Numerical irreducibility criteria for handlebody links

NUMERICAL IRREDUCIBILITY CRITERIA FOR HANDLEBODY LINKS

GIOVANNI BELLETTINI, MAURIZIO PAOLINI, AND YI-SHENG WANG

Abstract. In this paper we define a set of numerical criteria for a handlebody link to be irreducible. It provides an effective, easy-to-implement method to determine the irreducibility of handlebody links; particularly, it recognizes the irreducibility of all handlebody knots in the Ishii-Kishimoto-Moriuchi-Suzuki knot table and most handlebody links in the Bellettini-Paolini-Paolini-Wang link table.

1. Introduction

A handlebody link HL is a union of finitely many handlebodies of positive genus embedded in the 3-sphere $\mathbb{S}^3$; two handlebody links are equivalent if they are ambient isotopic [12], [3]. Throughout the paper handlebody links are non-split unless otherwise specified.

A handlebody link HL is reducible if there exists a cutting 2-sphere $\mathfrak{S}$ in $\mathbb{S}^3$ such that $\mathfrak{S}$ and HL intersect transversally at an incompressible disk D in HL (Fig. 1.1); otherwise it is irreducible. Note that a cutting sphere $\mathfrak{S}$ of a reducible handlebody

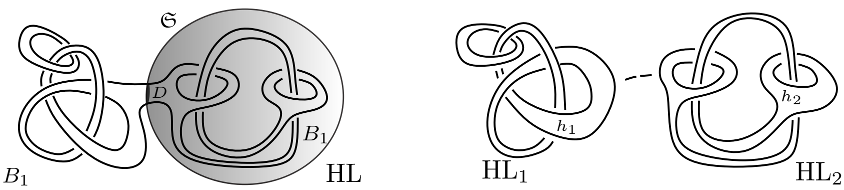


Figure 1.1. A reducible handlebody link and its factors

link HL factorizes it into two handlebody links $\mathrm{HL}_1, \mathrm{HL}_2$, where $\mathrm{HL}_i := \overline{\mathrm{HL} \cap B_i}$, and B_i, $i = 1, 2$, are the closures of components of the complement $\mathbb{S}^3 \setminus \mathfrak{S}$ (Fig. 1.1); the factorization is denoted by

$$\mathrm{HL} = (\mathrm{HL}_1, h_1)\text{--}(\mathrm{HL}_2, h_2), \tag{1.1}$$

and we call HL_i, $i = 1, 2$, a factor of the factorization, where h_1, h_2 are components of $\mathrm{HL}_1, \mathrm{HL}_2$ containing D, respectively.

Handlebody links are often studied and visualized via diagrams of their spines [3]; it is, however, not an easy task to detect the irreducibility of a handlebody link from its diagram. The complexity lies in the IH-move [3]. In fact, it is not known whether we have an affirmative answer to Conjecture 1.1 or Conjecture 1.2[1].

Conjecture 1.1. *Every reducible handlebody link admits a minimal diagram whose underlying plane graph is* 1*-edge-connected.*

2010 *Mathematics Subject Classification.* 57M25, 57M27.
Key words and phrases. reducibility, handlebody links, knot sum.

[1] Conj. 1.1 implies Conj. 1.2 in some special cases [14, Theorem 2] and [1, Theorem 6.1].

Conjecture 1.2. *The crossing number of a reducible handlebody link is the sum of crossing numbers of its factors:*

$$c\big((\mathrm{HL}_1, h_1)\text{--}(\mathrm{HL}_2, h_2)\big) = c(\mathrm{HL}_1) + c(\mathrm{HL}_2). \tag{1.2}$$

If either conjecture is true, it implies the reducible handlebody link table [1, Table 5] is complete, and thus, the irreducibility of all handlebody links in [1, Table 1] but 6_9 can be proved by simply comparing their ks_G*-invariants* [6]. The invariant $ks_G(\mathrm{HL})$ is the number of conjugacy classes of homomorphisms from the knot group G_{HL}, the fundamental group of HL's complement, to a finite group G—two homomorphisms are in the same conjugacy class if they are conjugate.

We do not purse these conjectures here but instead introduce some numerical criteria for a handlebody link to be irreducibile. Other irreducibility tests using quandle invariants have been developed by Ishii and Kishmoto [4], and are used in the classification of irreducible handlebody knots of genus 2 [5].

Main Results & Structure. A handlebody link HL is said to be of type $[n_1, n_2, ...n_m]$ if it consists of n_i handlebodies of genus i, $i = 1, \dots, m$, and a handlebody link is r-generator if its knot group is of rank r. Note that r is necessarily larger than or equal to the genus $g(\mathrm{HL})$ of HL, which is the sum $\sum_{i=1}^{m} i \cdot n_i$ of genera of components of HL. Let A_4, A_5 be alternating groups of degree $4, 5$, respectively.

Theorem 1.3 (**Necessary conditions for reducibility**–A_4)**.** *Let* HL *be a reducible handlebody link of genus* g*. If the trivial knot is a factor of some factorization of* HL*, then*

$$12 \mid \mathrm{ks}_{A_4}(\mathrm{HL}) + 6 \cdot 3^{g-1} + 2 \cdot 4^{g-1}; \tag{1.3}$$

if a 2-generator knot is a factor of some factorization of HL*, then*

$$12 + 24k \mid \mathrm{ks}_{A_4}(\mathrm{HL}) + (6 + 16k) \cdot 3^{g-1} + (2 + 6k) \cdot 4^{g-1}, k = 0 \text{ or } 1; \tag{1.4}$$

if a 2-generator link is a factor of some factorization of HL*, then*

$$48 + 24k \mid ks_{A_4}(\mathrm{HL}) + (26 + 16k) \cdot 3^{g-1} + (8 + 6k) \cdot 4^{g-1}, k = 0, 1, 2, 3 \text{ or } 4. \tag{1.5}$$

Theorem 1.4 (**Necessary conditions for reducibility**–A_5)**.** *Let* HL *be a reducible handlebody link of genus* g*. If the trivial knot is a factor of some factorization of* HL*, then*

$$60 \mid \mathrm{ks}_{A_5}(\mathrm{HL}) + 14 \cdot 4^{g-1} + 19 \cdot 3^{g-1} + 22 \cdot 5^{g-1}. \tag{1.6}$$

From these necessary conditions we derive the irreducibility test for handlebody knots of genus up to 3 and handlebody links of various types.

Corollary 1.5. *Given a* r*-generator handlebody knot* HL *of genus* g*, if* $r = g + 1$ *and* HL *fails to satisfy either* (1.3) *or* (1.6)*, then* HL *is irreducible; if* $r = g + 2$ *and* HL *fails to satisfy both* (1.3) *and* (1.4)*, then* HL *is irreducible.*

The situation with multi-component handlebody links is slightly more complicated as there are more possible combinations; thus we summarize it in a tabular format in Table 1, which is also a corollary of Theorems 1.3 and 1.4. The left two columns in Table 1 list criteria which if a handlebody link fails, it is irreducible. Be aware "& (i.e. and)" and "*or*" in those two columns.

The set of irreducibility criteria is put to test in Section 4; it detects the irreducibility of all handlebody knots, which are of type $[0, 1]$, in the Ishii-Kishimoto-Moruichi-Suzuki knot table [5] and the irreducibility of all handlebody links, which are of type $[1, 1], [2, 1]$ or $[3, 1]$, but two $(6_9, 6_{12})$, in the Bellettini-Paolini-Paolini-Wang link table [1], showing that it is highly sensitive to the irreducibility of a handlebody link.

The major constraint of the irreducibility test is that the rank of the knot group G_{HL} cannot be too large and the difference between the rank and the genus $g(\mathrm{HL})$

Table 1. Tests for irreducibility of handlebody links (more than one component)

no. of components	type	$r = g$	$r = g + 1$
		HL is irreducible if it fails criterion/criteria	
2	$[1, 1]$	(1.3) or (1.6)	(1.3) & (1.4)
	$[0, 2]$	(1.3) or (1.6)	(1.3) & (1.4)
	$[1, 0, 1]$	(1.3) or (1.6)	(1.3), (1.4) & (1.5)
	$[0, 1, 1]$	(1.3) or (1.6)	not applicable
3	$[2, 1]$	(1.3) & (1.5)	(1.3), (1.4) & (1.5)
	$[1, 2]$	(1.3) & (1.5)	not applicable
	$[2, 0, 1]$	(1.3) & (1.5)	
4	$[3, 1]$	(1.3) & (1.5)	not applicable

needs to be small; on the other hand, the criteria are easy to implement and can be computed by a code.

The paper is organized as follows: Section 2 recalls basic properties of handlebody links and knot groups. The necessary conditions for reducibility (Theorems 1.3 and 1.4) are proved in Section 3. Section 4 records results of the irreducibility test applying to various families of handlebody links. Lastly, the existence of irreducible handlebody links of any given type is proved by a concrete construction making use of a generalized knot sum for handlebody links.

2. Preliminaries

Throughout the paper we work in the piecewise linear category. We use HL to refer to general handlebody links (including handlebody knots), and use HK, K or L when referring specifically for handlebody knots, knots or links, respectively. $G_\bullet$ denotes the knot group of $\bullet = \mathrm{HL}, \mathrm{HK}, K$ or L; $\simeq$ stands for an isomorphism of groups. To begin with, we review some basic properties of reducible handlebody links and the free product of groups.

Definition 2.1. *The rank $rk(G)$ of a finitely generated group G is the smallest cardinality of a generating set of G.*

Definition 2.2. *A handlebody link is r-generator if its knot group is of rank r.*

The rank respects the free product of groups [2].

Lemma 2.1 (Grushko theorem)**.** *If $G = G_1 * G_2$, then*

$$rk(G) = rk(G_1) + rk(G_2).$$

Lemma 2.2. *A g-generator handlebody knot* HK *of genus g is trivial.*

Proof. By the exact sequence of group homology [10], the deficiency d of the knot group of HK is at most g; on the other hand, the Wirtinger presentation induces a presentation with deficiency g, so we have $d = g$. By [7, Satz 1], [11], the knot group is free, and therefore HK is trivial. □

The following are corollaries of Lemmas 2.1 and 2.2 and the fact that $\mathrm{HL} = (\mathrm{HL}_1, h_1)\text{--}(\mathrm{HL}_2, h_2)$ implies then $g(\mathrm{HL}) = g(\mathrm{HL}_1) + g(\mathrm{HL}_2)$. The corollaries, together with Theorems 1.3 and 1.4, give Corollary 1.5 and Table 1.

Corollary 2.3. *A $(g + 1)$-generator handlebody knot* HK *of genus $g = 2, 3$ is reducible if and only if the trivial knot is a factor of some factorization of* HK.

Corollary 2.4. *A 2-component, g-generator handlebody link* HL *of genus $g \leq 5$ is reducible if and only if the trivial knot is a factor of some factorization of* HL.

Corollary 2.5. *A genus g, $(g+1)$-generator handlebody link* HL *of type* $[1,1]$ *or* $[0,2]$ *is reducible if and only if the trivial knot or a* 2*-generator knot is a factor of some factorization of* HL.

Corollary 2.6. *A* 3*- or* 4*-component, g-generator handlebody link* HL *of genus* $g \leq 5$ *is reducible if and only if the trivial knot or a* 2*-generator link is a factor of some factorization of* HL.

Corollary 2.7. *A* 5*-generator handlebody link* HL *of type* $[1,0,1]$ *or* $[2,1]$ *is reducible if and only if the trivial knot,* 2*-generator knot, or* 2*-generator link is a factor of some factorization of* HL.

3. Irreducibility tests

3.1. **Homomorphisms to a finite group.**

Definition 3.1. *Given a handlebody link* HL *and a finite group G, $ks_G(\mathrm{HL})$ is the number of conjugacy classes of homomorphissm from G_{HL} to G, $ks^G_H(\mathrm{HL})$ is the number of conjugacy classes of homomorphisms from G_{HL} to a subgroup of G isomorphic to H, and $ks^w_G(\mathrm{HL})$ is the number of homomorphisms from G_{HL} to G.*

Lemma 3.1. *Suppose any subgroup of G either has trivial centralizer or is abelian, and any two maximal abelian subgroups of G have trivial intersection. Let H_i, $i = 1, \ldots, n$, be isomorphism types of maximal abelian subgroups of G, and l_i be the number of maximum abelian subgroups isomorphic to H_i. Then for any handlebody link* HL, *$ks_G(\mathrm{HL})$ can be expressed in terms of $ks^w_G(\mathrm{HL})$ and $ks^G_{H_i}(\mathrm{HL})$*

$$\begin{aligned} ks_G(\mathrm{HL}) &= ks^G_{H_1}(\mathrm{HL}) + \cdots + ks^G_{H_n}(\mathrm{HL}) - n + 1 \\ &\quad + \frac{ks^w_G(\mathrm{HL}) - l_1(ks^w_{H_1}(\mathrm{HL}) - 1) - \cdots - l_n(ks^w_{H_n}(\mathrm{HL}) - 1) - 1}{|G|}. \end{aligned} \tag{3.1}$$

Proof. The difference

$$ks_G(\mathrm{HL}) - \left(ks^G_{H_1}(\mathrm{HL}) + \cdots + ks^G_{H_n}(\mathrm{HL}) - n + 1\right) \tag{3.2}$$

is the number of conjugacy classes of homomorphisms $G_{\mathrm{HL}} \to G$ whose images have trivial centralizers. On the other hand, for such a homomorphism ϕ, we have

$$\phi \neq g \cdot \phi \cdot g^{-1},$$

for any non-trivial element $g \in G$, and hence the conjugacy class of ϕ contains $|G|$ members. Now, since the intersection of any two maximal abelian subgroups is trivial, the difference

$$ks^w_G(\mathrm{HL}) - l_1(ks^w_{H_1}(\mathrm{HL}) - 1) - \cdots - l_n(ks^w_{H_n}(\mathrm{HL}) - 1) - 1 \tag{3.3}$$

is the number of homomorphisms $G_{\mathrm{HL}} \to G$ whose images have trivial centralizers. Therefore dividing (3.3) by $|G|$ gives us (3.2), that is,

$$\begin{aligned} &\frac{ks^w_G(\mathrm{HL}) - l_1(ks^w_{H_1}(\mathrm{HL}) - 1) - \cdots - l_n(ks^w_{H_n}(\mathrm{HL}) - 1) - 1}{|G|} \\ &\qquad = ks_G(\mathrm{HL}) - \left(ks^G_{H_1}(\mathrm{HL}) + \cdots + ks^G_{H_n}(\mathrm{HL}) - n + 1\right), \end{aligned}$$

and this proves the formula (3.1). □

It is not difficult to check that A_4, A_5 satisfy conditions in Lemma 3.1, whence we derive the following formulas.

Corollary 3.2. *Let $\mathbb{Z}_n$ be the cyclic group of order n, and $V_4 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then*

$$ks_{A_4}(\mathrm{HL}) = ks^{A_4}_{V_4}(\mathrm{HL}) + ks^{A_4}_{\mathbb{Z}_3}(\mathrm{HL}) - 1 + \frac{ks^w_{A_4}(\mathrm{HL}) - 4(ks^w_{\mathbb{Z}_3}(\mathrm{HL}) - 1) - ks^w_{V_4}(\mathrm{HL})}{12} \tag{3.4}$$

$$ks_{A_5}(\mathrm{HL}) = ks^{A_5}_{V_4}(\mathrm{HL}) + ks^{A_5}_{\mathbb{Z}_3}(\mathrm{HL}) + ks^{A_5}_{\mathbb{Z}_5}(\mathrm{HL}) - 2 + \frac{ks^w_{A_5}(\mathrm{HL}) - 10(ks^w_{\mathbb{Z}_3}(\mathrm{HL}) - 1) - 5(ks^w_{V_4}(\mathrm{HL}) - 1) - 6(ks^w_{\mathbb{Z}_5}(\mathrm{HL}) - 1) - 1}{60}. \tag{3.5}$$

Given an injective homomorphism $H \xrightarrow{\iota} G$, then the number n_H of conjugacy classes of elements in G representable by elements in $\iota(H)$ is independent of ι if any two subgroups of G isomorphic to H are conjugate. If furthermore $\iota(H)$ is a maximal abelian subgroup with $\iota(H)$ being the centralizer of every element in $\iota(H)$, then $ks^w_H(\mathrm{HL}), ks^G_H(\mathrm{HL})$ can be computed explicitly.

Lemma 3.3. *Under the assumptions preceding the lemma, if $g(\mathrm{HL}) = g$, then*

$$ks^w_H(\mathrm{HL}) = |H|^g \quad \textbf{\textit{and}} \quad ks^G_H(\mathrm{HL}) = (n_H - 1) \cdot \frac{|H|^g - |H|}{|H| - 1} + n_H.$$

Proof. Firstly, since H is abelian, any homomorphism from G_{HL} to H factors through the abelianization of G_{HL}, which is the free abelian group $\mathbb{Z}^g$ of rank g. Especially, $ks^w_H(\mathrm{HL})$ (resp. $ks^G_H(\mathrm{HL})$) is equal to the numbers (resp. of conjugacy classes) of homomorphisms from $\mathbb{Z}^g$ to H. This implies the first identity.

For the second identity, we let

$$ks^G_H(\mathrm{HL}) = l_g$$

and $\mathrm{id}, h_2, \dots, h_{n_H} \in \iota(H) < G$ be selected representatives of the n_H conjugacy classes of elements in G. Note that if $g = 1$, we have $l_1 = n_H$.

For $g > 1$, up to conjugation, we may assume the g-th copy of $\mathbb{Z}^g$ is sent to $h \in \{\mathrm{id}, h_2 \dots, h_{n_H}\}$. There are l_{g-1} homomorphisms when $h = \mathrm{id}$, and $|H|^{g-1}$ homomorphisms when $h = h_i, i = 2, \dots, n_H$, because the centralizer of h_i is $\iota(H)$. As a result, we obtain the recursive formula

$$l_g = l_{g-1} + (n_H - 1) \cdot |H|^{g-1},$$

and hence

$$l_g - l_1 = \sum_{k=2}^{g}(l_k - l_{k-1}) = \sum_{k=2}^{g}(n_H - 1) \cdot |H|^{k-1} = (n_H - 1) \cdot \frac{|H|^g - |H|}{|H| - 1}. \tag{3.6}$$

This implies the second equality after we substitute $l_1 = n_H$ into (3.6). ☐

Maximal abelian subgroups of A_4, A_5 satisfy conditions assumed in Lemma 3.3, and hence we have the formulas:

$$ks^w_{\mathbb{Z}_3}(\mathrm{HL}) = 3^g; \quad ks^w_{V_4}(\mathrm{HL}) = 4^g; \quad ks^w_{\mathbb{Z}_5}(\mathrm{HL}) = 5^g, \tag{3.7}$$

$$ks^{A_4}_{\mathbb{Z}_3}(\mathrm{HL}) = 3^g; \quad ks^{A_4}_{V_4}(\mathrm{HL}) = \frac{4^g - 4}{3} + 2, \tag{3.8}$$

$$ks^{A_5}_{\mathbb{Z}_3}(\mathrm{HL}) = \frac{3^g - 3}{2} + 2; \quad ks^{A_5}_{V_4}(\mathrm{HL}) = \frac{4^g - 4}{3} + 2, \quad ks^{A_5}_{\mathbb{Z}_5}(\mathrm{HL}) = \frac{5^g - 5}{2} + 3, \tag{3.9}$$

Plugging (3.7), (3.8) into (3.4), and (3.7), (3.9) into (3.5) gives the following:

Corollary 3.4. *For a genus g handlebody link* HL, *we have*

$$ks^w_{A_4}(\mathrm{HL}) = 12ks_{A_4}(\mathrm{HL}) - 8 \cdot 3^g - 3 \cdot 4^g$$
$$ks^w_{A_5}(\mathrm{HL}) = 60ks_{A_5}(\mathrm{HL}) - 20 \cdot 3^g - 15 \cdot 4^g - 24 \cdot 5^g.$$

For the sake of convenience, we let $\mathbf{ks}_G(G')$ denote the set of conjugacy classes of homomorphisms from G' to G; especially, we have $ks_G(\mathrm{HL}) = |\mathbf{ks}_G(G_{\mathrm{HL}})|$.

Lemma 3.5. *For a 2-generator knot K, $ks_{A_4}(K) = 4$ or 6. In each case, $\mathbf{ks}_{A_4}(G_K)$ contains four conjugacy classes represented by homomorphisms whose images are abelian. If $ks_{A_4}(K) = 6$, the two additional conjugacy classes are represented by surjective homomorphisms.*

Proof. Since any non-surjective homomorphism $\phi : G_K \to A_4$ factors throught the abelianization of G_K, $\mathrm{Im}(\phi)$ is either trivial or isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. By (3.8), the number of conjugacy classes of non-surjective homomorphisms are

$$ks^{A_4}_{V_4}(K) + ks^{A_4}_{\mathbb{Z}_3}(K) - 1 = 3 + 2 - 1 = 4,$$

and hence $ks_{A_4}(K) \geq 4$.

Now, consider a two-generator presentation of G_K

$$< a, b \mid w(a, b) = 1 > \tag{3.10}$$

and its abelianization:

$$G_K \xrightarrow{\pi} G_K/[G_K, G_K] \simeq \mathbb{Z} =< g >; \tag{3.11}$$

let $g^{3n+l}, g^{3n'+l'}$ be the image of a, b under (3.11), respectively. Suppose both l and l' are non-zero, then either $3 \mid l' - l$ or $3 \mid l' - 2l$. If $3 \mid l' - l$, we replace b with b' by $b' = a^{-1}b$; this implies a new presentation of G_K:

$$G_K =< a, b' \mid w'(a, b') = 1 >,$$

where $w'(a, b') = w(a, ab')$, and b' vanishes under the composition

$$G_K \xrightarrow{\pi} G_K/[G_K, G_K] \simeq \mathbb{Z} \xrightarrow{\pm} \mathbb{Z}_3 \simeq A_4/[A_4, A_4].$$

Similarly, if $3 \mid 2l - l'$, we replace b with b'' by $b'' = a^{-2}b$ to get a new presentation

$$G_K =< a, b'' \mid w''(a, b'') = 1 >,$$

where $w''(a, b'') = w(a, a^2b'')$, and b'' vanishes under the composition

$$G_K \xrightarrow{\pi} G_K/[G_K, G_K] \simeq \mathbb{Z} \xrightarrow{\pm} \mathbb{Z}_3 \simeq A_4/[A_4, A_4].$$

Therefore, given a surjective homomorphism ϕ, we may assume $\phi(b)$ in (3.10) is in the commutator of A_4 and of order 2 and $\phi(a)$ is of order 3. Up to conjugation, there are only two such homomorphisms: one corresponds to $\phi(a) = (123)$, the other $\phi(a) = (132)$; note that every two elements of order 2 in A_4 are conjugate with respect to (123) or (132). This shows there are at most two surjective homomorphisms from G_K to A_4, and they always appear in pairs because there exists an automorphism of A_4 sending (123) to (132), namely

$$\begin{aligned} \Phi_{(23)} : A_4 &\to A_4 \\ x &\mapsto (23)x(23), \end{aligned} \tag{3.12}$$

□

Lemma 3.6. *If L is a 2-generator link, then $ks_{A_4}(L)$ is 14, 16, 18, 20 or 22. In each case, $\mathbf{ks}_{A_4}(G_L)$ contains 14 elements represented by homomorphisms whose images are abelian. If $ks_{A_4}(L) > 14$, then any additional conjugacy class is represented by surjective homomorphisms.*

Proof. Suppose $\phi : G_L \to A_4$ is non-surjective, then it factors through the abelianization of G_L, so by (3.8), the number of conjugacy classes of non-surjective homomorphism can be computed by

$$ks^{A_4}_{V_4}(K) + ks^{A_4}_{\mathbb{Z}_3}(K) - 1 = 9 + 6 - 1 = 14,$$

and particularly, $ks_{A_4}(L) \geq 14$.

Suppose $\phi : G_L \to A_4$ is onto, and

$$< a, b \mid w(a, b) = 1 >$$

is a presentation of G_L. Then either both $\phi(a)$ and $\phi(b)$ are of order 3 or one of them is of order 3 and the other order 2. In the former case, up to conjugation, there are four possibilities:

$$\begin{aligned}
\mathrm{I} : \phi(a) &= (123), & \phi(b) &= (124);\\
\mathrm{II} : \phi(a) &= (123), & \phi(b) &= (142);\\
\mathrm{III} : \phi(a) &= (132), & \phi(b) &= (124);\\
\mathrm{IV} : \phi(a) &= (132), & \phi(b) &= (142).
\end{aligned}$$

By (3.12) $w(\phi(a), 124) = 1$ if and only if $w(\Phi_{(23)}\big(\phi(a)\big), (142)) = 1$ since

$$w(\Phi_{(23)}(\phi(a)), (124)) = \Phi_{(23)}\big(w(\phi(a), (134))\big) = \Phi_{(23)}\Big((123)w\big(\phi(a), (142)\big)(132)\Big).$$

Therefore, I and IV appear in pair; so do II and IV, for a similar reason. Now, if one of $\phi(a)$ and $\phi(b)$ is of order 2, we also have four possibilities:

$$\begin{aligned}
\mathrm{I}' : \phi(a) &= (123), & \phi(b) &= (12)(34);\\
\mathrm{II}' : \phi(a) &= (132), & \phi(b) &= (12)(34);\\
\mathrm{III}' : \phi(a) &= (12)(34), & \phi(b) &= (123);\\
\mathrm{IV}' : \phi(a) &= (12)(34), & \phi(b) &= (132).
\end{aligned}$$

They appear in pairs as in the previous case. Thus, $ks_{A_4}(L)$ is an even integer between 14 and 22. □

3.2. **Necessary conditions for reducibility.** We divide the proof of Theorems 1.3 and 1.4 into three lemmas.

Lemma 3.7. *Given a reducible handlebody link* HL *of genus g, if the trivial knot is a factor of some factorization of* HL*, then*

$$12 \mid ks_{A_4}(\mathrm{HL}) + 6\cdot 3^{g-1} + 2\cdot 4^{g-1} \quad \textbf{\textit{and}} \quad 60 \mid ks_{A_5}(\mathrm{HL}) + 14\cdot 4^{g-1} + 19\cdot 3^{g-1} + 22\cdot 5^{g-1}.$$

Proof. By the assumption, the knot group G_{HL} is isomorphic to the free product $\mathbb{Z} * G_{\mathrm{HL}'}$, where HL' is a handlebody link of genus $g - 1$.

Recall that $\mathbf{ks}_{A_4}(\mathbb{Z})$ contains four elements by (3.8); let $\phi_1, \phi_2, \phi_3^1, \phi_3^2$ be homomorphism representing these four conjugacy classes with $\mathrm{Im}(\phi_1)$ trivial, $\mathrm{Im}(\phi_2)$ isomorphic to $\mathbb{Z}_2$, and $\mathrm{Im}(\phi_3^i), i = 1, 2$ isomorphic to $\mathbb{Z}_3$. Then observe that, given a homomorphism $\phi : G_{\mathrm{HL}} \to A_4$; by conjugating with some elements in A_4, we may assume its restriction $\phi|_{\mathbb{Z}}$ is one of

$$\{\phi_1, \phi_2, \phi_3^1, \phi_3^2\}.$$

Case 1: $\phi|_{\mathbb{Z}} = \phi_1$**.** Let $\phi, \psi : G_{\mathrm{HL}} \to A_4$ be two homomorphisms with

$$\phi|_{\mathbb{Z}} = \psi|_{\mathbb{Z}} = \phi_1.$$

Then they are in the same conjugacy class if and only if their restrictions $\phi|_{G_{\mathrm{HL}'}}, \psi|_{G_{\mathrm{HL}'}}$ are conjugate, so there are $ks_{A_4}(\mathrm{HL}')$ conjugacy classes in Case 1.

Case 2: $\phi|_{\mathbb{Z}_2} = \phi_2$**.** Let $\phi, \psi : G_{\mathrm{HL}} \to A_4$ be two homomorphisms with

$$\phi|_{\mathbb{Z}} = \psi|_{\mathbb{Z}} = \phi_2.$$

Then they are in the same conjugacy class if and only if

$$\phi|_{G_{\mathrm{HL}'}} = g \cdot \psi|_{G_{\mathrm{HL}'}} \cdot g^{-1}, \text{ \textbf{for some} } g \in V_4.$$

Hence in case 2, the number of conjugacy classes is

$$\frac{ks^w_{A_4}(\mathrm{HL}') - ks^w_{V_4}(\mathrm{HL}')}{4} + ks^w_{V_4}(\mathrm{HL}').$$

Case 3: $\phi|_{\mathbb{Z}} = \phi_3^i$**,** $i = 1$ **or** 2**.** Let $\phi, \psi : G_{\mathrm{HL}} \to A_4$ be two homomorphisms with

$$\phi|_{\mathbb{Z}} = \psi|_{\mathbb{Z}} = \phi_3^i, i = 1 \textbf{(resp. } 2\textbf{)}.$$

Then they are in the same conjugacy class if and only if

$$\phi|_{G_{\mathrm{HL}'}} = g \cdot \psi|_{G_{\mathrm{HL}'}} \cdot g^{-1}, \textbf{ for some } g \in \mathrm{Im}(\phi_3^i), i = 1\textbf{(resp. } 2\textbf{)},$$

and therefore for each i, there are

$$\frac{ks^w_{A_4}(\mathrm{HL}') - ks^w_{\mathbb{Z}_3}(\mathrm{HL}')}{3} + ks^w_{\mathbb{Z}_3}(\mathrm{HL}')$$

conjugacy classes.

Summing the three cases up gives the formula of $ks_{A_4}(\mathrm{HL})$ in terms of the ks-invariants of HL':

$$ks_{A_4}(\mathrm{HL}) = ks_{A_4}(\mathrm{HL}') + \frac{ks^w_{A_4}(\mathrm{HL}') - ks^w_{V_4}(\mathrm{HL}')}{4} + ks^w_{V_4}(\mathrm{HL}') + 2 \cdot \left(\frac{ks^w_{A_4}(\mathrm{HL}') - ks^w_{\mathbb{Z}_3}(\mathrm{HL}')}{3} + ks^w_{\mathbb{Z}_3}(\mathrm{HL}') \right). \tag{3.13}$$

Combining (3.13) with (3.7) and Corollary 3.4, we get the equation

$$ks_{A_4}(\mathrm{HL}) = 12 \cdot ks_{A_4}(\mathrm{HL}') - 6 \cdot 3^{g-1} - 2 \cdot 4^{g-1},$$

which implies the first assertion.

$ks_{A_5}(\mathrm{HL})$ can be computed in a similar manner. First note that $\mathbf{ks}_G(\mathbb{Z})$ contains five elements by (3.9), and they are represented by homomorphisms

$$\phi_1, \phi_2, \phi_3, \phi_5^1, \phi_5^2, \tag{3.14}$$

with $\mathrm{Im}(\phi_1)$ trivial, $\mathrm{Im}(\phi_2)$ isomorphic to $\mathbb{Z}_2$, $\mathrm{Im}(\phi_3)$ isomorphic to $\mathbb{Z}_3$, and $\mathrm{Im}(\phi_5^i), i = 1, 2$, isomorphic to $\mathbb{Z}_5$. As with the case of A_4, given a homomorphism $\phi : G_{\mathrm{HL}} \to A_5$, by conjugating with some element in A_5, we may assume its restriction on $\mathbb{Z}$ is one of the representing homomorphisms in (3.14). The number of conjugacy classes of homomorphisms that restrict to ϕ_1 is $ks_{A_5}(L)$ and the number of conjugacy classes of homomorphisms that restrict to ϕ_2, ϕ_3, or $\phi_5^i, i = 1, 2$, is

$$\frac{ks^w_{A_5}(\mathrm{HL}') - ks^w_{V_4}(\mathrm{HL}')}{4} + ks^w_{V_4}(\mathrm{HL}'),$$
$$\frac{ks^w_{A_5}(\mathrm{HL}') - ks^w_{\mathbb{Z}_3}(\mathrm{HL}')}{3} + ks^w_{\mathbb{Z}_3}(\mathrm{HL}'),$$
$$\textbf{or } \frac{ks^w_{A_5}(\mathrm{HL}') - ks^w_{\mathbb{Z}_5}(\mathrm{HL}')}{5} + ks^w_{\mathbb{Z}_5}(\mathrm{HL}'),$$

respectively, and summing them up givues the formula of $ks_{A_5}(\mathrm{HL})$:

$$ks_{A_5}(\mathrm{HL}) = ks_{A_5}(\mathrm{HL}') + \frac{ks^w_{A_5}(\mathrm{HL}') - ks^w_{V_4}(\mathrm{HL}')}{4} + ks^w_{V_4}(\mathrm{HL}') + \frac{ks^w_{A_5}(\mathrm{HL}') - ks^w_{\mathbb{Z}_3}(\mathrm{HL}')}{3} + ks^w_{\mathbb{Z}_3}(\mathrm{HL}') + 2 \cdot \left(\frac{ks^w_{A_5}(\mathrm{HL}') - ks^w_{\mathbb{Z}_5}(\mathrm{HL}')}{5} + ks^w_{\mathbb{Z}_5}(\mathrm{HL}') \right). \tag{3.15}$$

The formula (3.15), together with (3.7) and Corollary 3.4, implies the identity:

$$ks_{A_5}(\mathrm{HL}) = 60 \cdot ks_{A_5}(\mathrm{HL}') - 19 \cdot 3^{g-1} - 14 \cdot 4^{g-1} - 22 \cdot 5^{g-1},$$

and thus the second assertion. □

Lemma 3.8. *Given a reducible handlebody link* HL *of genus* g*, if a* 2*-generator knot* K *is a factor of some factorization of* HL*, then*

$$12 + 24k \mid ks_{A_4}(\mathrm{HL}) + (6 + 16k) \cdot 3^{g-1} + (2 + 6k) \cdot 4^{g-1},$$

where $k = 0$ *or* 1.

Proof. By the assumption the knot group G_{HL} is isomorphic to the free product $G_K * G_{\mathrm{HL}'}$, where HL' is a handlebody link of genus $g-1$. By Lemma 3.5, $\mathbf{ks}_{A_4}(G_K)$ might have two more elements than $\mathbf{ks}_{A_4}(\mathbb{Z})$. Let ϕ_s^1, ϕ_s^2 be representing surjective homomorphisms of these two conjugacy classes. Then, since two homomorphisms

$$\phi, \psi : G_{\mathrm{HL}} \to A_4 \quad \textbf{with} \quad \phi|_{G_K} = \psi|_{G_K} = \phi_s^i, \quad i = 1 \textbf{ or } 2 \tag{3.16}$$

are conjugate if and only if

$$\phi|_{G'_{\mathrm{HL}}} = \psi|_{G_{\mathrm{HL}'}}.$$

there are $ks^w_{A_4}(\mathrm{HL}')$ conjugacy classes of homomorphisms with the property (3.16). Adding this to (3.13), we obtain

$$\begin{aligned} ks_{A_4}(\mathrm{HL}) = ks_{A_4}(\mathrm{HL}') &+ \frac{ks^w_{A_4}(L) - ks^w_{V_4}(\mathrm{HL}')}{4} + ks^w_{V_4}(\mathrm{HL}') \\ &+ 2 \cdot \left(\frac{ks^w_{A_4}(\mathrm{HL}') - ks^w_{\mathbb{Z}_3}(\mathrm{HL}')}{3} + ks^w_{\mathbb{Z}_3}(\mathrm{HL}') \right) + 2k \cdot ks^w_{A_4}(\mathrm{HL}'), \end{aligned} \tag{3.17}$$

where $k = 0$ or 1. Plugging (3.7) and Corollary 3.4 into (3.17) implies the identity:

$$ks_{A_4}(\mathrm{HL}) = (12 + 24k) \cdot ks_{A_4}(\mathrm{HL}') - (6 + 16k) \cdot 3^{g-1} - (2 + 6k) \cdot 4^{g-1}, k = 0 \textbf{ or } 1,$$

and therefore the assertion. □

Lemma 3.9. *Given a reducible handlebody link* HL *of genus* g*, if a* 2*-generator link* L *is a factor of some factorization of* HL*, then*

$$48 + 24k \mid ks_{A_4}(\mathrm{HL}) + (26 + 16k) \cdot 3^{g-2} + (8 + 6k) \cdot 4^{g-2},$$

where $k = 0, 1, 2, 3,$ *or* 4.

Proof. By the assumption, the knot group G_{HL} is isomorphic to the free product $G_L * G_{\mathrm{HL}'}$, where HL' is a handlebody link of genus $g-2$. By Lemma 3.6, $\mathbf{ks}_{A_4}(G_L)$ contains $14 + 2k$ elements, $k = 0, 1, 2, 3,$ or 4, where one conjugacy class for the trivial homomorphism, five for non-trivial homomorphisms whose images are in V_4, eight for homomorphisms whose images isomorphic to $\mathbb{Z}_3$, and $2k$ for surjective homomorphisms. The same argument as in the proof of Lemmas 3.7 and 3.8 gives

$$\begin{aligned} ks_{A_4}(\mathrm{HL}) = ks_{A_4}(\mathrm{HL}') &+ 5 \cdot \left(\frac{ks^w_{A_4}(\mathrm{HL}') - ks^w_{V_4}(\mathrm{HL}')}{4} + ks^w_{V_4}(\mathrm{HL}') \right) \\ &+ 8 \cdot \left(\frac{ks^w_{A_4}(\mathrm{HL}') - ks^w_{\mathbb{Z}_3}(\mathrm{HL}')}{3} + ks^w_{\mathbb{Z}_3}(\mathrm{HL}') \right) + 2k \cdot ks^w_{A_4}(\mathrm{HL}'), \end{aligned} \tag{3.18}$$

where $k = 0, 1, 2, 3,$ or 4. Plugging (3.7) and Corollary 3.4 into (3.18), we obtain

$$ks_{A_4}(\mathrm{HL}) = (48 + 24k) \cdot ks_{A_4}(\mathrm{HL}') - (26 + 16k) \cdot 3^{g-2} - (8 + 6k) \cdot 4^{g-2}$$

and hence the lemma. □

4. Examples

4.1. **Applications to handlebody knot/link tables.** Irreducibility of handlebody knots in [5] and handlebody links in [1] are examined here with the irreducibility criteria (Corollary 1.5 and Table 1). The ks_{A_4}-and ks_{A_5}-invariants of handlebody links are computed by the Appcontour [9]; the same software is also used to find an upper bound of the rank of each knot group. In many cases, the upper bound is identical to the rank.

Table 2. Irreducibility of Ishii, Kishimoto, Moriuchi and Suzuki's handlebody knots

handlebody knot	rank	ks_{A_4}	A_4-criterion (1.3)	ks_{A_5}	A_5-criterion (1.6)
HK 4_1	3	30	✓	156	✓
HK 5_1	3	22	?	111	✓
HK 5_2	3	30	✓	156	✓
HK 5_3	3	30	✓	105	✓
HK 5_4	3	22	?	365	✓
HK 6_1	3	30	✓	143	✓
HK 6_2	3	30	✓	105	✓
HK 6_3	3	22	?	83	✓
HK 6_4	3	22	?	111	✓
HK 6_5	3	22	?	97	✓
HK 6_6	3	22	?	97	✓
HK 6_7	3	30	✓	157	✓
HK 6_8	3	22	?	105	✓
HK 6_9	3	30	✓	146	✓
HK 6_{10}	3	22	?	195	✓
HK 6_{11}	3	22	?	73	✓
HK 6_{12}	3	30	✓	135	✓
HK 6_{13}	3	30	✓	156	✓
HK 6_{14}	3	46	?	353	✓
HK 6_{15}	3	46	?	353	✓
HK 6_{16}	3	22	?	267	✓

The results of the irreducibility test are recorded in Tables 2 and 3, where the check mark ✓ stands for the corresponding condition(s) not satisfied, and hence the handlebody link is irreducibile, and the question mark means the opposite, so its irreducibility is inconclusive. To avoid confusion, HK is added to the name of each handlebody knot in [5]; so is HL to the name of each handlebody link in [1].

Since all handlebody knots in [5] are 3-generator, by Corollary 1.5, if either 12 does not divide $ks_{A_4}(\mathrm{HK}) + 26$, or 60 does not divide $ks_{A_5}(\mathrm{HK}) + 223$, HK is irreducible. On the contrary, in Table 3 different criteria are required to test each case, depending on the rank and the number of component (the column "comp.") based on Table 1. For instance, for a 3-generator handlbody link of type $[1, 1]$, such as HL 4_1, if it fails either of (1.3) and (1.6), it is irreducible. But, for HL 5_1, which is possibly 4-generator, we need to have *both* (1.3) and (1.4) failed in order to draw a conclusion; also, the A_5 criterion is not applicable in this case.

4.2. **Irreducible handlebody links of a given type.** Here we present a construction of irreducible handlebody link of any given type. First we introduce the notion of $\mathcal{D}$-irreducibility for handlebody-link-disk pairs.

TABLE 3. Irreducibility of handlebody links in [1]

comp.	handlebody link	rank	ks_{A_4}	A_4-criterion	ks_{A_5}	A_5-criterion
2	HL 4_1	3	114	✓	600	✓
	HL 5_1	≤ 4	98	✓	660	not applicable
	HL 6_1	3	90	✓	600	✓
	HL 6_2	3	106	?	689	✓
	HL 6_3	3	90	✓	469	✓
	HL 6_4	3	106	?	689	✓
	HL 6_5	≤ 4	210	✓	4020	not applicable
	HL 6_6	3	130	?	1380	✓
	HL 6_7	≤ 4	98	✓	597	not applicable
	HL 6_8	3	114	✓	1401	✓
3	HL 6_9	4	310	?	1841	not applicable
	HL 6_{10}	4	326	✓	2636	not applicable
	HL 6_{11}	4	486	✓	5876	not applicable
	HL 6_{12}	4	502	?	5883	not applicable
	HL 6_{13}	4	822	✓	19308	not applicable
	HL 6_{14}	4	486	✓	5876	not applicable
4	HL 6_{15}	5	1242	✓	12072	not applicable

Definition 4.1 (**$\mathcal{D}$-irreducibility**). *A handlebody link* HL *is $\mathcal{D}$-irreducible if either its complement $\overline{\mathbb{S}^3 \setminus \mathrm{HL}}$ admits no incompressible disks or it is a trivial knot. A handlebody-link-disk pair* (HL, D) *is a handlebody link* HL *together with an incompressible disk $D \subset \mathrm{HL}$. The pair (HL, D) is $\mathcal{D}$-irreducible if there exists no incompressible disk D' in the complement $\overline{\mathbb{S}^3 \setminus \mathrm{HL}}$ with $D' \cap D = \emptyset$. An unknot with a meridian disk is the trivial $\mathcal{D}$-irreducible handlebody-link-disk pair.*

$\mathcal{D}$-irreducibility is equivalent to irreducibility for genus $g \leq 2$ handlebody knots [15] but stronger in general [13, Examples 5.5-6], [1, Remark 3.3]. Any $\mathcal{D}$-irreducible handlebody link with an incompressible disk is a $\mathcal{D}$-irreducible pair. On the other hand, the underlying handlebody link of a $\mathcal{D}$-irreducible handlebody-link-disk pair could be trivial (left handlebody-knot-disk pair in Fig. 4.2a).

Definition 4.2 (**Knot sum**). *The knot sum of two handlebody-link-disk pairs* $(\mathrm{HL}_1, D_1), (\mathrm{HL}_2, D_2)$ *is a handlebody link* $(\mathrm{HL}_1, D_1)\#(\mathrm{HL}_2, D_2)$ *obtained by gluing* $\mathrm{HL}_1, \mathrm{HL}_2$ *together as follows: first remove a 3-ball B_1 (resp. B_2) with $\mathring{B}_1 \cap \mathrm{HL}_1$ (resp. $\mathring{B}_2 \cap \mathrm{HL}_2$) a tubular neighborhood $N(D_1)$ of D_1 (resp. $N(D_2)$ of D_2) from $\mathbb{S}^3$, where $\overline{N(D_1)}$ (resp. $\overline{N(D_2)}$) can be identified with the oriented 3-manifold $D_1 \times [0,1]$ (resp. $D_2 \times [0,1]$) using the given orientation on D_1 (resp. D_2). Then the knot sum is given by gluing resultant 3-manifolds $\overline{\mathbb{S}^3 \setminus B_1}, \overline{\mathbb{S}^3 \setminus B_2}$ via an orientation-reversing homeomorphism $f : \partial B_1 \to \partial B_2$ with $f(D_1 \times \{i\}) = D_2 \times \{j\}$, $i - j \equiv 1$ mod 2.*

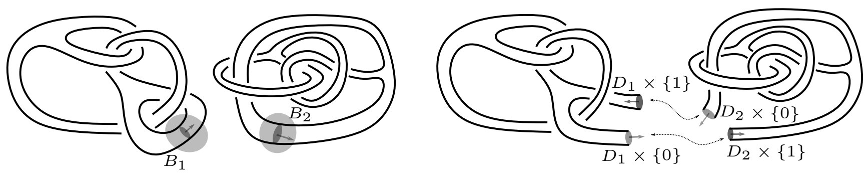


FIGURE 4.1. Knot sum of HK 4_1 and HK 5_1 with meridian disks

The knot sum resembles the order-2 connected sum of spatial graphs [8].

Theorem 4.1. *The knot sum of two non-trivial $\mathcal{D}$-irreducible handlebody-link-disk pairs* $(\mathrm{HL}_1, D_1), (\mathrm{HL}_2, D_2)$ *is $\mathcal{D}$-irreducible.*

Proof. We prove by contradiction. Suppose the knot sum

$$\mathrm{HL} \simeq (\mathrm{HL}_1, D_1)\#(\mathrm{HL}_2, D_2)$$

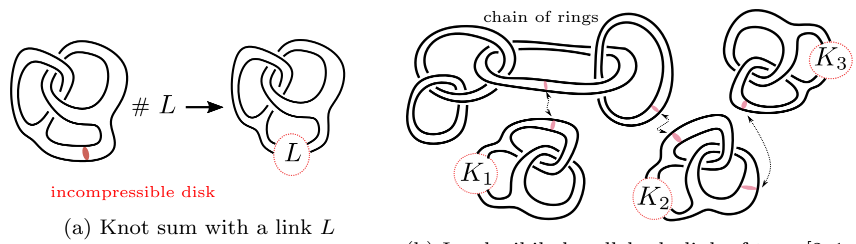
is not $\mathcal{D}$-irreducible, and D is an incompressible disk in $\overline{\mathbb{S}^3 \setminus \mathrm{HL}}$.

Let B be the 3-ball such that $B \cap \overline{\mathbb{S}^3 \setminus \mathrm{HL}}$ is the complement of HL_2, and denote the intersection annulus $\overline{\mathbb{S}^3 \setminus \mathrm{HL}} \cap \partial B$ by A. Isotopy D such that the number of components of $A \cap D$ is minimized.

Claim: $A \cap D = \emptyset$. Suppose the intersection is non-empty, then we can choose a component α of $A \cap D$ that is innermost in D. α must be an arc, for otherwise it would contradict either the $\mathcal{D}$-irreducibility of (HL_i, D_i) or the minimality. α cuts D into two disks, one of which, say D', has no intersection with A. Without loss of generality, we may assume D' is in $\overline{\mathbb{S}^3 \setminus B}$.

If α is essential in A, then HL_1 is equivalent to the union of a tubular neighborhood of α in B and $\overline{\mathbb{S}^3 \setminus B} \cap \mathrm{HL}$ in $\mathbb{S}^3$. Since $D' \cap \partial D$ is an arc connecting two sides of D_1 in HL_1, D_1 is not separating and therefore a meridian disk of HL_1. In addition, D' and ∂D_1 intersect at only one point, so (HL_1, D_1) is either trivial or not $\mathcal{D}$-irreducible, contradicting the assumption.

If α is inessential in A, let D'' be the disk cut off from A by α. Then $D' \cup D''$ is a compressing disk in HL_1. If $\partial(D' \cup D'')$ is inessential in $\partial\,\mathrm{HL}_1$, the intersection α can be removed—with other intersection arcs intact—by isotopying A. On the other hand, the $\mathcal{D}$-irreducibility of (HL_1, D_1) forces $\partial(D' \cup D'')$ to be inessential in $\partial\,\mathrm{HL}_1$. Thus, we have proved the claim, from which the theorem follows readily. □



(a) Knot sum with a link L

(b) Irreducibile handlebody link of type $[2, 1, 1]$

FIGURE 4.2

In Fig. 4.2, K_1, K_2, K_3, L are knots or links; if L in Fig. 4.2a is the composition of two Hopf links, the resulting knot sum is $\mathrm{HL}\,6_{12}$. Hence its irreducibility, which cannot be seen by our irreducibility test, follows from Theorem 4.1. The following corollary generalizes Suzuki's example [13, Theorem 5.2].

Corollary 4.2. *Given m non-negative integers $n_1, n_2, \ldots, n_m$ with $n := \sum n_i > 0$, there is an irreducible handlebody link of type $[n_1, n_2, \ldots, n_m]$.*

Proof. Consider a chain of rings with n-component—a knot sum of $n-1$ Hopf links (Fig. 4.2b). Label each ring with a number in $\{1, 2, \ldots, n\}$, and for the ring with label k,

$$\sum_{i=1}^{l-1} n_i < k \leq \sum_{i=1}^{l} n_i,$$

we consider its knot sum with an irreducible handlebody knot of genus l, which can be obtained by performing the knot sum operation iteratively on handlebody knots in [5] with meridian disks (Fig. 4.1). The resultant handlebody link is necessarily irreducible by Theorem 4.1 and of the prescribed type. □

Acknowledgements

The paper is benefited from the support of National Center for Theoretical Sciences.

References

[1] G. Bellettini, G. Paolini, M. Paolini, Y.-S. Wang: *Complete classification of* $(n, 1)$*-handlebody links up to six crossings*, to appear.
[2] I. A. Grushko: *On the bases of a free product of groups*, Matematicheskii Sbornik, **8** (1940), 169–182.
[3] A. Ishii: *Moves and invariants for knotted handlebodies*, Algebr. Geom. Topol. **8**, No. 3 (2008), 1403–1418.
[4] A. Ishii, K. Kishimoto: *The quandle coloring invariant of a reducible handlebody-knot*, Tsukuba J. Math. **35** (2011), 131–141.
[5] A. Ishii, K. Kishimoto, H. Moriuchi, M. Suzuki: *A table of genus two handlebody-knots up to six crossings*, J. Knot Theory Ramifications **21** (2012), 1250035.
[6] T. Kitano, M. Suzuki: *On the number of* $SL(2, \mathbb{Z}/p\mathbb{Z})$*-representations of knot groups* J. Knot Theory Ramifications **21** (2012), 1250035.
[7] W. Magnus: *Über freie Faktorgruppen und freie Untergruppen gegebener Gruppen*, Monatsh. Math. **47** (1939), 307–313.
[8] H. Moriuchi: *An enumeration of theta-curves with up to seven crossings* J. Knot Theory Ramifications **18** (2) (2009) 67–197.
[9] M. Paolini: Appcontour. Computer software. Vers. 2.5.3. Apparent contour. (2018) <`http://appcontour.sourceforge.net/`>.
[10] J. Stallings: *Homology and central series of groups*, J. Algebra **2** (1965), 170–181.
[11] U. Stammbach: *Ein neuer Beweis eines Satzes von Magnus*, Proc. Camb. Phil. Soc. **63** (1967), 929–930.
[12] S. Suzuki: *On linear graphs in 3sphere*, Osaka J. Math. **7** (1970), 375–396.
[13] S. Suzuki: *On surfaces in 3-sphere: prime decompositions*, Hokkaido Math. J. **4** (1975), 179–195.
[14] Y. Tsukui: *On surfaces in 3-space*, Yokohama Math. J. **18** (1970), 93–104.
[15] Y. Tsukui: *On a prime surface of genus* 2 *and homeomorphic splitting of* 3*-sphere*, The Yokohama Math. J. **23** (1975), 63–75.

Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, 53100 Siena, Italy, and International Centre for Theoretical Physics ICTP, Mathematics Section, 34151 Trieste, Italy
E-mail address: `bellettini@diism.unisi.it`

Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore, 25121 Brescia, Italy
E-mail address: `maurizio.paolini@unicatt.it`

National Center for Theoretical Sciences, Mathematics Division, Taipei 106, Taiwan
E-mail address: `yisheng@ncts.ntu.edu.tw`