\varepsilon| \le (\|v_1\|_{L^\infty(C^h_{\mathcal{R}_1})} + 1)|C^h_{\mathcal{R}_1}| < +\infty.$$

By compactness, there exists $u_{1*} \in BV(\Omega, [0,1])$ such that, up to a (not relabelled) subsequence, $u_1^\varepsilon \to u_{1*}$ in $L^1(\Omega)$ and a.e. in Ω as $\varepsilon \to 0^+$. Then any minimizer u_2 of $\mathcal{B}_{\beta_2}(\cdot, v_2, C)$ satisfies $u_{1*} \le u_2$ a.e. in Ω.

It remains to show that u_{1*} is a minimizer of $\mathcal{B}_{\beta_1}(\cdot, v_1, C)$. By (6.5) we may consider only those $u \in BV(\Omega, [0,1])$ with $u = 0$ a.e. in $\Omega \setminus C^h_{\mathcal{R}_1}$ as a competitor. In this case, the continuity of $u \mapsto \int_{C^h_{\mathcal{R}_1}} uv\,dx$, the minimality of u_1^ε and the lower semicontinuity of $\mathsf{C}_\beta(\cdot, \Omega)$ imply

$$\begin{aligned}
\mathcal{B}_{\beta_1}(u, v_1, C) &= \lim_{\varepsilon\to 0^+} \mathcal{B}_{\beta_1}(u, v_1+\varepsilon, C) \ge \liminf_{\varepsilon\to 0^+} \mathcal{B}_{\beta_1}(u_1^\varepsilon, v_1+\varepsilon, C) \\
&\ge \liminf_{\varepsilon\to 0^+} \mathsf{C}_{\beta_1}(u_1^\varepsilon, \Omega) + \lim_{\varepsilon\to 0^+} \int_{C^h_{\mathcal{R}_1}} u_1^\varepsilon (v_1+\varepsilon)\,dx \\
&\ge \mathsf{C}_{\beta_1}(u_{1*}, \Omega) + \int_{C^h_{\mathcal{R}_1}} u_{1*} v_1\,dx = \mathcal{B}_{\beta_1}(u_{1*}, v_1, C).
\end{aligned}$$

b) can be proven in a similar manner. □

Proof of Theorem 6.1. Let $R := \max\{R(E_0), R(F_0)\}$, where $R(E_0)$ and $R(F_0)$ are defined as in (4.4). Then by Theorem 4.1 any minimizer E_λ (resp. F_λ) of $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$ (resp. $\mathcal{A}_{\beta_2}(\cdot, F_0, \lambda)$) is contained in the cylinder $C := \hat{B}_R \times (0, H)$, where

$$H = 1 + \max\left\{ \max_{(x', x_{n+1}) \in \overline{E_0}} x_{n+1},\ \max_{(x', x_{n+1}) \in \overline{F_0}} x_{n+1} \right\}.$$

Set $v_1 := v_1(\lambda, E_0) = \lambda \tilde{d}_{E_0}$ and $v_2 := v_2(\lambda, F_0) = \lambda \tilde{d}_{F_0}$. Since $E_0 \subseteq F_0 \subset \Omega$, we have $\tilde{d}_{E_0} \ge \tilde{d}_{F_0}$. Moreover, by (4.3) there exists a cylinder $C := C^H_D$ such that $v_2 \ge 0$ in $\Omega \setminus C$.

a) Since $v_1 \geq v_2$ and $\beta_1 \leq \beta_2$, by Proposition 6.8 b) there exists a minimizer $u_2^* := u_2^*(\lambda, F_0)$ of $\mathcal{B}_{\beta_2}(\cdot, v_2, C)$ such that any minimizer u_1 of $\mathcal{B}_{\beta_1}(\cdot, v_1, C)$ satisfies

$$u_1 \leq u_2^*. \tag{6.7}$$

By Remark 6.5 there exists $t \in (0,1)$ such that $\chi_{\{u_2^* > t\}}$ is a minimizer of $\mathcal{B}_{\beta_2}(\cdot, v_2, C)$. Then, recalling the expression of v_2, by Lemma 6.6 $F_\lambda^* := \{u_2^* > t\}$ is a minimizer of $\mathcal{A}_{\beta_2}(\cdot, F_0, \lambda)$. Moreover, if E_λ is a minimizer of $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$, then by Lemma 6.6 χ_{E_λ} is a minimizer of $\mathcal{B}_{\beta_1}(\cdot, v_1, C)$, and by (6.7) $\chi_{E_\lambda} \leq u_2^*$. In particular,

$$E_\lambda = \{\chi_{E_\lambda} > t\} \subseteq \{u_2^* > t\} =: F_\lambda^*.$$

b) is analogous to a) using Proposition 6.8 a).

The last assertion follows with the same arguments from Lemma 6.6 and Proposition 6.7, since (6.1) implies that $\tilde{d}_{E_0} > \tilde{d}_{F_0}$. □

One useful case is when E_0 is a constrained minimizer of $C_\beta(\cdot, \Omega)$ in $\mathcal{E}(E_0)$: in this case E_0 acts as a barrier for minimizers of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$.

Proposition 6.9. *Assume that* E_0, β_1, β_2 *satisfy* (4.3). *Let* $\beta_1 \leq \beta_2$, E_0 *be a constrained minimizer of* $C_{\beta_2}(\cdot, \Omega)$ *in* $\mathcal{E}(E_0)$ *and* $E_\lambda \in BV(\Omega, \{0,1\})$ *be a minimizer of* $\mathcal{A}_{\beta_1}(\cdot, E_0, \lambda)$. *Then* $E_\lambda \subseteq \overline{E_0}$.

Proof. Comparing E_λ with $E_0 \cap E_\lambda$ we get

$$P(E_\lambda, \Omega) + \lambda \int_{E_\lambda \setminus E_0} \tilde{d}_{E_0}\, dx \leq P(E_\lambda \cap E_0, \Omega) + \int_{\partial\Omega} \beta_1 \chi_{E_\lambda \setminus E_0}\, d\mathcal{H}^n.$$

From the constrained minimality of E_0 we have $C_{\beta_2}(E_0, \Omega) \leq C_{\beta_2}(E_0 \cup E_\lambda, \Omega)$, i.e.

$$P(E_0, \Omega) \leq P(E_0 \cup E_\lambda, \Omega) - \int_{\partial\Omega} \beta_2 \chi_{E_\lambda \setminus E_0}\, d\mathcal{H}^n.$$

Adding these inequalities we obtain

$$\begin{aligned} P(E_\lambda, \Omega) + P(E_0, \Omega) + \lambda \int_{E_\lambda \setminus E_0} \tilde{d}_{E_0}\, dx \leq & P(E_\lambda \cup E_0, \Omega) + P(E_\lambda \cap E_0, \Omega) \\ & + \int_{\partial\Omega} (\beta_1 - \beta_2) \chi_{E_\lambda \setminus E_0}\, d\mathcal{H}^n. \end{aligned}$$

Then the condition $\beta_1 \leq \beta_2$ and (2.2) yield that

$$\lambda \int_{E_\lambda \setminus E_0} \tilde{d}_{E_0}\, dx \leq 0.$$

Since $\tilde{d}_{E_0} > 0$ outside $\overline{E_0}$, the last inequality is possible only if $|E_\lambda \setminus \overline{E_0}| = 0$, i.e. $E_\lambda \subseteq \overline{E_0}$. □

Proposition 6.9 gives the following monotonicity principle.

Proposition 6.10 (Monotonicity). *Assume that* E_0, β *satisfy* (4.3), E_0 *is a constrained minimizer of* $C_\beta(\cdot, \Omega)$ *in* $\mathcal{E}(E_0)$ *such that* $|\overline{E_0} \setminus E_0| = 0$ *and* $E_\alpha \in BV(\Omega, \{0,1\})$ *is a minimizer of* $\mathcal{A}_\beta(\cdot, E_0, \alpha)$ *for* $\alpha \geq 1$. *Then* $E_\lambda \subseteq E_\mu$ *for any* $1 \leq \lambda < \mu$. *Moreover, every* E_α, $\alpha \geq 1$ *is also a constrained minimizer of* $C_\beta(\cdot, \Omega)$ *in* $\mathcal{E}(E_\alpha)$.

Proof. Comparison between E_λ and $E_\lambda \cap E_\mu$ gives

$$P(E_\lambda, \Omega) + \lambda \int_{E_\lambda \setminus E_\mu} \tilde{d}_{E_0}\, dx \le P(E_\lambda \cap E_\mu, \Omega) + \int_{\partial\Omega} \beta \chi_{E_\lambda \setminus E_\mu}\, d\mathcal{H}^n.$$

Similarly, for E_μ and $E_\lambda \cup E_\mu$ we have

$$P(E_\mu, \Omega) \le P(E_\lambda \cup E_\mu, \Omega) + \mu \int_{E_\lambda \setminus E_\mu} \tilde{d}_{E_0}\, dx - \int_{\partial\Omega} \beta \chi_{E_\lambda \setminus E_\mu}\, d\mathcal{H}^n.$$

Adding the above inequalities and using (2.2) we obtain

$$(\lambda - \mu) \int_{E_\lambda \setminus E_\mu} \tilde{d}_{E_0}\, dx \le 0. \tag{6.8}$$

By hypothesis $|\overline{E_0} \setminus E_0| = 0$, according to Proposition 6.9, $E_\lambda, E_\mu \subseteq E_0$, Thus $\tilde{d}_{E_0} \le 0$ in $E_\lambda \setminus E_\mu$. But since $\lambda < \mu$, (6.8) is possible only if $|E_\lambda \setminus E_\mu| = 0$, i.e. $E_\lambda \subseteq E_\mu$.

To prove the final assertion take any set $E \in \mathcal{E}(E_\alpha)$. Then using $\mathcal{A}_\beta(E_\alpha, E_0, \alpha) \le \mathcal{A}_\beta(E_\alpha \cap E_0, E_0, \alpha)$, $\alpha \int_{(E_0 \cap E) \setminus E_\alpha} d_{E_0} dx \ge 0$, and $E_\alpha \subseteq E_0 \cap E$, we get

$$C_\beta(E_\alpha, \Omega) \le C_\beta(E_\alpha, \Omega) + \alpha \int_{(E_0 \cap E) \setminus E_\alpha} d_{E_0} dx \le C_\beta(E \cap E_0, \Omega).$$

Moreover, since $C_\beta(E_0, \Omega) \le C_\beta(E \cup E_0, \Omega)$, from (2.2) we obtain

$$C_\beta(E_\alpha, \Omega) + C_\beta(E_0, \Omega) \le C_\beta(E_0 \cap E, \Omega) + C_\beta(E_0 \cup E, \Omega) \le C_\beta(E, \Omega) + C_\beta(E_0, \Omega),$$

i.e. $C_\beta(E_\alpha, \Omega) \le C_\beta(E, \Omega)$. □

Proposition 6.11 (Comparison between minimizers of C_β and $\mathcal{A}_\beta$). *Suppose that E_0 and β satisfy* (4.3).

a) Let $E^+ \in BV(\Omega, \{0,1\})$ be a constrained minimizer of $C_\beta(\cdot, \Omega)$ in $\mathcal{E}(E_0)$. Then every minimizer E_λ of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ satisfies $E_\lambda \subseteq \overline{E^+}$.

b) Let $E^+ \in BV(\Omega, \{0,1\})$ be a bounded constrained minimizer of $C_\beta(\cdot, \Omega)$ in $\mathcal{E}(E^+)$. Then for every $E_0 \subseteq E^+$ and for every minimizer E_λ of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$ one has $E_\lambda \subseteq \overline{E^+}$. Moreover, E^+ can be chosen such that $|\overline{E^+} \setminus E^+| = 0$.

Proof. a) By Proposition 4.4 E^+ is a constrained minimizer of $C_\beta(\cdot, \Omega)$ in $\mathcal{E}(E^+)$. Let E^+_λ be the maximal minimizer of $\mathcal{A}_\beta(\cdot, E^+, \lambda)$ (Definition 6.4). By Proposition 6.9 we have $E^+_\lambda \subseteq \overline{E^+}$. Take any minimizer E_λ of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$. Since $E_0 \subseteq E^+$, by Theorem 6.1 a) we have

$$E_\lambda \subseteq E^+_\lambda \subseteq \overline{E^+}.$$

b) The proof of the first part is exactly the same as the proof of a). To prove the second part, we take any $E'_0 \in BV(\Omega, \{0,1\})$ satisfying the hypotheses of Proposition 5.8 and containing E_0. By Theorem 4.4 there exists a constrained minimizer E^+ of $C_\beta(\cdot, \Omega)$ in $\mathcal{E}(E'_0)$. In particular, E^+ is bounded, and by Proposition 5.8 $\mathcal{H}^n(\partial E^+) = P(E^+) < +\infty$. Since $\overline{E^+} \setminus E^+ \subseteq \partial E^+$, we have $|\overline{E^+} \setminus E^+| = 0$. □

7. Existence of a generalized minimizing movement

Consider the functional $\widehat{\mathcal{A}_\beta} : BV(\Omega,\{0,1\})\times BV(\Omega,\{0,1\})\times[1,+\infty)\times\mathbb{Z} \to [-\infty,+\infty]$ given by

$$\widehat{\mathcal{A}_\beta}(F,G,\lambda,k) := \begin{cases} \mathcal{A}_\beta(F,G,\lambda) & \text{if } k>0,\\ |F\Delta G| & \text{if } k\le 0. \end{cases}$$

For any $k\in\mathbb{N}$ we build the family of sets $E_\lambda(k)$ iteratively as follows: $E_\lambda(0):=E_0$ and $E_\lambda(k)$, $k\ge 1$, is a minimizer of $\widehat{\mathcal{A}_\beta}(\cdot,E_\lambda(k-1),\lambda,k)$ in $BV(\Omega,\{0,1\})$; notice that existence of minimizers follows from Theorem 4.1.

From now on, we omit the dependence on k of $\widehat{\mathcal{A}_\beta}$, and we use the notation $\widehat{\mathcal{A}_\beta}(F,G,\lambda)$.

Theorem 7.1 (Existence). *Let E_0 and β satisfy (5.1). Then $GMM(E_0)$ is nonempty, i.e. there exist a map $t\in[0,+\infty)\mapsto E(t)\in BV(\Omega,\{0,1\})$ and a diverging sequence $\{\lambda_j\}\subset[1,+\infty)$ such that*

$$\lim_{j\to+\infty} |E_{\lambda_j}([\lambda_j t])\Delta E(t)| = 0, \qquad t\in[0,+\infty). \tag{7.1}$$

Moreover, every GMM $t\in[0,+\infty)\mapsto E(t)$ starting from E_0 is contained in a bounded set E^+ depending only on E_0 and β, and belongs to $C^{1/2}_{\mathrm{loc}}((0,+\infty),L^1(\Omega))$, in the sense that

$$|E(t)\Delta E(t')| \le \theta(n,\kappa)P(E_0)|t-t'|^{1/2} \qquad \textit{for all } t,t'>0,\ |t-t'|<1, \tag{7.2}$$

where $\theta(n,\kappa)=\frac{C_{n,\kappa}}{\kappa}+1$ and $C_{n,\kappa}$ is defined in (5.20). If in addition $|\overline{E_0}\setminus E_0|=0$, then (7.2) holds for any $t,t'\ge 0$ with $|t-t'|<1$. Finally,

$$\nu_{E_{\lambda_j}([\lambda_j t])}\mathcal{H}^n \llcorner \partial^* E_{\lambda_j}([\lambda_j t]) \overset{w^*}{\rightharpoonup} \nu_{E(t)}\mathcal{H}^n\llcorner\partial^* E(t) \quad \textit{for all } t\ge 0 \textit{ as } \lambda_j\to+\infty. \tag{7.3}$$

Proof. Given $k\ge 0$ set $d_k(\cdot):=\mathrm{dist}(\cdot,\Omega\cap\partial E_\lambda(k))$. Then for $k\ge 1$ the minimality of $E_\lambda(k)$ entails

$$\mathcal{A}_\beta(E_\lambda(k),E_\lambda(k-1),\lambda) \le \mathcal{A}_\beta(E_\lambda(k-1),E_\lambda(k-1),\lambda),$$

i.e.

$$C_\beta(E_\lambda(k),\Omega) + \lambda\int_{E_\lambda(k)\Delta E_\lambda(k-1)} d_{k-1}dx \le C_\beta(E_\lambda(k-1),\Omega). \tag{7.4}$$

In particular, the sequence $k\in\mathbb{N}\cup\{0\}\mapsto C_\beta(E_\lambda(k),\Omega)$ is nonincreasing and

$$C_\beta(E_\lambda(k),\Omega)\le C_\beta(E_\lambda(0),\Omega) = C_\beta(E_0,\Omega)\le P(E_0). \tag{7.5}$$

Let $t>0$ and set $k=[\lambda t]$. Then (3.9) yields

$$\kappa P(E_\lambda([\lambda t])) \le C_\beta(E_\lambda([\lambda t]),\Omega)\le P(E_0). \tag{7.6}$$

Take $t_1,t_2>0$, $t_1<t_2$ and let $\lambda\ge 1$ be large enough that for some $k_0,N\in\mathbb{N}$, $N\ge 3$

$$k_0=[\lambda t_1],\quad k_0+N-1=[\lambda t_2],$$

i.e.

$$\frac{k_0}{\lambda}\le t_1 < \frac{k_0+1}{\lambda} < \ldots < \frac{k_0+N-1}{\lambda}\le t_2 < \frac{k_0+N}{\lambda}.$$

Then

$$\frac{N-2}{\lambda} = \frac{k_0 + N - 1 - (k_0+1)}{\lambda} \le t_2 - t_1. \tag{7.7}$$

Since all $E_\lambda(s)$, $s \ge 1$ satisfy uniform density estimates (5.4)-(5.5) (Theorem 5.1), by Proposition 5.7 we have[3]

$$\begin{aligned} |E_\lambda([\lambda t_2])\Delta E_\lambda([\lambda t_1])| = |E_\lambda(k_0+N-1)\Delta E_\lambda(k_0)| \le \sum_{s=k_0}^{k_0+N-2} |E_\lambda(s)\Delta E_\lambda(s+1)| \\ \le C_{n,\kappa}\ell \sum_{s=k_0}^{k_0+N-2} P(E_\lambda(s)) + \frac{1}{\ell}\sum_{s=k_0}^{k_0+N-2} \int_{E_\lambda(s+1)\Delta E_\lambda(s)} d_{E_\lambda(s)}\,dx \end{aligned} \tag{7.8}$$

for any $\ell \in (0, \frac{\gamma(n,\kappa)}{\lambda^{1/2}})$. The first sum can be estimated using (7.6):

$$\sum_{s=k_0}^{k_0+N-2} P(E_\lambda(s)) \le \frac{P(E_0)}{\kappa}\,(N-1). \tag{7.9}$$

Moreover, for any $s \in \mathbb{N}$, by (7.4)

$$\int_{E_{\lambda_j}(s+1)\Delta E_\lambda(s)} d_{E_\lambda(s)}\,dx \le \frac{1}{\lambda}\Big(C_\beta(E_\lambda(s),\Omega) - C_\beta(E_\lambda(s+1),\Omega)\Big),$$

and thus

$$\begin{aligned} \sum_{s=k_0}^{k_0+N-2} \int_{E_\lambda(s+1)\Delta E_\lambda(s)} d_{E_\lambda(s)}\,dx \le & \frac{1}{\lambda}\sum_{s=k_0}^{k_0+N-2} \Big(C_\beta(E_\lambda(s),\Omega) - C_\beta(E_\lambda(s+1),\Omega)\Big) \\ = & \frac{1}{\lambda}\Big(C_\beta(E_\lambda(k_0),\Omega) - C_\beta(E_\lambda(k_0+N-1),\Omega)\Big). \end{aligned}$$

Using (7.5) and the nonnegativity of $C_\beta(\cdot,\Omega)$ we get

$$\sum_{s=k_0}^{k_0+N-2} \int_{E_\lambda(s+1)\Delta E_\lambda(s)} d_{E_\lambda(s)}\,dx \le \frac{P(E_0)}{\lambda}. \tag{7.10}$$

Thus, from (7.8), (7.9) and (7.10)

$$|E_\lambda([\lambda t_1])\Delta E_\lambda([\lambda t_2])| \le \frac{C_{n,\kappa}P(E_0)}{\kappa}\,(N-1)\ell + \frac{P(E_0)}{\lambda\ell}. \tag{7.11}$$

Now take λ so large that

$$t_2 - t_1 > \frac{1}{\gamma(n,\kappa)^2\,\lambda},$$

[3] Notice that at this point we use $t_1 > 0$; since a priori we do not know whether E_0 satisfies the density estimates, we cannot start summing from $s = 0 = k_0$.

so that Proposition 5.7 holds for $\ell = \frac{1}{\lambda|t_2-t_1|^{1/2}}$. From (7.11) and (7.7) we obtain

$$
\begin{aligned}
\left|E_\lambda([\lambda t_1])\Delta E_\lambda([\lambda t_2])\right| \leq & \frac{C_{n,\kappa}P(E_0)}{\kappa}\,\frac{N-2}{\lambda|t_2-t_1|^{1/2}} + \frac{1}{\lambda}\,\frac{C_{n,\kappa}P(E_0)}{\kappa|t_2-t_1|^{1/2}} + P(E_0)\,|t_2-t_1|^{1/2} \\
\leq & \theta(n,\kappa)P(E_0)\,|t_2-t_1|^{1/2} + \frac{1}{\lambda}\,\frac{C_{n,\kappa}P(E_0)}{\kappa|t_2-t_1|^{1/2}}.
\end{aligned}
\tag{7.12}
$$

By Proposition 6.11 b) there exists a constrained minimizer $E^+ \supseteq E_0$ of $C_\beta(\cdot,\Omega)$ in $\mathcal{E}(E^+)$ such that $|\overline{E^+}\setminus E^+| = 0$ and $E_\lambda(1) \subseteq E^+$. By induction, we can show that $E_\lambda(k) \subseteq E^+$ for all $k \geq 1$. Consider now an arbitrary diverging sequence $\{\lambda_j\}$. Compactness and a diagonal process yield the existence of a subsequence (still denoted by $\{\lambda_j\}$) such that $E_{\lambda_j}([\lambda_j t])$ converges in $L^1(\Omega)$ to a set $E(t)$ for any rational $t \geq 0$ as $j \to +\infty$.

If $t_1, t_2 \in \mathbb{Q}\cap(0,+\infty)$, with $0 < |t_1 - t_2| < 1$, letting $\lambda_j \to +\infty$ in (7.12) we get

$$
|E(t_1)\Delta E(t_2)| \leq \theta(n,\kappa)P(E_0)|t_2-t_1|^{1/2}. \tag{7.13}
$$

By completeness of $L^1(\Omega)$ we can uniquely extend $\{E(t) : t \in \mathbb{Q}\cap(0,+\infty)\}$ to a family $\{E(t) : t \in (0,+\infty)\}$ preserving the Hölder continuity (7.13) in $(0,+\infty)$. Now we show (7.1). If $t = 0$, $E_0 = E_{\lambda_j}(0) \to E(0)$ in $L^1(\Omega)$ as $j \to +\infty$. If $t > 0$, take any $\varepsilon \in (0,1)$ and $t_\varepsilon \in \mathbb{Q}\cap(0,+\infty)$ such that $|t-t_\varepsilon| < \varepsilon$. By the choice of $\{\lambda_j\}$, (7.1) holds for t_ε and thus, using (7.12)-(7.13) we get

$$
\begin{aligned}
\limsup_{j\to+\infty}|E_{\lambda_j}([\lambda_j t])\Delta E(t)| \leq & \limsup_{j\to+\infty}|E_{\lambda_j}([\lambda_j t])\Delta E_{\lambda_j}([\lambda_j t_\varepsilon])| \\
& + \limsup_{j\to+\infty}|E_{\lambda_j}([\lambda_j t_\varepsilon])\Delta E(t_\varepsilon)| + |E(t_\varepsilon)\Delta E(t)| \\
\leq & 2\theta(n,\kappa)P(E_0)|t-t_\varepsilon|^{1/2} < 2\theta(n,\kappa)P(E_0)\sqrt{\varepsilon}.
\end{aligned}
$$

Therefore, letting $\varepsilon \to 0^+$ we get (7.1).

When $|\overline{E_0}\setminus E_0| = 0$, for any $t \in (0,1)$, choosing λ sufficiently large, from (7.12) we obtain

$$
\begin{aligned}
|E_\lambda([\lambda t])\Delta E(0)| \leq & |E_\lambda([\lambda t])\Delta E_\lambda(1)| + |E_\lambda(1)\Delta E_0| \\
\leq & \theta(n,\kappa)P(E_0)\left|t-\frac{1}{\lambda}\right|^{1/2} + \frac{1}{\lambda}\,\frac{C_{n,\kappa}P(E_0)}{\kappa|t-\frac{1}{\lambda}|^{1/2}} + |E_\lambda(1)\Delta E_0|.
\end{aligned}
\tag{7.14}
$$

By Lemma 4.6 a) the last term on the right hand side converges to 0 as $\lambda \to +\infty$. Hence letting $\lambda \to +\infty$ in (7.14) we get the $(1/2)$-Hölder continuity of $t \mapsto E(t)$ in $[0,+\infty)$.

Now let us prove (7.3). We need to show that for any $t \in [0,+\infty)$

$$
\lim_{j\to+\infty}\int_{\partial^* E_{\lambda_j}([\lambda_j t])} \phi\cdot\nu_{E_{\lambda_j}([\lambda_j t])}\,d\mathcal{H}^n = \int_{\partial^* E(t)} \phi\cdot\nu_{E(t)}\,d\mathcal{H}^n \quad \forall\phi \in C_c(\mathbb{R}^{n+1},\mathbb{R}^{n+1}).
$$

If $\phi \in C^1_c(\mathbb{R}^{n+1},\mathbb{R}^{n+1})$, by the generalized divergence formula (2.3) and by (7.1) we have

$$
\begin{aligned}
\lim_{j\to+\infty}\int_{\partial^* E_{\lambda_j}([\lambda_j t])} \phi\cdot\nu_{E_{\lambda_j}([\lambda_j t])}\,d\mathcal{H}^n = & \lim_{j\to+\infty}\int_{E_{\lambda_j}([\lambda_j t])} \operatorname{div}\phi\,d\mathcal{H}^n \\
= & \int_{E(t)} \operatorname{div}\phi\,d\mathcal{H}^n = \int_{\partial^* E(t)} \phi\cdot\nu_{E(t)}\,d\mathcal{H}^n.
\end{aligned}
\tag{7.15}
$$

In general, we approximate $\phi \in C_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ uniformly with $\phi_k \in C^1_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$, $k \geq 1$ and use the previous result.

Finally, if $\{E(t)\}_{t\geq 0} \in GMM(E_0)$, then by construction and Proposition 6.11 b) one has $E_{\lambda_j}([\lambda_j t]) \subseteq E^+$, where $E^+ := E^+(E_0, \beta)$ is a bounded minimizer of $C_\beta(\cdot, \Omega)$ in $\mathcal{E}(E^+)$; therefore $E(t) \subseteq E^+$ for all $t \geq 0$. $\square$

Definition 7.2 (**Maximal and minimal GMM**). *Let E_0, β satisfy* (5.1)*, and $\{\lambda_j\}$ be a diverging sequence such that*

$$E^*(t) := \lim_{j\to+\infty} E_{\lambda_j}([\lambda_j t])^* \qquad \forall t \geq 0$$

exist in $L^1(\Omega)$, where $E_{\lambda_j}([\lambda_j t])^*$ is the maximal minimizer of $\mathcal{A}_\beta(\cdot, E_{\lambda_j}([\lambda_j t]-1)^*, \lambda_j)$ with $(E_0)^* := E_0$ (Definition 6.4). We call $E^*(t)$ the maximal GMM associated to the sequence $\{\lambda_j\}$. Analogously,

$$E_*(t) := \lim_{j\to+\infty} E_{\lambda_j}([\lambda_j t])_* \qquad \forall t \geq 0,$$

obtained using the minimal minimizers $E_{\lambda_j}([\lambda_j t])_*$ of $\widehat{\mathcal{A}_\beta}(\cdot, E_{\lambda_j}([\lambda_j t]-1)_*, \lambda_j)$ with $(E_0)_* := E_0$, is called the minimal GMM associated to the sequence $\{\lambda_j\}$.

Observe that if $t \mapsto E(t)$ is any GMM obtained by the sequence $\{\lambda_j\}$, then according to the proof of Theorem 7.1 (possibly passing to nonrelabelled subsequences) there exist the maximal GMM $t \mapsto E^*(t)$ and the minimal GMM $t \mapsto E_*(t)$ associated to $\{\lambda_j\}$. Now by Remark 6.3 one has $E_*(t) \subseteq E(t) \subseteq E^*(t)$ for all $t \geq 0$.

Theorem 7.3 (**Comparison principle for maximal and minimal GMM**). *Let $E_0, F_0, \beta_1, \beta_2$ satisfy* (5.1) *with $E_0 \subseteq F_0$ and $\beta_1 \leq \beta_2$. If $E_*(t)$ and $F_*(t)$ are minimal GMMs associated to a sequence $\{\lambda_j\}$, then $E_*(t) \subseteq F_*(t)$ for all $t \geq 0$. Analogously, if $E^*(t)$ and $F^*(t)$ are maximal GMMs associated to $\{\lambda'_j\}$, then $E^*(t) \subseteq F^*(t)$ for all $t \geq 0$.*

Proof. Since $E_0 \subseteq F_0$, and $\beta_1 \leq \beta_2$, by definition of $E_\lambda(k)^*$ and $F_\lambda(k)^*$ (resp. $E_\lambda(k)_*$ and $F_\lambda(k)_*$) and by Theorem 6.1, we have $E_{\lambda*}(k) \subseteq F_{\lambda*}(k)$ (resp. $E^*_\lambda(k) \subseteq F^*_\lambda(k)$) which implies $E_*(t) \subseteq F_*(t)$ (resp. $E^*(t) \subseteq F^*(t)$) for all $t \geq 0$. $\square$

From the proof of Theorem 7.1 and Propositions 6.9 -6.10 we get the following result (compare with [11]), that could be applied, for instance, to E_0 as in Example 4.5.

Theorem 7.4. *Let E_0 be a constrained minimizer of $C_\beta(\cdot, \Omega)$ in $\mathcal{E}(E_0)$ such that $|\overline{E_0} \setminus E_0| = 0$. Then every maximal (minimal) GMM $t \mapsto E(t)$ starting from E_0 satisfies $E(t) \subseteq E(t')$ provided $t > t' \geq 0$.*

Proof. Applying Propositions 6.9 and 6.10 inductively to maximal minimizers $E_\lambda(k)^*$ of $\widehat{\mathcal{A}_\beta}(\cdot, E_\lambda(k-1)^*, \lambda)$ we get $E_\lambda(k)^* \subseteq E_\lambda(k-1)^*$ for all $k \geq 1$ and $\lambda \geq 1$. Hence, if $t > t' \geq 0$ then $E_\lambda([\lambda t])^* \subseteq E_\lambda([\lambda t'])^*$. Now the assertion of the theorem follows from (7.1). The arguments for minimal minimizers are the same. $\square$

8. GMM as a distributional solution

The aim of this section is to prove that under suitable assumptions GMM is in fact a distributional solution of (1.1)-(1.2). Let us start with the following

Definition 8.1 (Admissible variation). *A vector field* $X = (X', X_{n+1}) \in C^1_c(\overline{\Omega}, \mathbb{R}^{n+1})$ *is called admissible if* $X \cdot e_{n+1} = 0$ *on* $\partial\Omega$.

Observe that if $X \in C^1_c(\overline{\Omega}, \mathbb{R}^{n+1})$ is admissible, then for any $s \in (-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ sufficiently small, the vector field $f_s = \mathrm{Id} + sX$ is a C^1-diffeomorphism that satisfies $f_s(\Omega) = \Omega$, $f_s(\overline{\Omega}) = \overline{\Omega}$.

Proposition 8.2 (First variation of $\mathcal{A}_\beta$). *Suppose that* E_0, β *satisfy assumptions* (5.1) *and let* $E \in BV(\Omega, \{0,1\})$ *be bounded with* $\mathrm{Tr}(E) \in BV(\mathbb{R}^n, \{0,1\})$. *Then*

$$\begin{aligned}\frac{d}{ds}\mathcal{A}_\beta(f_s(E), E_0, \lambda)\Big|_{s=0} = &\int_{\Omega\cap\partial^* E} (\operatorname{div} X - \nu_E \cdot (\nabla X)\nu_E)\, d\mathcal{H}^n \\ &+ \lambda \int_{\Omega\cap\partial^* E} \tilde{d}_{E_0}\, X \cdot \nu_E \, d\mathcal{H}^n - \int_{\partial^*\mathrm{Tr}(E)} \beta X' \cdot \nu'_{\mathrm{Tr}(E)} d\mathcal{H}^{n-1},\end{aligned} \tag{8.1}$$

where $\partial^*\mathrm{Tr}(E)$ *is the essential boundary of* $\mathrm{Tr}(E)$ *on* $\partial\Omega$ *and* $\nu'_{\mathrm{Tr}(E)}$ *is the outer unit normal to* $\mathrm{Tr}(E) \subset \mathbb{R}^n$.

Proof. From [42, Theorem 17.5]

$$\frac{d}{ds}P(f_s(E), \Omega)\Big|_{s=0} = \int_{\Omega\cap\partial^* E} (\operatorname{div} X - \nu_E \cdot (\nabla X)\nu_E)\, d\mathcal{H}^n.$$

Moreover, [42, Theorem 17.8] and the admissibility of X imply that

$$\frac{d}{ds}\int_{f_s(E)} \tilde{d}_{E_0}\, dx\Big|_{s=0} = \int_{\partial^* E} \tilde{d}_{E_0}\, X \cdot \nu_E \, d\mathcal{H}^n = \int_{\Omega\cap\partial^* E} \tilde{d}_{E_0}\, X \cdot \nu_E \, d\mathcal{H}^n.$$

Finally, since $\mathrm{Tr}(E)$ is by assumption a set of finite perimeter in $\partial\Omega \equiv \mathbb{R}^n$, again using [42, Theorem 17.8] we get

$$\frac{d}{ds}\int_{\partial\Omega} \beta \chi_{f_s(E)} d\mathcal{H}^n\Big|_{s=0} = \int_{\partial^*\mathrm{Tr}(E)} \beta X' \cdot \nu'_{\mathrm{Tr}(E)}\, d\mathcal{H}^{n-1}.$$

□

Remark 8.3. Under assumptions (5.1) and $\beta \in \mathrm{Lip}(\partial\Omega)$, if E_λ is a minimizer of $\mathcal{A}_\beta(\cdot, E_0, \lambda)$, and if $\Omega\cap\partial E_\lambda$ is a C^2-manifold with $\mathcal{H}^{n-1}$-rectifiable boundary, then the mean curvature H_{E_λ} of $\Omega \cap \partial E_\lambda$ is equal to $-\lambda \tilde{d}_{E_0}$. Indeed, using the tangential divergence formula for manifolds with boundary we have

$$\int_{\Omega\cap\partial E_\lambda} (\operatorname{div} X - \nu_{E_\lambda} \cdot (\nabla X)\nu_{E_\lambda})\, d\mathcal{H}^n = \int_{\Omega\cap\partial E_\lambda} H_{E_\lambda} X \cdot \nu_{E_\lambda}\, d\mathcal{H}^n + \int_{\partial^*\mathrm{Tr}(E_\lambda)} X' \cdot \mathrm{n}^{\lambda\prime}\, d\mathcal{H}^{n-1},$$

where $\mathrm{n}^\lambda = (\mathrm{n}^{\lambda\prime}, \mathrm{n}^\lambda_{n+1})$ is the outer unit conormal to $\overline{\Omega \cap \partial E_\lambda}$ at $\overline{\Omega \cap \partial E_\lambda} \cap \partial\Omega$. By minimality of E_λ, we have $\frac{d}{ds}\mathcal{A}_\beta(f_s(E_\lambda), E_0, \lambda)\Big|_{s=0} = 0$, i.e.

$$\int_{\Omega\cap\partial E_\lambda} (H_{E_\lambda} + \lambda\tilde{d}_{E_0}) X \cdot \nu_{E_\lambda}\, d\mathcal{H}^n + \int_{\partial^*\mathrm{Tr}(E_\lambda)} X' \cdot (\mathrm{n}^{\lambda\prime} - \beta\nu'_{\mathrm{Tr}(E_\lambda)})\, d\mathcal{H}^{n-1} = 0.$$

This implies $H_{E_\lambda} = -\lambda\tilde{d}_{E_0}$ and $\mathrm{n}^{\lambda\prime} = \beta\nu'_{\mathrm{Tr}(E_\lambda)}$. Notice that from the latter in particular, we get

$$\beta = \mathrm{n}^\lambda \cdot (\nu'_{\mathrm{Tr}(E_\lambda)}, 0) = \nu_{E_\lambda} \cdot e_{n+1},$$

accordingly for instance with Theorem 5.3.

Remark 8.3 motivates the following definition [9, 42].

Definition 8.4 (Distributional mean curvature). *Let $E \in BV(\Omega, \{0,1\})$. The function $H_E \in L^1(\Omega \cap \partial^* E; \mathcal{H}^n \llcorner (\Omega \cap \partial^* E))$ is called distributional mean curvature of $\Omega \cap \partial^* E$ if for every $X \in C_c^1(\Omega, \mathbb{R}^{n+1})$ the generalized tangential divergence formula holds:*

$$\int_{\Omega \cap \partial^* E} (\operatorname{div} X - \nu_E \cdot (\nabla X)\nu_E)\, d\mathcal{H}^n = \int_{\Omega \cap \partial^* E} H_E\, X \cdot \nu_E\, d\mathcal{H}^n. \tag{8.2}$$

Given $x \in \mathbb{R}^{n+1}$ and $t > 0$ set

$$v_\lambda(t,x) := \begin{cases} -\lambda \tilde{d}_{E_\lambda([\lambda t]-1)}(x) & \text{if } t \geq \frac{1}{\lambda}, \\ 0 & \text{if } t \in [0, \frac{1}{\lambda}). \end{cases}$$

Remark 8.5. By Theorem 5.3, $\operatorname{Tr}(E_\lambda([\lambda t])) \in BV(\mathbb{R}^n, \{0,1\})$.

The next result relates GMM with distributional solutions of (1.1)-(1.2).

Theorem 8.6 (GMM is a distributional solution). *Let E_0, β satisfy* (5.1), $|\overline{E_0} \setminus E_0| = 0$, *$\{E(t)\}_{t \geq 0}$ be a GMM starting from E_0 obtained along the diverging sequence $\{\lambda_j\}$. Suppose that*

$$\mathcal{H}^n \llcorner (\Omega \cap \partial^* E_{\lambda_j}([\lambda_j t])) \overset{w^*}{\rightharpoonup} \mathcal{H}^n \llcorner (\Omega \cap \partial^* E(t)) \quad \text{as } j \to +\infty \text{ for a.e. } t \geq 0. \tag{8.3}$$

Then there exist a function $v : [0,+\infty) \times \Omega \to \mathbb{R}$ with

$$\int_0^{+\infty} \int_{\Omega \cap \partial^* E(t)} (v)^2\, d\mathcal{H}^n\, dt \leq \alpha(n,\kappa)\, P(E_0), \tag{8.4}$$

and a (not relabelled) subsequence such that

$$\lim_{j \to +\infty} \int_0^{+\infty} \int_{\Omega \cap \partial^* E_{\lambda_j}([\lambda_j t])} \phi v_{\lambda_j}\, d\mathcal{H}^n dt = \int_0^{+\infty} \int_{\Omega \cap \partial^* E(t)} \phi v\, d\mathcal{H}^n dt, \tag{8.5}$$

$$\lim_{j \to +\infty} \int_0^{+\infty} \int_{\Omega \cap \partial^* E_{\lambda_j}([\lambda_j t])} v_{\lambda_j}\, \nu_{E_{\lambda_j}([\lambda_j t])} \cdot \Psi\, d\mathcal{H}^n dt = \int_0^{+\infty} \int_{\Omega \cap \partial^* E(t)} v\, \nu_{E(t)} \cdot \Psi\, d\mathcal{H}^n dt \tag{8.6}$$

for any $\phi \in C_c(\Omega)$, $\Psi \in C_c([0,+\infty) \times \Omega, \mathbb{R}^{n+1})$, where $\alpha(n,\kappa) := \frac{75[(n+1)\omega_{n+1} + \omega_n]\mathrm{b}(n)}{(\kappa/2)^{n+1}\omega_{n+1}}$. Moreover, $\{E(t)\}_{t \geq 0}$ solves (1.1)-(1.2) *with initial datum E_0 in the following sense:*

(i) for a.e. $t \geq 0$ the set $\Omega \cap \partial^* E(t)$ has distributional mean curvature $H_{E(t)} = v$, and if $1 \leq n \leq 6$, for every $\phi \in C_c^1([0,+\infty) \times \Omega)$:

$$\int_0^{+\infty} \int_{E(t)} \partial_t \phi\, dx dt + \int_{E(0)} \phi(0,x)\, dx = \int_0^{+\infty} \int_{\Omega \cap \partial^* E(t)} \phi H_{E(t)}\, d\mathcal{H}^n dt; \tag{8.7}$$

(ii) if $\beta \in \operatorname{Lip}(\partial\Omega)$ and there exists $h \in L^1_{\mathrm{loc}}([0,+\infty))$ such that

$$P(\operatorname{Tr}(E_{\lambda_j}([\lambda_j t]))) \leq h(t) \qquad \text{for all } j \geq 1 \text{ and a.e. } t \geq 0, \tag{8.8}$$

then $\mathrm{Tr}(E(t)) \in BV(\mathbb{R}^n, \{0,1\})$ *for a.e.* $t > 0$ *and*

$$\begin{aligned}\int_{\Omega\cap\partial^* E(t)} & \Big(\operatorname{div} X - \nu_{E(t)} \cdot (\nabla X)\nu_{E(t)}\Big)\, d\mathcal{H}^n \\ &= \int_{\Omega\cap\partial^* E(t)} H_{E(t)}\, X\cdot \nu_{E(t)}\, d\mathcal{H}^n + \int_{\partial^*\mathrm{Tr}(E(t))} \beta\, X' \cdot \nu'_{\mathrm{Tr}(E(t))}\, d\mathcal{H}^{n-1}\end{aligned} \tag{8.9}$$

for every admissible $X \in C^1_c(\overline{\Omega}, \mathbb{R}^{n+1})$.

The need for assumption (8.3) is not surprising; see [41, 45] for conditional results obtained in other contexts in a similar spirit. We postpone the proof after several auxiliary results.

Proposition 8.7. *Assume that* E_0 *and* β *satisfy* (5.1). *Then for any* $\lambda \geq 1$ *and a.e.* $t \geq 1/\lambda$ *the function* $v_\lambda(t,\cdot)$ *is the distributional mean curvature of* $E_\lambda([\lambda t])$.

Proof. Set $E := E_\lambda([\lambda t])$. Remark 8.5 and (8.1) imply that

$$\int_{\Omega\cap\partial^* E} (\operatorname{div} X - \nu_E\cdot(\nabla X)\nu_E)\, d\mathcal{H}^n = \int_{\Omega\cap\partial^* E} v_\lambda\, X\cdot \nu_E\, d\mathcal{H}^n.$$

Hence, it suffices to prove $v_\lambda(t,\cdot) \in L^1(\Omega\cap\partial^* E; \mathcal{H}^n \llcorner \Omega\cap\partial^* E)$ for a.e. $t \in [1/\lambda, +\infty)$ and since $P(E(t),\Omega) < +\infty$, this follows from Lemma 8.9 below. □

Remark 8.8. From Definition 8.4, Proposition 8.7 and Lemma 8.9 it follows that

$$v_\lambda(t,x) = H_{E_\lambda([\lambda t])}(t,x) \quad \text{for a.e. } t\geq 1/\lambda \text{ and } \mathcal{H}^n\text{-a.e. } x\in\Omega\cap\partial E_\lambda([\lambda t]).$$

This is a discretized version of equation (1.1).

Lemma 8.9 (Uniform L^2-bound of the approximate velocities). *Under assumptions* (5.1) *the inequality*

$$\int_0^{+\infty}\int_{\Omega\cap\partial E_\lambda([\lambda t])} (v_\lambda)^2\, d\mathcal{H}^n dt \leq \alpha(n,\kappa)P(E_0)$$

holds.

Proof. The proof is analogous to the proof of [45, Lemma 3.6]. Given $\varepsilon > 0$ and $E \in BV(\Omega, \{0,1\})$ let

$$(\partial E)^+_\varepsilon := \{x \in \mathbb{R}^{n+1} :\ \operatorname{dist}(x, \Omega\cap\partial E) \leq \varepsilon\}.$$

For $t \in [\frac{1}{\lambda}, +\infty)$ and $\ell \in \mathbb{Z}$ such that $\ell \leq 1 + [\log_2(R(n,\kappa)\lambda^{1/2})]$, where $R(n,\kappa)$ is given by (5.2), define

$$K(\ell) = \Big\{x \in \Big(\partial E_\lambda([\lambda t]-1)\Big)^+_{R(n,\kappa)\lambda^{-1/2}} :\ 2^\ell < |v_\lambda(x,t)| \leq 2^{\ell+1}\Big\}.$$

By Proposition 5.5 $E_\lambda([\lambda t])\Delta E_\lambda([\lambda t]-1) \subseteq \cup_\ell K(\ell)$. Take $x \in K(\ell)\cap\Omega\cap\partial E_\lambda([\lambda t])$. Then $B_{\frac{2^{\ell-1}}{\lambda}}(x) \cap E_\lambda([\lambda t]-1) = \emptyset$ and hence, by Remark 5.4 the following density estimates hold:

$$\begin{aligned}|E_\lambda([\lambda t]) \cap B_{\frac{2^{\ell-1}}{\lambda}}(x)| &\geq \Big(\frac{\kappa}{2}\Big)^{n+1} \omega_{n+1}\Big(\frac{2^{\ell-1}}{\lambda}\Big)^{n+1}, \\ \mathcal{H}^n(B_{\frac{2^{\ell-1}}{\lambda}}(x)\cap\Omega\cap\partial E_\lambda([\lambda t])) &\leq \big[(n+1)\omega_{n+1}+\omega_n\big]\Big(\frac{2^{\ell-1}}{\lambda}\Big)^n.\end{aligned} \tag{8.10}$$

Using $2^{\ell-1} \le |v_\lambda(y,t)| \le 5\cdot 2^{\ell-1}$ for any $y \in B_{\frac{2^{\ell-1}}{\lambda}}(x)$, from (8.10) we deduce

$$\begin{aligned}\int_{B_{\frac{2^{\ell-1}}{\lambda}}(x)\cap\Omega\cap\partial E_\lambda([\lambda t])} (v_\lambda)^2\, d\mathcal{H}^n \le& 25[(n+1)\omega_{n+1}+\omega_n](2^{\ell-1})^2\left(\frac{2^{\ell-1}}{\lambda}\right)^n\\ \le& \frac{25[(n+1)\omega_{n+1}+\omega_n]}{(\kappa/2)^{n+1}\omega_{n+1}}\,\lambda\int_{B_{\frac{2^{\ell-1}}{\lambda}}(x)\cap(E_\lambda([\lambda t])\Delta E_\lambda([\lambda t]-1))}|v_\lambda|\,dx.\end{aligned}$$

Application of Besicovitch covering theorem to the collection of balls $\{B_{\frac{2^{\ell-1}}{\lambda}}(x):\ x\in K(\ell)\cap \partial E_\lambda([\lambda t])\}$ gives

$$\int_{K(\ell)\cap\Omega\cap\partial E_\lambda([\lambda t])} (v_\lambda)^2\, d\mathcal{H}^n \le \frac{25[(n+1)\omega_{n+1}+\omega_n]\mathfrak{b}(n)}{(\kappa/2)^{n+1}\omega_{n+1}}\,\lambda\int_{\{2^{\ell-1}\le|v_\lambda|\le 2^{\ell+2}\}\cap(E_\lambda([\lambda t])\Delta E_\lambda([\lambda t]-1))}|v_\lambda|\,dx.$$

Now summing up these inequalities over $\ell\in\mathbb{Z}$ with $\ell \le 1+[\log_2(R(n,\kappa)\lambda^{1/2})]$, and using the properties of $K(\ell)$ and the definition of $\alpha(n,\kappa)$ we get

$$\int_{\Omega\cap\partial E_\lambda([\lambda t])} (v_\lambda)^2\, d\mathcal{H}^n \le \alpha(n,\kappa)\,\lambda\int_{E_\lambda([\lambda t])\Delta E_\lambda([\lambda t]-1)}|v_\lambda|\,dx.$$

Observe that by (7.4) for any $t\ge 1/\lambda$ one has

$$\int_{E_\lambda([\lambda t])\Delta E_\lambda([\lambda t]-1)}|v_\lambda|\,dx \le C_\beta(E_\lambda([\lambda t]-1),\Omega) - C_\beta(E_\lambda([\lambda t]),\Omega).$$

Thus

$$\int_{\Omega\cap\partial E_\lambda([\lambda t])} (v_\lambda)^2\, d\mathcal{H}^n \le \alpha(n,\kappa)\,\lambda\Big(C_\beta(E_\lambda([\lambda t]-1),\Omega) - C_\beta(E_\lambda([\lambda t]),\Omega)\Big).$$

Fixing $T>0$ and integrating this inequality in $t\in[0,T]$ we get

$$\begin{aligned}\int_0^T\int_{\Omega\cap\partial E_\lambda([\lambda t])} (v_\lambda)^2\, d\mathcal{H}^n dt \le&\alpha(n,\kappa)\sum_{k=1}^{[T\lambda]+1}\Big(C_\beta(E_\lambda(k-1),\Omega)-C_\beta(E_\lambda(k),\Omega)\Big)\\ \le&\alpha(n,\kappa)\,C_\beta(E_0,\Omega)\le \alpha(n,\kappa)\,P(E_0),\end{aligned}$$

where we used (3.9). Now letting $T\to+\infty$ completes the proof. □

Proposition 8.10. *Let E_0,β satisfy* (5.1), *$\lambda\ge 1$ and E^+ be as in Proposition 6.11. Then*

$$\lambda\int_{1/\lambda}^T |E_\lambda([\lambda t])\Delta E_\lambda([\lambda t]-1)|\,dt \le |E^+| + \frac{P(E_0)}{\gamma(n,\kappa)} + \frac{2^{n+1}\omega_{n+1}\gamma(n,\kappa)\mathfrak{b}(n)}{\kappa c(n,\kappa)}\,P(E_0)\,T \tag{8.11}$$

for any $T>\frac{1}{\lambda}$. Here $\mathfrak{b}(n),\gamma(n,\kappa),c(n,\kappa)$ are defined in Section 5.

Proof. Let $[\lambda T]=N.$ Clearly,

$$\lambda\int_{1/\lambda}^T |E_\lambda([\lambda t])\Delta E_\lambda([\lambda t]-1)|\,dt = \sum_{k=1}^N |E_\lambda(k)\Delta E_\lambda(k-1)|.$$

We recall that $E_\lambda(k) \subset E^+$ for all $\lambda \geq 1$ and $k \geq 0$, by Proposition 6.11.

If $k = 1$, then

$$|E_\lambda(1) \Delta E_\lambda(0)| \leq |E^+|. \tag{8.12}$$

Now if $k \geq 2$, we write $E_\lambda(k) \Delta E_\lambda(k-1)$ as a union of A_k and B_k, where

$$A_k = \Big\{x \in E_\lambda(k) \Delta E_\lambda(k-1) : \ d_{E_\lambda(k-1)}(x) > \ell\Big\},$$

$$B_k = \Big\{x \in E_\lambda(k) \Delta E_\lambda(k-1) : \ d_{E_\lambda(k-1)}(x) \leq \ell\Big\}.$$

where $\ell := \frac{\gamma(n,\kappa)}{\lambda}$. By Chebyshev inequality $|A_k|$ can be estimated using (7.4) as

$$|A_k| \leq \frac{\lambda}{\gamma(n,\kappa)} \int_{E_\lambda(k)\Delta E_\lambda(k-1)} d_{E_\lambda(k-1)}\, dx \leq \frac{1}{\gamma(n,\kappa)} \Big(C_\beta(E_\lambda(k), \Omega) - C_\beta(E_\lambda(k), \Omega)\Big).$$

Hence, by (7.6)

$$\sum_{k=2}^N |A_k| \leq \frac{1}{\gamma(n,\kappa)} \sum_{k=2}^N \Big(C_\beta(E_\lambda(k), \Omega) - C_\beta(E_\lambda(k), \Omega)\Big) \leq \frac{P(E_0)}{\gamma(n,\kappa)}.$$

Moreover, by definition B_k can be covered by the family of balls $\{B_{2\ell}(x),\ x \in \partial E_\lambda(k-1)\}$. Thus, by Besicovitch covering theorem we can find at most countably many balls $\{B_\ell(x_j),\ x_j \in \partial E_\lambda(k-1)\}$ covering $\Omega \cap \partial E_\lambda(k-1)$. Hence, the lower density estimate (5.5) for $E_\lambda(k-1)$ used with ℓ implies

$$|B_{2\ell}(x_j) \cap B_k| \leq (2^{n+1}\omega_{n+1}\ell)\ell^n \leq \frac{2^{n+1}\omega_{n+1}}{c(n,\kappa)}\, \ell\, P(E_\lambda(k-1), B_\ell(x_j)),$$

from which it follows that

$$\begin{aligned}\sum_{k=2}^N |B_k| &\leq \sum_{k=2}^N \sum_{j\geq 1} |B_{2\ell}(x_j) \cap B_k| \leq \frac{2^{n+1}\omega_{n+1}}{c(n,\kappa)}\, \ell\, \sum_{k=2}^N \sum_{j\geq 1} P(E_\lambda(k-1), B_\ell(x_j)) \\ &\leq \frac{2^{n+1}\mathfrak{b}(n)\omega_{n+1}}{c(n,\kappa)}\, \ell\, \sum_{k=2}^N P(E_\lambda(k-1), \Omega).\end{aligned}$$

Therefore, using (7.6) and $N \leq \lambda T$, we get

$$\sum_{k=2}^N |B_k| \leq \frac{2^{n+1}\mathfrak{b}(n)\omega_{n+1}\gamma(n,\kappa)}{\kappa c(n,\kappa)}\, P(E_0)\, T. \tag{8.13}$$

Finally, (8.11) follows from (8.12)-(8.13). $\square$

The following error estimate is similar to error estimates shown in [41, 45].

Proposition 8.11 (Error estimate). *Let $1 \leq n \leq 6$. Under assumption (4.3), for every $\phi \in C_c^1([0,+\infty) \times \Omega)$ the following error-estimate holds:*

$$\lim_{j\to+\infty} \int_{1/\lambda_j}^{+\infty} \lambda_j \bigg(\int_\Omega (\chi_{E_{\lambda_j}([\lambda_j t])} - \chi_{E_{\lambda_j}([\lambda_j t]-1)})\phi\, dx - \int_{\Omega\cap\partial E_{\lambda_j}([\lambda_j t])} \tilde{d}_{E_{\lambda_j}([\lambda_j t]-1)}\, \phi\, d\mathcal{H}^n \bigg) dt \to 0. \tag{8.14}$$

Proof. Let us assume that $\operatorname{supp}\phi \subset\subset [0,T)\times\Omega^\varepsilon$, $\Omega^\varepsilon := \mathbb{R}^n\times(\varepsilon,+\infty)\subset\Omega$ for some $\varepsilon, T>0$. Let us take j so large that

$$4R\lambda_j^{-1/2} < \varepsilon, \tag{8.15}$$

where $R := R(n,\kappa)$ is defined in (5.2).

Given an integer $k\geq 1$ set

$$\Delta_k(j) := \int_{k/\lambda_j}^{(k+1)/\lambda_j} \lambda_j \bigg(\int_\Omega (\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)})\,\phi\,dx - \int_{\Omega\cap\partial E_{\lambda_j}(k)} \tilde d_{E_{\lambda_j}(k-1)}\,\phi\, d\mathcal{H}^n\bigg)dt.$$

We need to estimate

$$\int_{1/\lambda_j}^{T} \lambda_j \bigg(\int_{\Omega^\varepsilon} (\chi_{E_{\lambda_j}([\lambda_j t])} - \chi_{E_{\lambda_j}([\lambda_j t]-1)})\,\phi\,dx - \int_{\Omega^\varepsilon\cap\partial E_{\lambda_j}([\lambda_j t])} \tilde d_{E_{\lambda_j}([\lambda_j t]-1)}\,\phi\, d\mathcal{H}^n\bigg)dt = \sum_{k=1}^{N_j}\Delta_k(j),$$

where $N_j = [\lambda_j T]$.

First consider $\Delta_1(j)$. By virtue of Proposition 5.5 and (7.6),

$$\begin{aligned}
|\Delta_1(j)| =& \bigg| \int_{1/\lambda_j}^{2/\lambda_j} \lambda_j \bigg(\int_{\Omega^\varepsilon} (\chi_{E_{\lambda_j}(1)} - \chi_{E_{\lambda_j}(0)})\,\phi\,dx - \int_{\Omega^\varepsilon\cap\partial E_{\lambda_j}(1)} \tilde d_{E_{\lambda_j}(0)}\,\phi\, d\mathcal{H}^n\bigg)dt\bigg| \\
\leq & \|\phi\|_\infty\bigg(|E_{\lambda_j}(1)\Delta E_0| + \frac{R(n,\kappa)}{\sqrt{\lambda_j}}\,P(E_{\lambda_j}(1),\Omega)\bigg) \\
\leq & \|\phi\|_\infty\bigg(|E_{\lambda_j}(1)\Delta E_0| + \frac{R(n,\kappa)}{\kappa\sqrt{\lambda_j}}\,P(E_0)\bigg).
\end{aligned} \tag{8.16}$$

Hence, by Lemma 4.6 a), $\Delta_1(j)\to 0$ as $j\to+\infty$.

Recall that by Theorem 7.1 there exists a bounded set $E^+\subset\Omega$ (depending only on E_0 and β) such that $E_{\lambda_j}(k)\subseteq E^+$ for all $j\geq 1$ and $k\geq 1$.

Our aim is now to show that given $\sigma_2 := \frac{2n+5}{4(n+2)} \in (1/2,1)$, $\sigma_1\in(1/2,\sigma_2)$, there exists an increasing function $w\in C([0,\infty))$ with $w(0)=0$, such that for any $k\in\{2,\dots,N_j\}$,

$$\begin{aligned}
|\Delta_k(j)| \leq & C(n)\Big(\lambda_j^{-\sigma_2}\|\nabla\phi\|_\infty + (6w(1/\lambda_j) + C(n)\lambda_j^{\sigma_1-\sigma_2})\,\|\phi\|_\infty\Big)\,|E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1)| \\
& + \|\phi\|_\infty C(n,\kappa,\operatorname{diam}(E^+))\lambda_j^{-1/4}\Big(C_\beta(E_{\lambda_j}(k-1),\Omega) - C_\beta(E_{\lambda_j}(k),\Omega)\Big),
\end{aligned} \tag{8.17}$$

provided j is large enough, where $C(n)$ and $C(n,\kappa,\operatorname{diam}(E^+))$ are universal constants.

We may suppose that $E_{\lambda_j}(k)\neq\emptyset$ for any $k=2,\dots,N_j$. We divide the proof of (8.17) into four steps, and in the first three steps we deal with the regions of "low-curvature" (assumption (8.18)). In the final step we estimate the error in the "high-curvature" regions.

Step 1. For every $\sigma_1\in(1/2,\sigma_2)$ there exists an increasing function $w\in C([0,\infty))$ with $w(0)=0$ such that, if $k\in\{2,\dots,N_j\}$ and $x\in\Omega^\varepsilon\cap\partial E_{\lambda_j}(k)$ satisfy

$$d_{E_{\lambda_j}(k-1)}(y) \leq \lambda_j^{-\sigma_2}, \qquad y\in B_{R\lambda_j^{-1/2}}(x)\cap(E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1)), \tag{8.18}$$

then there is $\nu_k := \nu_k(x)\in\mathbb{S}^n$ such that

$$|\nu_{E_{\lambda_j}(s)}(y) - \nu_k| \leq w(1/\lambda_j), \qquad y\in B_{\lambda_j^{-\sigma_1}}(x)\cap\partial E_{\lambda_j}(s), \qquad s=k,k-1, \tag{8.19}$$

provided j is large enough satisfying also (8.15).

Indeed, fix $r \in (0, \lambda_j^{\sigma_1-1/2})$. By (8.15), the ball $B_{r\lambda_j^{-\sigma_1}}(x)$ does not intersect $\partial\Omega$, and hence

$$P(E_{\lambda_j}(k), B_{r\lambda_j^{-\sigma_1}}(x)) \le P(F, B_{r\lambda_j^{-\sigma_1}}(x)) + \lambda_j \int_{F\Delta E_{\lambda_j}(k)} d_{E_{\lambda_j}(k-1)}dy$$

for every $F \in BV(\Omega, \{0,1\})$ with $F\Delta E_{\lambda_j}(k) \subset\subset B_{r\lambda_j^{-\sigma_1}}(x)$. Since $d_{E_{\lambda_j}(k-1)}(\cdot)$ is 1-Lipschitz, by virtue of Proposition 5.5 (applied with $E_{\lambda_j}(k-1)$ instead of E_0),

$$d_{E_{\lambda_j}(k-1)}(y) \le d_{E_{\lambda_j}(k-1)}(x) + |x-y| \le R\lambda_j^{-1/2} + r\lambda_j^{-\sigma_1} \le \frac{R+1}{\lambda_j^{1/2}}, \qquad y \in F\Delta E_{\lambda_j}(k),$$

whence

$$\lambda_j \int_{F\Delta E_{\lambda_j}(k)} d_{E_{\lambda_j}(k-1)}dy \le (R+1)\lambda_j^{1/2}|F\Delta E_{\lambda_j}(k)|$$

and

$$P(E_{\lambda_j}(k), B_{r\lambda_j^{-\sigma_1}}(x)) \le P(F, B_{r\lambda_j^{-\sigma_1}}(x)) + (R+1)\lambda_j^{1/2}|F\Delta E_{\lambda_j}(k)|. \tag{8.20}$$

Let $z_o \in \partial E_{\lambda_j}(k-1)$ be such that $|x - z_o| = d_{E_{\lambda_j}(k-1)}(x)$. As $k \ge 2$,

$$P(E_{\lambda_j}(k-1), B_{r\lambda_j^{-\sigma_1}}(z_o)) \le P(F, B_{r\lambda_j^{-\sigma_1}}(z_o)) + (R+1)\lambda_j^{1/2}|F\Delta E_{\lambda_j}(k-1)| \tag{8.21}$$

whenever $F \in BV(\Omega, \{0,1\})$ satisfies $F\Delta E_{\lambda_j}(k-1) \subset\subset B_{r\lambda_j^{-\sigma_1}}(z_o)$. Set

$$E_{\lambda_j}^{\sigma_1}(k) := \frac{E_{\lambda_j}(k) - x}{\lambda_j^{-\sigma_1}}, \qquad E_{\lambda_j}^{\sigma_1}(k-1) := \frac{E_{\lambda_j}(k-1) - z_o}{\lambda_j^{-\sigma_1}}.$$

By virtue of (8.20)-(8.21) these sets satisfy

$$P(E_{\lambda_j}^{\sigma_1}(s), B_r(0)) \le P(F, B_r(0)) + \lambda_j^{1/2-\sigma_1}(R+1)|F\Delta E_{\lambda_j}^{\sigma_1}(s)|$$

for any $r \in (0, \lambda_j^{\sigma_1-1/2})$ and $F\Delta E_{\lambda_j}^{\sigma_1}(s) \subset\subset B_r(0)$, $s = k, k-1$. Hence $E_{\lambda_j}^{\sigma_1}(s)$, $s = k, k-1$, is an $((R+1)\lambda_j^{1/2-\sigma_1}, \lambda_j^{\sigma_1-1/2})$-minimizer of the perimeter (see [42, Section 23]). Since $\sigma_1 > 1/2$, $\lambda_j^{1/2-\sigma_1}(R+1) \to 0$ as $j \to +\infty$, and therefore, by compactness [42, Proposition 23.13], up to a (not relabelled) subsequence,

$$E_{\lambda_j}^{\sigma_1}(s) \to E^{\sigma_1}(s) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}) \text{ as } j \to +\infty, \ s = k, k-1,$$

where $E^{\sigma_1}(s)$, $s = k, k-1$, is a local minimizer of the perimeter in $\mathbb{R}^{n+1}$. Since $n \le 6$, by virtue of [33, Theorem 17.3] $E^{\sigma_1}(k)$ and $E^{\sigma_1}(k-1)$ are half-spaces. Moreover, by hypothesis (8.18),

$$d_{E_{\lambda_j}^{\sigma_1}(k-1)}(z) \le \lambda_j^{\sigma_1-\sigma_2}, \qquad z \in B_{R\lambda_j^{\sigma_1-1/2}}(0) \cap (E_{\lambda_j}^{\sigma_1}(k)\Delta E_{\lambda_j}^{\sigma_1}(k-1)),$$

and, therefore, $E^{\sigma_1}(k) = E^{\sigma_1}(k-1)$, i.e. there exists $\nu_k \in \mathbb{S}^n$ such that

$$E^{\sigma_1}(k) = E^{\sigma_1}(k-1) = \{z \in \mathbb{R}^{n+1} : z \cdot \nu_k < 0\}.$$

By [42, Theorem 26.6] $\nu_{E^{\sigma_1}_{\lambda_j}}(s) \to \nu_k$ uniformly in $B_1(0)$ as $j \to +\infty$, and the existence of w and the validity of (8.19) follow.

Besides (8.15), from now on we suppose j to be so large that $w(1/\lambda_j) < 1/4$.

Step 2. If $k \in \{2, \ldots, N_j\}$ and $x \in \Omega^\varepsilon \cap \partial E_{\lambda_j}(k)$ satisfy (8.18), then for any $\psi \in C^1_c(C^\rho_\rho(x,\nu))$,

$$\begin{aligned} \Big| \int_{C^\rho_\rho(x,\nu)} (\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)})\psi dy - \int_{C^\rho_\rho(x,\nu)\cap\partial E_{\lambda_j}(k)} \tilde{d}_{E_{\lambda_j}(k-1)}\, \psi \, d\mathcal{H}^n \Big| \\ \le \left(\lambda_j^{-\sigma_2} \|\nabla\psi\|_\infty + 6w(1/\lambda_j)\,\|\psi\|_\infty\right) \int_{C^\rho_\rho(x,\nu)} |\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)}| dy, \end{aligned} \tag{8.22}$$

where

$$\rho := \lambda_j^{-\sigma_1}/2,$$

$\nu = \nu_k$ is as in Step 1, and

$$C^\rho_\rho(x,\nu) := \{y \in \mathbb{R}^{n+1} :\ |(y-x)\cdot\nu| < \rho,\ \sqrt{|y-x|^2 - |(y-x)\cdot\nu|^2} < \rho\}.$$

For simplicity, suppose $x = 0$, $\nu = e_{n+1}$, and set $C^\rho_\rho := C^\rho_\rho(x,\nu)$. By Theorem 5.3, there exist $f, g \in C^2(\hat{B}_\rho)$ such that

$$C^\rho_\rho \cap \partial E_{\lambda_j}(k) = \{(\hat{x}, f(\hat{x})) :\ \hat{x} \in \hat{B}_\rho\} \qquad \text{and} \qquad C^\rho_\rho \cap \partial E_{\lambda_j}(k-1) = \{(\hat{x}, g(\hat{x})) :\ \hat{x} \in \hat{B}_\rho\},$$

where $\hat{B}_\rho$ is the ball in $\mathbb{R}^n$ centered at 0 of radius ρ. Let us show that

$$|f(\hat{x}) - g(\hat{x}) - \tilde{d}_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x}))| \le 2\, w(1/\lambda_j) d_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x})), \qquad \hat{x} \in \hat{B}_\rho. \tag{8.23}$$

Since the outer unit normal of $\partial E_{\lambda_j}(s)$, $s = k, k-1$, in C^ρ_ρ is "almost close" to e_{n+1}, $f(\hat{x}) - g(\hat{x})$ and $\tilde{d}_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x}))$ have the same sign and thus, we may suppose that $f(\hat{x}) \ge g(\hat{x})$ and $\tilde{d}_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x})) = d_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x}))$. Let $(\hat{y}, g(\hat{y})) \in \partial E_{\lambda_j}(k-1)$ be such that

$$d_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x})) = \sqrt{|\hat{x} - \hat{y}|^2 + |f(\hat{x}) - g(\hat{y})|^2}. \tag{8.24}$$

Denoting by $\alpha \in (0, \pi/2)$ the angle between e_{n+1} and $\nu_{E_{\lambda_j}(k-1)}(\hat{y}, g(\hat{y}))$, by (8.19) we have

$$\cos\alpha \ge 1 - 1/2w^2(1/\lambda_j) \tag{8.25}$$

and

$$d_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x})) \sin\alpha = |\hat{x} - \hat{y}|. \tag{8.26}$$

From this and (8.24) it follows that

$$d_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x})) \cos\alpha = f(\hat{x}) - g(\hat{y}). \tag{8.27}$$

By virtue of (8.19), $\|\nabla g\|_\infty \le w(1/\lambda_j)$, thus by (8.27) and (8.25),

$$\begin{aligned} |f(\hat{x}) - g(\hat{x}) - \tilde{d}_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x}))| \le |f(\hat{x}) - g(\hat{y}) - d_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x}))| + |g(\hat{y}) - g(\hat{x})| \\ \le w(1/\lambda_j) d_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x})) + \|\nabla g\|_\infty |\hat{x} - \hat{y}| \le 2w(1/\lambda_j) d_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x})), \end{aligned}$$

since $|\hat{x}-\hat{y}| \le d_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x}))$ by (8.26) and (8.23) is proven.

Now using $w(1/\lambda_j) \le 1/4$, from (8.23) we deduce

$$d_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x})) \le 2|f(\hat{x}) - g(\hat{x})|, \tag{8.28}$$

thus

$$|f(\hat{x}) - g(\hat{x}) - \tilde{d}_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x}))| \le 4\, w(1/\lambda_j)|f(\hat{x}) - g(\hat{x})|, \qquad \hat{x} \in \hat{B}_\rho. \tag{8.29}$$

As in the proof of [45, Corollary 4.2.2], the left-hand-side of (8.22) is represented as

$$\Gamma := \int_{\hat{B}_\rho} \Big(\int_{g(\hat{x})}^{f(\hat{x})} \psi(\hat{x}, z) dz - \tilde{d}_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x}))\, \psi(\hat{x}, f(\hat{x}))\, \sqrt{1 + |\nabla f(\hat{x})|^2} \Big)\, d\hat{x}. \tag{8.30}$$

Now we estimate (8.30) as follows:

$$\begin{aligned} |\Gamma| \le & \Big| \int_{\hat{B}_\rho} \int_{g(\hat{x})}^{f(\hat{x})} \Big(\psi(\hat{x}, z) - \psi(\hat{x}, f(\hat{x})\Big) dz d\hat{x} \Big| \\ &+ \int_{\hat{B}_\rho} |\psi(\hat{x}, f(\hat{x}))|\, |f(\hat{x}) - g(\hat{x}) - \tilde{d}_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x}))| d\hat{x} \\ &+ \int_{\hat{B}_\rho} d_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x}))\, |\psi(\hat{x}, f(\hat{x}))| \Big(\sqrt{1 + |\nabla f(\hat{x})|^2} - 1 \Big)\, d\hat{x} =: \Gamma_1 + \Gamma_2 + \Gamma_3. \end{aligned}$$

By the Lipschitz continuity of ψ,

$$\begin{aligned} \Gamma_1 \le & \|\nabla\psi\|_\infty \int_{\hat{B}_\rho} \int_{g(\hat{x})}^{f(\hat{x})} |z - f(\hat{x})| dz d\hat{x} = \frac{1}{2} \|\nabla\psi\|_\infty \int_{\hat{B}_\rho} |f(\hat{x}) - g(\hat{x})|^2 d\hat{x} \\ \le & \lambda^{-\sigma_2} \|\nabla\psi\|_\infty \int_{\hat{B}_\rho} |f(\hat{x}) - g(\hat{x})| d\hat{x}, \end{aligned} \tag{8.31}$$

since by virtue of (8.23) and (8.18),

$$|f(\hat{x}) - g(\hat{x})| \le (1 + 2w(1/\lambda_j)) d_{E_{\lambda_j}(k-1)}(\hat{x}, f(\hat{x})) \le 2\lambda_j^{-\sigma_2}.$$

By (8.29),

$$\Gamma_2 \le 4\, w(1/\lambda_j)\, \|\psi\|_\infty \int_{\hat{B}_\rho} |f(\hat{x}) - g(\hat{x})|\, d\hat{x}. \tag{8.32}$$

Finally, since $\|\nabla f\|_\infty \le w(1/\lambda_j)$ by (8.19), from the elementary inequality $\sqrt{1 + |a|^2} \le 1 + |a|$ and (8.28) we obtain

$$\Gamma_3 \le 2w(1/\lambda_j)\, \|\psi\|_\infty \int_{\hat{B}_\rho} |f(\hat{x}) - g(\hat{x})|\, d\hat{x}. \tag{8.33}$$

Now (8.22) follows from the inequality $|\Gamma| \le \Gamma_1 + \Gamma_2 + \Gamma_3$, (8.31)-(8.33) and the relation

$$\int_{\hat{B}_\rho} |f(\hat{x}) - g(\hat{x})|\, d\hat{x} = \int_{C_\rho^\rho} |\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)}| dy.$$

Step 3. Set

$$\mathcal{O} := \left\{C^{\rho}_{\rho}(x, \nu_k(x)) : \quad x \in \partial E_{\lambda_j}(k) \ \text{ satisfies (8.18)}\right\} \qquad \text{and} \qquad O := \bigcup_{C^{\rho}_{\rho}(x,\nu_k(x))\in\mathcal{O}} C^{\rho}_{\rho}(x, \nu_k(x)),$$

where

$$\rho := \lambda_j^{-\sigma_1}/2.$$

Note that $\mathcal{O}$ is a covering for the region $O \cap (E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))$ with "low curvature". By Besicovitch covering theorem, we can extract a finite subcover $\{C^{\rho}_{\rho}(x_l, \nu(x_l))\} \subseteq \mathcal{O}$ such that each point of $O \cap (E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))$ belongs to at most $\mathtt{b}(n)$ cylinders. Let $\{\eta_l\}$ be an associated partition of unity, i.e.

- $\eta_l \in C^{\infty}_c(C^{\rho}_{\rho}(x_l, \nu(x_l)), [0, 1]), \ \ l = 1, 2, \ldots;$
- $\sum\limits_l \eta_l = 1$ in $O \cap (E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))$.

By (8.22) we have

$$\begin{aligned}
\Delta^{\text{low}}_k :=& \Big| \int_O (\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)})\phi dx - \int_{O\cap\partial E_{\lambda_j(k)}} \phi \tilde{d}_{E_{\lambda_j}(k-1)}\, d\mathcal{H}^n \Big| \\
\le& \sum_l \Big| \int_{C^{\rho}_{\rho}(x_l,\nu(x_l))} (\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)})\, \phi\eta_l\, dx - \int_{C^{\rho}_{\rho}(x_l,\nu(x_l))\cap\partial E_{\lambda_j(k)}} \tilde{d}_{E_{\lambda_j}(k-1)}\, \phi\eta_l\, d\mathcal{H}^n \Big| \\
\le& \sum_l \Big(\lambda_j^{-\sigma_2} \, \|\nabla(\eta_l\phi)\|_\infty + 6w(1/\lambda_j)\|\phi\|_\infty \Big) \int_{C^{\rho}_{\rho}(x_l,\nu(x_l))} |\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)}| dy.
\end{aligned}$$

Since $\|\nabla(\phi\eta_l)\|_\infty \le \|\nabla\phi\|_\infty + C(n)\|\phi\|_\infty/\rho$ and $\|\phi\eta_l\|_\infty \le \|\phi\|_\infty$, by the property of the covering,

$$\Delta^{\text{low}}_k \le \mathtt{b}(n) \Big(\lambda_j^{-\sigma_2} \, \|\nabla\phi\|_\infty + (6w(1/\lambda_j) + C(n)\, \lambda_j^{\sigma_1-\sigma_2})\|\phi\|_\infty \Big) |E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1)|. \tag{8.34}$$

Step 4. Now we estimate the error in $\Omega^{\varepsilon} \setminus O$. By Proposition 5.5 (applied with $E_{\lambda_j}(k-1)$ instead of E_0), $(\Omega^{\varepsilon} \setminus O) \cap (E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))$ can be covered by the family of balls

$$\mathtt{B} := \{B_r(x) : \ x \in \Omega^{\varepsilon} \cap \partial E_{\lambda_j}(k-1)\}, \qquad r := R\lambda_j^{-1/2}.$$

By Besicovitch covering theorem we can extract a finite collection $\{B_r(x_i)\} \subset \mathtt{B}$ such that each point of $\Omega^{\varepsilon} \cap (E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))$ is covered with at most $\mathtt{b}(n)$ elements of $\{B_r(x_i)\}$. First we handle the error in each $B_r(x_i)$. By the definition, there exists $y_o \in B_{R\lambda_j^{-1/2}}(x_i) \cap (E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))$ such that $d_{E_{\lambda_j}(k-1)}(y_o) \ge \lambda_j^{-\sigma_2}$. Then clearly,

$$d_{E_{\lambda_j}(k-1)}(z) \ge \lambda_j^{-\sigma_2}/2, \qquad z \in B_{\lambda_j^{-\sigma_2}/2}(y_o).$$

Note that if $y_o \in E_{\lambda_j}(k) \setminus E_{\lambda_j}(k-1)$, then $E_{\lambda_j}(k-1) \cap B_{\lambda_j^{-\sigma_2}/2}(y_o) = \emptyset$, and by Remark 5.4 a),

$$|(E_{\lambda_j}(k) \setminus E_{\lambda_j}(k-1)) \cap B_{\lambda_j^{-\sigma_2}/2}(y_o)| = |E_{\lambda_j}(k) \cap B_{\lambda_j^{-\sigma_2}/2}(y_o)| \ge (\kappa/4)^{n+1}\omega_{n+1}\lambda_j^{-\sigma_2(n+1)},$$

and if $y_o \in E_{\lambda_j}(k-1) \setminus E_{\lambda_j}(k)$, then $B_{\lambda_j^{-\sigma_2}/2}(y_o) \subset E_{\lambda_j}(k-1)$, and by Remark 5.4 b),

$$|(E_{\lambda_j}(k-1) \setminus E_{\lambda_j}(k)) \cap B_{\lambda_j^{-\sigma_2}/2}(y_o)| = |B_{\lambda_j^{-\sigma_2}/2}(y_o) \setminus E_{\lambda_j}(k)| \ge (\kappa/4)^{n+1}\omega_{n+1}\lambda_j^{-\sigma_2(n+1)},$$

hence, by the choice of σ_2,

$$\int_{B_{\lambda_j^{-\sigma_2}/2}(y_o)\cap(E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))} d_{E_{\lambda_j}(k-1)}\,dz \ge \left(\frac{\kappa}{4}\right)^{n+1}\ \frac{\omega_{n+1}}{2\lambda_j^{\sigma_2(n+2)}} = \left(\frac{\kappa}{4}\right)^{n+1}\ \frac{\omega_{n+1}}{2}\,\lambda_j^{-\frac{n+5/2}{2}}.$$

The definition of r and this inequality imply

$$\begin{aligned}
\int_{B_r(x_i)} |\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)}|\,dx &\le \omega_{n+1}\left(R\lambda_j^{-1/2}\right)^{n+1} \\
&\le \left(\frac{4R}{\kappa}\right)^{n+1} 2\lambda_j^{3/4}\int_{B_{\lambda_j^{-\sigma_2}/2}(y)\cap(E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))} d_{E_{\lambda_j}(k-1)}\,dz \\
&\le \left(\frac{4R}{\kappa}\right)^{n+1} 2\lambda_j^{3/4}\int_{B_r(x_i)\cap(E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))} d_{E_{\lambda_j}(k-1)}\,dz,
\end{aligned} \tag{8.35}$$

where we used $B_{\lambda_j^{-\sigma_2}/2}(y_o) \subseteq B_r(x_i)$.

Since $\mathcal{A}_\beta(E_{\lambda_j}(k), E_{\lambda_j}(k-1), \lambda_j) \le \mathcal{A}_\beta(E_{\lambda_j}(k)\setminus B_r(x_i), E_{\lambda_j}(k-1), \lambda_j)$, $B_r(x_i) \subset\subset \Omega$ and $E_{\lambda_j}(k) \subseteq E^+$, we have

$$P(E_{\lambda_j}(k), B_r(x_i)) \le C(n, \mathrm{diam}(E^+))(R\lambda_j^{-1/2})^n,$$

whence, by Proposition 5.5,

$$\begin{aligned}
&\int_{B_r(x_i)\cap\partial E_{\lambda_j}(k)} d_{E_{\lambda_j}(k-1)}\,d\mathcal{H}^n \le C(n, \mathrm{diam}(E^+))(R\lambda_j^{-1/2})^{n+1} \\
&\le \frac{C(n, \mathrm{diam}(E^+))}{\omega_{n+1}}\left(\frac{4R}{\kappa}\right)^{n+1} 2\lambda_j^{3/4}\int_{B_r(x_i)\cap(E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))} d_{E_{\lambda_j}(k-1)}\,dz.
\end{aligned} \tag{8.36}$$

From (8.35) and (8.36) we get

$$\begin{aligned}
&\left|\int_{B_r(x_i)} (\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)})\phi\,dy - \int_{B_r(x_i)\cap\partial E_{\lambda_j}(k)} \tilde{d}_{E_{\lambda_j}(k-1)}\phi\,d\mathcal{H}^n\right| \\
&\qquad \le \|\phi\|_\infty C(n, \kappa, \mathrm{diam}(E^+))\lambda_j^{3/4}\int_{B_r(x_i)\cap(E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))} d_{E_{\lambda_j}(k-1)}\,dz.
\end{aligned} \tag{8.37}$$

Inequality (8.37) and the property of the covering yield

$$\begin{aligned}
\Delta_k^{\mathrm{high}} &:= \left|\int_{\Omega^\varepsilon\setminus O} (\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)})\phi\,dy - \int_{(\Omega^\varepsilon\setminus O)\cap\partial E_{\lambda_j}(k)} \tilde{d}_{E_{\lambda_j}(k-1)}\,\phi\,d\mathcal{H}^n\right| \\
&\le \mathfrak{b}(n)\|\phi\|_\infty C(n, \kappa, \mathrm{diam}(E^+))\lambda_j^{3/4}\int_{E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1)} d_{E_{\lambda_j}(k-1)}\,dz.
\end{aligned}$$

By (7.4)

$$\lambda_j^{3/4}\int_{E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1)} d_{E_{\lambda_j}(k-1)}\,dz \le \lambda^{-1/4}\Big(C_\beta(E_{\lambda_j}(k-1), \Omega) - C_\beta(E_{\lambda_j}(k), \Omega)\Big),$$

and thus we have also

$$\Delta_k^{\mathrm{high}} \le \mathfrak{b}(n)\|\phi\|_\infty C(n,\kappa,\mathrm{diam}(E^+))\lambda_j^{-1/4}\Big(C_\beta(E_{\lambda_j}(k-1),\Omega) - C_\beta(E_{\lambda_j}(k),\Omega)\Big). \tag{8.38}$$

Now (8.17) follows from the inequality $|\Delta_k(j)| \le \Delta_k^{\mathrm{low}} + \Delta_k^{\mathrm{high}}$, (8.34) and (8.38).

Since $N_j \le T\lambda_j < +\infty$, by Proposition 8.10,

$$\sum_{k=2}^{N_j} |E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1)| \le C(n,\kappa,P(E_0),T)$$

and by (7.6),

$$\sum_{k=2}^{N_j}\Big(C_\beta(E_{\lambda_j}(k-1),\Omega) - C_\beta(E_{\lambda_j}(k),\Omega)\Big) \le P(E_0).$$

Hence, from (8.17) we deduce

$$\begin{aligned}\sum_{k=2}^{N_j} |\Delta_k(j)| \le & C(n,\kappa,P(E_0),T)\Big(\lambda_j^{-\sigma_2}\|\nabla\phi\|_\infty + (w(1/\lambda_j) + \lambda_j^{\sigma_1-\sigma_2})\,\|\phi\|_\infty\Big)\\ &+ \|\phi\|_\infty\, C(n,\kappa,\mathrm{diam}(E^+))\,\lambda_j^{-1/4}\,P(E_0).\end{aligned} \tag{8.39}$$

Now the error estimate (8.14) follows from (8.16) and (8.39). □

Proof of Theorem 8.6. Lemma 8.9, (8.3) and [36, Theorem 4.4.2] imply that there exist a (not relabelled) subsequence and a function $v : [0,+\infty)\times\Omega \to \mathbb{R}$ satisfying (8.4)-(8.6). In particular, from (8.4) it follows that $H_{E(t)} := v(t,\cdot)\big|_{\Omega\cap\partial^* E(t)} \in L^2(\Omega\cap\partial^* E(t), \mathcal{H}^n \llcorner (\Omega\cap\partial^* E(t)))$ for a.e. $t > 0$. Let us prove that $H_{E(t)}$ is the distributional mean curvature of $E(t)$ for a.e. $t \ge 0$. Fixing $t \ge 0$, by the divergence formula (2.3) for any $\phi \in C_c^1(\mathbb{R}^{n+1},\mathbb{R}^{n+1})$ one has

$$\int_{E_{\lambda_j}([\lambda_j t])} \mathrm{div}\,\phi dx - \int_{\Omega\cap\partial^* E_{\lambda_j}([\lambda_j t])} \phi\cdot\nu_{E_{\lambda_j}([\lambda_j t])}\, d\mathcal{H}^n = \int_{\partial\Omega\cap\partial^* E_{\lambda_j}([\lambda_j t])} \phi_{n+1} d\mathcal{H}^n.$$

Hence, from (7.1) and (7.3) we get

$$\int_{E(t)} \mathrm{div}\,\phi dx - \int_{\Omega\cap\partial^* E(t)} \phi\cdot\nu_{E(t)}\, d\mathcal{H}^n = \lim_{j\to+\infty}\int_{\mathrm{Tr}(E_{\lambda_j}([\lambda_j t]))} \phi_{n+1} d\mathcal{H}^n. \tag{8.40}$$

The left-hand-side of (8.40) is $\int_{\mathrm{Tr}(E(t))}\phi_{n+1}d\mathcal{H}^n$, therefore,

$$\mathcal{H}^n \llcorner \mathrm{Tr}(E_{\lambda_j}([\lambda_j t])) \overset{w^*}{\rightharpoonup} \mathcal{H}^n \llcorner \mathrm{Tr}(E(t)) \qquad \text{as } j\to+\infty. \tag{8.41}$$

Combining this with (8.3) we get

$$\mathcal{H}^n \llcorner \partial^* E_{\lambda_j}([\lambda_j t]) \overset{w^*}{\rightharpoonup} \mathcal{H}^n \llcorner \partial^* E(t) \quad \text{as } j\to+\infty \text{ for a.e. } t\ge 0.$$

Take $\eta \in C_c^1([0,+\infty))$ and an admissible $X \in C_c^1(\overline{\Omega},\mathbb{R}^{n+1})$. By (8.3) and [45, formula (4.2)] for a.e. $t \ge 0$ and for every $F \in C_c(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$ one has

$$\lim_{j\to+\infty}\int_{\Omega\cap\partial^* E_{\lambda_j}([\lambda_j t])} F(x,\nu_{E_{\lambda_j}([\lambda_j t])}(x))\, d\mathcal{H}^n = \int_{\Omega\cap\partial^* E(t)} F(x,\nu_{E(t)}(x))\, d\mathcal{H}^n. \tag{8.42}$$

In particular, taking $F \in C_c(\overline{\Omega} \times \mathbb{R}^{n+1})$ such that $F(x,\xi) = \operatorname{div} X(x) - \xi \cdot \nabla X(x)\xi$ in $\Omega \times \{|\xi| \le 2\}$, by the dominated convergence theorem, (8.2) and (8.6), for $\Psi(t,x) = \eta(t)X(x)$ we establish

$$\begin{aligned}
\int_0^{+\infty} \eta(t) \int_{\Omega\cap\partial^* E(t)} F(x, \nu_{E(t)}(x)) d\mathcal{H}^n dt &= \lim_{j\to+\infty} \int_0^{+\infty} \int_{\Omega\cap\partial^* E_{\lambda_j}([\lambda_j t])} \eta(t) F(x, \nu_{E_{\lambda_j}([\lambda_j t])}) d\mathcal{H}^n dt \\
&= \lim_{j\to+\infty} \int_0^{+\infty} \int_{\Omega\cap\partial^* E_{\lambda_j}([\lambda_j t])} v_{\lambda_j} \nu_{E_{\lambda_j}([\lambda_j t])} \cdot \Psi(t,x) d\mathcal{H}^n dt \\
&= \int_0^{+\infty} \int_{\Omega\cap\partial^* E(t)} v \nu_{E(t)} \cdot \Psi(t,x) d\mathcal{H}^n dt = \int_0^{+\infty} \eta(t) \int_{\Omega\cap\partial^* E(t)} H_{E(t)} \nu_{E(t)} \cdot X \, d\mathcal{H}^n dt.
\end{aligned}$$

Since $\eta \in C_c^1([0,+\infty))$ is arbitrary, for a.e. $t \ge 0$ we get

$$\int_{\Omega\cap\partial^* E(t)} (\operatorname{div} X - \nu_{E(t)} \cdot (\nabla X)\nu_{E(t)}) d\mathcal{H}^n = \int_{\Omega\cap\partial^* E(t)} H_{E(t)} \nu_{E(t)} \cdot X \, d\mathcal{H}^n,$$

hence $H_{E(t)}$ is the generalized mean curvature of $\Omega \cap \partial^* E(t)$.

Let us show (8.7). Take $\phi \in C_c^1([0,+\infty) \times \Omega)$. By a change of variables we have

$$\begin{aligned}
&\int_{1/\lambda_j}^{+\infty} \Big[\int_{E_{\lambda_j}([\lambda_j t])} \phi dx - \int_{E_{\lambda_j}([\lambda_j t]-1)} \phi dx \Big] dt \\
&\qquad = \int_{1/\lambda_j}^{+\infty} \int_{E_{\lambda_j}([\lambda_j t])} (\phi(t,x) - \phi(t+1/\lambda_j, x)) dx dt - \frac{1}{\lambda_j} \int_{E(0)} \phi(x,0)\, dx.
\end{aligned}$$

Since $E(0) = E_0$, from (7.13) we get

$$\lim_{j\to+\infty} \int_{1/\lambda_j}^{+\infty} \lambda_j \Big[\int_{E_{\lambda_j}([\lambda_j t])} \phi dx - \int_{E_{\lambda_j}([\lambda_j t]-1)} \phi dx \Big] dt = - \int_0^{+\infty} \int_{E(t)} \frac{\partial\phi}{\partial t}(t,x)\, dx dt - \int_{E_0} \phi(x,0) dx.$$

Therefore, (8.14), (8.5) and the definition of $H_{E(t)}$ imply

$$\begin{aligned}
\int_0^{+\infty} \int_{E(t)} \partial_t \phi \, dx dt + \int_{E_0} \phi(x,0) dx &= \lim_{j\to+\infty} \int_0^{+\infty} \int_{\Omega\cap\partial E_{\lambda_j}([\lambda_j t])} v_{\lambda_j} \phi \, d\mathcal{H}^n dt \\
&= \int_0^{+\infty} \int_{\Omega\cap\partial^* E(t)} H_{E(t)} \phi \, d\mathcal{H}^n dt.
\end{aligned}$$

(ii) Take an admissible $X \in C_c^1(\overline{\Omega}, \mathbb{R}^{n+1})$ and $\eta \in C_c^1([0,+\infty))$. From (8.1)

$$\begin{aligned}
&\int_0^{+\infty} \eta(t) \int_{\Omega\cap\partial^* E_{\lambda_j}([\lambda_j t])} \Big(\operatorname{div} X - \nu_{E_{\lambda_j}([\lambda_j t])} \cdot (\nabla X) \nu_{E_{\lambda_j}([\lambda_j t])} \Big) d\mathcal{H}^n dt \\
&\qquad - \int_0^{+\infty} \eta(t) \int_{\Omega\cap\partial^* E_{\lambda_j}([\lambda_j t])} v_{\lambda_j} X \cdot \nu_{E_{\lambda_j}([\lambda_j t])} \, d\mathcal{H}^n dt \\
&\quad = \int_0^{+\infty} \eta(t) \int_{\partial^* \mathrm{Tr}(E_{\lambda_j}([\lambda_j t]))} \beta X' \cdot \nu'_{\mathrm{Tr}(E_{\lambda_j}([\lambda_j t]))} \, d\mathcal{H}^{n-1}.
\end{aligned} \tag{8.43}$$

Let $\{\lambda_{j_l}\}_{l\ge1}$ be any subsequence of $\{\lambda_j\}$. By the uniform bound (8.8) on the perimeters and by compactness there exists a further subsequence $\{\lambda_{j_{l_k}}\}_{k\ge1}$ of $\{\lambda_{j_l}\}_{l\ge1}$ and a set $\hat{F} \in BV(\mathbb{R}^n, \{0,1\})$

such that $\mathrm{Tr}(E_{j_{l_k}}([j_{l_k}t])) \to \hat F$ in $L^1(\mathbb{R}^n)$ and[4]

$$\nu'_{\mathrm{Tr}(E_{\lambda_{j_{l_k}}}([\lambda_{j_{l_k}}t]))}\mathcal{H}^{n-1}\llcorner\partial^*\mathrm{Tr}(E_{\lambda_{j_{l_k}}}([\lambda_{j_{l_k}}t])) \overset{w^*}{\rightharpoonup} \nu'_{\hat F}\mathcal{H}^{n-1}\llcorner\partial^*\hat F \quad \text{as } k\to+\infty$$

for a.e. $t\ge 0$. By (8.41) for every $\phi\in C_c(\mathbb{R}^n)$ we have

$$\int_{\mathrm{Tr}(E(t))}\phi\, d\mathcal{H}^n = \lim_{k\to+\infty}\int_{\mathrm{Tr}(E_{\lambda_{j_{l_k}}}([\lambda_{j_{l_k}}t]))}\phi\, d\mathcal{H}^n = \int_{\hat F}\phi\, d\mathcal{H}^n.$$

Whence, $\hat F = \mathrm{Tr}(E(t))$. Therefore

$$\nu'_{\mathrm{Tr}(E_{\lambda_j}([\lambda_j t]))}\mathcal{H}^{n-1}\llcorner\partial^*\mathrm{Tr}(E_{\lambda_j}([\lambda_j t])) \overset{w^*}{\rightharpoonup} \nu'_{\mathrm{Tr}(E(t))}\mathcal{H}^{n-1}\llcorner\partial^*\mathrm{Tr}E(t) \quad \text{as } j\to+\infty.$$

Now taking limit in (8.43), using (8.42), (8.6) and applying the dominated convergence theorem on the right-hand-side we get (8.9). □

Appendix A. Existence of minimizers for functionals of the form $C_\beta + \mathcal{V}$

In this appendix we prove an existence result for minimum problems of type

$$\inf_{E\in BV(\Omega,\{0,1\})}\mathcal{G}_\beta(E), \qquad \mathcal{G}_\beta(E) := C_\beta(E,\Omega)+\mathcal{V}(E), \tag{A.1}$$

where $\mathcal{V}: BV(\Omega,\{0,1\})\to(-\infty,+\infty]$. Since $C_\beta(\cdot,\Omega)$ is finite in $BV(\Omega,\{0,1\})$, the functional $\mathcal{G}_\beta$ is well-defined in $BV(\Omega,\{0,1\})$. We study (A.1) under the following hypotheses on $\mathcal{V}$:

Hypothesis A.1. (a) $\mathcal{V}$ is bounded from below in $BV(\Omega,\{0,1\})$ and there exists a cylinder $C_r^K\subset\Omega$, $K>1$ such that $\mathcal{V}(C_r^K)<+\infty$;

(b) $\mathcal{V}(E)\ge\mathcal{V}(E\cap C_\rho^l)$ for any $E\in BV(\Omega,\{0,1\})$, $\rho\in(r,+\infty]$, and $l\in(K-1,K+1)$;

(c) $\mathcal{V}(E)\ge\mathcal{V}(E\setminus(C_{\rho_1}^K\setminus\overline{C_{\rho_2}^K}))$ for any $E\in BV(\Omega,\{0,1\})$ and $r<\rho_2<\rho_1<+\infty$;

(d) $\mathcal{V}$ is $L^1(\Omega)$-lower semicontinuous in $BV(\Omega,\{0,1\})$.

Example A.2. Besides (4.8) the following functionals $\mathcal{V}: BV(\Omega,\{0,1\})\to(-\infty,+\infty]$ satisfy Hypothesis A.1:

1) given $f\in L^1_{\mathrm{loc}}(\Omega)$ with $f\ge0$ a.e. in $\Omega\setminus C_r^l$ for some $r,l>0$,

$$\mathcal{V}(E)=\int_E f\,dx.$$

In particular, we may take $f=\lambda\tilde d_{E_0}$ with $\emptyset\ne E_0\in BV(\Omega,\{0,1\})$ and $E_0\subset C_r^h$ so that by (4.2) $\mathcal{G}_\beta$ coincides with $\mathcal{A}_\beta(\cdot,E_0,\lambda)+\int_{E_0}\tilde d_{E_0}dx$.

2) Given a bounded set $E_0\in BV(\Omega,\{0,1\})$, $\mathcal{V}(E)=|E\Delta E_0|^p$, $p>0$.

[4] Arguing, for example, as in (7.15).

Given $\mathcal{V}$ satisfying Hypothesis A.1 set

$$\mathfrak{a} := \kappa^{-1}\left(\sup_{R>r}\inf_{E\in BV(C_R^K,\{0,1\})}\mathcal{G}_\beta(E) - \inf\mathcal{V}\right).$$

Clearly, $\kappa\mathfrak{a} \le \mathcal{G}_\beta(C_r^K) - \inf\mathcal{V},$ hence $\inf\mathcal{G}_\beta < +\infty.$

In view of the previous observation, once we prove the next theorem, the proof of Theorem 4.1 follows.

Theorem A.3 (**Existence of minimizers and uniform bound**). *Suppose that Hypothesis A.1 holds. Suppose also $\beta \in L^\infty(\partial\Omega)$ and there exists $\kappa \in (0, \frac{1}{2}]$ such that $-1 \le \beta \le 1-2\kappa$ $\mathcal{H}^n$-a.e on $\partial\Omega$. Then the minimum problem*

$$\inf_{E\in BV(\Omega,\{0,1\})}\mathcal{G}_\beta(E)$$

has a solution. Moreover, any minimizer is contained in $C^K_{\mathcal{R}_0}$, where[5]

$$\mathcal{R}_0 := r + 1 + \max\left\{8^{n^2+n+1}\,\mathfrak{a}^{\frac{n+1}{n}},\, 4\mu(\kappa,n)\right\} \tag{A.2}$$

and $\mu(\kappa,n)$ is defined in Section 4.1.

Remark A.4. In case of Example A.2 1) with $f = \lambda\tilde{d}_{E_0}$ for some $C_r^K \supseteq E_0$,

$$\kappa\mathfrak{a} \le \kappa\sup_{R>r}\inf_{E\in BV(C_R^K,\{0,1\})}\mathcal{A}_\beta(E,E_0,\lambda) \le \kappa\mathcal{A}_\beta(E_0,E_0,\lambda) = \kappa C_\beta(E_0,\Omega) \le \kappa P(E_0).$$

Hence, $\mathcal{R}_0 \le R_0$, where R_0 is defined in (4.4). The same is true if $\mathcal{V}$ is as in (4.8).

The assumption on β and the $L^1(\Omega)$-lower semicontinuity of $C_\beta(\cdot,\Omega)$ (Lemma 3.5) imply the $L^1(\Omega)$-lower semicontinuity of $\mathcal{G}_\beta$. Moreover, the coercivity (3.9) of $C_\beta(\cdot,\Omega)$, Hypothesis A.1 (a) and (3.11) imply the coercivity of $\mathcal{G}_\beta$:

$$\mathcal{G}_\beta(E) \ge \kappa P(E) + \inf\mathcal{V} \qquad \forall E \in BV(\Omega,\{0,1\}). \tag{A.3}$$

The main problem in the proof of existence of minimizers of $\mathcal{G}_\beta$ is the lack of compactness due to the unboundedness of Ω. However, for every $R > 0$ inequality (A.3), the compactness theorem in $BV(C_R^K,\{0,1\})$ (see for instance [7, Theorems 3.23 and 3.39]) and the lower semicontinuity of $\mathcal{G}_\beta$ imply that there exists a solution $E^R \in BV(C_R^K,\{0,1\})$ of

$$\inf_{E\in BV(C_R^K,\{0,1\})}\mathcal{G}_\beta(E).$$

To prove Theorem A.3 we mainly follow [17, Section 4], where the existence of volume preserving minimizers of $C_\beta(\cdot,\Omega)$ has been shown. We need two preliminary lemmas. As in [17, Section 3] first we show that one can choose a minimizing sequence consisting of bounded sets.

[5] One could refine the expression of $\mathcal{R}_0$ using the isoperimetric inequality [24], but we do not need this here.

Lemma A.5 (**Truncations with horizontal hyperplanes and vertical cylinders**). *Suppose that Hypothesis A.1 holds. Then*

$$\inf_{E \in BV(\Omega,\{0,1\})} \mathcal{G}_\beta(E) = \inf_{R>0} \inf_{E \in BV(C_R^K,\{0,1\})} \mathcal{G}_\beta(E). \tag{A.4}$$

Proof. We need two intermediate steps. The first step concerns truncations with horizontal hyperplanes.

Step 1. We have

$$\inf_{E \in BV(\Omega,\{0,1\})} \mathcal{G}_\beta(E) = \inf_{E \in BV(\Omega_K,\{0,1\})} \mathcal{G}_\beta(E). \tag{A.5}$$

Indeed, it suffices to show that if $E \setminus \Omega_{K-\frac{1}{4}} \neq \emptyset$, then

$$\mathcal{G}_\beta(E) \geq \mathcal{G}_\beta(E \cap \overline{\Omega_{K-\frac{1}{2}}}).$$

Clearly, E and $E \cap \overline{\Omega_{K-\frac{1}{2}}}$ have the same trace on $\partial\Omega$ and thus

$$\int_{\partial\Omega} [1+\beta]\chi_E \, d\mathcal{H}^n = \int_{\partial\Omega} [1+\beta]\chi_{E\cap\overline{\Omega_{K-\frac{1}{2}}}} \, d\mathcal{H}^n.$$

From the comparison theorem of [6, page 216] we have

$$P(E) > P(E \cap \overline{\Omega_{K-\frac{1}{2}}}).$$

By Hypothesis A.1 (b) we have also

$$\mathcal{V}(E) \geq \mathcal{V}(E \cap \overline{\Omega_{K-\frac{1}{2}}}),$$

therefore from the definition of $\mathcal{G}_\beta$ we get even the strict inequality

$$\mathcal{G}_\beta(E) > \mathcal{G}_\beta(E \cap \overline{\Omega_{K-\frac{1}{2}}}). \tag{A.6}$$

The second step is more delicate and concerns truncations with the lateral boundary of vertical cylinders.

Step 2. For any $\varepsilon \in (0,1)$ there exists $R_\varepsilon > r$ and $E_\varepsilon \in BV(C^K_{R_\varepsilon}, \{0,1\})$ such that

$$\mathcal{G}_\beta(E_\varepsilon) \leq \inf_{E \in BV(\Omega_K,\{0,1\})} \mathcal{G}_\beta(E) + \varepsilon.$$

Indeed, according to Step 1 and Hypothesis A.1 (a), given $\varepsilon > 0$ there exists $F_\varepsilon \in BV(\Omega_K, \{0,1\})$ with $F_\varepsilon \subset \overline{\Omega_{K-\frac{1}{4}}}$ such that

$$\mathcal{G}_\beta(F_\varepsilon) < \inf_{E \in BV(\Omega,\{0,1\})} \mathcal{G}_\beta(E) + \frac{\varepsilon}{2} < +\infty.$$

Since $|F_\varepsilon| < +\infty$, for sufficiently large $R > r$ one has

$$|F_\varepsilon \cap (C^K_{R+1} \setminus C^K_R)| = \int_R^{R+1} \mathcal{H}^n(F_\varepsilon \cap \partial C^K_\rho) \, d\rho < \frac{\varepsilon}{2}.$$

Hence there exists $R_\varepsilon \in (R, R+1)$ such that

$$\mathcal{H}^n(F_\varepsilon \cap \partial C^K_{R_\varepsilon}) \leq \frac{\varepsilon}{2}, \qquad \mathcal{H}^n(\Omega \cap \partial^* F_\varepsilon \cap \partial C^K_{R_\varepsilon}) = 0.$$

Now, let $E_\varepsilon := F_\varepsilon \cap C^K_{R_\varepsilon}$. Since $\mathcal{H}^n(\Omega \cap \partial^* F_\varepsilon \cap \partial C^K_{R_\varepsilon}) = 0$, we have

$$\begin{aligned}P(E_\varepsilon, \Omega) =& P(E_\varepsilon, \Omega_K) = P(F_\varepsilon, \Omega_K) + \mathcal{H}^n(F_\varepsilon \cap \partial C^K_{R_\varepsilon}) - P(F_\varepsilon, \Omega_K \setminus \overline{C^K_{R_\varepsilon}}) \\ =& P(F_\varepsilon, \Omega) + \mathcal{H}^n(F_\varepsilon \cap \partial C^K_{R_\varepsilon}) - P(F_\varepsilon, \Omega_K \setminus \overline{C^K_{R_\varepsilon}}).\end{aligned} \tag{A.7}$$

By Hypothesis A.1 (a), $\mathcal{V}(F_\varepsilon) \ge \mathcal{V}(E_\varepsilon)$, thus employing (A.7) we get

$$\mathcal{G}_\beta(F_\varepsilon) \ge \mathcal{G}_\beta(E_\varepsilon) - \mathcal{H}^n(F_\varepsilon \cap \partial C^K_{R_\varepsilon}) + P(F_\varepsilon, \Omega_K \setminus \overline{C^K_{R_\varepsilon}}) - \int_{\partial\Omega} \beta \chi_{F_\varepsilon \setminus C^K_{R_\varepsilon}}\, d\mathcal{H}^n.$$

By Lemma 3.1 applied with $E = F_\varepsilon$ and $A = \Omega_K \setminus \overline{C^K_{R_\varepsilon}}$, we have

$$P(F_\varepsilon, \Omega_K \setminus \overline{C^K_{R_\varepsilon}}) - \int_{\partial\Omega} \beta \chi_{F_\varepsilon \setminus C^K_{R_\varepsilon}}\, d\mathcal{H}^n \ge 0.$$

Consequently, from the choice of F_ε and R_ε we get

$$\mathcal{G}_\beta(E_\varepsilon) \le \mathcal{G}_\beta(F_\varepsilon) + \mathcal{H}^n(F_\varepsilon \cap \partial C^K_{R_\varepsilon}) < \inf_{E \in BV(\Omega,\{0,1\})} \mathcal{G}_\beta(E) + \varepsilon.$$

This concludes the proof of Step 2.

Now, observe that

$$\inf_{E \in BV(\Omega,\{0,1\})} \mathcal{G}_\beta(E) \le \inf_{R>0} \inf_{E \in BV(C^K_R,\{0,1\})} \mathcal{G}_\beta(E).$$

On the other hand, since the mapping

$$R \in (0, +\infty) \mapsto \inf_{E \in BV(C^K_R,\{0,1\})} \mathcal{G}_\beta(E)$$

is nonincreasing, Step 2 implies

$$\inf_{E \in BV(\Omega,\{0,1\})} \mathcal{G}_\beta(E) \ge \inf_{R>0} \inf_{E \in BV(C^K_R,\{0,1\})} \mathcal{G}_\beta(E),$$

therefore (A.4) follows. □

As in [17, Lemma 3] the following lemma holds.

Lemma A.6 (Good choice of a radius). *Suppose that β satisfies (4.3) and Hypothesis A.1 holds. Let E^R be a minimizer of $\mathcal{G}_\beta$ in $BV(C^K_R, \{0,1\})$. Then for any $R > \mathcal{R}_0$ there exists $t_R \in [r+1, \mathcal{R}_0]$ such that*

$$\mathcal{H}^n(E^R \cap \partial C^K_{t_R}) = 0.$$

Hence

$$P(E^R, \Omega) = P(E^R \setminus \overline{C^K_{t_R}}, \Omega) + P(E^R \cap C^K_{t_R}, \Omega). \tag{A.8}$$

Proof. The idea of the proof is to cut the E^R with vertical cylinders, similarly to [17, Lemma 5] where cuts with horizontal hyperplanes are performed.

For $R > \mathcal{R}_0$ by the isoperimetric-type inequality [23, Theorem VI], (A.3), the minimality of E^R and by the definition of $\mathfrak{a}$ we have

$$|E^R|^{\frac{n}{n+1}} \le P(E^R) \le \frac{\mathcal{G}_\beta(E^R) - \inf \mathcal{V}}{\kappa} = \frac{1}{\kappa}\left(\inf_{E \in BV(C_R^K,\{0,1\})} \mathcal{G}_\beta(E) + \inf \mathcal{V}\right) \le \mathfrak{a}.$$

Thus, for any $0 < a < b$ one has

$$|E^R \cap (C_b^K \setminus C_a^K)| \le \mathfrak{a}^{\frac{n+1}{n}}. \tag{A.9}$$

Take $r + 1 < r_1 < r_2 < r_3 < \mathcal{R}_0$ such that

$$\mathcal{H}^n(\Omega \cap \partial^* E^R \cap \partial C_{r_i}^K) = 0, \quad i = 1, 2, 3,$$

and set

$$v_1 = |E^R \cap (C_{r_2}^K \setminus C_{r_1}^K)|, \qquad v_2 = |E^R \cap (C_{r_3}^K \setminus C_{r_2}^K)|,$$
$$m = \max_{i=1,2,3} \mathcal{H}^n(E^R \cap \partial C_{r_i}^K).$$

Step 1. We claim that

$$\min\{v_1, v_2\} \le \mu m^{\frac{n+1}{n}}, \tag{A.10}$$

where $\mu := \mu(\kappa, n) > 0$.

It suffices to prove that

$$v_1^{\frac{n}{n+1}} + v_2^{\frac{n}{n+1}} \le 2\mu^{\frac{n}{n+1}} m.$$

We have

$$\begin{aligned} v_1^{\frac{n}{n+1}} \le & P(E^R \cap (C_{r_2}^K \setminus \overline{C_{r_1}^K})) \le P(E^R, C_{r_2}^K \setminus \overline{C_{r_1}^K}) + \mathcal{H}^n(E^R \cap \partial C_{r_1}^K) \\ & + \mathcal{H}^n(E^R \cap \partial C_{r_2}^K) + \int_{\partial\Omega} \chi_{E^R \cap (C_{r_2}^K \setminus \overline{C_{r_1}^K})}\, d\mathcal{H}^n \\ \le & P(E^R, C_{r_2}^K \setminus \overline{C_{r_1}^K}) + \int_{\partial\Omega} \chi_{E^R \cap (C_{r_2}^K \setminus \overline{C_{r_1}^K})}\, d\mathcal{H}^n + 2m. \end{aligned}$$

Similarly,

$$v_2^{\frac{n}{n+1}} \le P(E^R, C_{r_3}^K \setminus \overline{C_{r_2}^K}) + \int_{\partial\Omega} \chi_{E^R \cap (C_{r_3}^K \setminus \overline{C_{r_2}^K})}\, d\mathcal{H}^n + 2m.$$

Hence

$$v_1^{\frac{n}{n+1}} + v_2^{\frac{n}{n+1}} \le P(E^R, C_{r_3}^K \setminus \overline{C_{r_1}^K}) + \int_{\partial\Omega} \chi_{E^R \cap (C_{r_3}^K \setminus \overline{C_{r_1}^K})}\, d\mathcal{H}^n + 4m. \tag{A.11}$$

Comparing $E^R \setminus (C_{r_3}^K \setminus \overline{C_{r_1}^K})$ with E^R, we get $\mathcal{G}_\beta(E^R) \le \mathcal{G}_\beta(E^R \setminus (C_{r_3}^K \setminus \overline{C_{r_1}^K}))$, therefore from Hypothesis A.1 (c) we obtain

$$P(E^R) \le P(E^R \setminus (C_{r_3}^K \setminus \overline{C_{r_1}^K})) + \int_{\partial\Omega} [1 + \beta] \chi_{E^R \cap (C_{r_3}^K \setminus \overline{C_{r_1}^K})}\, d\mathcal{H}^n. \tag{A.12}$$

Inserting in (A.12) the identity

$$P(E^R \setminus (C^K_{r_3} \setminus \overline{C^K_{r_1}})) = P(E^R) + \mathcal{H}^n(E^R \cap \partial C^K_{r_1}) + \mathcal{H}^n(E^R \cap \partial C^K_{r_3})$$
$$- P(E^R, C^K_{r_3} \setminus \overline{C^K_{r_1}}) - \int_{\partial\Omega} \chi_{E^R \cap (C^K_{r_3} \setminus \overline{C^K_{r_1}})}\, d\mathcal{H}^n,$$

we get

$$P(E^R, C^K_{r_3} \setminus \overline{C^K_{r_1}}) - \int_{\partial\Omega} \beta \chi_{E^R \cap (C^K_{r_3} \setminus \overline{C^K_{r_1}})}\, d\mathcal{H}^n \le 2m. \tag{A.13}$$

By Lemma 3.1 applied with $A = C^K_{r_3} \setminus \overline{C^K_{r_1}}$ and $E = E^R$, the left-hand-side of (A.13) is not less than

$$\kappa P(E^R, C^K_{r_3} \setminus \overline{C^K_{r_1}}) + \kappa \int_{\partial\Omega} \chi_{E^R \cap (C^K_{r_3} \setminus \overline{C^K_{r_1}})}\, d\mathcal{H}^n,$$

hence

$$P(E^R, C^K_{r_3} \setminus \overline{C^K_{r_1}}) + \int_{\partial\Omega} \chi_{E^R \cap (C^K_{r_3} \setminus \overline{C^K_{r_1}})}\, d\mathcal{H}^n \le \frac{2m}{\kappa}.$$

Then from (A.11) it follows that

$$v_1^{\frac{n}{n+1}} + v_2^{\frac{n}{n+1}} \le \left(\frac{2m}{\kappa} + 4m\right) = 2\mu^{\frac{n}{n+1}} m.$$

This finishes the proof of Step 1.

Before going to Step 2 we need some preliminaries. Choose any $R \ge \mathcal{R}_0$. Let $a_0 = r + 1$, $b_0 = \mathcal{R}_0$. Given $r + 1 \le a_k \le b_k \le \mathcal{R}_0$, $k \in \mathbb{N}$, define

$$v_k = |E^R \cap (C^K_{b_k} \setminus C^K_{a_k})|.$$

By (A.6) $E^R \setminus \Omega_{K-\frac{1}{4}} = \emptyset$, hence

$$|E^R \cap (C^K_b \setminus C^K_a)| = \int_a^b \mathcal{H}^n(E^R \cap \partial C^K_\rho)\, d\rho, \quad 0 \le a < b.$$

Therefore, for $h_k = \frac{b_k - a_k}{4}$ it is possible to find $r_{k,1} \in (a_k, a_k + h_k)$, $r_{k,2} \in (\frac{a_k+b_k}{2} - \frac{h_k}{2}, \frac{a_k+b_k}{2} + \frac{h_k}{2})$ and $r_{k,3} \in (b_k - h_k, b_k)$ such that

$$\mathcal{H}^n(E^R \cap \partial C^K_{r_{k,i}}) \le \frac{v_k}{h_k}, \quad \mathcal{H}^n(\Omega \cap \partial^* E^R \cap \partial C^K_{r_{k,i}}) = 0 \quad \text{for } i = 1, 2, 3\,. \tag{A.14}$$

We choose

$$(a_{k+1}, b_{k+1}) = \begin{cases} (r_{k,1}, r_{k,2}) & \text{if } |E^R \cap (C^K_{r_{k,1}} \setminus C^K_{r_{k,2}})| \le |E^R \cap (C^K_{r_{k,2}} \setminus C^K_{r_{k,3}})|, \\ (r_{k,2}, r_{k,3}) & \text{if } |E^R \cap (C^K_{r_{k,1}} \setminus C^K_{r_{k,2}})| > |E^R \cap (C^K_{r_{k,2}} \setminus C^K_{r_{k,3}})|. \end{cases}$$

Let

$$m_k = \max_{i=1,2,3} \mathcal{H}^n(E^R \cap \partial C^K_{r_{k,i}}).$$

Step 2. Using the definition of $\mathcal{R}_0$ we show that

$$m_k \le \left(\frac{1}{2}\right)^{(\frac{n+1}{n})^k}. \tag{A.15}$$

Indeed, according to (A.10), (A.14) and the definition of (a_k, b_k) one has

$$v_{k+1} \le \mu m_k^{\frac{n+1}{n}}, \qquad m_k \le \frac{v_k}{h_k}.$$

By construction, $b_{k+1} - a_{k+1} \ge \frac{b_k - a_k}{8}$, i.e. $h_{k+1} \ge \frac{h_k}{8}$. By induction one can check that

$$m_k \le \left(8^{\sum_{j=1}^{k} j\alpha^j} \left(\frac{\mu}{h_0}\right)^{\sum_{j=1}^{k} \alpha^j} \frac{v_0}{h_0} \right)^{1/\alpha^k}, \tag{A.16}$$

where $\alpha := \frac{n}{n+1}$. Note that

$$\sum_{j=1}^{k} j\alpha^j \le \alpha \sum_{j\ge1} j\alpha^{j-1} = \frac{\alpha}{(1-\alpha)^2} = n(n+1).$$

Since $h_0 = \frac{\mathcal{R}_0 - r - 1}{4}$ and $v_0 \le \mathfrak{a}^{\frac{n+1}{n}}$ by (A.9), the choice of $\mathcal{R}_0$ in (A.2) implies $8^{n(n+1)} v_0/h_0 \le 1/2$. Moreover $\left(\frac{\mu}{h_0}\right)^{\sum_{j=1}^{k} \alpha^j} \le 1$, since $\frac{\mu}{h_0} = \frac{4\mu}{R_0 - r - 1} \le 1$. Now (A.15) follows from these estimates and (A.16).

Step 3. Let $i_k \in \{1, 2, 3\}$ be such that $m_k = \mathcal{H}^n(E^R \cap \partial C^K_{r_{k,i_k}})$. Since $a_k \le r_{k,i_k} \le b_k$, $\{a_k\}$ is nondecreasing and $\{b_k\}$ is nonincreasing, there exists $t_R \in [r+1, \mathcal{R}_0]$ such that $r_{k,i_k} \to t_R$ (possibly up to a subsequence). Then, by Step 2,

$$\mathcal{H}^n(E^R \cap \partial C^K_{t_R}) = \lim_{k\to+\infty} m_k = 0,$$

which concludes the proof of the lemma. □

Proof of Theorem A.3. Let us prove the existence of a minimizer of $\mathcal{G}_\beta$. For $R > \mathcal{R}_0$ let $t_R \in [r+1, \mathcal{R}_0]$ be as in Lemma A.6. Then from (A.8) and $\mathcal{V}(E^R) \ge \mathcal{V}(E^R \cap C^K_{t_R})$ we get

$$\mathcal{G}_\beta(E^R) \ge \mathcal{G}_\beta(E^R \cap C^K_{t_R}) + P(E^R \setminus \overline{C^K_{t_R}}, \Omega) - \int_{\partial\Omega} \beta \chi_{E^R \setminus \overline{C^K_{t_R}}} d\mathcal{H}^n. \tag{A.17}$$

By (3.9) and the isoperimetric-type inequality

$$P(E^R \setminus \overline{C^K_{t_R}}, \Omega) - \int_{\partial\Omega} \beta \chi_{E^R \setminus \overline{C^K_{t_R}}} d\mathcal{H}^n \ge \kappa P(E^R \setminus \overline{C^K_{t_R}}) \ge \kappa \left|E^R \setminus \overline{C^K_{t_R}}\right|^{\frac{n}{n+1}}. \tag{A.18}$$

Thus from (A.17)

$$\mathcal{G}_\beta(E^R) \ge \mathcal{G}_\beta(E^R \cap C^K_{t_R}).$$

Hence, $F^R := E^R \cap C^K_{t_R} \subseteq C^K_{\mathcal{R}_0}$ satisfies

$$\min_{E \in BV(C^K_R, \{0,1\})} \mathcal{G}_\beta(E) = \mathcal{G}_\beta(F^R).$$

From (3.9) and the minimality of F^R we get

$$\kappa P(F^R) \leq C_\beta(F^R, \Omega) \leq \mathcal{G}_\beta(F^R) - \inf \mathcal{V} \leq \kappa \mathfrak{a},$$

and thus, by compactness there exists $E \in BV(C^K_{\mathcal{R}_0}, \{0,1\})$ such that (up to a subsequence) $F^R \to E$ in $L^1(\Omega)$ as $R \to +\infty$. From the $L^1(\Omega)$-lower semicontinuity of $\mathcal{G}_\beta$ and from (A.4) we conclude that E is a minimizer of $\mathcal{G}_\beta$.

Now we prove that any minimizer E of $\mathcal{G}_\beta$ satisfies $E \subseteq C^K_{\mathcal{R}_0}$. Arguing as in the proof of (A.6) one can show that $E \subseteq \overline{\Omega_{K-\frac{1}{4}}}$.

Claim. There exists $R > r+1$ (possibly depending on $\mathcal{V}$ and r) such that $E \subseteq C^K_R$.

For any $\rho > 1$ such that $\mathcal{H}^n(\Omega \cap \partial^* E \cap \partial C^K_\rho) = 0$, by the minimality of E we have $\mathcal{G}_\beta(E) \leq \mathcal{G}_\beta(E \cap C^K_\rho)$, i.e.

$$P(E, \Omega_K \setminus \overline{C^K_\rho}) - \int_{\partial\Omega} \beta \chi_{E \setminus C^K_\rho} \, d\mathcal{H}^n \leq \mathcal{H}^n(E \cap \partial C^K_\rho). \tag{A.19}$$

By Lemma 3.1

$$P(E, \Omega_K \setminus \overline{C^K_\rho}) - \int_{\partial\Omega} \beta \chi_{E \setminus C^K_\rho} \, d\mathcal{H}^n \geq \kappa\Big(P(E, \Omega_K \setminus \overline{C^K_\rho}) + \int_{\partial\Omega} \chi_{E \setminus C^K_\rho} \, d\mathcal{H}^n\Big). \tag{A.20}$$

Moreover, by the isoperimetric-type inequality,

$$|E \setminus C^K_\rho|^{\frac{n}{n+1}} \leq P(E, \Omega_K \setminus \overline{C^K_\rho}) + \mathcal{H}^n(E \cap \partial C^K_\rho) + \int_{\partial\Omega} \chi_{E \setminus C^K_\rho E} \, d\mathcal{H}^n.$$

therefore, (A.19) and (A.20) imply

$$|E \setminus C^K_\rho|^{\frac{n}{n+1}} \leq \frac{\kappa+1}{\kappa} \mathcal{H}^n(E \cap \partial C^K_\rho). \tag{A.21}$$

Set $m(\rho) = |E \setminus C^K_\rho|$. Clearly, $m : (1, +\infty) \to [0, |E|]$. Moreover, m is absolutely continuous, nonincreasing, $\lim\limits_{\rho \to +\infty} m(\rho) = 0$ and $\mathcal{H}^n(E \cap \partial C^K_\rho) = -m'(\rho)$ for a.e. $\rho > r+1$. By (A.21) $-m'(\rho) \geq \frac{\kappa+1}{\kappa} (n+1) m(\rho)^{\frac{n}{n+1}}$. If E is unbounded, then $m(\rho) > 0$ for any $\rho > r+1$, and thus, for any $\rho_1, \rho_2 > r+1$, $\rho_1 < \rho_2$ we have

$$m(\rho_1)^{\frac{1}{n+1}} - m(\rho_2)^{\frac{1}{n+1}} \geq \frac{\kappa+1}{\kappa} (\rho_2 - \rho_1).$$

Now letting $\rho_2 \to +\infty$ we obtain $m(\rho_1) = +\infty$, a contradiction. Consequently, there exists $R > r+1$ such that $m(R) = 0$, i.e. $E \subseteq C^K_R$.

From the claim it follows that E is a minimizer of $\mathcal{G}_\beta$ also in $BV(C^K_R, \{0,1\})$. By Lemma A.6 we can find $t_R \in [r+1, \mathcal{R}_0]$ such that $\mathcal{H}^n(E \cap \partial C^K_{t_R}) = 0$. Then using $\mathcal{V}(E) \geq \mathcal{V}(E \cap C^K_{t_R})$, the relations (A.17) - (A.18) applied with E in place of E^R imply

$$\mathcal{G}_\beta(E) \geq \mathcal{G}_\beta(E \cap C^K_{t_R}) + \kappa \Big|E \setminus \overline{C^K_{t_R}}\Big|^{\frac{n}{n+1}}.$$

Therefore, the minimality of E yields $\Big|E \setminus \overline{C^K_{t_R}}\Big| = 0$, i.e. $E \subseteq C^K_{t_R}$. Since $t_R \leq \mathcal{R}_0$, the conclusion follows. □

Appendix B. Local well-posedness

In this appendix we sketch the proof of short time existence and uniqueness of smooth hypersurfaces moving with normal velocity equal to their mean curvature in Ω and meeting the boundary $\partial\Omega$ at a prescribed (not necessarily constant) angle. The following theorem is a generalization of [38, Theorem 1], where short time existence and uniqueness have been proven for constant β.

Theorem B.1 (**Short time existence and uniqueness**). *Let $\beta \in C^{1+\alpha}(\partial\Omega)$, $\|\beta\|_\infty \le 1-2\kappa$, $\kappa \in (0,\frac{1}{2}]$ and $E_0 \subset \Omega$ be a bounded open set such that $\Gamma_0 = \overline{\Omega \cap \partial E_0}$ is a $C^{3+\alpha}$-hypersurface, $\alpha \in (0,1)$. Assume that $\mathcal{U} \subset \mathbb{R}^n$ is a bounded open set with $C^{3+\alpha}$-boundary, $p^0 \in C^{3+\alpha}(\overline{\mathcal{U}}, \mathbb{R}^{n+1})$ is a parametrization of Γ_0 such that $p^0_{n+1} > 0$ in $\mathcal{U}$, $p^0_{n+1} = 0$ on $\partial\mathcal{U}$, and*

$$-e_{n+1} + \beta(p^0)\nu_0 = Dp^0[n^0] \quad \text{on } \partial\mathcal{U}, \tag{B.1}$$

where $n^0 = (n^0_1, \dots, n^0_n)$ is the outward unit normal to $\partial\mathcal{U}$, $\nu_0 = \nu(p^0)$ is the outward unit normal to Γ_0 at p^0 and $Dp^0[n^0] = \sum_{j=1}^n n^0_j p^0_{\sigma_j}$. Then there exists $T_0 = T_0(\|\beta\|_{C^{1+\alpha}}, \|p^0\|_{C^{3+\alpha}}) > 0$, a unique family of bounded open sets $\{E(t) \subset \Omega : t \in [0,T_0]\}$ with a parametrization $p \in C^{1+\alpha/2,2+\alpha}([0,T_0] \times \overline{\mathcal{U}}, \mathbb{R}^{n+1})$ of $\Gamma(t) = \overline{\Omega \cap \partial E(t)}$ solving the parabolic system

$$p_t = \operatorname{trace}((Dp \cdot (Dp)^T)^{-1} D^2 p) \ \text{ in } (0,T_0) \times \mathcal{U}, \tag{B.2}$$

where $(Dp \cdot (Dp)^T)_{ij} = p_{\sigma_i} \cdot p_{\sigma_j}$ and $(D^2 p)_{ij} = p_{\sigma_i\sigma_j}$, coupled with the initial condition $p(0,\cdot) = p^0$, the boundary conditions

$$\begin{cases} p_{n+1}(t,\cdot) = 0 & \text{on } \partial\mathcal{U} \text{ for any } t \in [0,T_0], \\ e_{n+1} \cdot \nu(p(t,\cdot)) = \beta(p(t,\cdot)) & \text{on } \partial\mathcal{U} \text{ for any } t \in [0,T_0], \end{cases} \tag{B.3}$$

and the orthogonality conditions

$$Dp^0[n^0] \cdot \tau_{0i} = 0 \text{ on } [0,T_0] \times \partial\mathcal{U} \text{ for every } i = 1, \dots, n-1, \tag{B.4}$$

where $\nu(p(t,\cdot))$ is the outward unit normal to $\Gamma(t)$ at $p(t,\cdot)$ and $\tau_{01}, \dots, \tau_{0n-1} \in \mathbb{R}^n \times \{0\}$ is a basis for the tangent space to $\Gamma_0 \cap \partial\Omega$ at p^0.

Remark B.2. Assumption (B.1) on p^0 is not restrictive. Indeed, if $q : \partial\mathcal{U} \to \Gamma_0 \cap \partial\Omega$ is a $C^{3+\alpha}$ parametrization of the contact set, we may extend it to a sufficiently small tubular neighborhood $S := \{x \in \mathcal{U} : \operatorname{dist}(x, \partial\mathcal{U}) < \varepsilon\}$ of $\partial\mathcal{U}$ in $\mathcal{U}$ with the properties that q is a $C^{3+\alpha}$ diffeomorphism, $q(S) \subset \Gamma_0$ and

$$q(\sigma) = q(\varsigma) + |\sigma - \varsigma|(e_{n+1} - \beta(q(\varsigma))\nu_0(q(\varsigma))) + O(|\sigma - \varsigma|^2),$$

where ς is the projection of $\sigma \in S$ on $\partial\mathcal{U}$. Since $\sigma = \varsigma - |\sigma - \varsigma| n^0(\varsigma)$, it follows

$$\nabla q(\varsigma)\, n^0(\varsigma) = -e_{n+1} + \beta(q(\varsigma))\nu_0(q(\varsigma)),$$

which is (B.1). Now we may arbitrarily extend q to a $C^{3+\alpha}$ diffeomorphism in $\overline{\mathcal{U}}$ such that $q(\overline{\mathcal{U}}) = \Gamma_0$.

Remark B.3. The unit normal to $\Gamma(t)$ at the point $p(t,\sigma_1,\dots,\sigma_n)\in\Gamma(t)$ can be written with an abuse of notation $\nu=\nu(p(t,\sigma_1,\dots,\sigma_n))=\frac{\tilde\nu}{|\tilde\nu|}$, where

$$\tilde\nu := \tilde\nu(p_\sigma) = \det\begin{bmatrix} e_1 & e_2 & \dots & e_n & e_{n+1} \\ & & p_{\sigma_1} & & \\ & & p_{\sigma_2} & & \\ & & \vdots & & \\ & & p_{\sigma_n} & & \end{bmatrix}.$$

Proof of Theorem B.1. The idea of the proof is standard: first we linearize the equation around the initial condition, then prove existence for the linearized system and finally we use a fixed point argument.

Step 1. Let us linearize system (B.2) fixing some $t_0>0$. Let $X(t_0)\subset C^{1+\alpha/2,2+\alpha}([0,t_0]\times\overline{\mathcal{U}},\mathbb{R}^{n+1})$ be the nonempty convex set consisting of all functions $w\in C^{1+\alpha/2,2+\alpha}([0,t_0]\times\overline{\mathcal{U}},\mathbb{R}^{n+1})$ such that

1) $w(0,\cdot)=p^0$,

2) $w_{n+1}(t,\cdot)=0$ on $\partial\mathcal{U}$ for any $t\in[0,t_0]$,

3) $\sum_{j=1}^n n_j^0 w_{\sigma_j}\cdot\tau_{0i}=0$ on $[0,t_0]\times\partial\mathcal{U}$ for every $i=1,\dots,n-1$.

For $w\in X(t_0)$ set $f(t,w):=\mathrm{trace}\big[((Dw\cdot(Dw)^T)^{-1}-(Dp^0\cdot(Dp^0)^T)^{-1})D^2w\big]$. Then (B.2) is equivalent to

$$w_t=\mathrm{trace}\big[(Dp^0\cdot(Dp^0)^T)^{-1}D^2w\big]+f(t,w).$$

Notice that

$$|f(t,w)|\le c(\|p^0\|_{C^1(\overline{\mathcal{U}})})\|w\|_{C^{0,2}([0,t_0]\times\overline{\mathcal{U}})}\|w-p_0\|_{C^{0,1}([0,t_0]\times\overline{\mathcal{U}})},$$

where $c(\|p^0\|_{C^1(\overline{\mathcal{U}})})>0$. Now we linearize the contact angle condition. Since we have $e_{n+1}\cdot\nu(p^0)=\beta(p^0)$, from Remark B.3 it follows that

$$e_{n+1}\cdot(\tilde\nu(w_\sigma)-\tilde\nu(p^0_\sigma))=\beta(w)|\tilde\nu(w_\sigma)|-\beta(p^0)|\tilde\nu(p^0_\sigma)|. \tag{B.5}$$

Let $H_1(t,w):=\tilde\nu(w_\sigma)-\tilde\nu(p^0_\sigma)-D\tilde\nu(p^0_\sigma)[w_\sigma-p^0_\sigma]$, where

$$D\tilde\nu=\begin{bmatrix} D_{p_{\sigma_1}}\tilde\nu^1 & D_{p_{\sigma_2}}\tilde\nu^1 & \dots & D_{p_{\sigma_n}}\tilde\nu^1 \\ D_{p_{\sigma_1}}\tilde\nu^2 & D_{p_{\sigma_2}}\tilde\nu^2 & \dots & D_{p_{\sigma_n}}\tilde\nu^2 \\ \vdots & \vdots & \dots & \vdots \\ D_{p_{\sigma_1}}\tilde\nu^{n+1} & D_{p_{\sigma_2}}\tilde\nu^{n+1} & \dots & D_{p_{\sigma_n}}\tilde\nu^{n+1}\end{bmatrix},\qquad q_\sigma=\begin{bmatrix} q_{\sigma_1}\\ q_{\sigma_2}\\ \vdots\\ q_{\sigma_n}\end{bmatrix}=\begin{bmatrix} (q_1)_{\sigma_1} & \dots & (q_{n+1})_{\sigma_1}\\ (q_1)_{\sigma_2} & \dots & (q_{n+1})_{\sigma_2}\\ \dots & \vdots & \dots\\ (q_1)_{\sigma_n} & \dots & (q_{n+1})_{\sigma_n}\end{bmatrix}$$

and

$$D\tilde\nu[q_\sigma]=\begin{bmatrix} \sum_{i=1}^n D_{p_{\sigma_i}}\tilde\nu^1\cdot q_{\sigma_i}\\ \sum_{i=1}^n D_{p_{\sigma_i}}\tilde\nu^2\cdot q_{\sigma_i}\\ \vdots\\ \sum_{i=1}^n D_{p_{\sigma_i}}\tilde\nu^{n+1}\cdot q_{\sigma_i}\end{bmatrix}=\begin{bmatrix} \sum_{i=1}^n\sum_{j=1}^{n+1} D_{(p_j)_{\sigma_i}}\tilde\nu^1\cdot(q_j)_{\sigma_i}\\ \sum_{i=1}^n\sum_{j=1}^{n+1} D_{(p_j)_{\sigma_i}}\tilde\nu^2\cdot(q_j)_{\sigma_i}\\ \vdots\\ \sum_{i=1}^n\sum_{j=1}^{n+1} D_{(p_j)_{\sigma_i}}\tilde\nu^{n+1}\cdot(q_j)_{\sigma_i}\end{bmatrix}.$$

Clearly, $|H_1(t,w)| = O\Big(\|w-p^0\|^2_{C^{0,1}([0,t_0]\times\overline{\mathcal{U}})}\Big)$. Moreover,

$$|\tilde{\nu}(w_\sigma)| = |\tilde{\nu}(p^0_\sigma)| + \nu(p^0)\cdot D\tilde{\nu}(p^0_\sigma)[w_\sigma - p^0_\sigma] + H_2(t,w)$$

with $|H_2(t,w)| = O\Big(\|w-p^0\|^2_{C^{0,1}([0,t_0]\times\overline{\mathcal{U}})}\Big)$. Finally, since $\beta \in C^{1+\alpha}(\partial\Omega)$ we have

$$\beta(w)|\tilde{\nu}(w_\sigma)| - \beta(p^0)|\tilde{\nu}(p^0_\sigma)| = \beta(p^0)\nu(p^0)\cdot D\tilde{\nu}(p^0_\sigma)[w_\sigma - p^0_\sigma] + H_3(t,w),$$

where $H_3(t,w) = O\Big(\|w-p^0\|^2_{C^{0,1}([0,t_0]\times\overline{\mathcal{U}})}\Big)$. Thus, (B.5) is equivalent to

$$(e_{n+1} - \beta(p^0)\nu(p^0))\cdot D\tilde{\nu}(p^0_\sigma)[w_\sigma] = (e_{n+1} - \beta(p^0)\nu(p^0))\cdot D\tilde{\nu}(p^0_\sigma)[p^0_\sigma] + H_4(t,w),$$

where $H_4(t,w) = O\Big(\|w-p^0\|^2_{C^{0,1}([0,t_0]\times\overline{\mathcal{U}})}\Big)$.

Thus we have the following linear parabolic system of equations

$$\mathcal{L}(\sigma,\partial_t,\partial_\sigma)w = f \text{ in } (0,t_0)\times\mathcal{U}$$

subject to the boundary conditions $\mathcal{B}_\beta(\varsigma,\partial_\sigma)w = F(t,\varsigma)$ on $[0,t_0]\times\partial\mathcal{U}$, where

$$F(t,\varsigma) = \left[0,\ (e_{n+1} - \beta(p^0)\nu(p^0))\cdot D\tilde{\nu}(p^0_\sigma)[p^0_\sigma] + H_4(t,w),\ \underbrace{0,\ \dots\ ,0}_{(n-1)-\text{times}}\right]^T$$

and, under the notation $\{g_0\}^{ij} = \{p^0_{\sigma_i}\cdot p^0_{\sigma_j}\}^{-1}$, $\tilde{\nu}_0 = \tilde{\nu}(p^0_\sigma)$, $\beta_0 = \beta(p^0)$ the $(n+1)\times(n+1)$-matrices $\mathcal{L}(\sigma,t,\xi,\zeta)$ and $\mathcal{B}_\beta(\varsigma,\xi)$, $\xi\in\mathbb{R}^n$, $\zeta\in\mathbb{C}$ are defined as follows:

$$\mathcal{L}(\sigma,\zeta,\xi) := \operatorname{diag}\left(\zeta - \sum_{i,j=1}^n g_0^{ij}\xi_i\xi_j, \zeta - \sum_{i,j=1}^n g_0^{ij}\xi_i\xi_j, \dots, \zeta - \sum_{i,j=1}^n g_0^{ij}\xi_i\xi_j\right),$$

$$\mathcal{B}_\beta(\varsigma,\xi) := \begin{bmatrix} 0 & \dots & 1 \\ \sum_{k=1}^{n+1}\sum_{i=1}^n(-\delta_{k,n+1} - \beta_0\nu_0^k)D_{(p_1)_{\sigma_i}}\tilde{\nu}_0^k\xi_i & \dots & \sum_{k=1}^{n+1}\sum_{i=1}^n(-\delta_{k,n+1} - \beta_0\nu_0^k)D_{(p_{n+1})_{\sigma_i}}\tilde{\nu}_0^k\xi_i \\ \tau_0{}_1^1\sum_{i=1}^n n_i^0\xi_i & \dots & \tau_0{}_1^{n+1}\sum_{i=1}^n n_i^0\xi_i \\ \vdots & \vdots & \vdots \\ \tau_0{}_{n-1}^1\sum_{i=1}^n n_i^0\xi_i & \dots & \tau_0{}_{n-1}^{n+1}\sum_{i=1}^n n_i^0\xi_i \end{bmatrix},$$

where the first row must be intended as $[0,\dots,0,1]$.

Step 2. Now we check the compatibility conditions [50]. Take any $\varsigma\in\partial\mathcal{U}$ and let θ be in the tangent space of $\partial\mathcal{U}$ at ς. Let $\lambda_0 := \lambda_0(\varsigma,\zeta,\theta)$ be a solution of the quadratic equation

$$h(\lambda;\varsigma,\zeta,\theta) := \zeta + \sum_{i,j=1}^n g_0^{ij}\theta_i\theta_j - 2\lambda\sum_{i,j=1}^n g_0^{ij}\theta_i n_j^0 + \lambda^2\sum_{i,j=1}^n g_0^{ij}n_i^0 n_j^0 = 0$$

in $\lambda\in\mathbb{C}$ with positive imaginary part. Notice that $\det\mathcal{L} = (h(\lambda;\varsigma,\zeta,\theta))^{n+1}$ and

$$\hat{\mathcal{L}} = (\det\mathcal{L})\mathcal{L}^{-1} = \operatorname{diag}((h(\lambda;\varsigma,\zeta,\theta))^n,\dots,(h(\lambda;\varsigma,\zeta,\theta))^n).$$

In order to prove the compatibility conditions we should prove that the rows of the matrix

$$\mathcal{B}_\beta(\varsigma, i(\theta - \lambda n^0))\hat{\mathcal{L}}(x, \zeta, i(\theta - \lambda n^0))$$

are linearly independent modulo the polynomial $(\lambda - \lambda_0)^{n+1}$ whenever $\Re(\zeta) \geq 0$, $|\zeta| > 0$. According to the definitions of $\mathcal{L}$ and $\mathcal{B}_\beta$ one checks [38] that the compatibility conditions are equivalent to the conditions

$$c_1 e_{n+1} + c_2 \tilde{\nu}(p^0) + \sum_{i=1}^{n-1} c_{i+2}\tau_{0i} = 0 \iff c_1 = c_2 = \ldots = c_{n+1} = 0.$$

Since a basis of the tangent space $\{\tau_{0i}\}_{i=1}^{n-1}$ of $\Gamma_0 \cap \partial\Omega$ belongs to the horizontal subspace of $\mathbb{R}^{n+1}$ and $\tilde{\nu}(p^0)$ is normal to $\Gamma_0 \cap \partial\Omega$ at p^0 we have $c_3 = \ldots = c_{n+1} = 0$. Moreover, since $|\beta| \leq 1 - 2\kappa$, and Γ_0 satisfies the contact angle condition, e_{n+1} and $\tilde{\nu}(p^0)$ are linearly independent, i.e. $c_1 = c_2 = 0$.

Step 3. By [50, Theorem 4.9] since $\partial\mathcal{U} \in C^{3+\alpha}$, $\beta \in C^{1+\alpha}(\partial\Omega)$ and the compatibility conditions hold, for any $\tilde{f}, \tilde{F} \in C^{0,\alpha}([0,t_0] \times \overline{\mathcal{U}})$, $p^0 \in C^{3+\alpha}(\overline{\mathcal{U}})$ there exists a unique solution $w \in C^{1+\alpha/2,2+\alpha}([0,t_0] \times \overline{\mathcal{U}})$ such that

$$\begin{aligned}
&w_t = \operatorname{tr}((Dp^0 \cdot (Dp^0)^t)^{-1} D^2 w) + \tilde{f},\\
&w(0,\cdot) = p^0,\\
&w_{n+1}(t,\cdot) = 0 \qquad \text{on } \partial\mathcal{U} \text{ for any } t \in [0,t_0],\\
&(e_{n+1} - \beta(p^0)\nu(p^0)) \cdot D\tilde{\nu}(p^0)[w_\sigma] = (e_{n+1} - \beta(p^0)\nu(p^0)) \cdot D\tilde{\nu}(p^0)[p^0_\sigma] + \tilde{F}(t,x) \quad \text{on } [0,t_0] \times \partial\mathcal{U},\\
&\left(\sum_{j=1}^{n} n^0_j w_{\sigma_j}\right) \cdot \tau_{0i} = 0 \qquad \text{on } [0,t_0] \times \partial\mathcal{U} \text{ and } i = 1,\ldots,n-1.
\end{aligned}$$

Step 4. Finally, mimicking [28] we can prove the existence of and uniqueness of a solution to (B.2)-(B.4) in a time interval $[0, T_0]$ for some sufficiently small $T_0 > 0$ depending on $\|\beta\|_{C^{1+\alpha}}$ and $\|p^0\|_{C^{3+\alpha}}$. □

We call $E(t)$ the smooth flow starting from E_0.

Proposition B.4 (Comparison for strong solutions). *Let $\beta_i \in (-1,1)$, $E_0^{(i)} \subset \Omega$ be bounded sets such that $\overline{\Omega \cap \partial E_0^{(i)}}$ are $C^{3+\alpha}$ hypersurfaces, and the smooth flows $E^{(i)}(t)$ starting from $E_0^{(i)}$ exist in $[0,T_0]$, $i = 1,2$. If $\beta_1 \leq \beta_2$ and $\operatorname{dist}(\Omega \cap \partial E_0^{(1)}, \Omega \cap \partial E_0^{(2)}) > 0$, then $\operatorname{dist}(\Omega \cap \partial E^{(1)}(t), \Omega \cap \partial E^{(2)}(t)) > 0$ for all $t \in [0,T_0]$.*

Proof. The proof is an adaptation of the classical one (see for instance [10]). □

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