The Tempered Discrete Linnik distribution

Statistical Methods and Applications manuscript No.
(will be inserted by the editor)

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Received: date / Accepted: date

Abstract We introduce a new family of integer-valued distributions by considering a tempered version of the Discrete Linnik law. The proposal is actually a generalization of the well-known Poisson-Tweedie law. The suggested family is extremely flexible since it contains a wide spectrum of distributions ranging from light-tailed laws (such as the Binomial) to heavy-tailed laws (such as the Discrete Linnik). The main theoretical features of the Tempered Discrete Linnik distribution are explored by providing a series of identities in law, which describe its genesis in terms of mixture Poisson distribution and compound Negative Binomial distribution - as well as in terms of mixture Poisson-Tweedie distribution. Moreover, we give a manageable expression and a suitable recursive relationship for the corresponding probability function. Finally, an application to scientometric data - which deals with the scientific output of the researchers of the University of Siena - is considered.

Keywords Lévy-Khintchine representation · Positive Stable distribution · Linnik distribution · Discrete Stable distribution · Discrete Linnik distribution · Mixture Poisson distribution · Hirsh index.

1 Introduction

In recent years, heavy-tailed models - *in primis*, stable distributions - have been used in a variety of fields, such as statistical physics, mathematical finance and financial econometrics (see *e.g.*. Rachev et al., 2011, and references therein). However, these models may be partially appropriate to provide a good fit to data, since their tails are too "fat" to describe empirical distributions, as remarked by Klebanov and Slámová (2015). In order to overcome this drawback, the so-called "tempered" versions of heavy-tailed distributions

have been successfully introduced (see *e.g.*. Rosínski, 2007). Indeed, tempering allows for models that are similar to original distributions in some central region, even if possess lighter - i.e. tempered - tails. Klebanov and Slámová (2015) have suitably discussed these issues and have suggested various tempering techniques.

In the framework of integer-valued distributions, the Discrete Stable is a well-known heavy-tailed law originally suggested by Steutel and van Harn (1979). Its tempered counterpart is the so-called Poisson-Tweedie law independently introduced by Gerber (1991) and Hougaard et al. (1997). The Poisson-Tweedie distribution encompasses classical families - such as the Poisson - as well as large families - such as the Generalized Poisson Inverse Gaussian and the Poisson-Pascal. Indeed, this law has been adopted for modelling integer-valued data arising in a plethora of frameworks - for example, clinical experiments (Hougaard et al., 1997), environmental studies (El-Shaarawi et al., 2011) and scientometric analysis (Baccini et al., 2016, Zhu and Joe, 2009). Hence, tempering yields a versatile statistical model which is extensively utilized - in contrast to the Discrete Stable distribution which is scarcely considered in practical applications.

Christoph and Schreiber (1998) emphasize that the Discrete Stable law may be seen as the special case - for the limiting value of a parameter - of the so-called Discrete Linnik law introduced by Devroye (1993) and Pakes (1995). Hence, owing to the extra parameter, the Discrete Linnik is a heavy-tailed distribution family which is more flexible than the Discrete Stable. Therefore, it is worth proposing a tempered version of the Discrete Linnik law in order to generalize the Poisson-Tweedie family. However, Klebanov and Slámová (2015) have remarked that tempering is not univocally characterized. Hence, with the aim of obtaining a suitably-defined Tempered Discrete Linnik law, we consider the suggestion - outlined by Barabesi and Pratelli (2014a) - for obtaining integer-valued families of mixture Poisson distributions. A succinct description of the pattern adopted for introducing the new law is as follows. First, we remark that the Discrete Stable may be seen as a mixture Poisson distribution with a mixturing Positive Stable distribution, as well as the Poisson-Tweedie may be expressed as a mixture Poisson distribution with a mixturing Tempered Positive Stable distribution. Subsequently, by generalizing these results and by noticing that the Discrete Linnik may be seen as a "scale" mixture of Discrete Stable distributions with a mixturing Gamma distribution, we similarly introduce the Tempered Discrete Linnik as a "scale" mixture of Poisson-Tweedie distributions. The achieved law is actually a tempered distribution with a tempering of geometric nature - analogously to the Poisson-Tweedie law.

The Tempered Discrete Linnik distribution has an intrinsic interest, since it encompasses - or provides valuable generalization - of many well-known laws. As a matter of fact, different parameter choices give rise to classical distributions such as the Binomial or the Negative Binomial laws, or provide extended versions of Generalized Poisson Inverse Gaussian and the Poisson-Pascal laws. In any case, owing to its flexibility, the Tempered Discrete Linnik law is also

potentially appealing for practical applications as a statistical model, since it displays several advantages with respect to the Poisson-Tweedie law. Indeed, we show that the Tempered Discrete Linnik law may cover an extended range of dispersion and kurtosis. Moreover, the Tempered Discrete Linnik distribution may fit rather involved integer-valued data, which show challenging features such an excess of zeroes and a long tail at the same time - in contrast to the Poisson-Tweedie law. As an example, these features may be interesting for modelling scientometric and bibliometric data, which frequently display zero inflation and long tails owing to the nature of the scientific production process. The authors have been precisely motivated to introduce the Tempered Discrete Linnik distribution in order to model such a kind of data - for which the Poisson-Tweedie distribution produces an unsatisfactory fitting in several cases.

The paper is organized as follows. In Section 2 we revise the issues suggested by Barabesi and Pratelli (2014a) for introducing families of integer-valued distributions and we survey the main features of the Discete Stable, the Poisson-Tweedie and the Discrete Linnik distributions. Section 3 contains our proposal for the Tempered Discrete Linnik distribution, while in Section 4 we consider its main properties. An example with real data dealing with a scientometric application is considered in Section 5. Finally, some conclusions are drawn in Section 6.

2 Some families of mixture Poisson distributions

As to a repeatedly-adopted notation in the present paper, if X represents a random variable (r.v.), the corresponding Laplace transform is given by $L_X(s) = \mathrm{E}[\exp(-sX)]$. In addition, if X is an integer-valued r.v., the corresponding probability generating function (p.g.f.) is given by $g_X(s) = \mathrm{E}[s^X]$. We also describe the suggestion by Barabesi and Pratelli (2014a) for devising integer-valued distribution families as mixtures Poisson laws in the next Sections. Let ν be a measure on $\mathbb{R}^+ =]0, \infty[$ in such a way that $\int_{\mathbb{R}^+} \min(1, x)\nu(\mathrm{d}x) < \infty$. From the Lévy-Khintchine representation (see *e.g.* Sato 1999, p.197) there exists a positive r.v. Y with Laplace transform given by

$$L_Y(t) = \exp(-\eta\psi(t)), \ \mathrm{Re}(t) > 0,$$

where $\eta \in \mathbb{R}^+$ and

$$\psi(t) = \int_{\mathbb{R}^+} (1 - \exp(-tx))\nu(dx).$$

Moreover, let $X_P := X_P(\lambda)$ represent a Poisson r.v. with parameter λ, *i.e.* the p.g.f. of $X_P(\lambda)$ is given by $g_{X_P}(s) = \exp(-\lambda(1 - s))$ with $s \in [0, 1]$. Hence, if the r.v.'s X_P and Y are independent, the Mixture Poisson r.v.

$$X_{MP} \stackrel{\mathcal{L}}{=} X_P(Y)$$

displays the p.g.f. given by

$$g_{X_{MP}}(s) = \mathrm{E}[\exp(-Y(1-s))] = L_Y(1-s) = \exp(-\eta\psi(1-s)),\ s \in [0,1]. \quad (1)$$

Hence, by suitably selecting the measure ν - and consequently the r.v. Y - a family of mixture Poisson distributions may be obtained from (1).

2.1 The Discrete Stable law and related distributions

First, we introduce the absolutely-continuous Positive Stable r.v., say X_{PS}, with Laplace transform given by

$$L_{X_{PS}}(t) = \exp(-\lambda t^{\gamma}),\ \mathrm{Re}(t) > 0, \quad (2)$$

where $(\gamma, \lambda) \in]0,1] \times \mathbb{R}^+$ (see *e.g.* Zolotarev 1986, p.114). For a discussion of the laws connected to the Positive Stable law, see Devroye and James (2014), Favaro and Nipoti (2014), Lijoi and Prunster (2014). In order to emphasize the dependence on γ and λ, we eventually adopt the notation $X_{PS} := X_{PS}(\gamma, \lambda)$.

The integer-valued counterpart of the Positive Stable r.v. is the Discrete Stable r.v. X_{DS} proposed by Steutel and van Harn (1979) with p.g.f. given by

$$g_{X_{DS}}(s) = \exp(-\lambda(1-s)^{\gamma}),\ s \in [0,1], \quad (3)$$

where in turn $(\gamma, \lambda) \in]0,1] \times \mathbb{R}^+$. For a survey of the properties of this distribution, see *e.g.* Marcheselli et al. (2008) and Christoph and Schreiber (2001). Similarly to the Positive Stable r.v., we also write $X_{DS} := X_{DS}(\gamma, \lambda)$.

The p.g.f. (3) of the Discrete Stable r.v. $X_{DS}(\gamma, \lambda)$ is obtained from expression (1) when a stable subordinator is considered, i.e. $\frac{\nu(dx)}{dx} \propto x^{-\gamma-1}\mathbf{1}_{\mathbb{R}^+}(x)$. Moreover, since from (1) it also holds

$$g_{X_{DS}}(s) = \mathrm{E}[\exp(-X_{PS}(1-s))] = L_{X_{PS}}(1-s),$$

it also follows $Y \stackrel{\mathcal{L}}{=} X_{PS}(\gamma, \lambda)$ and

$$X_{DS}(\gamma, \lambda) \stackrel{\mathcal{L}}{=} X_P(X_{PS}(\gamma, \lambda)), \quad (4)$$

which is actually equivalent to the identity in distribution emphasized by Devroye (1993, Theorem in Section 1). Identity (4) is very suitable for random variate generation. Indeed, many generators for Poisson variates are available in statistical literature, while Positive Stable variates are readily obtained by means of the well-known Kanter's representation (Kanter, 1975).

On the basis of expression (4), a general "scale" mixture of Discrete Stable r.v.'s, say X_{MDS}, with a mixturing absolutely-continuous positive r.v. V having Laplace transform L_V, may be achieved by considering the identity in distribution

$$X_{MDS} \stackrel{\mathcal{L}}{=} X_{DS}(\gamma, V) \stackrel{\mathcal{L}}{=} X_P(X_{PS}(\gamma, V)). \quad (5)$$

Obviously, (4) is achieved from (5) by assuming a degenerate distribution for V, *i.e.* $P(V = \lambda) = 1$. Moreover, from (5), it is apparent that the p.g.f. of the r.v. X_{MDS} turns out to be

$$g_{X_{MDS}}(s) = \mathrm{E}[\exp(-V(1-s)^{\gamma})] = L_V((1-s)^{\gamma}),\ s \in [0,1]. \tag{6}$$

Hence, families of mixture of Discrete Stable r.v.'s can be generated by means of (5) and (6) by suitably selecting the r.v. V.

We conclude with a final remark on the p.g.f. (6). Let X_S be a Sibuya r.v. (as named by Devroye 1993, since it is a special case of the Negative Binomial Beta r.v. proposed by Sibuya, 1979) with p.g.f.

$$g_{X_S}(s) = 1 - (1-s)^{\gamma},\ s \in [0,1],$$

where $\gamma \in]0,1]$ (for a survey of this law, see Huillet 2016). In the following, the Sibuya r.v. is also denoted by $X_S := X_S(\gamma)$. Therefore, expression (6) may be also interestingly reformulated as

$$g_{X_{MDS}}(s) = L_V(1 - g_{X_S}(s)),\ s \in [0,1]. \tag{7}$$

Thus, if L_V displays a suitable structure, expression (7) eventually gives rise to a representation of the r.v. X_{MDS} in terms of a compound r.v. with a compounding Sibuya r.v. As a quite easy example, the r.v. X_{DS} may be also expressed as a compound Poisson r.v. as

$$g_{X_{DS}}(s) = \exp(-\lambda(1 - g_{X_S}(s))),\ s \in [0,1],$$

and hence

$$X_{DS}(\gamma, \lambda) \stackrel{\mathcal{L}}{=} \sum_{i=1}^{Z} W_i,$$

where $Z \stackrel{\mathcal{L}}{=} X_P(\lambda)$ and the W_i's are i.i.d. r.v.'s such that $W_i \stackrel{\mathcal{L}}{=} X_S(\gamma)$ - which are in turn independent of Z.

2.2 The Poisson-Tweedie law and related distributions

We preliminarly provide some issues on the so-called Tweedie distribution (Hougaard, 1986) which is actually a Tempered Positive Stable distribution. For this reason, we denote the Tweedie r.v. as X_{TPS}. With a slight change in the parameterization proposed by Hougaard (1986), the Laplace transform of the r.v. X_{TPS} is given by

$$L_{X_{TPS}}(t) = \exp(\mathrm{sgn}(\gamma)\lambda(\theta^{\gamma} - (\theta + t)^{\gamma})),\ \mathrm{Re}(t) > 0, \tag{8}$$

where $(\gamma, \lambda, \theta) \in \{\,]-\infty, 1] \times \mathbb{R}^+ \times \mathbb{R}^+\} \cup \{\,]0,1] \times \mathbb{R}^+ \times \{0\}\}$. In this case θ represents the tempering parameter and the Positive Stable distribution is obtained for $\theta = 0$. It should be remarked that the tempering extends the range of values for the parameter γ with respect to the Positive Stable distribution.

Indeed, as shown by Aalen (1992), for $\gamma \in \mathbb{R}^-$ where $\mathbb{R}^- =]-\infty, 0[$, it is immediate to reformulate the r.v. X_{TPS} as a compound Poisson of Gamma r.v.'s. More precisely, let the r.v. $X_G := X_G(\lambda, \delta)$ be distributed according to a Gamma law with corresponding Laplace transform given by

$$L_{X_G}(t) = (1 + \lambda t)^{-\delta}, \ \mathrm{Re}(t) > 0,$$

where $(\lambda, \delta) \in \mathbb{R}^+ \times \mathbb{R}^+$. Hence, on the basis of (8), the following identity in distribution holds for $\gamma \in \mathbb{R}^-$

$$X_{TPS}(\gamma, \lambda, \theta) \stackrel{\mathcal{L}}{=} X_G(1/\theta, -\gamma X_P(\lambda\theta^\gamma)). \tag{9}$$

By extending the issues of Section 2.1, if a tempered stable subordinator is adopted in (1), i.e. $\frac{\nu(dx)}{dx} \propto \exp(-\theta x)x^{-\gamma-1}\mathbf{1}_{\mathbb{R}^+}(x)$, the Poisson-Tweedie distribution - which is actually a Tempered Discrete Stable distribution - is achieved. For more details about the Poisson-Tweedie law, see Baccini et al. (2016) and El-Shaarawi et al. (2011). Indeed, if X_{TDS} denotes the Poisson-Tweedie r.v., X_{TDS} has the following p.g.f.

$$g_{X_{TDS}}(s) = \exp(\mathrm{sgn}(\gamma)\lambda(\theta^\gamma - (\theta + 1 - s)^\gamma)), \ s \in [0, 1],$$

where in turn $(\gamma, \lambda, \theta) \in \{\,]-\infty, 1] \times \mathbb{R}^+ \times \mathbb{R}^+\} \cup \{\,]0, 1] \times \mathbb{R}^+ \times \{0\}\}$. Hence - as expected from (1) - it holds $g_{X_{TDS}}(s) = L_{X_{TPS}}(1 - s)$. In order to be consistent with the existing literature and for practical convenience, we prefer to reparameterize the previous p.g.f. by assuming that $\gamma = a$, $\lambda = bc^a$ and $\theta = 1/c - 1$, in such a way that

$$g_{X_{TDS}}(s) = \exp(\mathrm{sgn}(a)b((1 - c)^a - (1 - cs)^a)), \ s \in [0, 1], \tag{10}$$

where $(a, b, c) \in \{\,]-\infty, 0] \times \mathbb{R}^+ \times [0, 1[\,\} \cup \{\,]0, 1] \times \mathbb{R}^+ \times [0, 1]\}$. The p.g.f. (10) is provided as a slight modification of the formulation suggested by El-Shaarawi et al. (2011). Obviously, the p.g.f of the Discrete Stable r.v. $X_{DS}(a, b)$ is obtained for $c = 1$. Moreover, it is worth noting that c actually represents the "tempering" parameter. Indeed, for $a \in]0, 1]$ and by considering the r.v. $X_{DS}(a, b)$, the following identity

$$g_{X_{TDS}}(t) = \frac{g_{X_{DS}}(cs)}{g_{X_{DS}}(c)}$$

emphasizes the geometric "nature" of the tempering. Similarly to the Tweedie distribution, tempering extends the range of parameter values (with respect to the Discrete Stable distribution) for the parameter a - which may assume negative values, even if c must be strictly less than unity in such a case. Finally, from (1) with $Y \stackrel{\mathcal{L}}{=} X_{TPS}(a, bc^a, 1/c - 1)$, it also holds

$$X_{TDS}(a, b, c) \stackrel{\mathcal{L}}{=} X_P(X_{TPS}(a, bc^a, 1/c - 1)), \tag{11}$$

which constitutes the identity in distribution remarked by Hougaard et al. (1997) and which generalizes expression (4).

On the basis of expression (11), it is at once apparent that a "scale" mixture of Tempered Discrete Stable r.v.'s, say X_{MTDS}, with a mixturing absolutely-continuous positive r.v. V having Laplace transform L_V, may be achieved by considering the following identity in distribution which generalizes (5)

$$X_{MTDS} \stackrel{\mathcal{L}}{=} X_{TDS}(a, V, c) \stackrel{\mathcal{L}}{=} X_P(X_{TPS}(a, c^a V, 1/c - 1)). \tag{12}$$

Obviously, (11) is achieved from (12) by assuming a degenerate distribution for V, *i.e.* $P(V = b) = 1$. Moreover, the corresponding p.g.f. turns out to be

$$\begin{aligned} g_{X_{MTDS}}(s) &= \mathrm{E}[\exp(\mathrm{sgn}(a)V((1-cs)^a - (1-c)^a))] \\ &= L_V(\mathrm{sgn}(a)((1-cs)^a - (1-c)^a)),\ s \in [0,1]. \end{aligned} \tag{13}$$

Hence, families of mixture of Tempered Discrete Stable r.v.'s can be generated by means of (12) by suitably selecting the r.v. V.

In order to reformulate (13) similarly to (7) when $a \in]0,1]$, let $X_{GDS} := X_{GDS}(\gamma, \tau)$ be the Geometric Down-weighted Sibuya r.v. introduced by Zhu and Joe (2009) with the p.g.f.

$$g_{X_{GDS}}(s) = 1 - g_{X_S}(\tau) + g_{X_S}(\tau s) = 1 + (1-\tau)^\gamma - (1-\tau s)^\gamma,$$

where $(\gamma, \tau) \in]0,1] \times]0,1]$. Hence, by considering the r.v. $X_{GDS}(a, c)$, for $a \in]0,1]$ expression (13) may be also rewritten as

$$g_{X_{MTDS}}(s) = L_V(1 - g_{X_{GDS}}(s)),\ s \in [0,1]. \tag{14}$$

Moreover, let $X_{NB} := X_{NB}(\pi, \delta)$ be a Negative Binomial r.v. with p.g.f. given by

$$g_{X_{NB}}(s) = \left(\frac{1-\pi}{1-\pi s}\right)^\delta,\ s \in [0,1],$$

where $(\pi, \delta) \in]0,1[\times \mathbb{R}^+$. Thus, when $a \in \mathbb{R}^-$ and by considering the Negative Binomial r.v. $X_{NB}(c, -a)$, expression (13) may be also rewritten as

$$g_{X_{MTDS}}(s) = L_V((1-c)^a(1 - g_{X_{NB}}(s))),\ s \in [0,1]. \tag{15}$$

Therefore, if L_V displays a suitable structure, expression (14) and (15) eventually gives rise to a representation of the r.v. X_{MDS} in terms of a compound r.v. with compounding Sibuya or Negative Binomial r.v.'s, respectively, according to $a \in]0,1[$ or $a \in \mathbb{R}^-$.

2.3 The Discrete Linnik law and related distributions

By following the formulation adopted by Christoph and Schreiber (1998), the p.g.f. of the r.v. X_{DL} distributed according to the Discrete Linnik law is given by

$$g_{X_{DL}}(s) = (1 + \lambda(1-s)^\gamma/\delta)^{-\delta},\ s \in [0,1], \tag{16}$$

where $(\gamma, \lambda, \delta) \in]0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$. For a detailed description of the main features of the law, see Christoph and Schreiber (1998, 2001). It is apparent that the p.g.f. of the Discrete Stable r.v. X_{DS} is achieved as $\delta \to \infty$. In addition, it should be remarked that the Discrete Linnik law is defined for some negative δ in such a way that $\lambda \leq |\delta|(1-\gamma)$ and, in this case, the distribution is also named Generalized Sibuya (Huillet 2016). However, the results of this Section are mainly given for positive δ. It is apparent that the Discrete Linnik law is very flexible and may encompass a large variety of distribution families ranging from light-tailed laws (*e.g.* the Binomial law for suitable λ and negative integer δ when $\gamma = 1$) to heavy-tailed laws (*e.g.* the Discrete Mittag-Leffler law for $\delta = 1$). In order to emphasize the dependence on the parameters, we also adopt the notation $X_{DL} := X_{DL}(\gamma, \lambda, \delta)$.

For $\delta \in \mathbb{R}^+$ and on the basis of the remarks provided in Section 2.1, it is promptly proven that the Discrete Linnik r.v. is a "scale" mixture of Discrete Stable r.v.'s obtained by selecting $V \stackrel{\mathcal{L}}{=} X_G(\lambda/\delta, \delta)$ in (5). In this case, the p.g.f. (16) is obtained by means of (6), while from (5) it also holds

$$X_{DL}(\gamma, \lambda, \delta) \stackrel{\mathcal{L}}{=} X_{DS}(\gamma, X_G(\lambda/\delta, \delta)) \stackrel{\mathcal{L}}{=} X_P(X_{PS}(\gamma, X_G(\lambda/\delta, \delta))), \tag{17}$$

which is actually similar to the identity in distribution obtained by Devroye (1993, Section 2). A further remark is in order, since expression (17) may be also rephrased in terms of the absolutely-continuous Positive Linnik r.v. $X_{PL} := X_{PL}(\gamma, \lambda, \delta)$ with Laplace transform given by

$$L_{X_{PL}}(t) = (1 + \lambda t^\gamma/\delta)^{-\delta}, \ \mathrm{Re}(t) > 0,$$

where $(\gamma, \lambda, \delta) \in]0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$ - according to the formulation provided by Christoph and Schreiber (2001). The absolutely-continuous Positive Linnik has been introduced by Pakes (1995) by extending the former proposal by Linnik (1962, p.67). Obviously, the Laplace transform (1) of the Positive Stable r.v. $X_{PS}(\gamma, \lambda)$ is achieved as $\delta \to \infty$. For more details on the Positive Linnik distribution, see *e.g.* Jose et al. (2010) and Barabesi et al. (2016a). Since it easily proven that

$$X_{PL}(\gamma, \lambda, \delta) \stackrel{\mathcal{L}}{=} X_{PS}(\gamma, X_G(\lambda/\delta, \delta)),$$

by means of expression (17) it promptly follows that

$$X_{DL}(\gamma, \lambda, \delta) \stackrel{\mathcal{L}}{=} X_P(X_{PL}(\gamma, \lambda, \delta)), \tag{18}$$

which actually generalizes (4). Indeed, expression (4) is recovered from expression (18) as $\delta \to \infty$. Finally, we remark that the identity in distribution (17) is very suitable for random variate generation similarly to identity (4).

For $\delta \in \mathbb{R}^+$, the r.v. X_{DL} may be also expressed as a compound Negative Binomial r.v. with a compounding Sibuya r.v. indeed, from expression (7) with $V \stackrel{\mathcal{L}}{=} X_G(\lambda/\delta, \delta)$, it follows that

$$g_{X_{DL}}(s) = (1 + \lambda(1 - g_{X_S}(s))/\delta)^{-\delta}, \ s \in [0,1],$$

and hence

$$X_{DL}(\gamma, \lambda, \delta) \stackrel{\mathcal{L}}{=} \sum_{i=1}^{Z} W_i,$$

where $Z \stackrel{\mathcal{L}}{=} X_{NB}(\lambda/(\delta+\lambda), \delta)$ and the W_i's are i.i.d. r.v.'s such that $W_i \stackrel{\mathcal{L}}{=} X_S(\gamma)$ - which are in turn independent of Z.

3 The Tempered Discrete Linnik distribution

On the basis of the issues discussed in Section 2, we are ready to introduce a tempered version of the Discrete Linnik distribution. Indeed, in a complete parallelism with Section 2.3, where the Discrete Linnik r.v. is given as a "scale" mixture of the Discrete Stable r.v., the Tempered Discrete Linnik r.v. may be introduced as a "scale" mixture of the Poisson-Tweedie r.v. by selecting $V \stackrel{\mathcal{L}}{=} X_G(bd, 1/d)$ in (12). In this case, if the Tempered Discrete Linnik r.v. is denoted by X_{TDL}, from expression (13) the corresponding p.g.f. turns out to be

$$g_{X_{TDL}}(s) = (1 + \mathrm{sgn}(a) bd((1-cs)^a - (1-c)^a))^{-1/d}, \ s \in [0,1], \tag{19}$$

with $(a, b, c, d) \in \{\,]-\infty, 0] \times \mathbb{R}^+ \times [0, 1[\, \times \mathbb{R}^+\} \cup \{\,]0, 1] \times \mathbb{R}^+ \times [0, 1] \times \mathbb{R}^+\}$.

Several comments on the parameterization are in order. First, similarly to the Discrete Linnik law, the Tempered Discrete Linnik law is also defined for some negative d - a distribution that could be named Tempered Generalized Sibuya by extending the definition by Huillet (2016). As an example, the Binomial law is obtained for $a = 1$, $b \in]0, 1]$ and $1/d \in \mathbb{Z}^-$. Secondly, the adopted formulation allows to achieve the Poisson-Tweedie distribution as d approaches zero - and this could be preferable with respect to a limit at infinity. Hence, the special cases encompassed by the Poisson-Tweedie law are obtained in the limit. As an example, as $d \to 0$, the Poisson law is obtained for $a = 1$, the Generalized Poisson Inverse Gaussian law for $a \in]0, 1[$ and the Poisson-Pascal law for $a \in \mathbb{R}^-$. Moreover, for $a = 0$ the r.v. X_{TDL} degenerates at 0.

According to the previous remarks, the p.g.f. (19) seems a natural generalization of the p.g.f. (10). As usual, we also adopt the notation $X_{TDL} := X_{TDL}(a, b, c, d)$. Obviously, it promptly holds $X_{DL}(a, b, 1/d) \stackrel{\mathcal{L}}{=} X_{TDL}(a, b, 1, d)$ for $a \in]0, 1]$, *i.e* the Discrete Linnik law is a special case of the Tempered Linnik law. Moreover, similarly to the Poisson-Tweedie law, c represents the "tempering" parameter. Indeed, for $a \in]0, 1]$ and by considering the r.v. $X_{DL}(a, b_0, 1/d)$, if $b = b_0/(1 + b_0 d(1-c)^a)$, for the r.v. $X_{TDL}(a, b, c, d)$ the following identity holds

$$g_{X_{TDL}}(t) = \frac{g_{X_{DL}}(cs)}{g_{X_{DL}}(c)}.$$

From the previous expression, the geometric "nature" of the tempering is in turn apparent. Analogously to the Poisson-Tweedie distribution, it is worth

noting that tempering extends the range of parameter values for the parameter a with respect to the Discrete Linnik distribution. In the following, we focus on the case $d \in \mathbb{R}^+$.

For $d \in \mathbb{R}^+$, similarly to (11) and by means of (12), the following identity in distribution holds

$$\begin{aligned} X_{TDL}(a,b,c,d) &\stackrel{\mathcal{L}}{=} X_{TDS}(a, X_G(bd, 1/d), c) \\ &\stackrel{\mathcal{L}}{=} X_P(X_{TPS}(a, X_G(bdc^a, 1/d), 1/c - 1) \end{aligned} \tag{20}$$

which actually generalizes (17). If $a \in \mathbb{R}^-$, it should be remarked that on the basis of (9) identity (20) may be rewritten as

$$X_{TDL}(a,b,c,d) \stackrel{\mathcal{L}}{=} X_P(X_G(c/(1-c), -aX_P(X_G(bd(1-c)^a, 1/d)))). \tag{21}$$

A further interesting mixture representation for the Tempered Discrete Linnik r.v. can be obtained. Indeed, the absolutely-continuous Tempered Positive Linnik r.v. $X_{TPL} := X_{TPL}(\gamma, \lambda, \theta, \delta)$ is defined by means of the Laplace transform

$$L_{X_{TPL}}(t) = (1 + \mathrm{sgn}(\gamma)\lambda((\theta + t)^\gamma - \theta^\gamma)/\delta)^{-\delta}, \ \mathrm{Re}(t) > 0,$$

where $(\gamma, \lambda, \theta, \delta) \in \{]-\infty, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+\} \cup \{]0, 1] \times \mathbb{R}^+ \times \{0\} \times \mathbb{R}^+\}$. For more details on this law, see Barabesi et al. (2016a). The Laplace transform (8) of the Tempered Positive Stable r.v. $X_{TPS}(\gamma, \lambda)$ is achieved from the previous expression as $\delta \to \infty$. Since it holds that

$$X_{TPL}(\gamma, \lambda, \theta, \delta) \stackrel{\mathcal{L}}{=} X_{TPS}(\gamma, X_G(\lambda/\delta, \delta), \theta),$$

from expression (20) it promptly follows that

$$X_{TDL}(a,b,c,d) \stackrel{\mathcal{L}}{=} X_P(X_{TPL}(a,b,c,1/d)),$$

which actually generalizes (18).

For $d \in \mathbb{R}^+$, on the basis of (14) and (15), the r.v. X_{TDL} may be also expressed as a compound distribution. Indeed, when $a \in]0, 1]$ from (14) it holds

$$g_{X_{TDL}}(s) = (1 + bd(1 - g_{X_{GDS}}(s)))^{-1/d}, \ s \in [0, 1],$$

and hence

$$X_{TDL}(a,b,c,d) \stackrel{\mathcal{L}}{=} \sum_{i=1}^{Z} W_i, \tag{22}$$

where $Z \stackrel{\mathcal{L}}{=} X_{NB}(bd/(1+bd), 1/d)$ and the W_i's are i.i.d. r.v.'s such that $W_i \stackrel{\mathcal{L}}{=} X_{GDS}(a, c)$ - which are in turn independent of Z. Moreover, when $a \in \mathbb{R}^-$ expression (15) may be rewritten as

$$g_{X_{TDL}}(s) = (1 + bd(1-c)^a(1 - g_{X_{NB}}(s)))^{-1/d}, \ s \in [0, 1],$$

and hence (22) holds with $Z \stackrel{\mathcal{L}}{=} X_{NB}(bd(1-c)^a/(1+bd(1-c)^a), 1/d)$ and the W_i's are i.i.d. r.v.'s such that $W_i \stackrel{\mathcal{L}}{=} X_{NB}(c, -a)$ - which are in turn independent of Z. Owing to the reproductive property of the Negative Binomial law, expression (22) also provides

$$X_{TDL}(a, b, c, d) \stackrel{\mathcal{L}}{=} X_{NB}(c, -aX_{NB}(bd(1-c)^a/(1+bd(1-c)^a), 1/d)). \quad (23)$$

It is worth remarking that identities (21) and (23) are straightforwardly equivalent since $X_{NB}(\pi, \delta) \stackrel{\mathcal{L}}{=} X_P(X_G(\pi/(1-\pi), \delta))$.

It should be remarked that random variate generation for the Tempered Discrete Linnik distribution may be achieved on the basis of the identities in distribution introduced in the present Section. More precisely, we consider two approaches which depend on expressions (20) and (22), respectively.

The first approach deals with the Poisson mixture representation (20). When $a \in \mathbb{R}^-$, since in such a case identity (20) reduces to identity (21), the generation of a Tempered Discrete Linnik variate actually requires two Poisson variates and two Gamma variates. Since fast Poisson and Gamma variate generators are commonly available in statistical packages, ad hoc algorithms are not necessary in this setting. When $a \in \mathbb{R}^+$, on the basis of identity (20), a Tempered Discrete Linnik variate may be obtained by generating: i) a Tempered Discrete Stable variate and a Gamma variate, or ii) a Poisson variate, a Tempered Positive Stable variate and a Gamma variate. If strategy i) is adopted, the generation of a Tempered Discrete Stable variate is considered by Baccini et al. (2016), where some suitable algorithms are discussed - in particular, Algorithms 2 and 4 of their paper seem to conjugate computational simplicity and efficiency. If strategy ii) is considered, the focus boils down to the generation of a Tempered Positive Stable variate. To this aim, an algorithm has been recently introduced by Barabesi et al. (2016a). Their proposal is simpler and more efficient than the algorithm previously suggested by Devroye (2009) - for more details, see Barabesi et al. (2016). In any case, an easy-to-implement and quite efficient algorithm - which may be alternatively adopted - has been proposed by Hofert (2011a) (see also Hofert, 2011b).

The second approach is related to the compound Negative Binomial representation (22). When $a \in \mathbb{R}^-$, since identity (22) reduces to identity (23), the generation of a Tempered Discrete Linnik variate requires two Negative Binomial variates - which in turn are commonly available in statistical packages. Hence, in this parameter range, expression (23) could be preferable to the equivalent expression (21) in order to obtain Tempered Discrete Linnik variates, since two Negative Binomial variates are solely needed. When $a \in \mathbb{R}^+$, on the basis of identity (22), a Tempered Discrete Linnik variate may be obtained as a Negative-Binomial stopped sum of Geometric Down-weighted Sibuya variates. Since Geometric Down-weighted Sibuya variates are readily obtained (see e.g. Baccini et al., 2016), the algorithm is easy-to-implement - even if it could be relatively efficient for some parameter choices.

Therefore, in both the approaches, the generation of a Tempered Discrete Linnik variate may be carried out by means of generators which are well-established in literature and no new techniques are demanded.

4 Further properties of the Tempered Discrete Linnik distribution

A closed form for the probability function (p.f.) of the Tempered Discrete Linnik r.v. is achieved by means of Result 1 provided in the Appendix. Indeed, by suitably adapting Result 1 to the Tempered Discrete Linnik distribution, it is easily shown that the p.f. $p_{X_{TDL}}$ of the r.v. X_{TDL} may be expressed as the following finite sum

$$p_{X_{TDL}}(k) = p_{X_{TDL}}(0)\,\frac{(-c)^k}{k!}\sum_{m=0}^{k}(-1/d)_m\,K^m C(k,m,a) \tag{24}$$

for $k \in \mathbb{N}$, where

$$p_{X_{TDL}}(0) = (1+\mathrm{sgn}(a)bd(1-(1-c)^a))^{-1/d},$$

while

$$K = \frac{\mathrm{sgn}(a)bd}{1+\mathrm{sgn}(a)bd(1-(1-c)^a)}.$$

In addition,

$$C(k,m,a) = \frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(aj)_k$$

is the generalized factorial coefficient (for more details on this combinatorial quantity, see Charalambides and Singh, 1988, and Charalambides, 2005, p.96), while $(t)_k = t(t-1)\ldots(t-k+1)$ with $t \in \mathbb{R}$, $k \in \mathbb{N}^+$ and $(t)_0 = 1$ represents the falling factorial. We would recognize that a referee has suggested reformulating expression (24) by adopting the generalized factorial coefficients. The referee has also remarked that the evaluation of the generalized factorial coefficient may be suitably (and efficiently) implemented by means of the triangular recursion

$$C(k,m,a) = (am-k+1)C(k-1,m,a) + aC(k-1,m-1,a)$$

with the initial conditions $C(0,0,a) = 1$, $C(k,0,a) = 0$ for $k \in \mathbb{N}^+$ and $C(0,m,a) = 0$ for $m \in \mathbb{N}^+$ (see Charalambides and Singh, 1988).

Obviously, on the basis of expression (24) and for $c = 1$, the p.f. of the Discrete Linnik r.v. $X_{DL}(a,b,1/d)$ is promptly obtained. In addition, in turn, on the basis of Result 1 in the Appendix, the p.f. $p_{X_{TDS}}$ of the Poisson-Tweedie r.v. X_{TDS} may be expressed as

$$p_{X_{TDS}}(k) = p_{X_{TDS}}(0)\,\frac{(-c)^k}{k!}\sum_{m=0}^{k}(-\mathrm{sgn}(a)b)^m C(k,m,a) \tag{25}$$

for $k \in \mathbb{N}$, where

$$p_{X_{TDS}}(0) = \exp(-\mathrm{sgn}(a)b(1-(1-c)^a)).$$

Expression (25) is actually equivalent to the formula provided by Baccini et al. (2016) for $p_{X_{TDL}}$.

It is worth noting that $p_{X_{TDL}}$ may be also evaluated by means of a recursive relationship which generalizes the expression proposed by El-Shaarawi et al. (2011) for the Poisson-Tweedie law. Indeed, it holds that

$$p_{X_{TDL}}(k+1) = \frac{1}{k+1}\left(Kc\Big(\frac{a}{d}+k\Big)p_{X_{TDL}}(k)+(1-K)\sum_{j=1}^{k} jr_{k-j+1}p_{X_{TDL}}(j)\right), \tag{26}$$

where $r_1 = (1-a)c$, while

$$r_{k+1} = \frac{k-1+a}{k+1}\, cr_k$$

for $k = 1, 2, \ldots$ (for the proof, see Result 2 in the Appendix).

In order to emphasize the flexibility of the Tempered Discrete Linnik distribution, we show the plots of the p.f. (24) in the Figures 1 and 2 for some parameter choices. More precisely, Figures 1 and 2 display the plots of $p_{X_{TDL}}$ for selected positive and negative values of a, respectively. In each plot, the parameters a, b, and c are fixed and several values of d are considered - hence, the distributions have the same mean, see the expression of $\mathrm{E}[X_{TDL}]$ given below. From the analysis of these plots, it is apparent that $p_{X_{TDL}}$ may exhibit very different morphologies with varying d. As an example, by examining the plot at the right-bottom of Figure 1, $p_{X_{TDL}}$ may be decreasing with a quite large probability at zero - as well as unimodal with a small probability at zero - as d changes. This behaviour is more evident for negative values of a. Indeed, by considering some plots of Figure 2, $p_{X_{TDL}}$ may be even bimodal with a mode at zero and a quite heavy tail. Hence, the Tempered Discrete Linnik distribution is also a suitable candidate for modelling zero-inflated count data - even in the presence of a long right tail. These issues suggest that the Tempered Discrete Linnik distribution can fit a substantially larger data range with respect to the Poisson-Tweedie distribution thanks to the extra parameter d.

Some comments on the special cases of the Tempered Discrete Linnik distribution for $d \in \mathbb{R}^+$ are worthwhile. To this aim, Table I reports the main laws encompassed by the proposed distribution. First, for $a = 1$ the Tempered Discrete Linnik distribution includes the Negative Binomial distribution and the Poisson distribution (as a limiting case). Indeed, when $a = 1$, it holds that $X_{TDL}(1, b, c, d) \stackrel{\mathcal{L}}{=} X_{NB}(bcd/(1+bcd), 1/d)$ and hence the r.v. $X_P(bc)$ is obtained as $d \to 0$. It should be noticed that the Negative Binomial is also achieved when $a \in]0, 1]$ and $bd(1-c)^a = 1$. Moreover, as $d \to 0$, the proposed law covers the Generalized Poisson-Inverse Gaussian and the Poisson-Pascal distributions - which are the usually-adopted appellations of

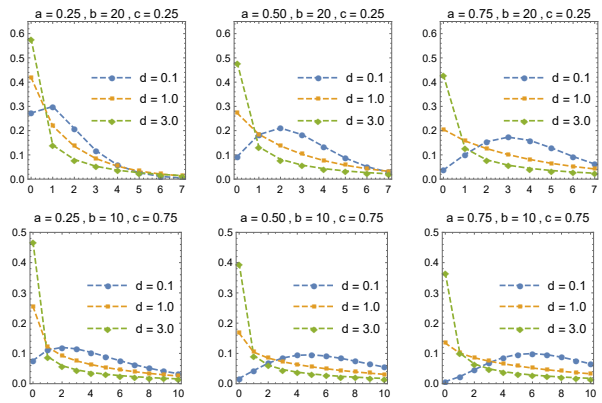

Fig. 1 Plot of $p_{X_{TDL}}$ for some positive a and some choices of b, c and d.

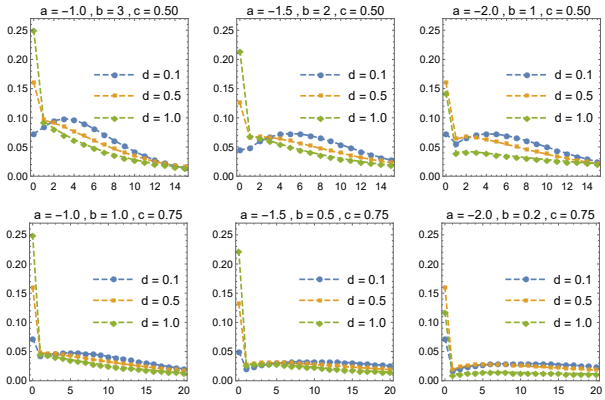

Fig. 2 Plot of $p_{X_{TDL}}$ for some negative a and some choices of b, c and d.

the Poisson-Tweedie distribution when $a \in]0,1]$ and $a \in \mathbb{R}^-$, respectively (see El-Shaarawi et al., 2011). Hence, for a general $d \in \mathbb{R}^+$, the broader versions of these two distributions are named in Table I as Generalized Negative Binomial-Inverse Gaussian and Generalized Poisson-Pascal. Obviously, the usual Discrete Linnik distribution is obtained for $c = 1$, which encompasses the Mittag-Leffler distribution when $d = 1$ (see Huillet, 2016). For $a = 1/2$, the Tempered Discrete Linnik law is actually a generalization of the Poisson Inverse Gaussian law (for a discussion of this law, see e.g. Johnson et al., 2005, p.484), which is obtained as $d \to 0$. Thus, this new distribution is named in Table 1 as Negative Binomial Inverse Gaussian. It is worth noting that in this special case expression (24) remarkably reduce to a single sum, since

$$C(k,m,1/2) = (-1)^{k-m} 2^{-2k+m} \frac{(k-1)!}{(m-1)!} \binom{2k-m-1}{k-1}$$

Table 1 Synopsis of the main laws encompassed by the Tempered Discrete Linnik distribution for the different parameter choices (zero values for the parameter d should be interpreted in the limit).

law	a	b	c	d
Poisson	1	$\mathbb{R}^+$	$]0,1]$	0
Negative Binomial	1	$\mathbb{R}^+$	$]0,1]$	$\mathbb{R}^+$
Discrete Stable	$]0,1]$	$\mathbb{R}^+$	1	0
Poisson-Tweedie (Generalized Poisson-Inverse Gaussian)	$]0,1]$	$\mathbb{R}^+$	$[0,1]$	0
Generalized Negative Binomial-Inverse Gaussian	$]0,1]$	$\mathbb{R}^+$	$[0,1]$	$\mathbb{R}^+$
Poisson-Tweedie (Poisson-Pascal)	$]-\infty,0]$	$\mathbb{R}^+$	$]0,1[$	0
Generalized Poisson-Pascal	$]-\infty,0]$	$\mathbb{R}^+$	$]0,1[$	$\mathbb{R}^+$
Discrete Linnik	$]0,1]$	$\mathbb{R}^+$	1	$\mathbb{R}^+$
Discrete Mittag-Leffler	$]0,1]$	$\mathbb{R}^+$	1	1
Generalized Discrete Mittag-Leffler	$]0,1]$	$\mathbb{R}^+$	$]0,1]$	1
Poisson Inverse Gaussian	$1/2$	$\mathbb{R}^+$	$]0,1]$	0
Negative Binomial-Inverse Gaussian	$1/2$	$\mathbb{R}^+$	$]0,1]$	$\mathbb{R}^+$
Polyá-Aeppli	-1	$\mathbb{R}^+$	$]0,1[$	0
Generalized Polyá-Aeppli	-1	$\mathbb{R}^+$	$]0,1[$	$\mathbb{R}^+$

for $k \in \mathbb{N}^+$, while $C(0,m,1/2) = 1$. The simplification of $C(k,m,1/2)$ is a known result for the generalized factorial coefficients (see *e.g.* Lijoi et al., 2007). As emphasized by a referee, analogous simplifications hold for a rational $a \in]0,1[$. When $a = -1$, the Tempered Discrete Linnik is in turn a generalization of the Polyá-Aeppli law (for more about this distribution, see *e.g.* Johnson et al., 2005, p.410), which is obtained as $d \to 0$. In this case also, expression (24) remarkably reduce to a single sum, since

$$C(k,m,-1) = (-1)^k \frac{k!}{m!} \binom{k-1}{m-1}$$

for $k \in \mathbb{N}^+$, while $C(0,m,-1) = 1$. It should be remarked that in this special case the generalized factorial coefficients actually reduce to the Lah numbers, i.e. $L(k,m) = C(k,m,-1)$ (see *e.g.* Charalambides, 2005, p.97).

As to the main descriptive indexes, on the basis of (19) and after tedious algebra, it follows that the expectation and the variance of the r.v. X_{TDL} are respectively given by

$$\mu = \mathrm{E}[X_{TDL}] = |a|bc(1-c)^{a-1}$$

and

$$\sigma^2 = \mathrm{Var}[X_{TDL}] = d\mu^2 + \frac{(1-ac)\mu}{1-c}.$$

It is worth noting that μ does not depend on the parameter d - i.e., the Tempered Discrete Linnik r.v. and the Poisson-Tweedie r.v. actually display the same expectation. However, since the dispersion index is given by

$$D = \frac{\sigma^2}{\mu} = d\mu + \frac{1-ac}{1-c},$$

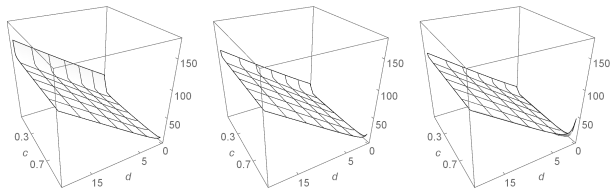

Fig. 3 Three-dimensional plot of κ_4 as a function of (c, d) for $a = 0.25, 0.50, 0.75$ (from left to right) and $b = 10$.

the Poisson-Tweedie distribution may solely display over-dispersion (since $D \geq 1$ when $d = 0$), while the Tempered Discrete Linnik distribution may accommodate for under-dispersion (for some admissible values $d < 0$), as well as for over-dispersion (when $d > 0$). Hence, the Tempered Discrete Linnik law substantially extend the range of the dispersion index with respect to the Poisson-Tweedie law.

After further tedious algebra, it also follows

$$m_3 = \mathrm{E}[(X_{TDL} - \mu)^3] = \frac{\sigma^4}{\mu} + d\mu\sigma^2 + \frac{c(1-a)\mu}{(1-c)^2}$$

and

$$m_4 = \mathrm{E}[(X_{TDL} - \mu)^4] = 3(2d+1)\sigma^4 + \frac{(4c(1-a) + (1-ac)^2)\sigma^2}{(1-c)^2} + \frac{c^2(1-a^2)\mu}{(1-c)^3},$$

from which the skewness and kurtosis indexes may be promptly expressed as

$$\kappa_3 = \frac{m_3}{\sigma^3} = \frac{D}{\sigma} + \frac{d\sigma}{D} + \frac{c(1-a)}{(1-c)^2\sigma D}$$

and

$$\kappa_4 = \frac{m_4}{\sigma^4} = 3(2d+1) + \frac{4c(1-a) + (1-ac)^2}{(1-c)^2\sigma^2} + \frac{c^2(1-a^2)}{(1-c)^3\sigma^2 D}.$$

On the basis of these expressions, we provide a further confirmation of the flexibility of the Tempered Discrete Linnik distribution in Figures 3-4. Indeed, these Figures display the three-dimensional plots of κ_4 as a function of c and d for some positive and negative a, respectively, and when $b = 10$. It is at once apparent that the Tempered Discrete Linnik distribution substantially extends the range of the kurtosis index κ_4 with respect to the Poisson-Tweedie distribution. Indeed, for any given c, the kurtosis index increases as d increases for each selected values of a, either positive or negative. In addition, since the probability at zero increases as d increases, a distribution with a bold mass at zero and heavy tails can be even achieved, as previously remarked.

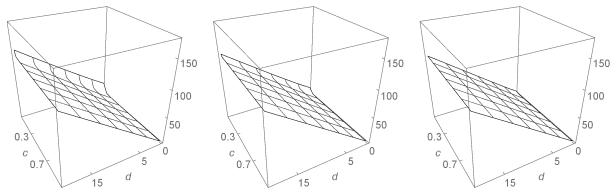

Fig. 4 Three-dimensional plot of κ_4 as a function of (c, d) for $a = -0.5, -1.0, -5.0$ (from left to right) and $b = 10$.

5 An application to scientometric data

In order to highlight the versatility of the proposed law, we consider the data collected by Baccini et al. (2014). This dataset deals with the scientific productivity of 942 permanent researchers of the University of Siena (Italy) during the period 2008-2010. Three different bibliometric indicators were considered for each scholar, i.e. the number of publications in the institutional research repository of the University of Siena (RI), the number of publications in Scopus (SI) and the h-index score (H). More precisely, as to RI, it actually represents the number of authored or co-authored scientific outputs, recorded in the repository during the period 2008-2010 and classified as articles, books, chapters in books and conference proceedings. As to SI, it consists of the number of authored or co-authored publications registered in the Scopus database during the period 2008-2010. Finally, as to H, it represents the value of the index proposed by Hirsh (2005) extracted from the Scopus database on December 31, 2010 (for more details on the inference for the h-index, see Pratelli et al., 2012). Obviously, the three indexes are integer-valued.

Scientometricians have often suggested mixture Poisson laws for modelling the distribution of bibliometric indicators (see *e.g.* Burrell, 2014, and Burrell and Fenton, 1993). Indeed, it is widely accepted by scientometricians that a scientific production process may be seen as a population of sources which randomly produces items over time. A reasonable model for this framework could be based on the assumption that the individual sources produce items according to a counting process (typically, a Poisson process), in such a way that the production rates vary over the single sources (i.e. a mixing distribution is assumed). Hence, on the basis of results obtained in Section 3, these issues suggest that the Tempered Discrete Linnik distribution could be a natural candidate for modelling the scientific production output.

From Figure 5, the empirical distribution of RI seems to show a mild zero excess. In contrast, as it can be assessed from Figures 6 and 7, the empirical distributions of SI and H display a marked zero inflation. A moderately-heavy right-tail is quite apparent in the empirical distributions of RI and H, since the empirical kurtoses are given by $\bar{\kappa}_4 = 15.46$ for RI and $\bar{\kappa}_4 = 7.69$ for H, respectively. In the case of SI, the excess of kurtosis is more evident since the empirical kurtosis is given by $\bar{\kappa}_4 = 57.74$. In turn, on the basis of the findings

Table 2 Maximum likelihood estimates for the Tempered Discrete Linnik and Poisson-Tweedie distributions and corresponding model performance indexes for the three bibliometric indicators.

Indicator	Model	$\widehat{a}$	$\widehat{b}$	$\widehat{c}$	$\widehat{d}$	$\max \ell$	AIC	χ^2
Repository	TDL	−2.40	24.43	0.10	0.89	−2979.62	5967.35	56.12
	TDS	0.10	11.60	0.90	–	−2981.82	5969.63	59.56
Scopus	TDL	0.10	26.25	0.73	2.50	−2418.64	4845.28	81.08
	TDS	−0.38	0.55	0.92	–	−2419.82	4845.63	89.26
H-index	TDL	−0.97	0.50	0.76	0.70	−2525.16	5054.75	64.67
	TDS	−0.57	0.50	0.87	–	−2535.03	5076.29	82.15

given in Section 4, these issues validate the Tempered Discrete Linnik law as a suitable model for the considered data. Indeed, the Tempered Discrete Linnik distribution may be effectively able to capture both the quite long tail and the considerable mass at zero of the empirical distributions.

In order to achieve the maximum likelihood estimation of the parameters of the Tempered Discrete Linnik distribution, we have performed computations through an algorithm implemented as a compiled FORTRAN routine. The method of moments were adopted in order to achieve the initial estimates for the numerical maximization of the likelihood function. However, since the Tempered Discrete Linnik r.v. has a p.g.f. with a simple structure, more advanced method based on the empirical p.g.f. - which do not require extensive computation - could be adopted to obtaining the parameter starting values (see *e.g.* Dowling and Nakamura, 1997). The maximum likelihood estimation was implemented by using the recursive relationship (26) in order to compute the p.f. Indeed, expression (24) seems to suffer of a loss of precision for large argument values when implemented as a FORTRAN routine - even if the triangular recursion was adopted for achieving the evaluation of the generalized factorial coefficient - while the recursive expression (26) is more stable.

The maximum likelihood estimates of the parameters of the Tempered Discrete Linnik and the Poisson-Tweedie distributions are reported in Table 2. In the same table, the maximum value of the log-likelihood function ($\max \ell$), the Akaike Information Criterion (AIC) and the chi-square index (χ^2) are reported. Moreover, the empirical distributions, as well as the fitted Tempered Discrete Linnik and Poisson-Tweedie distributions are plotted in Figures 5, 6 and 7 for the three indicators, respectively. From these figures it is even visually apparent that the Tempered Discrete Linnik law provides the best fit. As a matter of fact, the Tempered Discrete Linnik law is preferable to the Poisson-Tweedie law according to the model performance indexes considered in Table 2.

A further argument leads to argue similar conclusions. Indeed, the estimated values of the parameter d are not about zero, i.e. Poisson-Tweedie law - considered a sub-model of the Tempered Discrete Linnik law - is not sufficient for capturing the joint presence of zero excess and long tails. On the basis of

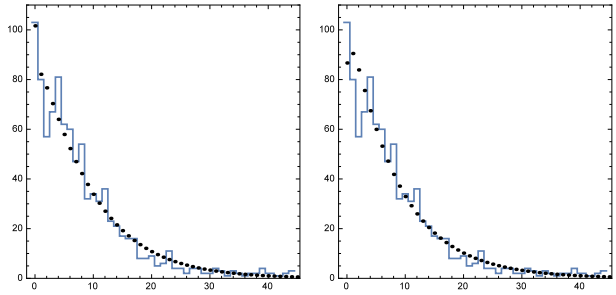

Fig. 5 Empirical distribution of RI with the fitted Tempered Discrete Linnik distribution (left plot) and the fitted Poisson-Tweedie distribution (right plot).

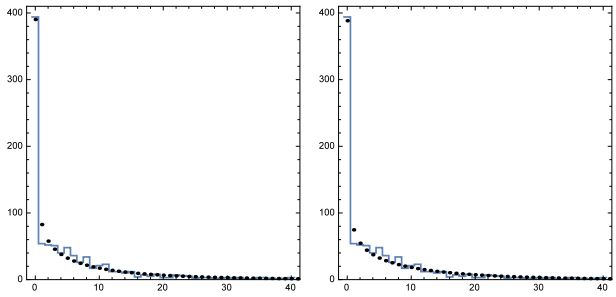

Fig. 6 Empirical distribution of SI with the fitted Tempered Discrete Linnik distribution (left plot) and the fitted Poisson-Tweedie distribution (right plot).

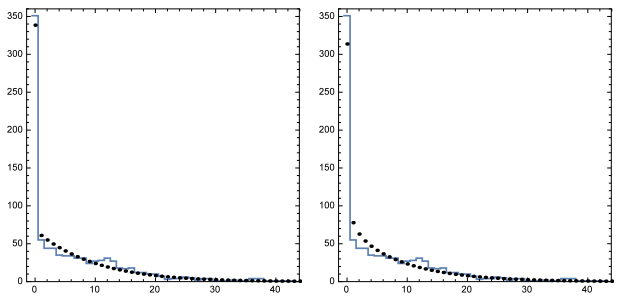

Fig. 7 Empirical distribution of H with the fitted Tempered Discrete Linnik distribution (left plot) and the fitted Poisson-Tweedie distribution (right plot).

Table 1, similar considerations exclude further sub-models such as the Poisson and Negative Binomial laws (the estimated values of the parameter a are not about one), as well as the heavy-tailed Discrete Stable and the Discrete Linnik laws (the estimated values of the parameter c are not about one).

6 Conclusions and future directions

We have proposed a tempered version of the Discrete Linnik law, which actually generalizes the celebrated Poisson-Tweedie law. Indeed, the Tempered Discrete Linnik distribution can be expressed as a mixture of a Poisson-Tweedie distribution with respect to a mixturing Gamma distribution. In addition, the law may be also seen as a mixture of a Poisson distribution with respect to a mixturing Tempered Positive Linnik distribution and it also has a compound Negative Binomial representation. Hence, the Tempered Discrete Linnik distribution is well established from a probabilistic viewpoint.

On the basis of the properties obtained in the Sections 3 and 4, the Tempered Discrete Linnik distribution also gives rise to a versatile statistical model for dealing with count data. Indeed, as a first issue, the Tempered Discrete Linnik law may fit empirical distributions displaying whether light or long tails. Moreover, tempering allows to obtain distributions which are similar to the Discrete Linnik or the Discrete Stable distributions, but not so heavy-tailed - a desirable feature for modelling real-world data. As a second issue, the Tempered Discrete Linnik distribution can manage an extended range of kurtosis with respect to the Poisson-Tweedie distribution, as emphasized in Section 4. In addition, it is noteworthy that the Tempered Discrete Linnik distribution may actually model zero-inflated long-tailed empirical distributions - which may be even bimodal. This property has been illustrated in Section 5 by considering some datasets dealing with the scientific production of researchers, which commonly displays these rather complex features.

As to the forthcoming research, a possible target is the full analysis of the properties of the Tempered Discrete Linnik distribution when the parameter d is negative. This issue is interesting, since in such a case the Tempered Discrete Linnik distribution displays under-dispersion, in contrast to the Poisson-Tweedie distribution which may solely show over-dispersion - see the dispersion index provided in Section 4. However, the structure of the parameter space seems to be rather complex for negative d. As an easy example for $a = 1$, the Tempered Discrete Linnik law obviously reduces to a Binomial law if $1/d \in \mathbb{Z}^-$, but expression (19) is not a proper p.g.f. for a negative $1/d \in \mathbb{Q}$. A further goal consists in developing the Tempered Discrete Linnik distribution as a basis for a generalized linear model for count data when some explanatory variables are available. Since the distribution automatically adapts to zero-inflated and - eventually - long-tailed count data, this generalized linear model could be extremely flexible without the need to introduce zero-inflated or hurdle components. In addition, this class of models would

contain the commonly-adopted generalized linear models for count data - such as the Poisson and Negative Binomial - as special cases.

Acknowledgements The authors would like to thank Prof. Luca Pratelli for many useful advices. We also thank two anonymous reviewers for their careful reading of our manuscript and their insightful comments and suggestions.

Appendix

Result 1. We provide a result on the p.f. of a family of r.v.'s displaying a very general type of p.g.f., which encompasses (13) - and hence (3), (10), (16) and (19) - as special cases. Let us consider an integer-valued r.v. X with p.g.f. given by

$$g_X(s) = \varphi(\alpha + \beta(1-\phi s)^\gamma), \ \ s \in [0,1],$$

where $\alpha, \beta, \gamma, \phi \in \mathbb{R}$ are parameters in such a way that $\phi \in [0,1]$, while $\varphi : \mathbb{R} \mapsto [0,1]$ is a suitable function. As an example, the p.g.f. (19) of the r.v. X_{TDL} is obtained by setting $\alpha = 1 - \text{sgn}(a)bd(1-c)^a$, $\beta = \text{sgn}(a)bd$, $\phi = c$ and $\gamma = a$ and by assuming that $\varphi(x) = x^{-1/d}$. As a further example, the p.g.f. (10) of the r.v. X_{TDS} is achieved by setting $\alpha = \text{sgn}(a)b(1-c)^a$, $\beta = -\text{sgn}(a)b$, $\phi = c$ and $\gamma = a$, while $\varphi(x) = \exp(x)$. If the function φ is analytic in a neighbourhood of $(\alpha+\beta)$, for $k \in \mathbb{N}$ the p.f. p_X of the r.v. X may be expressed as

$$\begin{aligned} p_X(k) &= \frac{1}{k!} \left. \frac{d^k g_X(s)}{ds^k} \right|_{s=0} = \frac{1}{k!} \left. \frac{d^k \varphi(\alpha+\beta+\beta((1-\phi s)^\gamma - 1)}{ds^k} \right|_{s=0} \\ &= \frac{1}{k!} \sum_{m=0}^{k} \frac{\beta^m}{m!} \left. \frac{d^m \varphi(s)}{ds^m} \right|_{s=\alpha+\beta} \left. \frac{d^k((1-\phi s)^\gamma - 1)^m}{ds^k} \right|_{s=0} \end{aligned}$$

and, by means of the Binomial Theorem, it follows that

$$\begin{aligned} p_X(k) &= \frac{1}{k!} \sum_{m=0}^{k} \frac{\beta^m}{m!} \left. \frac{d^m \varphi(s)}{ds^m} \right|_{s=\alpha+\beta} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \left. \frac{d^k (1-\phi s)^{\gamma j}}{ds^k} \right|_{s=0} \\ &= \frac{(-\phi)^k}{k!} \sum_{m=0}^{k} \beta^m \left. \frac{d^m \varphi(s)}{ds^m} \right|_{s=\alpha+\beta} \frac{1}{m!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (\gamma j)_k \\ &= \frac{(-\phi)^k}{k!} \sum_{m=0}^{k} \beta^m \left. \frac{d^m \varphi(s)}{ds^m} \right|_{s=\alpha+\beta} C(k,m,\gamma), \end{aligned}$$

where the generalized factorial coefficient and the falling factorial are defined in Section 4.

Result 2. On the basis of expression (19), it turns out that

$$g'_{X_{TDL}}(s) = (1 + \text{sgn}(a)bd((1-cs)^a - (1-c)^a))^{-1} |a| bc (1-cs)^{a-1} g_{X_{TDL}}(s),$$

from which

$$(K(1-cs) + (1-K)(1-cs)^{1-a}) g'_{X_{TDL}}(s) = \frac{Kac}{d} \, g_{X_{TDL}}(s), \qquad (27)$$

where K is defined in Section 4. It is worth noting that

$$(1-cs)^{1-a} = 1 - \sum_{k=1}^{\infty} r_k s^k,$$

where in turn the r_k's are defined in Section 4. Hence, from (27) it follows that

$$\left(1 - Kcs - (1-K)\sum_{k=1}^{\infty} r_k s^k\right) \sum_{k=1}^{\infty} k p_{X_{TDL}}(k) s^{k-1} = \frac{Kac}{d} \sum_{k=0}^{\infty} p_{X_{TDL}}(k) s^k. \tag{28}$$

Since

$$\left(1 - Kcs - (1-K)\sum_{k=1}^{\infty} r_k s^k\right) \sum_{k=1}^{\infty} k p_{X_{TDL}}(k) s^{k-1} =$$

$$\sum_{k=0}^{\infty} \left((k+1) p_{X_{TDL}}(k+1) - Kck p_{X_{TDL}}(k) - (1-K) \sum_{j=1}^{k} j r_{k-j+1} p_{X_{TDL}}(j) \right) s^k$$

equating the coefficients of the powers of s in expression (28) the recursive relation (26) is obtained.

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