Effective inseparability, lattices, and preordering relations

EFFECTIVE INSEPARABILITY, LATTICES, AND PRE-ORDERING RELATIONS

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Abstract. We study effectively inseparable (e.i.) pre-lattices (i.e. structures of the form $L = \langle \omega, \wedge, \vee, 0, 1, \leqslant_L \rangle$ where ω denotes the set of natural numbers and the following hold: $\wedge, \vee$ are binary computable operations; $\leqslant_L$ is a c.e. pre-ordering relation, with $0 \leqslant_L x \leqslant_L 1$ for every x; the equivalence relation $\equiv_L$ originated by $\leqslant_L$ is a congruence on L such that the corresponding quotient structure is a non-trivial bounded lattice; the $\equiv_L$-equivalence classes of 0 and 1 form an effectively inseparable pair), and show (Theorem 30, solving a problem in [17]), that if L is an e.i. pre-lattice then $\leqslant_L$ is universal with respect to all c.e. pre-ordering relations, i.e. for every c.e. pre-ordering relation R there exists a computable function f such that, for all x, y, $x \mathrel{R} y$ if and only if $f(x) \leqslant_L f(y)$; in fact (Corollary 43) $\leqslant_L$ is locally universal, i.e. for every pair $a <_L b$ and every c.e. pre-ordering relation R one can find a reducing function f from R to $\leqslant_L$ such that the range of f is contained in the interval $\{x : a \leqslant_L x \leqslant_L b\}$. Also (Theorem 47) $\leqslant_L$ is uniformly dense, i.e. there exists a computable function f such that for every a, b if $a <_L b$ then $a <_L f(a, b) <_L b$, and if $a \equiv_L a'$ and $b \equiv_L b'$ then $f(a, b) \equiv_L f(a', b')$. Some consequences and applications of these results are discussed: in particular (Corollary 55) for $n \geqslant 1$ the c.e. pre-ordering relation on Σ_n sentences yielded by the relation of provable implication of any c.e. consistent extension of Robinson's Q or R is locally universal and uniformly dense; and (Corollary 56) the c.e. pre-ordering relation of provable implication of Heyting Arithmetic is locally universal and uniformly dense.

1. Introduction

Given binary relations R, S on the set ω of natural numbers, we say that R is *reducible* to S (notation: $R \leq S$) if there is a computable function f (called a *computable embedding* of R into S) such that

$$(\forall x, y)[x \mathrel{R} y \Leftrightarrow f(x) \mathrel{S} f(y)].$$

Moreover, given a relation R and a class $\mathcal{A}$ of relations on ω, we say that R is $\mathcal{A}$*-universal* if $R \in \mathcal{A}$ and $S \leq R$ for every $S \in \mathcal{A}$.

An interesting case, to which a rapidly growing amount of literature is being devoted, is when one takes $\mathcal{A}$ to be a class of equivalence relations in some fixed complexity measure: the notion of universality becomes in this case a useful tool to measure the computational complexity of classification problems in computable mathematics (problems which are in fact equivalence relations): for instance it is shown in [8] that the isomorphism relation for various familiar classes of computable groups is Σ^1_1-complete. An even more restricted case is when one considers only computably enumerable (henceforth abbreviated as c.e.) equivalence relations (c.e. equivalence relations will be abbreviated as *ceers* throughout the paper), see e.g. [2]. Interest in ceers is motivated by their importance in logic and algebra: for instance, modulo suitable Gödel numberings identifying the

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various objects with numbers, the relation of provable equivalence between formulas of a formal system, and word problems for finitely presented groups and semigroups, are ceers. When studying ceers, particular attention has been given to Σ^0_1-universal ceers: see for instance [1] for a survey.

It is very natural to move from equivalence relations to the wider class consisting of the pre-ordering relations. Indeed, many interesting classification problems in computable mathematics occur as pre-ordering relations (for instance embedding problems, instead of isomorphism problems, for various classes of computable groups). Restriction to c.e. pre-ordering relations (i.e. to pre-orderings R such that the set $\{\langle x, y\rangle : x\ R\ y\}$ is c.e.) is again mostly motivated by their importance in logic: for instance if one considers a strong enough formal system, then the relation of provable implication between formulas of a formal system is a Σ^0_1-universal pre-ordering relation by [17], i.e. it is c.e. and every c.e. pre-ordering relation is reducible to it.

In this paper we investigate some classes of Σ^0_1-universal (henceforth called simply *universal*) pre-ordering relations. Our quest for interesting universal c.e. pre-ordering relations is based on the notion of effective inseparability for lattices (or, better, pre-lattices, as we will call them in this paper). In this regard, this paper can be viewed as a generalization of some aspects of the work done by Montagna and Sorbi [17], Nies [19] and Shavrukov [26], concerning effectively inseparable Boolean algebras.

To make more sense of this, we first need some definitions. In general, given a type $\tau = \langle \{f_i\}_{i\in I}, \{R_j\}_{j\in J}\rangle$ for operations and relations, then a *pre-structure of type* τ is a structure $A = \langle \omega, \{f_i^A\}_{i\in I}, \{R_j^A\}_{j\in I}, \leqslant_A\rangle$ with operations and relations interpreting the type, such that $\leqslant_A$ is a pre-ordering and the equivalence relation $\equiv_A$ induced by $\leqslant_A$ (i.e. $x \equiv_A y$ if $x \leqslant_A y$ and $y \leqslant_A y$) is a congruence for the operations and the relations. If in addition operations and relations are finitary, $\leqslant_A$ is c.e., each operation f_i^A (interpreting f_i) is computable uniformly in i, and each relation R_j^A (interpreting R_j) is c.e. uniformly in i, then we say that the pre-structure is a *c.e. pre-structure of type* τ.

Definition 1. If A is a pre-structure then the quotient $A_{/\equiv_A}$ will be called the *quotient structure associated with* A. (Notice that $A_{/\equiv_A}$ is partially ordered by the partial ordering originated by $\leqslant_A$.)

For instance, a pre-ordering relation can be just viewed as a pre-structure $R = \langle \omega, \leqslant_R\rangle$ (called a *pre-order*) with empty type, whose associated quotient structure is a partial order. It will be customary in the following to identify pre-ordering relations with their corresponding pre-orders, so that the terms "pre-ordering relation" and "pre-order" will be treated as synonymous.

Definition 2. A *c.e. pre-lattice* is a c.e. pre-structure $L = \langle \omega, \wedge, \vee, \leqslant_L\rangle$ such that the associated quotient structure is a lattice (with $\wedge$ and $\vee$ inducing in the quotient the operations of meet and join, respectively).

Having in mind this definition, it is now clear what *c.e. Boolean pre-algebras*, *c.e. distributive pre-lattices*, *c.e. Heyting pre-algebras*, *c.e. pre-semilattices*, etc., are. In particular a *c.e. bounded pre-lattice* is a c.e. pre-structure $L = \langle \omega, \wedge, \vee, 0, 1, \leqslant_L\rangle$ such that the associated quotient structure is a bounded lattice for which the $\equiv_L$-equivalence classes $[0]_L$ and $[1]_L$, of 0 and 1, respectively, are the least element and the greatest element. (It is clear that up to a computable permutation of ω we may assume that in a c.e. bounded pre-lattice L one has that $0 \leqslant_L x \leqslant_L 1$ for every x, so that our choice of 0 and 1 as representatives of the least element and the greatest element, respectively, is no loss of generality.)

Definition 3. By a *computable isomorphism* between pre-structures A, B we mean a computable function f which reduces $\leqslant_A \preceq \leqslant_B$, and the assignment on equivalence classes $[x]_A \mapsto [f(x)]_B$ is a well defined isomorphism between the corresponding quotient structures of A and B.

Remark 4. Our terminology follows largely [17] except for the fact that what we call here "c.e. pre-structures" are therein called *positive pre-structures*. In the Russian literature, a *positively numbered structure* is a pair $\langle A^-, \nu\rangle$, where A^- has universe ω with computable operations and c.e. relations such that $\nu : \omega \to A$ is a surjective function and the relation "$\nu(x) = \nu(y)$" is a c.e. equivalence relation inducing a congruence θ on A^-. So, what we call in this paper "c.e. pre-lattices", "c.e. Boolean pre-algebras" etc., in the Russian literature would be certain "positively numbered lattices", "positively numbered Boolean algebras", etc.). Our "e.i. Boolean pre-algebras" are e.i. Boolean algebras in the sense of [18, 26]. See also [23, 11, 12, 9] for an approach to c.e structures similar to the one taken in this paper.

Remark 5. One could give a seemingly more general definition of a c.e. pre-structure of type τ, by asking that the universe of the structure be just a c.e. set instead of ω. Up to computable isomorphisms, nothing is however gained in generality by doing so. Let us briefly give a justification of this claim in the case of c.e. pre-lattices. Let $L = \langle U \wedge, \vee, \leqslant\rangle$ be a c.e. pre-lattice, where U is a c.e. set. Let $h : \omega \longrightarrow U$ be a computable surjection. This induces a c.e. pre-lattice $L' = \langle \omega, \wedge', \vee', \leqslant_{L'}\rangle$ where $x \leqslant_{L'} y$ if and only if $h(x) \leqslant_L h(y)$, and (letting h^{-1} be any partial computable function such that $h(h^{-1}(x)) = x$ for every $x \in U$) $x \wedge' y = h^{-1}(h(x) \wedge h(y))$, $x \vee' y = h^{-1}(h(x) \vee h(y))$. The assignment on equivalence classes $[x]_A \mapsto [h(x)]_B$ is clearly a well-defined isomorphism between the corresponding quotient structures of A and B. Notice that if U is infinite, then we can take h to be a computable permutation of ω. On the other hand, if the equivalence classes of $\equiv_L$ are infinite then from any computable isomorphism we can effectively go to a computable isomorphism induced by a computable permutation.

The following notion from computability theory will play a fundamental role in this paper. We recall that a disjoint pair (A, B) of sets of numbers is called an *effectively inseparable* (or simply, *e.i.*) pair if there exists a partial computable function $\varphi(u, v)$ (called a *productive function* for the pair) such that

$$(\forall u, v)[A \subseteq W_u \,\&\, B \subseteq W_v \,\&\, W_u \cap W_v = \varnothing \Rightarrow \varphi(u, v)\downarrow\notin W_u \cup W_v].$$

It is known, see e.g. [29, p.44], that every e.i. pair of c.e. sets has a total productive function: in fact, one can uniformly go from any index of productive function to an index of a total productive function. Also if (A_0, A_1), (X_0, X_1) are disjoint pairs of c.e. sets, with (A_0, A_1) e.i. and either $(A_0, B_1) \subseteq (B_0, B_1)$ (meaning $A_0 \subseteq B_0$ and $A_1 \subseteq B_1$) or $(A_0, B_1) \leqslant_m (B_0, B_1)$ (meaning that there is a computable function simultaneously m-reducing $A_0 \leqslant_m B_0$ and $A_1 \leqslant_m B_1$) then (B_0, B_1) is e.i. as well.

Definition 6. An *effectively inseparable* (or *e.i.*) *pre-lattice* is a c.e. bounded pre-lattice $L = \langle \omega, \wedge, \vee, 0, 1 \leqslant_L\rangle$ such that the pair of equivalence classes $([0]_L, [1]_L)$ is e.i.. (The definition of course specializes to c.e. Boolean pre-algebras (defining *e.i. Boolean pre-algebras*) and c.e. Heyting pre-algebras (defining *e.i. Heyting pre-algebras*), which are examples of c.e. bounded pre-lattices.)

It is shown in [17] (based on [21]; see also [14]) that if B is an e.i. Boolean pre-algebra, and $\leqslant_B$ is the c.e. pre-ordering relation relative to B, then $\leqslant_B$ is universal. Ianovski et al. [14] have extended this result throughout the arithmetical hierarchy by showing that for every $n \geqslant 1$, the pre-ordering relation of a Σ^0_n-effectively inseparable Boolean pre-algebra (called Σ^0_n-*effectively inseparable Boolean algebra*) is a Σ^0_n-universal pre-ordering relation.

Moving in this paper from e.i. Boolean pre-algebras to e.i. pre-lattices, we show that the pre-ordering relation of any e.i. pre-lattice is universal. In fact, we show that every c.e. pre-order can be computably embedded in any nonempty interval of any e.i. pre-lattice (this property will be called *local universality*: see Corollary 43). Moreover every e.i. pre-lattice L is uniformly dense

(Theorem 47), meaning that there is a computable function $f(a,b)$ such that for every a, b if $a <_L b$ then $a <_L f(a,b) <_L b$, and if $a \equiv_L a'$ and $b \equiv_L b'$ then $f(a,b) \equiv_L f(a',b')$. Remarkably, these results on universality, local universality and uniform density, do not use distributivity (on the other hand Example 28 shows that e.i. non-distributive pre-lattices do exist), but do not hold in general if we move from e.i. pre-lattices to e.i. pre-semilattices, i.e. c.e. bounded pre-semilattices $S = \langle \omega, \vee, 0, 1, \leqslant_S \rangle$ such that the pair $([0]_S, [1]_S)$ is e.i. (in Observation 35 we exhibit an e.i. upper pre-semilattice whose pre-ordering relation is not universal). We also point out (Theorem 58 and Theorem 59) several distinct natural computable isomorphism types of e.i. pre-lattices.

The greater generality of considering pre-lattices instead of Boolean pre-algebras enables us to extend to consistent extensions of very weak systems of arithmetic such as Robinson's systems Q or R some of the results of Shavrukov and Visser [27] stated for consistent extensions of Elementary Arithmetic EA (see Corollary 55): in particular (Corollary 55) for $n \geqslant 1$ the c.e. pre-ordering relation on Σ_n sentences yielded by the relation of provable implication of any c.e. consistent extension of Robinson's Q or R is locally universal and uniformly dense.

Since our results apply to every e.i. lattice, they hold of e.i. Heyting pre-algebras as well (for instance, the Heyting pre-algebra of the Lindenbaum sentence algebra of any consistent intuitionistic extension of Heyting Arithmetic, as defined in Definition 23) to which one can therefore extend the above mentioned results showing for instance (Corollary 56) that the c.e. pre-ordering relation of provable implication of Heyting Arithmetic is locally universal and uniformly dense.

In contrast with what happens for e.i. Boolean pre-algebras which are universal with respect to the class of all c.e. Boolean pre-algebras (a c.e. Boolean pre-algebra B is *universal* with respect to the class of all c.e. Boolean pre-algebras if for every c.e. Boolean pre-algebra B' one has that $\leqslant_{B'} \preceq \leqslant_B$ via a reduction f which induces a monomorphism between the associated quotient structures) we show that, although guaranteeing universality of its pre-ordering relation, being an e.i. pre-lattice does not suffice to guarantee universality with respect to the class of all c.e. pre-lattices: Theorem 52 provides an e.i. distributive pre-lattice such that not all c.e. distributive pre-lattices reduce to it via a reduction inducing a monomorphisms between the corresponding quotient structures.

1.1. **Further notations and terminology.** If $R = \langle \omega, \leqslant_R \rangle$ is a pre-order then we use the following notations. We write $x <_R y$ for $x \leqslant_R y$ but $y \not\leqslant_R x$; the symbol $|_R$ denotes $\leqslant_R$-incomparability, i.e. $x \mid_R y$ if $x \not\leqslant_R y$ and $y \not\leqslant_R x$; if $a \in \omega$ and $X \subseteq \omega$ then we write $a \leqslant_R X$ if $a \leqslant_R x$, for all $x \in X$ (similarly define $a <_R X$, $X \leqslant_R a$, and $X <_R a$). We also use interval notation, i.e. $[x,y]_{\leqslant_R} = \{a : x \leqslant_R a \leqslant_R y\}$, and other possible interval variations such as for instance $(x,y)_{\leqslant_R}$.

In some of the proofs, we will make use of infinite computable lists of fixed points as given by the Recursion Theorem. The use of these infinite lists can be formally justified as follows: we fix a single index j so that we control φ_j by the Recursion Theorem, and we then take a computable list $(j_i)_{i\in\omega}$ of indices for the columns $\varphi_{j_i}(k) = \varphi_j(\langle i,k \rangle)$. We can then control the φ_{j_i} in any order we wish, as we are simply controlling the single function φ_j. An alternative formal justification to arguments using infinite lists of fixed points is also provided by the Case Functional Recursion Theorem [6]: see also [20] for useful comments about this theorem.

All pre-orders considered in this paper are assumed to be non-trivial, i.e. the associated quotient structures do not collapse to a single element.

2. Ceers, pre-orders and effective inseparability

From the theory of ceers we recall the following definitions (see e.g. [2]). Particularly important for us is "uniform finite precompleteness", a notion due to Montagna [16], and widely used in [5, 25].

Definition 7. Let R be a non-trivial ceer (i.e. R is a ceer with at least two equivalence classes):

- R is *effectively inseparable* (abbreviated as *e.i.*) if every pair of distinct equivalence classes is e.i.;
- R is *uniformly effectively inseparable* (abbreviated as *u.e.i.*) if there exists a partial computable function $\chi(x, y, u, v)$ such that if $x \not R y$ then the partial computable function $\psi(u, v) = \chi(x, y, u, v)$ is a productive function for the pair of equivalence classes $([x]_R, [y]_R)$;
- R is *precomplete* if there exists a computable function $f(e, x)$ such that for all e, x if $\varphi_e(x)\downarrow$ then $f(e, x) \mathrel{R} \varphi_e(x)$;
- R is *uniformly finitely precomplete* (abbreviated as *u.f.p.*) if there exists a computable function $k(D, e, x)$ (called a *totalizer*, where D is a finite set given by its canonical index) such that for all D, e, x if $\varphi_e(x)$ converges to a number R-equivalent to some $d \in D$ then $k(D, e, x) \mathrel{R} \varphi_e(x)$.

Remark 8. Notice that since for pairs of c.e. sets it is possible to uniformly go from productive functions to total productive functions (see e.g. [29, p.44]), it is easy to see that it is possible for ceers to uniformly go from uniform productive functions witnessing u.e.i.-ness to total uniform productive functions witnessing u.e.i.-ness. Thus we can assume in the following that our productive functions witnessing u.e.i.-ness for ceers are in fact total.

Fact 9. *For ceers we have: precomplete $\Rightarrow$ u.f.p. $\Rightarrow$ u.e.i. $\Rightarrow$ e.i.. It is not known whether the implication u.e.i. $\Rightarrow$ u.f.p. holds; the other implications are strict. Every u.e.i. ceer R is Σ^0_1-universal as an equivalence relation, i.e. every ceer can be reduced to it.*

Proof. For a proof see [2]. The fact that precomplete ceers are u.e.i. was first noticed in [30]. □

Via the equivalence relation induced by a pre-ordering relation we export Definition 7 to pre-orders and pre-structures, as follows.

Definition 10. A c.e. pre-order $R = \langle \omega, \leqslant_R \rangle$ is called *effectively inseparable*, (respectively: *uniformly effectively inseparable*, *precomplete*, *uniformly finitely precomplete*), if so is the associated ceer $\equiv_R$.

Remark 11. A warning on the terminology introduced in the previous definition: to be meticulous, one could find the definition of an e.i. pre-lattice not coherent with Definition 10, for which an e.i. pre-order is a c.e. pre-order in which all pairs of distinct equivalence classes are e.i., whereas for e.i. pre-lattices we only require, following [17], effective inseparability of the equivalence classes of $0, 1$. It will turn out in any case (Theorem 29) that e.i. pre-lattices are also e.i. pre-orders in the sense of Definition 10, in fact they are u.e.i. .

The following observation shows a simple application of effective inseparability to c.e. pre-orders and c.e. pre-lattices.

Observation 12. *If $R = \langle \omega, \leqslant_R \rangle$ is a c.e. pre-order so that $([u]_R, [v]_R)$, where $u <_R v$, is an e.i. pair of sets then for every non-empty c.e. set X one can find (uniformly from u, v and a c.e. index of X) an element y such that if $u <_R X <_R v$ then $y |_R X$. If in addition R is a c.e. pre-lattice then we can find y as above such that if $u <_R X <_R v$ then $y |_R X$ and $u <_R y <_R v$.*

Proof. Assume that u, v, X are as in the statement of the observation. Let p be a productive function for the pair $([u]_R, [v]_R)$ of e.i. sets. Let

$$U = \{y : (\exists x \in X)[y \leqslant_R x]\} \quad \text{and} \quad V = \{y : (\exists x \in X)[y \geqslant_R x]\},$$

(thus $u \in U$ and $v \in V$) and by the Reduction Principle (see [22]) let $U' \subseteq U$, $V' \subseteq V$ be c.e. sets such that $U' \cap V' = \varnothing$ and $U \cup V = U' \cup V'$. If $u <_R X <_R v$ then clearly $[u]_R \subseteq U'$ and

$[v]_R \subseteq V'$. If u', v' are c.e. indices of U', V' then $p(u', v') \notin U \cup V$ and thus $p(u', v')|_R x$ with all $x \in X$. To show uniformity, use the fact that u', v' can be uniformly found starting from u, v, and a c.e. index of X.

If instead of just a c.e. pre-order we start with a c.e. pre-lattice $R = \langle \omega, \wedge, \vee, \leqslant_R \rangle$ then define

$$U = \{y : (\exists x \in X)[(u \vee y) \wedge v \leqslant_R x]\} \quad \text{and} \quad V = \{y : (\exists x \in X)[(u \vee y) \wedge v \geqslant_R x]\}.$$

Arguing as above, by the Reduction Principle get from U, V disjoint c.e. sets U', V' with indices u', v' such that $U \cup V = U' \cup V'$: then it is easy to see that $y = (u \vee p(u', v')) \wedge v$ satisfies the claim. □

3. Examples of e.i. pre-lattices

In this section we point out several examples of e.i. pre-lattices, many of them coming from the already existing literature.

Definition 13. If S is a c.e. set S of formulas of a first order language L, by a *Gödel numbering of S* we will mean *any* computable 1-1 function from S to ω.

3.1. **E.i. Boolean pre-algebras.** E.i. Boolean pre-algebras appear very naturally in logic.

Example 14. Consider any c.e. consistent extension T of Robinson's systems R or Q for first order arithmetic (as presented for instance in Definitions III.6.10 and III.8.2 of [28]), with the usual first order language L for arithmetic. Our choice of R is motivated by the fact that we would like to be as general as possible, and indeed R is often recognized as a prototypical example of a weak theory of arithmetic, and although Q interprets R (see e.g. [28, Theorem III.8.3]) we explicitly mention Q as well as an example of a weak finitely axiomatizable theory.

Let us fix a Gödel numbering γ of $\mathrm{Sent}(L)$ (all sentences of L) *onto* ω so that $\gamma(\neg 0 = 0) = 0$ and $\gamma(0 = 0) = 1$.

Definition 15. The *Boolean pre-algebra of T relatively to* γ is the Boolean pre-algebra

$$B_T^\gamma = \langle \omega, \wedge^\gamma, \vee^\gamma, \neg^\gamma, 0, 1, \leqslant_T^\gamma \rangle$$

where $\wedge^\gamma, \vee^\gamma, \neg^\gamma$ are (via γ) the usual propositional connectives (i.e. $x \wedge^\gamma y = \gamma(\gamma^{-1}(x) \wedge \gamma^{-1}(x))$, etc.), and $x \leqslant_T^\gamma y$ if $\vdash_T \gamma^{-1}(x) \to \gamma^{-1}(y)$.

Remark 16. It is easy to see that up to computable isomorphisms induced by computable permutations of ω, the Boolean pre-algebra B_T^γ does not depend on the chosen γ, so it is fair to refer to B_T^γ as just

$$B_T = \langle \omega, \wedge, \vee, \neg, 0, 1, \leqslant_T \rangle,$$

without mentioning, unless when really needed, the chosen Gödel numbering. Notice that B_T is not trivial by consistency of T.

Theorem 17. *B_T is an e.i. Boolean pre-algebra.*

Proof. This immediately follows from the well known fact (see for instance [28]) that the sets of theorems and anti-theorems of T (where σ is an *anti-theorem* if $\neg\sigma$ is a theorem) form an e.i. pair of c.e. sets. We give however a somewhat more detailed proof which will be exploited again for Theorem 22 below. Fix a γ as above such that $B_T = B_T^\gamma$, and let (A_0, A_1) be a disjoint pair of c.e. sets which are e.i.. Consider (see for instance [28, Corollary III.6.20]) two strict Σ_1 formulas $H_0(v), H_1(v)$ (i.e. each $H_i(v)$ is of the form $(\exists w)\psi_i(w, v)$ with ψ_i containing no quantifiers or only quantifiers which are bounded by variables) such that $A_i = \{m : \mathbb{N} \models H_i(\overline{m})\}$ (where $\overline{m}$ is the numeral term corresponding to the number m) and let $H(v)$ be the formula $(\exists w)(\psi_0(w, v) \wedge (\forall w' \leqslant$

$w)\neg\psi_{\Gamma}(w',v))$. Using [28, Lemma III.6.27] one can show that if $m \in A_0$ then $\vdash_U H(\overline{m})$ and if $m \in A_1$ then $\vdash_U \neg H(\overline{m})$, where U is either R or Q. Let now (X_0, X_1) be the disjoint pair of c.e. sets $X_0 = \{m : \vdash_T \neg H(\overline{m})\}$ and $X_1 = \{m : \vdash_T H(\overline{m})\}$: since $(X_0, X_1) \supseteq (A_1, A_0)$ we have that (X_0, X_1) is e.i.: on the other hand the computable function $m \mapsto \gamma(H(\overline{m}))$ gives a 1-reduction of (X_0, X_1) to the pair $([0]_T, [1]_T)$ (where $[x]_T$ denotes the equivalence class of a number x under the equivalence relation induced by $\leqslant_T$): therefore the pair $([0]_T, [1]_T)$ is e.i. too, and thus B_T is an e.i. Boolean pre-algebra. □

Moreover, the associated quotient structure of B_T is isomorphic to the Lindenbaum sentence algebra of T, which leads us to the following definition.

Definition 18. B_T as above is called the *Boolean pre-algebra of the Lindenbaum sentence algebra of* T.

For introductions to e.i. Boolean pre-algebras and related topics, see (listed in chronological order) [18, 17, 19, 26].

3.2. **C.e. precomplete pre-lattices and c.e. pre-lattices of sentences.** If L is a c.e. precomplete bounded pre-lattice, then $\equiv_L$ is a precomplete ceer and thus by Fact 9 yields a partition of ω such that all pairs of distinct equivalence classes are e.i.. Therefore L is an e.i. pre-lattice.

3.3. **C.e. pre-lattices of sentences.** Let us consider a decidable first order language L and a c.e. consistent theory T on L, and let $\mathcal{C} \subseteq \mathrm{Sent}(L)$ be an infinite c.e. set closed under the connectives $\wedge$ and $\vee$, and intersecting both the set of theorems and the set of anti-theorems of T. Fix a Gödel numbering γ of $\mathcal{C}$ onto ω for which 0 and 1 are the Gödel numbers of a theorem and an anti-theorem, respectively.

Definition 19. The *pre-lattice of* $\mathcal{C}$*-sentences relatively to* γ is the c.e. bounded pre-lattice

$$L^{\gamma}_{\mathcal{C}/T} = \langle \omega, \wedge^{\gamma}, \vee^{\gamma}, 0, 1, \leqslant^{\gamma}_{\mathcal{C}/T} \rangle$$

where $x \leqslant^{\gamma}_{\mathcal{C}/T} y$ if and only if $\vdash_T \gamma^{-1}(x) \to \gamma^{-1}(y)$, and the operations are given (via γ) by the propositional connectives. A *pre-lattice of sentences* is a pre-lattice of the form $L^{\gamma}_{\mathcal{C}/T}$, for some choice of T, $\mathcal{C}$, and γ.

Once again it is immediate to see that up to computable isomorphisms induced by computable permutations of ω the definition does not depend on the choice of γ, and thus, unless when strictly necessary for definiteness, reference to the chosen γ will always be omitted in the following.

Also, up to a computable isomorphism provided by a computable permutation of ω it does not matter whether we consider $\mathcal{C}$ or its closure under provable equivalence in T: indeed, suppose that $\mathcal{C}$ is a set of sentences providing the pre-lattice of sentences $L_{\mathcal{C}/T}$ (relatively, say, to γ), let

$$[\mathcal{C}]_T = \{\sigma \in \mathrm{Sent}(L) : (\exists \gamma \in \mathcal{C})[\vdash_T \gamma \leftrightarrow \sigma]\},$$

and let us consider the pre-lattice of sentences $L_{[\mathcal{C}]_T/T}$ (relatively say to δ). It is easy to see that $h = \delta \circ \gamma^{-1}$ is a computable function satisfying for every x, y,

$$x \leqslant_{\mathcal{C}/T} y \Leftrightarrow h(x) \leqslant_{[\mathcal{C}]_T/T} h(y)$$

and for every y there exists x such that $h(x) \equiv_{[\mathcal{C}]_T/T} y$. By a standard back-and-forth argument, and using that the equivalence classes of both $\equiv_{\mathcal{C}/T}$ and $\equiv_{[\mathcal{C}]_T/T}$ are infinite, it is easy to see that from h one can construct a computable permutation of ω inducing an isomorphism of $L_{\mathcal{C}/T}$ with $L_{[\mathcal{C}]_T/T}$: At stage 0 we define $f_0 = \varnothing$. At stage $2s+1$, if $s \in \mathrm{dom}(f_{2s})$ then let $f_{2s+1} = f_{2s}$; otherwise effectively search for the first y for which $h(s) \equiv_{[\mathcal{C}]_T/T} y$ and $y \notin \mathrm{range}(f_{2s})$, and define

$f_{2s+1} = f_{2s} \cup \{(s, y)\}$. At stage $2s + 2$, if $s \in \text{range}(f_{2s+1})$ then let $f_{2s+2} = f_{2s+1}$; otherwise effectively search for the first pair $\langle u, x\rangle$ for which $h(u) \equiv_{[\mathcal{C}]_T/T} s$, $x \equiv_{\mathcal{C}/T} u$, and $x \notin \text{dom}(f_{2s+1})$, and define $f_{2s+1} = f_{2s} \cup \{(x, s)\}$.

Pre-lattices of sentences for c.e. arithmetical first order theories T have been extensively studied by Shavrukov and Visser (see for instance [26] and [27]), in particular taking $\mathcal{C}$ to be the set of Σ_n-sentences for some $n \geqslant 1$. The reader is referred to [28] for a clear introduction to the classes Σ_n: the definition of Σ_n is purely syntactic (and in particular each class is decidable), depending only on the language and not on any specific theory T, but our previous observation shows that whether we take Σ_n or its closure under provable equivalence in a chosen T we get pre-lattices that are isomorphic via an isomorphism provided by a computable permutation of ω as long as in either pre-lattice the pre-ordering relation is provided by $\vdash_T$.

It is easy to see that the quotient structure associated to $L_{\Sigma_n/T}$ is isomorphic to the Lindenbaum lattice of Σ_n-sentences of T, i.e. the bounded sublattice of the Lindenbaum sentence algebra of T, restricted to the equivalence classes of Σ_n-sentences. This motivates the following definition.

Definition 20. $L_{\Sigma_n/T}$ will be called the *pre-lattice of the Lindenbaum lattice of Σ_n-sentences of T*.

For the following theorem (where EA, called Elementary Arithmetic, is the theory $I\Delta_0 + \text{Exp}$) see also [27, Example 4.1].

Theorem 21 ([30])**.** *If T is a c.e. consistent extension of elementary arithmetic EA then, for $n \geqslant 1$, $L_{\Sigma_n/T}$ is precomplete.*

Proof. If $\ulcorner\ \urcorner$ is a conventional Gödel numbering encoding the syntactic objects and formulas of L as normally used in the arithmetization of metamathematics (representability of partial computable functions, c.e. sets, etc.: see e.g. [28] or [13] for details), and φ is a partial computable function then from [30] we can see that there exists a computable function f_φ which always land at $\ulcorner\ \urcorner$-numbers of Σ_n sentences, and if $\varphi(x)\downarrow= \ulcorner\sigma\urcorner$, with $\sigma \in \Sigma_n$, then $f_\varphi(x) = \ulcorner\tau\urcorner$ with $\vdash_T \sigma \leftrightarrow \tau$. Let γ be a Gödel numbering of the Σ_n-sentences onto ω so that $L_{\Sigma_n/T} = L^\gamma_{\Sigma_n/T}$. Now, given a partial computable function φ, consider the partial computable function ψ with the same domain as φ and $\psi(x) = \ulcorner\sigma\urcorner$ if $\sigma = \gamma^{-1}(\varphi(x))$, and let $g = \gamma(\tau)$ if $f_\psi(x) = \ulcorner\tau\urcorner$: it is immediate to see that g makes φ total modulo $\equiv^\gamma_T$, showing that $\equiv^\gamma_T$ is precomplete. □

As every c.e. bounded precomplete lattice is e.i., it follows that if T is a c.e. consistent extension of elementary arithmetic EA then $L_{\Sigma_n/T}$ is an e.i. pre-lattice. For Robinson's weaker systems Q and R, it is not known if $L_{\Sigma_n/T}$ is precomplete. The following theorem shows however that we still have an e.i. pre-lattice.

Theorem 22. *If T is a c.e. consistent extension of Q or R then $L_{\Sigma_n/T}$ is an e.i. pre-lattice.*

Proof. Let $n \geqslant 1$ be given, and fix a suitable Gödel numbering γ of the Σ_n-sentences onto ω so that $L_{\Sigma_n/T} = L^\gamma_{\Sigma_n/T}$. By consistency of T, $L_{\Sigma_n/T}$ is not trivial. Notice that the formula $H(v)$ used in the proof of Theorem 17 is Σ_1: therefore if (X_0, X_1) is the e.i. pair of c.e. sets as in that proof then the computable function $m \mapsto \gamma(H(\overline{m}))$ gives a 1-reduction of (X_0, X_1) to the pair $([0]_{\Sigma_n/T}, [1]_{\Sigma_n/T})$ which is therefore e.i. . □

3.4. **E.i. Heyting pre-algebras.** Examples of e.i. Heyting pre-algebras which are not Boolean pre-algebras come from intuitionistic logic. For instance, Let iT be any c.e. consistent intuitionistic extension of Heyting Arithmetic HA in the usual arithmetical first order language L, and, having fixed a Gödel numbering γ from $\text{Sent}(L)$ onto ω with $\gamma(\neg 0 = 0) = 0$ and $\gamma(0 = 0) = 1$, let

$\mathrm{iT}^\gamma = \langle \omega, \wedge^\gamma, \vee^\gamma, \to^\gamma, \neg^\gamma, 0, 1, \leqslant^\gamma_{iT}\rangle$ be the c.e. Heyting pre-algebra where the operations are the connectives (via Gödel numbers given by γ) and $x \leqslant^\gamma_{iT} y$ if and only if $\vdash_{iT} \gamma^{-1}(x) \to \gamma^{-1}(y)$. Once again, up to computable isomorphisms induced by computable permutations of ω, the definition is independent of γ, so the superscript γ will be omitted unless when strictly necessary for a clearer understanding of the text. As the Lindenbaum sentence algebra of iT is isomorphic with the quotient structure of iT it is fair to give the following definition.

Definition 23. The Heyting pre-algebra iT is called the *Heyting pre-algebra of the Lindenbaum sentence algebra of iT*.

By an argument similar to the proof of Theorem 17, using similar arithmetization tools (available in Heyting Arithmetic), one can show effective inseparability of iT. We choose however a different argument, based on the Gödel-Gentzen double-negation translation, which we particularly like for its simplicity and clarity.

Lemma 24. iT *is an e.i. Heyting pre-algebra.*

Proof. Let PA denote Peano Arithmetic, and let B_{PA} be the Boolean pre-algebra of the Lindenbaum sentence algebra of PA. The Gödel-Gentzen double-negation translation (see e.g. [3] for mathematical and historical details) ensures that there is a computable mapping G transforming each arithmetical sentence σ into an arithmetical sentence $G(\sigma)$ such that $\vdash_{PA} \sigma$ if and only if $\vdash_{HA} G(\sigma)$. Moreover $G(\neg\sigma) = \neg G(\sigma)$, thus $\vdash_{PA} \neg\sigma$ if and only if $\vdash_{HA} \neg G(\sigma)$ as well. From this one easily gives a 1-reduction $([0]_{PA}, [1]_{PA}) \leqslant_1 ([0]_{HA}, [1]_{HA})$ of disjoint pairs of sets, implying that $([0]_{HA}, [1]_{HA})$ is e.i. as so is $([0]_{PA}, [1]_{PA})$: therefore the pair $([0]_{iT}, [1]_{iT})$ is e.i. too as it contains the pair $([0]_{HA}, [1]_{HA})$. $\square$

Remark 25. There is of course a much wider range of examples of e.i. Heyting pre-algebras than the ones covered by Lemma 24: forthcoming work will address the issue of examples of e.i. Heyting pre-algebras arising from intuitionistic versions of weak fragments of arithmetic.

3.5. **E.i. free distributive pre-lattices.** Useful examples (and also counterexamples) of e.i. pre-structures come from free pre-structures.

Definition 26. Define a pre-structure to be *free on a countably infinite set of generators* if the corresponding associated quotient structure is free on a countably infinite set of generators.

Consider a computable presentation of the form $F^d_{0,1}(X) = \langle \omega, \wedge, \vee, 0, 1\rangle$ of the free bounded distributive lattice on a decidable infinite set X (with $0, 1 \notin X$)) which is computably listed without repetitions by $\{x_i : i \in \omega\}$. Fix a pair (U, V) of e.i. c.e. sets, and let α be the c.e. congruence on $F^d_{0,1}(X)$ generated by the set of pairs $\{(x_i, 0) : i \in U\} \cup \{(x_i, 1) : i \in V\}$. (This argument to encode effective inseparability in a c.e. quotient of $F^d_{0,1}(X)$ has been suggested in private communication by Shavrukov.) Define $L^d_{01} = \langle \omega, \wedge, \vee, 0, 1, \leqslant_{L^d_{01}}\rangle$ where $\wedge, \vee$ are the same operations as in $F^d_{0,1}(X)$, and $x \leqslant_{L^d_{01}} y$ if $[x]_\alpha = [x \wedge y]_\alpha$. Throughout the remainder of this section leading to Example 27, for simplicity denote $L = L^d_{01}$. It is easy to see that L is a c.e. pre-lattice whose associated quotient structure is $F^d_{0,1}(X)_{/\alpha}$. On the other hand we can argue that $F^d_{0,1}(X)_{/\alpha}$ is isomorphic with $F^d_{0,1}(\{x_i : i \notin U \cup V\})$: if $f : X \longrightarrow F^d_{0,1}(\{x_i : i \notin U \cup V\})$ is the function mapping $x_i \mapsto 0$ if $i \in U$, $x_i \mapsto 1$ if $i \in V$, and $x_i \mapsto x_i$ if $i \notin U \cup V$, then by freeness f extends to a (unique) homomorphism $g : F^d_{0,1}(X) \longrightarrow F^d_{0,1}(\{x_i : i \notin U \cup V\})$ which is onto as f is onto a generating set, and it is not difficult to see that $\ker(g) = \alpha$, i.e. $g(x) = g(y)$ if and only if $x \;\alpha\; y$. Thus by the First Isomorphism Theorem of universal algebra we see that $F^d_{0,1}(X)_{/\alpha} \simeq F^d_{0,1}(\{x_i : i \notin U \cup V\})$. In conclusion, L is free bounded distributive on a countably infinite set of generators; moreover the function $f(i) = x_i$ provides a 1-reduction of the e.i. pair (U, V) to the pair $([0]_L, [1]_L)$, giving that L is e.i..

We have shown:

Example 27. L^d_{01} is an e.i. distributive pre-lattice, which is a free bounded distributive pre-lattice on countably many generators.

3.6. **E.i. non-distributive pre-lattices.** The same construction as in Section 3.5 but starting with the free bounded non-distributive lattice $F^{01}_{nd}(X)$ on X, leads to a c.e. pre-lattice L^{nd}_{01}.

Example 28. L^{nd}_{01} is an e.i. non-distributive pre-lattice which is free bounded on countably many generators.

4. Effectively inseparable pre-lattices and universality

Montagna and Sorbi [17, Theorem 2.1] proved that if L is an e.i. pre-lattice and $R = \langle \omega, \leqslant_R \rangle$ is a decidable pre-partial order then $\leqslant_R \preceq \leqslant_L$, but left open ([17, Remark 2.1]) whether such a pre-partial order $\leqslant_L$ is universal. In the next section we show that this is so.

4.1. **E.i. pre-lattices vs. uniform finite precompleteness.** We first show that effective inseparability of the pair $([0]_L, [1]_L)$ makes the ceer $\equiv_L$ (and thus $\leqslant_L$, according to Definition 10) u.f.p..

Theorem 29. *Let L be a c.e. pre-lattice. Then L is e.i. if and only if the ceer $\equiv_L$ is u.f.p..*

Proof. If $\equiv_L$ is u.f.p. then by Fact 9 all pairs of distinct equivalence classes are e.i..

Suppose now that L is e.i.. We need to define a computable function $k(D, e, x)$ such that if $\varphi_e(x) \downarrow \equiv_L d$ for some $d \in D$ then $k(D, e, x) \equiv_L \varphi_e(x)$.

Let p be a productive function for the pair $([0]_L, [1]_L)$. Let

$$\{u_{d,D,e,x}, v_{d,D,e,x} : \ D \text{ finite subset of } \omega,\ d \in D,\ e, x \in \omega\}$$

be a computable set of indices we control by the Recursion Theorem. For a pair $(u_{d,D,e,x}, v_{d,D,e,x})$ in this set let $c_{d,D,e,x} = p(u_{d,D,e,x}, v_{d,D,e,x})$ and $a_{d,D,e,x} = d \wedge c_{d,D,e,x}$. Define

$$k(D, e, x) = \bigvee_{d \in D} a_{d,D,e,x}.$$

We now specify how to computably enumerate $W_{u_{d,D,e,x}}$ and $W_{v_{d,D,e,x}}$ for $d \in D$. We wait for $\varphi_e(x)$ to converge to some y which is $\equiv_L$ to some element in D, and while waiting, we let $W_{u_{d,D,e,x}}$ and $W_{v_{d,D,e,x}}$ enumerate $[0]_L$ and $[1]_L$, respectively. If we wait forever then for all $d \in D$ we end up with $W_{u_{d,D,e,x}} = [0]_L$ and $W_{v_{d,D,e,x}} = [1]_L$. If the wait terminates, let $d_0 \in D$ be the first seen so that $\varphi_e(x) \equiv_L d_0$, and then enumerate $c_{d_0,D,e,x}$ into $W_{u_{d_0,D,e,x}}$, while keeping on with $W_{u_{d_0,D,e,x}}$ enumerating $[0]_L$ and with $W_{v_{d_0,D,e,x}}$ enumerating $[1]_L$: this ends up with $W_{u_{d_0,D,e,x}} = [0]_L \cup \{c_{d_0,D,e,x}\}$ and $W_{v_{d_0,D,e,x}} = [1]_L$, thus forcing $c_{d_0,D,e,x} \in [1]_L$ and thus $a_{d_0,D,e,x} \equiv_L d_0$. For all $d \in D$ with $d \neq d$, we let $W_{u_{d,D,e,x}} = [0]_L$ and $W_{v_{d,D,e,x}} = [1]_L \cup \{c_{d,D,e,x}\}$: this forces $c_{d,D,e,x} \in [0]_L$ and thus $a_{d,D,e,x} \equiv_L 0$ for each such d.

In conclusion, if $\varphi_e(x)$ converges to a number that is $\equiv_L$-equivalent to some $d \in D$ then, for the number d_0 for which we see this for the first time, we have

$$k(D, e, x) = \bigvee_{d \in D} a_{d,D,e,x} \equiv_L d_0,$$

as desired. $\square$

4.2. **E.i. pre-lattices are universal.** We now show that effective inseparability for a c.e. pre-lattice implies universality with respect to the class of all c.e. pre-orders.

Theorem 30. *If $L = \langle \omega, \wedge, \vee, 0, 1, \leqslant_L \rangle$ is an e.i. pre-lattice then $\leqslant_L$ is universal.*

Proof. By Theorem 29 let $j(D, e, x)$ be a function witnessing that $\equiv_L$ is u.f.p..

Claim 31. *There is a computable function $k(a, b, D, e, x)$ so that if $a \leqslant_L b$, then $k(a, b, D, e, x) \in [a, b]_{\leqslant_L}$, and if $\varphi_e(x) \equiv_L d \in D$ for some $d \in D$ then $k(a, b, D, e, x) \equiv_L (d \wedge b) \vee a$. In particular, if $\varphi_e(x) \equiv_L d \in D$ for some $d \in D$ with $d \in [a, b]_{\leqslant_L}$, then $k(a, b, D, e, x) \equiv_L d$.*

Proof. Let $k(a, b, D, e, x) = (j(D, e, x) \wedge b) \vee a$. Then if $a \leqslant_L b$, then $k(a, b, D, e, x) \in [a, b]_{\leqslant_L}$. If $\varphi_e(x) \equiv_L d \in D$, then $k(a, b, D, e, x) \equiv_L (d \wedge b) \vee a$. Thus, if $d \in [a, b]_{\leqslant_L}$, then $k(a, b, D, e, x) \equiv_L d$. □

Since we know that $\equiv_L$ has this behavior with regard to k, we use a "speedup" of the enumeration of $\equiv_L$ so that this effect happens immediately. That is, whenever at a stage s we need to define, given a, b, D, some computation $\varphi_e(x)$ to converge to some $d \in D$ relying on the fact that this entails $k(a, b, D, e, x) \equiv_L (d \wedge b) \vee a$, we momentarily stop the construction until we in fact see $\equiv_L$ to give this equivalence, and only after that we proceed with the construction: we may therefore pretend that when we make $\varphi_e(x)$ converge then at the same stage we also see $k(a, b, D, e, x) \equiv_L (d \wedge b) \vee a$.

Let $R = \langle \omega, \leqslant_R \rangle$ be any c.e. pre-order on ω. We must construct a computable function $f : \omega \to \omega$ so that $n \leqslant_R m$ if and only if $f(n) \leqslant_L f(m)$. We fix an infinite set E of indices that we control via the Recursion Theorem. The idea is then to define the map f reducing $\leqslant_R$ to $\leqslant_L$ inductively. Having defined $f(j)$ for each $j < n$, we define $f(n)$ as follows. We consider first the partial order of the set T_n^0 of all possible pre-partial orders on the set $\{j \mid j < n\}$ ordered by $O_1 \preceq_n^0 O_2$ if O_2 extends O_1, that is, if $i \leqslant_{O_1} j$ implies $i \leqslant_{O_2} j$ for each $i, j < n$. This defines a finite partial order $(T_n^0, \preceq_n^0)$. Next we consider the partial order $(T_n, \preceq_n)$ where

$$\begin{aligned} T_n = \{(O, X, Y) \mid O \in T_n^0, X, Y \subseteq \{j \mid j < n\} \\ \text{and } a \leqslant_O b \text{ for every } a \in X \text{ and } b \in Y \\ \text{and if } i \leqslant_O a \in X, \text{ then } i \in X, \text{ and if } j \geqslant_O b \in Y \text{ then } j \in Y\}. \end{aligned}$$

In other words, T_n is comprised of pre-partial orders of the set $\{j \mid j < n\}$ with a chosen "interval". We say $(O_1, X_1, Y_1) \preceq_n (O_2, X_2, Y_2)$ if $O_1 \preceq_n^0 O_2$ and $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. In other words, we have refined both the order and the interval. Once again, $(T_n, \preceq_n)$ is a finite partial order.

For each element $\tau = (O, X, Y) \in T_n$, we define an $x_\tau \in \omega$. We begin by defining x_τ for the leaf υ in T_n. Note that the leaf corresponds to a pre-partial order where all elements are O-equivalent, $X = Y = \{j \mid j < n\}$. We define x_υ to be

$$x_\upsilon = k(\bigvee_{i \in X} f(i), \bigwedge_{j \in Y} f(j), \{f(j) \mid j < n\} \cup \{0, 1\}, e_\upsilon, 0)$$

where $e_\upsilon \in E$ is distinct from any previously mentioned element of E.

For every node $\tau \in T_n$, we define

$$x_\tau = k(\bigvee_{i \in X} f(i), \bigwedge_{j \in Y} f(j), \{x_\sigma \mid \tau \preceq_n \sigma\} \cup \{0, 1\}, e_\tau, 0)$$

where $e_\tau \in E$ is distinct from any previously mentioned element of E.

We fix notation that $\tau = (O_\tau, X_\tau, Y_\tau)$ and $x_\tau = k(a_\tau, b_\tau, S_\tau, e_\tau, 0)$. Let λ be the root in T_n, i.e. the partial order given by an anti-chain and $X = Y = \varnothing$.

We define $f(n)$ to be the element x_λ, where λ is the root in T_n ($\lambda = \lambda(n)$ depends in fact on n). The idea is to use in a construction in stages the indices e_τ for various τ to ensure that: If the elements $\{j \mid j < n\}$ in $\leqslant_R$ have order type O, and $X = \{j \mid j < n \wedge j \leqslant_R n\}$ and $Y = \{j \mid j < n \wedge j \geqslant_R n\}$ at stage s, then $x_\lambda \equiv_L x_{(O,X,Y)}$. Since there are only finitely many elements of T_n, we will only need to act at finitely many stages to ensure this is true ($(O, X, Y) = \tau(n, s)$ depends in fact on n and s).

We work with computable approximations $\{\leqslant_{R,s}: s \in \omega\}$, $\{\leqslant_{L,s}: s \in \omega\}$ to $\leqslant_R$, $\leqslant_L$ respectively (and consequent approximations $\{\equiv_{R,s}: s \in \omega\}$, $\{\equiv_{L,s}: s \in \omega\}$ to $\equiv_R$, $\equiv_L$ respectively): we may assume that these sequences are increasing sequences of pre-orders $\leqslant_{R,s}$, $\leqslant_{L,s}$ with unions R, S respectively, such that the predicates, in x, y, s, $x \leqslant_{R,s} y$, $x \leqslant_{L,s} y$ are decidable and $\leqslant_{R,0}, \leqslant_{L,0}$ are the identity pre-orders on ω. In the construction below it is understood that at each stage s, when we monitor the c.e. progress of the two c.e pre-orderings and of the corresponding equivalence relations, we work with the corresponding approximations at that stage.

Recall that $f(n) = x_{\lambda(n)}$ where $\lambda(n)$ is the root in T_n.

Stage 0. Do nothing.

Stage $s+1$. Let n be given, and suppose we have dealt already with all $j < n$. We now deal with n. Recall that $f(n) = x_\lambda$. Let O_1 represent the order type of R restricted to $\{j \mid j < n\}$ at stage s and O_2 be the order type of R restricted to $\{j \mid j < n\}$ at stage $s+1$. Let X_1 be the set of elements in $\{j \mid j < n\}$ which are $\leqslant_R n$ at stage s and let Y_1 be the set of elements in $\{j \mid j < n\}$ which are $\geqslant_R n$ at stage s. Let X_2, Y_2 be defined similarly at stage $s+1$. Then $(O_1, X_1, Y_1) \preceq_n (O_2, X_2, Y_2)$.

If $(O_1, X_1, Y_1) \neq (O_2, X_2, Y_2)$ then make $\varphi_{e_{(O_1,X_1,Y_1)}}(0)$ converge to equal $x_{(O_2,X_2,Y_2)}$. Note that this causes (by our speedup) that $x_{(O_1,X_1,Y_1)} \equiv_L x_{(O_2,X_2,Y_2)}$ also at stage $s+1$. (We will show by induction that then $f(n) = x_\lambda$ will become $\equiv_L$-equivalent to $x_{(O_2,X_2,Y_2)}$, which will ensure that $f(n) \leqslant_L f(j)$ for every $j \in Y_2$ and $f(n) \geqslant_L f(j)$ for every $j \in X_2$.)

If $(O_1, X_1, Y_1) = (O_2, X_2, Y_2)$ then go to $n + 1$.

If ever we see, at any stage, $f(n) \leqslant_L f(m)$ when we have not yet seen $n \leqslant_R m$, then we halt the entire construction and call action $\diamondsuit$. Action $\diamondsuit$, if ever called, will cause a contradiction, which will guarantee that this will never happen and action $\diamondsuit$ will never in fact be called.

Action $\diamondsuit$. We now describe action $\diamondsuit$. Suppose that at stage s of the construction, we see $f(n) \leqslant_L f(m)$ and we have not yet seen $n \leqslant_R m$. In particular, at this stage $f(n) \equiv_L x_\tau$ and $f(m) \equiv_L x_\sigma$ where $\tau = \tau(n, s)$ and $\sigma = \tau(m, s)$ are as given for n and m by Claim 32, so that we have not as yet caused $\varphi_{e_\tau}(0)$ or $\varphi_{e_\sigma}(0)$ to converge. Recall that $a_\tau = \bigvee_{i \in X_\tau} f(i)$ and $b_\tau = \bigwedge_{j \in Y_\tau} f(j)$, and similarly for σ.

Now, for every $z \in \{n\} \cup Y_\tau$, we perform the following action, which we call One(z): take $\rho = \tau(z, s)$ as given for z again by Claim 32 (in particular $x_\rho \equiv_L f(z)$ and $\varphi_{e_\rho}(0)\uparrow$ at this stage). Now, make $\varphi_{e_\rho}(0)$ converge to equal 1. This forces $f(z) \equiv_L b_\rho \vee a_\rho$ (by our speedup this happens at s). In other words, we cause at s $f(z)$ to become as large as possible.

Also, for every $z \in \{m\} \cup X_\sigma$, we perform the following action, which we call Zero(z): take ρ as given for z again by Claim 32 (in particular $x_\rho \equiv_L f(z)$ and $\varphi_{e_\rho}(0)\uparrow$ at this stage). Now, make $\varphi_{e_\rho}(0)$ converge to equal 0, which causes $f(z) \equiv_L a_\rho$ (by our speedup this happens at s). In other words, we cause at s $f(z)$ to become as small as possible.

Verification. We now verify that the above construction yields a function $f : \omega \to \omega$ so that $n \leqslant_R m$ if and only if $f(n) \leqslant_L f(m)$.

Claim 32. *Suppose that we have not called action $\diamondsuit$ at any stage $< s$, and let m be given. Let $\tau(m,s) = (O, X, Y) \in T_m$ be the triple defined by: O is the order type of $\{j \mid j < m\}$ in R at stage s, $X = \{j \mid j < m$ and $j \leqslant_R m\}$ at stage s, and $Y = \{j \mid j < m$ and $j \geqslant_R m\}$ at stage s. Then (letting for simplicity $\tau = \tau(m,s)$) $f(m) \equiv_L x_\tau$ at stage s and $\varphi_{e_\tau}(0)\uparrow$ at stage s, and f preserves the pre-partial order given by R at stage s on the interval $\{j : j \leqslant m\}$.*

Proof. At stage 0, we have $f(m) = x_\lambda$, so the claim holds for every m at stage 0.

Assume by induction that the claim holds for every m at stage s and for every $j < m$ at stage $s+1$.

Notice that we can act at stage $s+1$, as the construction has not been stopped due to $\diamondsuit$. Let $\tau(m,s) = (O, X, Y)$ be the triple defined as in the statement of the Claim, i.e. O is the order type of $\{j \mid j < m\}$ in R at stage s, $X = \{j \mid j < m$ and $j \leqslant_R m\}$ at stage s and $Y = \{j \mid j < m$ and $j \geqslant_R m\}$ at stage s. By induction, if $\tau = \tau(m,s)$ then at s $f(m) \equiv_L x_\tau$. Let $\sigma = \tau(m, s+1)$ be the triple defined analogously at stage $s+1$, so we make $\varphi_{e_\tau}(0)\downarrow = x_\sigma$. But then, by our speedup, at $s+1$

$$x_\tau = k(a_\tau, b_\tau, S_\tau, e_\tau, 0) \equiv_L (x_\sigma \wedge b_\tau) \vee a_\tau.$$

Moreover, by inductive hypothesis at $s+1$ (on f preserving at $s+1$ the pre-order of R on numbers smaller than m), $a_\tau \leqslant_L b_\tau$ and $a_\sigma \leqslant_L b_\sigma$: this implies by property of function k that $a_\tau \leqslant_L x_\tau \leqslant_L b_\tau$ and $a_\sigma \leqslant_L x_\sigma \leqslant_L b_\sigma$. Now, since $X_\tau \subseteq X_\sigma$ and $Y_\tau \subseteq Y_\sigma$, we get (all the following $\equiv_L$-equivalences by our speedup may be assumed to be seen at $s+1$)

$$a_\tau \leqslant_L a_\sigma \leqslant_L b_\sigma \leqslant_L b_\tau,$$

and so at stage $s+1$ we have that $f(m) \equiv_L x_\tau \equiv_L x_\sigma$: indeed $x_\sigma \wedge b_\tau \equiv_L x_\sigma$ and $x_\sigma \vee a_\tau \equiv_L x_\sigma$. Further, there is no condition which could have caused φ_{e_σ} to have converged before this stage, so we still have $\varphi_{e_\sigma}(0)\uparrow$ at stage $s+1$.

We finally show that f preserves the pre-order given at stage $s+1$ by R on $\{j : j \leqslant m\}$. By inductive hypothesis, the pre-order given at stage $s+1$ by R on $\{j : j < m\}$ is preserved. Also, at $s+1$, since $f(m) \equiv_L x_\sigma$ we have that $a_\sigma \leqslant_L f(m) \leqslant_L b_\sigma$, so all the inequalities which occur between m and any $j < m$ in R are preserved in $\leqslant_L$ at $s+1$. □

It now remains to show that we never call action $\diamondsuit$. Then, Claim 32 shows that f preserves the pre-order from R and the fact that we do not call action $\diamondsuit$ shows that we never have any more inequalities holding in $\leqslant_L$ than in $\leqslant_R$. Thus f is a reduction of $\leqslant_R$ to $\leqslant_L$.

Claim 33. *We never call action $\diamondsuit$.*

Proof. Suppose we call action $\diamondsuit$ at stage s because we see $f(n) \leqslant_L f(m)$ and we have not yet seen that $n \leqslant_R m$. By calling action One(z) for every $z \in \{n\} \cup Y_\tau$, we arrange that for each of these elements z, $f(z)$ become (by our speedup this happens at s) $\equiv_L$-equivalent to $b_\rho \vee a_\rho$, as in the construction (with ρ, τ, σ given for m, n, z at s by Claim 32), $f(m) \equiv_L x_\tau$, $f(n) \equiv_L x_\sigma$, $f(z) \equiv_L x_\rho$ and we have not as yet caused either of $\varphi_{x_\tau}(0)$, $\varphi_{x_\sigma}(0)$, $\varphi_{x_\rho}(0)$ to converge. But since at s the order in L is the image of the order in R (since we have not as yet called action $\diamondsuit$), we get that $f(z) \equiv_L b_\rho$, since $b_\rho \geqslant_L a_\rho$. We now show that for every $z \in \{n\} \cup Y_\tau$, $f(z)$ becomes $\equiv_L$-equivalent to 1 (in the following $\equiv_L$ and $\leqslant_R$ are understood to be approximated at stage s). Suppose towards a contradiction that for some element in this set, the image under f is not $\equiv_L$-equivalent to 1, and let z be $\leqslant_R$-maximal among those such that $f(z)$ is not L-equivalent to 1. Subject to this restraint, we further choose z to be minimal (as a number in ω). Now, if $y \in Y_\rho$ (so that $y < z$ and thus $y < n$ as $z \leqslant n$ being $z \in Y_\tau \cup \{n\}$) then $y \geqslant_R z \geqslant_R n$ (if $z = n$ then at s $z \geqslant_R n$ by our assumptions on the approximations to $\leqslant_R$) and thus $y \in Y_\tau$: by R-maximality of z, we see that either $y \equiv_R z$

or $f(y) \equiv_L 1$. In the former case, we have $f(y) \equiv_L f(z)$ (since at s f preserves the pre-order) and that z was not minimal in $\{n\} \cup Y_\tau$ with $f(z) \not\equiv_L 1$. Thus, we always have the latter case, and we see that for every $y \in Y_\rho$ we have $f(y) \equiv_L 1$. But $f(z) \equiv_L b_\rho$ which is the meet of the images of the members of Y_ρ, and thus $\equiv_L 1$. Thus, $f(z) \equiv_L 1$.

The same proof shows that for every $z \in \{m\} \cup X_\sigma$, $f(z)$ becomes $\equiv_L$-equivalent to 0. Thus we have that $1 \equiv_L f(n) \leqslant_L f(m) \equiv_L 0$ (taking in turn $z = n$ and $z = m$), which is a contradiction. Thus, the action $\diamondsuit$ is never called. □

This ends the proof. □

Remark 34. It should be noted that the proofs of Theorem 29 and Theorem 30 do not require distributivity in their assumptions. On the other hand, for a c.e. pre-lattice distributivity is not a consequence of being e.i.: Example 28 shows in fact that there are e.i. non-distributive pre-lattices.

Notice however that the proofs of Theorem 29 and Theorem 30 make use of the existence of both operations of $\wedge$ and $\vee$. The following observation shows that this is indeed necessary. A bounded c.e. upper pre-semilattice $U = \langle \vee, 0, 1, \leqslant_U \rangle$ is called *effectively inseparable* (*e.i*) if, again, the pair $([0]_U, [1]_U)$ is effectively inseparable.

Observation 35. *There exists an e.i. upper pre-semilattice whose pre-ordering relation is not universal.*

Proof. The same construction as in Section 3.5 but starting with the free bounded upper pre-semilattice on X leads to a c.e. bounded upper pre-semilattice U^-_{01} which is e.i. and as a bounded upper pre-semilattice is free on countably many generators.

We claim that the c.e. order $\langle \omega, \leqslant^* \rangle$ where $x \leqslant^* y$ if $x \geqslant y$ can not be reduced to $\leqslant_{U^-}$, thus showing that $\leqslant_{U^-_{01}}$ is not universal. This is so because there is no infinite descending chain in U^-_{01}, as every element other than 1 is the join of a unique finite set of generators, which are atoms in U^-_{01}. □

4.3. **Doing without boundedness.** We have shown (Theorem 29) that every e.i. pre-lattice is u.f.p. and thus u.e.i. by Fact 9. In this section we reverse the last implication by showing that for c.e. pre-lattices u.e.i.-ness, even without boundedness, is enough to have u.f.p.-ness.

Lemma 36. *Let $L = \langle \omega, \wedge, \vee, 0, 1, \leqslant_L \rangle$ be a c.e. pre-lattice, with a, b such that $a <_L b$ and the pair $([a]_L, [b]_L)$ is e.i.. Then there exists a productive function p for this pair such that if $[a]_L \subseteq W_u$, $[b]_L \subseteq W_v$ and $W_u \cap W_v = \varnothing$ then $p(u, v) \in [a, b]_{\leqslant_L}$.*

Proof. Let q be a productive function for the pair $([a]_L, [b]_L)$. We use two indices u', v' (uniformly depending on u, v respectively) which we control by the Recursion Theorem. We program enumerations of $W_{u'}$ and $W_{v'}$ as follows. Let $j = (a \vee q(u', v')) \wedge b$. Then $W_{u'}$ and $W_{v'}$ enumerate $[a]_L$ and $[b]_L$ respectively: if we see $j \in W_u$ then we also enumerate $q(u', v')$ into $W_{u'}$; if we see $j \in W_v$ then we also enumerate $q(u', v')$ into $W_{v'}$. In other words,

$$W_{u'} = \begin{cases} [a]_L, & \text{if } j \notin W_u, \\ [a]_L \cup \{q(u'v')\}, & \text{otherwise,} \end{cases}$$

$$W_{v'} = \begin{cases} [b]_L, & \text{if } j \notin W_v, \\ [b]_L \cup \{q(u'v')\}, & \text{otherwise.} \end{cases}$$

Assume now that $[a]_L \subseteq W_u$ and $[b]_L \subseteq W_v$ and $W_u \cap W_v = \varnothing$. Then $j \notin W_u \cup W_v$: if for instance $j \in W_u$ then we have forced $q(u', v') \in [b]_L$ giving $j \in [b]_L \subseteq W_v$, thus $W_u \cap W_v \neq \varnothing$, contradiction. Thus, $p(u, v) = j$ is our desired productive function. □

Lemma 37. *Let $L = \langle \omega, \wedge, \vee, 0, 1, \leqslant_L \rangle$ be a c.e. pre-lattice, with a, b such that $a <_L b$ and $([a]_L, [b]_L)$ is e.i.. Then the interval $[a, b]_L$ is computably isomorphic to an e.i. pre-lattice.*

Proof. We first observe that the interval $[a, b]_L$ is infinite, as the equivalence classes of a and b are infinite. We argue as in Remark 5. Let $h : \omega \longrightarrow [a, b]_L$ be a computable bijection with $h(0) = a$ and $h(1) = b$. This induces a c.e. pre-lattice $L' = \langle \omega, \wedge, \vee, 0, 1, \leqslant_{L'} \rangle$ where $x \leqslant_{L'} y$ if and only if $h(x) \leqslant_L h(y)$. By Lemma 36 the equivalence classes $[0]_{L'}, [1]_{L'}$ form an e.i. pair: if $[0]_{L'} \subseteq W_u$ and $[1]_{L'} \subseteq W_v$, and $W_u \cap W_v$ are disjoint, then $h[W_u]$ and $h[W_v]$ are disjoint supersets of $[a]_L, [b]_L$ respectively, so that by the lemma (which provides a productive function which always lands in $[a, b]_L$ when applied to indices of disjoint c.e. supersets of $[a]_L, [b]_L$) we can effectively find $y \in [a, b]_L$ so that $h^{-1}(y)$ is defined but $y \notin h[W_u] \cup h[W_v]$, and thus $h^{-1}(y) \notin W_u \cup W_v$. ☐

Theorem 38 and Observation 39 below answer questions raised in private communication by Shavrukov.

Theorem 38. *A c.e. pre-lattice is u.e.i. if and only if it is u.f.p..*

Proof. The implication $\Leftarrow$ follows from Fact 9. For the other implication, let $L = \langle \omega, \wedge, \vee, 0, 1, \leqslant_L \rangle$ be a c.e. pre-lattice which is u.e.i.. By Remark 8 from any pair a, b we can uniformly find an index of a total computable function q which is a productive function for the pair $([a]_L, [b]_L)$ if the two equivalence classes are distinct. A close look at Lemmata 36 and 37 shows that their proofs are uniform: namely, (Lemma 36) if L is u.e.i then from any pair a, b we can uniformly find a computable function $q_{a,b}$ (use totality of q) which has the property therein stated if $a <_L b$; and thus (Lemma 37) from any pair a, b we can uniformly go to an index of a partial computable function $h_{a,b}$, partial computable binary functions $\wedge_{a,b}, \vee_{a,b}$, a c.e. relation $\leqslant_{a,b}$, and a computable function $p_{a,b}$ such that if $a \nleqslant_L b$ then $h_{a,b}$ never converges; if $a <_L b$ then $L_{a,b} = \langle \omega, \wedge_{a,b}, \vee_{a,b}, 0, 1, \leqslant_{a,b} \rangle$ is an e.i. pre-lattice such that $h_{a,b} : [a, b]_L \longrightarrow \omega$ is a bijection giving an order-theoretic isomorphism of the sublattice of L having universe $[a, b]_L$ with $L_{a,b}$, and $p_{a,b}$ is a productive function for $([0]_{L_{a,b}}, [1]_{L_{a,b}})$; if $a \equiv_L b$, then $h_{a,b}$ is a bijection between $[a]_L$ and ω. It follows that if $a \leqslant_L b$ then $h_{a,b}$ is a bijection between $[a, b]_L$ and ω, and thus from the canonical index of a finite $D \subseteq [a, b]_L$ we can effectively find the canonical index of the image $h_{a,b}[D]$. Finally, by the proof of Theorem 29 (which is uniform) from a, b, using totality of $p_{a,b}$, we can uniformly find an index of a total computable function $k_{a,b}(D, e, x)$ which witnesses that $L_{a,b}$ is u.f.p. if $a <_L b$.

Given now any finite set D let $a_D = \bigwedge D$ and $b_D = \bigvee D$ where the meet and the join are taken in L: notice that $a_D \leqslant_L b_D$, thus h_{a_D,b_D} is a bijection between $[a_D, b_D]_L$ and ω; and let f be a computable function such that $\varphi_{f(e,a,b)} = h_{a_D,b_D} \circ \varphi_e$. But then it is easy to see that the computable function

$$k(D, e, x) = h^{-1}_{a_D,b_D}(k_{a_D,b_D}(h_{a_D,b_D}[D], f(e, a_D, b_D), x))$$

witnesses that L is u.f.p.: if $a_D <_L b_D$ this follows from the fact that k_{a_D,b_D} witnesses that the e.i. pre-lattice L_{a_D,b_D} is u.f.p.; if $a_D \equiv_L b_D$ (when L_{a_D,b_D} collapses to one point) this follows from the fact that $[D]_L = [a_D]_L$, and in this case $h^{-1}_{a_D,b_D}$ always lands in $[a_D]_L$. ☐

All the examples seen so far of u.e.i. (or, equivalently, u.f.p.) c.e. pre-lattices are in fact e.i. pre-lattices, which by the very definition, are bounded. The following observation answers the natural question of whether u.e.i.-ness entails boundedness.

Observation 39. *There exist c.e. pre-lattices which are u.e.i. but not bounded.*

Proof. Fix L to be any e.i. pre-lattice. Let M be $\bigoplus_{i \in \omega} L$, that is the direct sum of ω copies of L: M has universe the set of elements in L^ω with only finitely many non-0 entries, and the operations and pre-ordering relation of M are component-wise; clearly this universe can be coded as ω. M is not bounded from above, though it is not hard to show that M is u.e.i. In fact, we claim that for any pair of u.e.i. c.e. pre-lattices A and B, the direct sum $A \oplus B$ is a u.e.i. c.e. pre-lattice, and an index for a uniform productive function for $A \oplus B$ can be found uniformly from indices for uniform productive functions for A and for B. Before proving the claim, let us conclude the argument to show that M is u.e.i.: since every pair of elements from M appear in a finite sum of copies of L, their $\equiv_M$-equivalence classes are e.i., and it is uniform to find the index witnessing this, so M is u.e.i. If we want an example which is neither bounded above nor below, consider $M \oplus M^*$, where M^* is the same as M but with the order reversed.

We are only left to show the claim, that if A and B are u.e.i. c.e. lattices then the c.e. pre-lattice $A \oplus B$, which equals $A \times B$, is u.e.i. as well. We code $A \oplus B$ as ω by using the Cantor pairing function, and let π_0, π_1 be the computable projections associated with the Cantor pairing function. Since u.e.i. implies u.f.p. (Fact 9) we know that A and B are u.f.p.: pick two totalizers K^A, K^B for $\equiv_A$ and $\equiv_B$, respectively. In order to show that $A \oplus B$ is u.f.p., for every D, e, x let $u^A_{D,e}, u^B_{D,e}$ be indices of partial computable functions as follows: to compute $\varphi_{u^A_{D,e}}(x)$ wait until $\varphi_e(x) \downarrow \equiv_{A\oplus B} d$ for some $d \in D$: if and when this happens then make $\varphi_{u^A_{D,e,x}}(0)$ converge to $\pi_0(d)$; otherwise $\varphi_{u^A_{D,e,x}}(0)$ diverges. The computation $\varphi_{u^B_{D,e,x}}(0)$ is defined similarly, outputting $\pi_1(d)$ when it converges. Define now

$$k(D, e, x) = \langle k^A(\pi_0[D], u^A_{D,e}, x), k^B(\pi_1[D], u^B_{D,e}, x)\rangle.$$

Using that K^A, K^B are totalizers for $\equiv_A$ and $\equiv_B$, respectively, it is straightforward to see that k is a totalizer for $\equiv_{A\oplus B}$. Since (Fact 9) u.f.p. implies u.e.i. we have that $A \oplus B$ is u.e.i.. Uniformity from indices of uniform productive functions for A and B, respectively, to indices of a productive function for $A \oplus B$ follows from the following facts: the implications stated in Fact 9 are uniform, i.e. one can uniformly go from an index of a uniform productive function for a u.e.i. ceer to an index of a totalizer of that ceer, and vice versa; and the proof of the above claim shows that one can uniformly go from indices of k^A, k^B to indices of k. $\square$

Remark 40. Notice that M and $M \oplus M^*$ are distributive if and only if L is distributive.

5. Local universality and uniform density

When dealing with ordered or pre-ordered structures, one of the most immediate and important tasks is to study density questions, in particular whether a given structure is dense or where density locally holds or fails. For pre-lattices, effective inseparability turns out to be, as far as density goes, a powerful tool of investigation.

5.1. Density.

We first look at the case in which a c.e. pre-lattice L need not be bounded but it has a nontrivial interval $[a, b]_L$ such that the pair $([a]_L, [b]_L)$ is e.i.. Before proving in Corollary 43 that the bounded interval $[a, b]_L$ enjoys the universality property already shown of any e.i. pre-lattice, namely every c.e. pre-order can be computably embedded into the interval $[a, b]_L$, we first notice, by the following example, that such a c.e. pre-lattice, even if bounded, need not be u.e.i..

Example 41. Let L be a u.e.i. pre-lattice. Form L' by placing a new element called 0 below every element of L and a new element called 1 above every element of L. Formally, let h be a computable bijection of ω with $\omega \setminus \{0, 1\}$, and define $\leqslant_{L'}$ be the c.e. pre-ordering given by $x \leqslant_{L'} y$ if and only if

$$x = 0 \vee y = 1 \vee [\{x, y\} \cap \{0, 1\} = \varnothing \,\&\, h^{-1}(x) \leqslant_L h^{-1}(y)].$$

It is now immediate to see that there is a c.e. bounded pre-lattice $L' = \langle \omega, \wedge, \vee, 0, 1, \leqslant_{L'} \rangle$ such that $[0]_{L'} = \{0\}$, $[1]_{L'} = \{1\}$, and thus the pair $([0]_{L'}, [1]_{L'})$ is not e.i.. On the other hand all disjoint pairs $([a]_{L'}, [b]_{L'})$ with $\{a, b\} \cap \{0, 1\} = \varnothing$ are e.i..

Definition 42. A c.e. pre-order R is *locally universal* if for every pair a, b such that $a <_R b$ one can computably embed any c.e. pre-order in the interval $[a, b]_R$. A c.e. pre-lattice (Boolean pre-algebra, Heyting pre-algebra, etc.) is *locally universal* if so is its pre-ordering relation.

Corollary 43. *If $L = \langle \omega, \wedge, \vee, 0, 1, \leqslant_L \rangle$ is a c.e. pre-lattice and $([a]_L, [b]_L)$ are e.i. with $a <_L b$ then one can reduce any c.e. pre-order to the interval $[a, b]_L$.*

Proof. Trivial, since by Lemma 37 $[a, b]_L$ is computably isomorphic to an e.i. pre-lattice. □

Corollary 44. *Every u.e.i. c.e. pre-lattice is locally universal.*

Proof. This is immediate by Corollary 43 since in a u.e.i. c.e. pre-lattice, all pairs of distinct equivalence classes are e.i.. □

Remark 45. If L is a u.e.i. c.e. pre-lattice then local universality is uniform: from a, b and a c.e. index for a c.e. pre-order R one can uniformly find an index of a computable function f which is a computable embedding of R into $[a, b]_L$ if $a <_L b$. This follows from the fact that the proof of Theorem 30 is uniform in an index of $\leqslant_R$, and from the observations on uniformity of Lemmata 36 and 37 made in the proof of Theorem 38.

5.2. **Uniform density.** Uniform density for c.e. pre-lattices appearing in logic and for c.e. precomplete pre-lattices is treated in detail in [27], to which the reader is referred also for motivations and historical remarks. We recall the definition.

Definition 46. A c.e. pre-ordering $\leqslant_R$ is *uniformly dense* if there exists a computable function $f(a, b)$ so that for all a, b if $a <_R b$ then $a <_R f(a, b) <_R b$, and for all pairs a', b' such that $a \equiv_R a'$ and $b \equiv_R b'$ we have that $f(a, b) \equiv_R f(a', b')$. A c.e. pre-lattice (Boolean pre-algebra, Heyting pre-algebra, etc.) is *uniformly dense* if so is its pre-ordering relation.

The proof of the next theorem follows, via a natural generalization, along the lines of the proof of uniform density for c.e. precomplete pre-lattices in [27].

Theorem 47. *Every u.e.i. c.e. pre-lattice is uniformly dense.*

Proof. Let L be a u.e.i. pre-lattice, and let $k(D, e, x)$ be a computable function witnessing that $\equiv_L$ is u.f.p.: see Theorem 38.

Let $\{e_{a,b} : a, b \in \omega\}$ be a computable list of indices we control by the Recursion Theorem. Denote by $\equiv_L^2$ the equivalence relation

$$(a, b) \equiv_L^2 (a', b') \Leftrightarrow \left[a \equiv_L a' \mathbin{\&} b \equiv_L b'\right],$$

and define

$$f(a, b) = (a \vee j(a, b)) \wedge b,$$

where $j(a, b) = k(D_{a,b}, e_{a,b}, 0))$ and $D_{a,b} = \{a, b, j(a', b') : \langle a', b' \rangle < \langle a, b \rangle\}$. (Here and in the following, the reader is invited to make sure to distinguish between the natural orderings of ω for which we use the symbols $\leqslant, <$, and the pre-ordering or the strict pre-ordering of L for which we use the symbols $\leqslant_L, <_L$).

Now we specify how to compute the various $\varphi_{e_{a,b}}(0)$.

At step 0 all computations $\varphi_{e_{a,b}}(x)$ are undefined.

At step $s + 1$ we consider all pairs (a, b) with $\langle a, b \rangle \leqslant s$, for which $\varphi_{e_{a,b}}(0)$ is still undefined:

(1) for each such pair (a,b) let $(a_m,b_m) \equiv_L^2 (a,b)$ (at s) and $\langle a_m,b_m\rangle < \langle a,b\rangle$ is least with this property, if there is any such pair. If so, define $\varphi_{e_{a,b}}(0) = j(a_m,b_m)$. By the properties of the totalizer k, this makes $j(a_m,b_m) \equiv_L j(a,b)$ and thus $f(a_m,b_m) \equiv_L f(a,b)$;

(2) if after this, (a,b) is still a pair such that $\varphi_{e_{a,b}}(0)$ is undefined and $a \leqslant_L b$ at this stage then
   (a) if $(a \vee j(a,b)) \wedge b) \equiv_L a$ (at s) then define $\varphi_{e_{a,b}}(0) = b$: this forces $j(a,b) \equiv_L b$ and thus $a \equiv_L b$;
   (b) if $(a \vee j(a,b)) \wedge b) \equiv_L b$ (at s) then define $\varphi_{e_{a,b}}(0) = a$: this forces $j(a,b) \equiv_L a$ and thus, again, $a \equiv_L b$.

This ends the construction. Notice that $\varphi_{e_{a,b}}(0)$ is eventually defined if there is (a',b') in the same $\equiv_L^2$-class as (a,b) and $\langle a',b'\rangle < \langle a,b\rangle$.

We now show that f is extensional. As $f(a,b) \equiv_L a$ if $a \equiv_L b$, then extensionality trivially holds when $a \equiv_L b$. Suppose that $a \not\equiv_L b$. Then $\varphi_{e_{a,b}}(0)$ has not been defined through either subclause of clause (2), as this would give $a \equiv_L b$. We claim that in this case $f(a,b) \equiv f(a_m,b_m)$ where (a_m,b_m) is in the same $\equiv_L^2$-class as (a,b) and $\langle a_m,b_m\rangle$ is least with this property. We prove by induction on $\langle a',b'\rangle$ that this is true for all (a',b') which is in the same $\equiv_L^2$-class as (a_m,b_m). The claim is trivial when $(a',b') = (a_m,b_m)$. Otherwise $\varphi_{e_{a,b}}(0)$ is defined at some stage (not through (2) as we have already seen). Thus we have forced $j(a,b) \equiv_L j(a',b')$ with (a',b') in the same $\equiv_L^2$-class as (a,b) but $\langle a',b'\rangle < \langle a,b\rangle$. This makes $f(a,b) \equiv_L f(a',b')$. By inductive assumption, $f(a',b') \equiv_L f(a_m,b_m)$: hence $f(a,b) \equiv_L f(a_m,b_m)$.

Finally we show that if $a <_L b$ then $a <_L f(a,b) <_L b$: notice that we have $a \leqslant_L f(a,b) \leqslant_L b$ for free as $a \leqslant_L b$. Indeed by extensionality we may assume that $\langle a,b\rangle$ is least among the pairs in the $\equiv_L^2$-class of (a,b). If $a <_L f(a,b) <_L b$ does not hold then sooner or later clause (2) would force $a \equiv_L b$, a contradiction. □

6. Reducibilities inducing monomorphisms

Given a class $\mathcal{A}$ of pre-ordered structures of the same type, it is natural to study reductions (called *$\mathcal{A}$-reductions*) between the structures in $\mathcal{A}$ which *respect* the operations. For instance:

Definition 48. Let $\mathcal{L}$ be the class of c.e. pre-lattices , and assume that $L_1 = \langle \omega, \vee_1, \wedge_1, \leqslant_1\rangle$ and $L_2 = \langle \omega, \vee_2, \wedge_2, \leqslant_2\rangle$ lie in $\mathcal{L}$. We say that a computable functions f *$\mathcal{L}$-reduces* L_1 to L_2 (notation: $L_1 \preceq_{\mathcal{L}} L_2$) if f reduces $\leqslant_{L_1}$ to $\leqslant_{L_2}$ and for all $x, y \in L_1$,

$$f(x \vee_1 y) \equiv_{L_2} f(x) \vee_2 f(y),$$
$$f(x \wedge_1 y) \equiv_{L_2} f(x) \wedge_2 f(y).$$

Clearly f induces a monomorphism between the associated quotient structures. Given a class $\mathcal{A}$ of c.e. pre-structures of the same type, we say that a c.e. pre-structure L is *$\mathcal{A}$-universal* (or *universal with respect to* $\mathcal{A}$) if L has a type containing the type of the pre-structures in $\mathcal{A}$ and $A \preceq_{\mathcal{A}} L^{\text{red}}$ for every $A \in \mathcal{A}$ (where L^{red} is the pre-structure in $\mathcal{A}$ obtained from L by restricting its type to the type of $\mathcal{A}$).

In addition to using the symbol $\mathcal{L}$ to denote the class of c.e. pre-lattices, in the following we will use the following notations: $\mathcal{B}$ is the class of c.e. Boolean pre-algebras; $\mathcal{L}_{0,1}^d$ comprises the c.e. distributive bounded pre-lattices; $\mathcal{L}_0^d$ comprises the c.e. distributive pre-lattices with a least element; $\mathcal{L}^d$ is the class of c.e. distributive pre-lattices. We will keep the term "universal" without specifications, to mean universality with respect to the c.e. pre-orders: thus a c.e. pre-ordered structure L with c.e. pre-ordering relation $\leqslant_L$ is *universal* if $\leqslant_R \preceq \leqslant_L$ for every c.e. pre-order R.

Lemma 49 ([17]). *$\mathcal{L}^d$-universality implies universality; $\mathcal{L}^d_{0,1}$-universality implies $\mathcal{L}^d$-universality; $\mathcal{B}$-universality implies $\mathcal{L}^d_{0,1}$-universality.*

Proof. The first claim follows from the fact that every c.e. pre-order can be reduced to the pre-ordering relation of some c.e. distributive pre-lattice, see for instance the proof of [17, Theorem 3.1]. The second claim is trivial. For the third claim see [17, Theorem 3.2]. □

Theorem 50. [17] *Every e.i. Boolean pre-algebra is $\mathcal{B}$-universal (by Lemma 49, this implies universality and $\mathcal{A}$-universality for the classes $\mathcal{A}$ specified in the Lemma).*

In [31] Visser introduced a c.e. extension of Heyting Arithmetic, called HA^* (which is obtained, as described in [7], as HA plus the Completeness Principle for HA^* by a fixed point construction). Let HA^* be the Heyting pre-algebra of the Lindenbaum sentence algebra of HA^* (i.e $\mathrm{HA}^* = \mathrm{i}(HA^*)$ in the notation of Section 3.4)): clearly HA^* is an e.i. Heyting pre-algebra as HA^* extends HA.

Building on some proof techniques due to Shavrukov [24], Visser [32], and Zambella [33], De Jongh and Visser [7] prove the following $\mathcal{A}$-universality result, where in this case $\mathcal{A}$ is the class of all c.e. prime Heyting pre-algebras (a Heyting algebra is *prime* if its greatest element is join-irreducible).

Theorem 51 ([7]). HA^* *is $\mathcal{A}$-universal.*

6.1. **An e.i. pre-lattice which not $\mathcal{L}^d_{0,1}$-universal.** Although every e.i. pre-lattice is universal, and despite the fact that the e.i. Boolean pre-algebras are $\mathcal{B}$-universal by Theorem 50, and also the fact there exist e.i. distributive pre-lattices which are $\mathcal{L}^d_{0,1}$-universal as we will observe in Section 7, the following theorem shows that one should not be led to conclude that all e.i. pre-lattices are $\mathcal{L}$-universal, or even that all e.i. distributive pre-lattices are $\mathcal{L}^d$-universal.

Theorem 52. *There exist e.i. distributive pre-lattices that are not $\mathcal{L}^d$-universal.*

Proof. Since we know from Example 27 that there is an e.i. distributive pre-lattice L^d_{01} which is a free bounded distributive pre-lattice on a countably infinite set of generators, it is enough to show that for every X not all countable distributive lattices which are isomorphic to quotient structures of c.e. pre-lattices (see Definition 1) can be lattice-embedded into $F^d_{0,1}(X)$: the claim then follows from the fact that by Example 27 the quotient structure corresponding to L^d_{01} is isomorphic to some such $F^d_{0,1}(X)$. To see this, take B to be any infinite Boolean algebra. Suppose that we have a lattice theoretic-theoretic embedding of B into $F^d_{0,1}(X)$. Clearly $0 \in B$ can not be mapped to $0 \in F^d_{0,1}(X)$, as the former 0 is meet-reducible whereas the latter one is not. So the least element $0 \in B$ must be mapped to an element $a \in F^d(X)$ (recall that $F^d_{0,1}(X) = 0 \oplus F^d(X) \oplus 1$ where $F^d(X)$ is the free distributive lattice on X, see [4]). Likewise, the greatest element $01 \in B$ must be mapped to an element $b \in F^d(X)$. Now each element $x \in F^d(X)$ can be written (in a unique way, called *normal form* of x) as $\sum_{i \in I_x} U_i^x$ where I_x is a nonempty finite set, $\sum$ denotes join, each U_i^x is a nonempty finite set of distinct generators, and for simplicity we identify (as in clause notation) such a finite set with the meet of its element (i.e., to be explicit, $x = \sum_{i \in I_x} \prod U_i^x$); it can be also assumed that the U_i^x are all $\subseteq$-incomparable (if $U_i^x \subseteq U_j^x$, just delete j from I_x); moreover given $x = \sum_{i \in I_x} U_i^x$ and $y = \sum_{i \in I_y} U_i^y$, we have

$$x \leqslant y \Leftrightarrow (\forall i \in I_x)(\exists j \in I_y)[U_j^y \subseteq U_i^x].$$

Now suppose that $a = x \wedge y$, and $b = x \vee y$: thus (representing a, b, x, y in normal form) we have

$$\sum_{i \in I_a} U_i^a = \sum_{(u,v) \in I_x \times I_y} (U_u^x \cup U_v^y),$$
$$\sum_{i \in I_b} U_i^b = \sum_{u \in I_x} U_u^x + \sum_{v \in I_y} U_v^y.$$

By selecting the pairs (u, v) for which $U_u^x \cup U_v^y$ is minimal under inclusion (so that its join is maximal) we can pick a subset $J \subseteq I_x \times I_y$ such that

$$\sum_{i \in I_a} U_i^a = \sum_{(u,v) \in J} (U_u^x \cup U_v^y). \tag{1}$$

and all the $U_u^x \cup U_v^y$, with $(u, v) \in J$, are $\subseteq$-incomparable. Using $\geqslant$ of (1) it follows that

$$(\forall (u, v) \in J)(\exists i \in I_a)[U_i^a \subseteq U_u^x \cup U_v^y];$$

on the other hand, using $\leqslant$ of (1) we have

$$(\forall i \in I_a)(\exists (u, v) \in J)[U_u^x \cup U_v^y \subseteq U_i^a].$$

It follows that

$$(\forall (u, v) \in J)(\exists i \in I_a)(\exists (u', v') \in J)[U_{u'}^x \cup U_{v'}^y \subseteq U_i^a \subseteq U_u^x \cup U_v^y] :$$

by incomparability, for all $(u, v), (u', v') \in J$ it follows that $U_{u'}^x \cup U_{v'}^y = U_u^x \cup U_v^y$, and thus U_u^x and U_v^y are subsets of $U = \bigcup_{i \in I_a} U_i^a$. Consider now the equality $b = \sum_{u \in I_x} U_u^x + \sum_{v \in I_y} U_v^y$ and let us pick subsets $H \subseteq I_x$, $K \subseteq I_y$ which select the sets U_u^x and V_v^y which are $\subseteq$-minimal and $b = \sum_{u \in H} U_u^x + \sum_{v \in K} U_v^y$. By arguing as before we can show that for every $u \in H$ and $k \in K$ we have that $U_h^x, Y_k^y \subseteq V = \bigcup_{i \in I_b} U_i^b$. Finally, suppose that some U_i^x or some U_j^y has been left out of the reduced family that joins to b (i.e. in the former case $i \notin H$ and the latter case $j \notin K$): assume for instance that $i_0 \in I_x \smallsetminus H$, for some i_0. Then (by $\subseteq$-incomparability the sets U_i^x) there is $j_0 \in I_y$ such that $V_{j_0}^y \subseteq U_{i_0}^x$; but then for the pair (i_0, j_0) we have $U_{i_0}^x \cup V_{j_0}^y = U_{i_0}^x$. If $(i_0, j_0) \in J$ then we know that $U_{i_0}^x \subseteq U$; on the other hand if $(i_0, j_0) \notin J$ then there exists a pair $(i_1, j_1) \in J$ such that $U_{i_1}^x \cup U_{j_1}^y \subseteq U_{i_0}^x$, which by incomparability implies $i_0 = i_1$ and thus we have that $U_{i_0}^x \subseteq U$, as $U_{i_1}^x \subseteq U$. If instead for some j_0 we have $j_0 \in I_y \smallsetminus K$ so that $U_{j_0}^y$ has been left out of the reduced family that joins to b then by a similar argument we conclude that $U_{j_0}^y \subseteq U$. In conclusion, for all $(u, v) \in I_x \times I_y$ we have that we have $U_u^x \cup U_v^y \subseteq U \cup V$. Thus there can be finitely only many pairs x, y that meet to a and join to b, since the normal forms of x, y may involve only generators in $U \cup V$. But being infinite, the Boolean algebra B has infinitely many distinct pairs of elements that meet to 0 and join to 1. Thus the images under the embedding of these pairs would provide an infinite set of pairs of elements that in $F_{0,1}^d(X)$ meet to a and join to b, contradiction. □

Remark 53. Things are even more dramatic if we consider non-distributive pre-lattices. Galvin and Jònsson [10] prove that a distributive lattice L can be embedded in a free lattice if and only if L is a countable linear sum of lattices where: each lattice in the linear sum is either a one element lattice, or an eight element Boolean algebra $2 \times 2 \times 2$, or a direct product of the two element chain with a countable chain.

Therefore no c.e. distributive pre-lattice with quotient structure isomorphic to $0 \oplus L \oplus 1$, where L is not isomorphic with any of the lattices characterized by Galvin and Jònsson as above, can be reduced to L_{01}^{nd} (whose quotient structure is isomorphic to a lattice of the form $0 \oplus F^{nd}(X) \oplus 1$ where $F^{nd}(X)$ is a free lattice) of Example 28 preserving $\wedge$ and $\vee$.

7. Some applications

In this section we give some applications of Theorem 30 and Theorem 47. Our results on universality, local universality, and uniform density hold of *all* e.i. pre-lattices, which form a large enough class to include some c.e. pre-structures of particular interest which have been widely studied and for which some or all of these results have been proved already, and also some interesting cases (listed in Section 7.1) where these were not known. In Section 7.1 we list some of these new cases. In Section 7.2 we will briefly review the literature on what was already known about e.i. pre-structures, and universality with respect to c.e. pre-orders, or even $\mathcal{A}$-universality for various classes $\mathcal{A}$.

7.1. **New applications.** In the next three corollaries we point out three applications of our theorems which, in their generality, seem to have gone unnoticed so far.

Corollary 54. *If L is a c.e. precomplete pre-lattice then L is locally universal, i.e. in any non-trivial interval $[a, b]_L$ one can embed every c.e. pre-order.*

Proof. By Corollary 43. □

Corollary 55. *If T is a consistent c.e. extension of R or Q then for every $n \geqslant 1$ the pre-lattice of sentences $L_{\Sigma_n/T}$ satisfies:*

(1) $L_{\Sigma_n/T}$ is locally universal;
(2) $L_{\Sigma_n/T}$ is uniformly dense.

Proof. Use Theorem 22: then Claim (1) comes from Corollary 43. Claim (2) comes from Theorem 47. □

Corollary 56. *If iT is a c.e. consistent intuitionistic theory extending HA then the Heyting pre-algebra* iT *of the Lindenbaum sentence algebra of iT satisfies:*

(1) iT *is locally universal;*
(2) iT *is uniformly dense.*

Proof. Claim (1) comes Corollary 43. Claim (2) comes from Theorem 47. □

7.2. **Reviewing (some of) the existing literature.** For e.i. Boolean pre-algebras, embedding results via computable functions yielding universality, were already known in the literature. We have already seen (Theorem 50) that each e.i. Boolean pre-algebra is $\mathcal{B}$-universal. This theorem alone is enough to establish local universality of the e.i. Boolean pre-algebras: in fact there is no need to directly use Corollary 43 as any non-trivial interval of an e.i. Boolean pre-algebra is itself computably isomorphic (use Lemma 37 and the fact that any such interval is a c.e. Boolean pre-algebra) to an e.i. Boolean pre-algebra, and thus any such interval is $\mathcal{B}$-universal. Moreover, uniform density holds in any e.i. Boolean pre-algebra, as all e.i. Boolean pre-algebras are computably isomorphic (see Theorem 57) and Shavrukov and Visser [27] show that uniform density holds in any B_T of Example 14.

Let us now move to c.e. precomplete pre-lattices and pre-lattices of sentences. Uniform density for c.e. precomplete pre-lattices was proved by Shavrukov and Visser in [27]. Now, if T is a c.e. consistent extension of elementary arithmetic EA then by Theorem 21 $L_{\Sigma_n/T}$ is precomplete, and thus uniformly dense. Moreover, it was already known that for such a T, and for $n \geqslant 1$, $L_{\Sigma_n/T}$ is locally $\mathcal{L}_0^d$-universal, and locally $\mathcal{L}_{0,1}^d$-universal if $n \geqslant 2$. Indeed $\mathcal{L}_0^d$-universality of any $L_{\Sigma_n/T}$ for $n \geqslant 1$, and $\mathcal{L}_{01}^d$-universality of any $L_{\Sigma_n/T}$ for $n \geqslant 2$ was first proved in [17] where it is shown that the reducing functions can in fact be taken of the form $x \mapsto \gamma(\psi(\overline{x}))$ with $\psi(v)$ a Σ_n formula, $\overline{x}$ the numeral term corresponding to x, and $\gamma(\psi(\overline{x}))$ the Gödel number of $\psi(\overline{x})$ (under the γ exploited in building the pre-lattice of sentences).

Shavrukov [26] proved that for $n \geqslant 1$ the c.e. Boolean pre-algebra $B_{\Delta_n/T}$ (having as domain the Δ_n-sentences, and regarded as a c.e. Boolean pre-algebra by Remark 5) is an e.i. Boolean pre-algebra unless $n = 1$ and T is Σ_1-sound; and if $n \geqslant 1$, α, β are Σ_n sentences, $\vdash_T \alpha \to \beta$, and $\nvdash_T \beta \to \alpha$, then $[\alpha, \beta]_{\Delta_n/T}$ (i.e. the Δ^0_n sentences in the interval) is an e.i. Boolean pre-algebra unless $n = 1$, T is Σ_1-sound, and $T \vdash \beta$. $\mathcal{A}$-universality of $L_{\Sigma_n/T}$ (for the relevant class $\mathcal{A}$) finally follows by $\mathcal{B}$-universality of the e.i. Boolean pre-algebras (Theorem 50) and Lemma 49.

8. Computable isomorphism types of universal c.e. pre-orders

It would be interesting to characterize "natural" classes of universal pre-structures falling in a single computable isomorphism type, as happens for instance for the e.i. Boolean pre-algebras:

Theorem 57 (([17, 18], based on [21]))**.** *All e.i. Boolean pre-algebras are computably isomorphic.*

Some natural computable isomorphism types have been pointed out for universal ceers. We recall that a *diagonal* function for an equivalence relation R is a computable function δ such that $x \not\mathrel{R} \delta(x)$ for all x. The following are known: All u.f.p. ceers endowed with a diagonal function are computably isomorphic, see [16] (and independently [15]); all precomplete ceers are computably isomorphic, see [15].

Nonetheless, there are several interesting different computable isomorphism types of universal c.e. pre-orderings, even if we restrict attention to the pre-orderings relative to e.i. (hence bounded by definition) pre-structures.

Theorem 58. *The following list provides classes of e.i. pre-structures such that for each pair $(\mathcal{A}, \mathcal{B})$ of distinct classes (i.e., (1), (2a), (2b) (2c), (3))) in the list there is a pair of objects $(A, B) \in (\mathcal{A} \times \mathcal{B})$, with A, B not computably isomorphic.*

(1) e.i. Boolean pre-algebras;
(2) e.i. distributive pre-lattices; inside this class, one can further list the following subclasses:
  (a) e.i. Heyting pre-algebras;
  (b) e.i. free distributive pre-lattices on a countably infinite set of generators;
  (c) c.e. precomplete distributive pre-lattices, including pre-lattices of sentences of consistent extensions of Robinson's Q or R;
(3) e.i. non-distributive pre-lattices.

Proof. Of course distinctions between some of the classes derive immediately from evident structural differences (for instance a free distributive lattice can not be isomorphic with a Boolean algebra, and so on) which immediately imply that for a given pair $(A, B) \in (\mathcal{A} \times \mathcal{B})$ of pre-structures no isomorphism may exist between the corresponding quotient structures, and thus no computable isomorphism may exist between the pre-structures. Apart from these obvious considerations, the proof follows from previous results proved in this paper or recalled in this paper from the literature. We list here some of the distinguishing properties. The c.e. precomplete pre-lattices do not have a diagonal function (this is a consequence of the celebrated Ershov Fixed Point Theorem for precomplete equivalence relations). If L is the Heyting pre-algebra of the Lindenbaum sentence algebra of HA then L is e.i. and has a diagonal function, so no computably isomorphic copy of it can lie in 2c, and can not lie in 2b either since in every free bounded distributive lattice the least element is meet-irreducible, but this does not hold in the Lindenbaum sentence algebra of HA. For the same reason, if $n \geqslant 1$ and T is a c.e. consistent extension of EA then no isomorphic copy of $L_{\Sigma_n/T}$ can lie in (2b). ☐

One must not think that each of the classes displayed in Theorem 58 contains only one computable isomorphism class. For instance, as already remarked, if T is Σ_1-sound then the precomplete distributive pre-lattice $L_{\Sigma_1/T}$ is not isomorphic to any $L_{\Sigma_n/T}$, for $n \geqslant 2$.

E.i. Boolean pre-algebras are free on infinitely many generators, since so is B_T of Example 14. However, even adding freeness, the analogue of Theorem 57 for e.i. distributive pre-lattices fails, as shown in next theorem.

Theorem 59. *There are e.i. distributive pre-lattices that are free on countably infinitely many generators but are not computably isomorphic.*

Proof. We construct two e.i. bounded distributive pre-lattices L_1, L_2, each of which is free on a countably infinite sets of generators, and such that for every i, the *requirement* R_i is satisfied, i.e. the partial computable function φ_i does not induce an isomorphism between the quotient structures associated with L_1 and L_2. We take L_2 to be the pre-lattice L^d_{01} of Example 27, but this time we show how to build this pre-lattice and the congruence α via a construction in stages, which we then synchronize with (part of) the construction of L_1.

As in the discussion in Section 3.5 consider a computable presentation $F^d_{0,1}(X) = \langle \omega, \wedge, \vee, 0, 1\rangle$ of the free bounded distributive lattice on a decidable infinite set X (with $0, 1 \notin X$) which is computably listed without repetitions by $\{x_i : i \in \omega\}$, and let (U, V) an e.i. pair of c.e. sets.

We start with specifying a construction of $L_2 = L^d_{01}$ in stages. Assume that $\{U_s : s \in \omega\}, \{V_s : s \in \omega\}$ are computable approximations to U, V respectively, i.e. computable sequences of finite sets giving $U = \bigcup_s U_s$ and $V = \bigcup_s V_s$. Let $X_s = \{x_i : i < s\}$, and let α_s be the congruence on $F^d_{01}(X_s)$ generated by the pairs $\{(x_i, 0) : x_i \in X_s \,\&\, i \in U_s\} \cup \{(x_i, 1) : x_i \in X_s \,\&\, i \in V_s\}$. It is easy to see (First Isomorphism Theorem) that $F^d_{01}(X_s)_{/\alpha_s} \simeq F^d_{01}(X_s \smallsetminus \{x_i : i \in U_s \cup V_s\})$, and if α is the congruence on $F^d_{01}(X)$ exhibited in Section 3.5 then $\alpha = \bigcup_s \alpha_s$. Taking $L_2 = L^d_{01}$ of Example 27 we know that L_2 is e.i., $L_{2/\alpha} = F^d_{01}(X)_{/\alpha} \simeq F^d_{01}(X \smallsetminus \{x_i : i \in U \cup V\})$, and L_2 is free on a countably infinite set of generators. We use the sequence $\{F^d_{01}(X_s)_{/\alpha_s} : s \in \omega\}$ to approximate $F^d_{01}(X)_{/\alpha}$ in the construction of L_1.

We now specify how to build L_1. Split X as $X = Y \cup Z$, with Y, Z decidable and infinite. Split also Z into two infinite decidable sets Z_1, Z_2 so that Z_1 is computably enumerated without repetitions by $\{z^1_i : i \in \omega\}$.

We define in stages sequences $\{X_{1,s} : s \in \omega\}$, $\{H_s : s \in \omega\}$, $\{\beta_s : s \in \omega\}$, where $H_s \subseteq X_{1,s}$, $X_{1,s}$ is a finite subset of X, β_s is a congruence on $F^d_{0,1}(X_{1,s})$; moreover let $G_s = X_{1,s} \smallsetminus H_s$. Recall that in a bounded lattice an element x is meet-irreducible if $x \neq 0, 1$ and is not the meet of any two strictly bigger elements.

Stage 0. Let $X_{1,0} = H_0 = \varnothing$ so that $F^d_{0,1}(X_{1,0})$ is the two-element bounded lattice, and let β_0 be the identity equivalence relation on $\{0, 1\}$.

Stage $s+1$, say $s = \langle i, t\rangle$. If R_i has already acted, or $z^1_i \notin X_{1,s}$, or $\varphi_i(z^1_i)$ has not as yet converged in fewer than s steps to an $x \in F^d_{0,1}(X_s)$, or $\varphi_i(z^1_i)$ has already converged to an $x \in F^d_{0,1}(X_s)$, such that $[x]_{\alpha_s}$ is meet-reducible in $F^d_{0,1}(X_s)_{/\alpha_s}$, then let $X_{1,s+1} = X_{1,s} \cup \{x_s\}$, $H_{s+1} = H_s$, and let β_{s+1} be the congruence on $F^d_{0,1}(X_{1,s+1})$ generated by the pairs in β_s. Otherwise, R_i *acts* by choosing the least two distinct elements $z^2_{k_1}, z^2_{k_2} \in Z_2$ not as yet used at any stage of the construction; let $X_{1,s+1} = X_{1,s} \cup \{x_s, z^2_{k_1}, z^2_{k_2}\}$, let β_{s+1} be the congruence on $F^d_{0,1}(X_{1,s+1})$ generated by β_s plus the pair $(z^1_i, z^2_{k_1} \wedge z^2_{k_2})$; let $H_{s+1} = H_s \cup \{z^1_j\}$.

The following can be easily checked. If $H = \bigcup_s H_s$ and $G = X \smallsetminus H$ then $G = \{g : (\exists t)(\forall s \geqslant t)[g \in G_s]\}$ and is comprised of all elements $g \in X$ such that g is not collapsed at some stage to the meet of two elements of Z_2 by the c.e. congruence β on $F^d_{0,1}(X)$ where $\beta = \bigcup_s \beta_s$; moreover $Y \subseteq G$ as $Y \cap H = \varnothing$. Let now α_1 be the c.e. congruence on $F^d_{0,1}(X)$ generated by the pairs in $\beta \cup \{(y_i, 0) : i \in U\} \cup \{(y_i, 1) : i \in V\}$. On numbers x, y define $x \leqslant_{L_1} y$ if $[x]_{\alpha_1} = [x \wedge y]_{\alpha_1}$. Then $L_1 = \langle \omega, \wedge, \vee, 0, 1, \leqslant_{L_1} \rangle$, where $\wedge, \vee$ are the same operations as in $F^d_{0,1}(X)$, is a c.e. pre-lattice whose associated quotient structure is $F^d_{0,1}(X)_{/\alpha_1}$. By an argument similar to the one used for α and L^d_{01} in Section 3.5, one can show that L_1 is e.i. and $F^d_{0,1}(X)_{/\alpha_1} \simeq F^d_{0,1}(G \smallsetminus \{y_i : i \in U \cup V\})$ so that L_1 is free on a countably infinite set of generators.

It remains to show that for every i, φ_i does not induce an isomorphism from $L_{1/\alpha_1}(= F^d_{0,1}(X)_{/\alpha_1})$ to $L_{2/\alpha}(= F^d_{0,1}(X)_{/\alpha})$. Assume $\varphi_i(z^1_i)$ converges and $\varphi_i(z^1_i) = x$: if $[x]_\alpha \in \{[0]_\alpha, [1]_\alpha\}$ or $[x]_\alpha$ is meet-irreducible then at some stage in the construction of β we introduce a pair $z^2_{k_1}, z^2_{k_2}$, and make $[z^1_i]_{\alpha_1}$ meet-reducible since we make $[z^1_i]_{\alpha_1} = [z^2_{k_1}]_{\alpha_1} \wedge [z^2_{k_2}]_{\alpha_1}$, and on the other hand $[z^2_{k_1}]_{\alpha_1}, [z^2_{k_2}]_{\alpha_1}$ are incomparable as under the isomorphism $F^d_{0,1}(X)_{/\alpha_1} \simeq F^d_{0,1}(G \smallsetminus \{y_i : i \in U \cup V\})$ they correspond to two distinct generators of $F^d_{0,1}(G \smallsetminus \{y_i : i \in U \cup V\})$. Similarly if $[x]_\alpha$ is meet-reducible then for every s with $x \in F^d_{01}(X_s)$ we have that $[x]_{\alpha_s}$ is meet-reducible, thus $[z^1_i]_{\alpha_1}$ is meet-irreducible in L_{1/α_1} as $[z^1_i]_{\alpha_1}$ corresponds (under the isomorphism) to a generator of $F^d_{0,1}(G \smallsetminus \{y_i : i \in U \cup V\})$ since z^1_i is never extracted from G. This shows that φ_i does not induce an isomorphism, since meet-reducibility is a property that is invariant under isomorphisms. □

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