# Passivity-based Analysis and Design of Multi-contact Haptic Systems via LMIs

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## 1. Introduction

Stability is a key feature in haptic interaction with virtual environments, since unwanted oscillations can impair realism and, most importantly, may be potentially harmful for the human operator. The issue of stability in this context has been addressed by several authors since the early 90's (Minsky et al. (1990)) and involves quite a few aspects, since the systems at hand are complex and some of their components, namely the human operators, are difficult to model. Stability has been considered from multiple viewpoints, and passivity has often been exploited in this context, since it provides a powerful tool for analyzing etherogeneous interconnected systems (Lozano et al. (2000)). The fundamental paper Colgate & Schenkel (1997) and more recent works such as Stramigioli et al. (2005) provide different approaches to the characterization of passivity in sampled-data systems and in particular in haptics. In Miller (2000); Miller et al. (1999; 2000; 2004) a discrete-time passivity framework is proposed to deal with stability analysis and controller (virtual coupling) design also in the presence of non-passive virtual environments, and in particular Miller et al. (2000) addresses the presence of nonlinearities. In Naghshtabrizi & Hespanha (2006), an $H_\infty$-type approach to the design of virtual couplings ensuring passivity and transparency is proposed. Most of the above contributions are mainly focused on the case of a single human operator interacting with a one-degree of freedom virtual environment, although also multivariable systems can be addressed to some extent.

In this chapter, we deal specifically with stability analysis and controller design for haptic systems involving several devices and human operators that interact with a common virtual environment through multiple points of contact. Multi-contact interaction is an important issue in virtual reality and haptics (Barbagli et al. (2005a)). Researchers from the computer haptics community, the branch of the haptic science closer to traditional robotics (Salisbury et al. (2004)), have investigated several aspects in this scenario such as friction modeling (Barbagli et al. (2004); Melder & Harwin (2003)) and interaction with deformable objects (Barbagli et al. (2005b)), but mostly neglected stability issues.

Our approach is related to the framework of Miller (2000); Miller et al. (1999; 2000; 2004) but exploits some features that are peculiar to multi-contact systems. Indeed, in a multi-contact scenario, several structural constraints arise due to the number, type and (physical or virtual) location of the devices that are coupled through the virtual environment. Moreover, virtual coupling implementation may be subject to structure constraints as well. As a matter of fact, it is often the case that the device and virtual coupling must be lumped together. More importantly, multi-contact systems may be physically distributed, and therefore the virtual coupling may share only limited information with the devices and virtual environment due to decentralization and limited communication requirements.

The contribution of this chapter is here summarized. First, we formalize the stability problem in a multi-user multi-contact setting. Then, we introduce a sufficient stability condition for the multidimensional haptic loop in the form of a single LMI problem. Moreover, we provide the parameterization of a class of stabilizing controllers satisfying structural constraints in terms of the solution of a sequence of LMI problems. This parameterization foreshadows the possibility of addressing several interesting performance problems in a computationally efficient way. Finally, the problem of optimizing controller transparency (in an $H_\infty$ sense) within the proposed stabilizing controller class is considered.
The rest of the chapter is organized as follows. In Section 2 we report some preliminary and specific results on passivity-based analysis of multi-dimensional haptic systems; in Section 3 we derive the sought LMI stability condition, while in Section 4 we address controller parameterization. In Section 5 we formulate the controller transparency problem using the given parameterization. Finally, section 6 reports two illustrative application examples, and conclusions are drawn in Section 7.

**Notation**
For a square matrix $X$, $X > 0$ ($X < 0$) denotes positive (negative) definiteness, $X^T$ denotes transpose and $\|X\|$ denotes some matrix norm of $X$. $I_m$ is the $m \times m$ identity matrix. $X = \text{blockdiag}(X_1, \ldots, X_N)$ denotes a block-diagonal matrix with diagonal blocks $X_1, \ldots, X_N$. With $\mathcal{BD}(m; m_1, \ldots, m_N)$ we denote the set of $m \times m$ block-diagonal matrices whose $N$ blocks have dimensions $m_1 \times m_1, \ldots, m_N \times m_N$, with $\sum_{i=1}^{N} m_i = m$. The latter notation is also used without ambiguity for block-diagonal transfer matrices of $m-$input, $m-$output linear systems and for generic $m-$input, $m-$output operators. With $\overline{\mathcal{BD}}(m_1 \times n_1, \ldots, m_N \times n_N)$ we indicate the set of non-square block-diagonal matrices with block sizes $m_i \times n_i$, $i = 1, \ldots, N$. For a transfer matrix $G$, $\|G\|_\infty$ denotes its $H_\infty$ norm.

## 2. Preliminaries

### 2.1 Passivity results

The approach to stability analysis and controller design presented here exploits a generalization of the passivity framework in Miller et al. (1999)-Miller et al. (2004), which is based upon the concepts of output strict passivity (OSP) and input strict passivity (ISP) (Byrnes & Lin (1994); Lozano et al. (2000)). We find it convenient to employ a slightly different characterization of the concepts OSP and ISP in both the continuous and the discrete-time context. Let us introduce the following two definitions.

**Definition 1.** (continuous-time passivity). Let $\Sigma$ be a continuous-time dynamical system with input vector $u(t) \in \mathbb{R}^m$, output vector $y(t) \in \mathbb{R}^m$, and state vector $\psi(t) \in \mathbb{R}^n$. If there exists a continuously differentiable positive definite function $V(\psi) : \mathbb{R}^n \to \mathbb{R}$ (called the *storage function*) and $m \times m$ symmetric matrices $\Delta$ and $\Phi$ such that along all system trajectories $(\psi(t), u(t), y(t))$, $t \in \mathbb{R}$, the following inequality holds

$$\dot{V}(\psi(t)) \leq y(t)^T u(t) - y(t)^T \Delta y(t) - u(t)^T \Phi u(t),$$

then, system $\Sigma$ is *passive* if $\Delta = \Phi = 0$, *output strictly passive with level* $\Delta$ ($\Delta-$OSP) if $\Delta > 0$, $\Phi = 0$, *input strictly passive with level* $\Phi$ ($\Phi-$ISP) if $\Delta = 0$, $\Phi > 0$, respectively.

**Definition 2.** (discrete-time passivity). Let $\Sigma_d$ be a discrete-time dynamical system with input vector $u(k) \in \mathbb{R}^m$, output vector $y(k) \in \mathbb{R}^m$, and state vector $\psi(k) \in \mathbb{R}^n$. If there exists a positive definite function $V(\psi) : \mathbb{R}^n \to \mathbb{R}$ and $m \times m$ symmetric matrices $\Delta$ and $\Phi$ such that along all system trajectories $(\psi(k), u(k), y(k))$, $k \in \mathbb{N}$, the following inequality holds

$$\begin{array}{l} \Delta V(\psi(k)) = V(\psi(k+1)) - V(\psi(k)) \\ \leq y(k)^T u(k) - y(k)^T \Delta y(k) - u(k)^T \Phi u(k), \end{array} \tag{1}$$

then the system is passive if $\Delta = \Phi = 0$, output strictly passive ($\Delta-$OSP) if $\Delta > 0$, $\Phi = 0$, input strictly passive ($\Phi-$ISP) if $\Delta = 0$, $\Phi > 0$, respectively.

**Remark 1.** Note that Definitions 1 and 2 differ from the standard notions of OSP/ISP in that the weights $\Delta$ and $\Phi$ are symmetric matrices instead of scalars.
Note that $\Delta$ and $\Phi$ need not necessarily be positive definite in this context: indeed, a dynamical system will be said to lack OSP (ISP) when the above definitions hold for non-positive definite $\Delta$ ($\Phi$).

Let $\Sigma_d$ be a discrete-time time-invariant linear system defined by the state space representation $(A, B, C, D)$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$. A straightforward extension of the standard Kalman-Yacubović-Popov lemma (Byrnes & Lin (1994)) applies.

**Lemma 1.** *System $\Sigma_d$ is passive ($\Delta-$OSP,$\Phi-$ISP) if and only if there exists a symmetric matrix $P \in \mathbb{R}^n$ such that the following two matrix inequalities hold:*

$$\begin{array}{c} P > 0 \\ \left[ \begin{array}{cc} A^T PA - P + C^T \Delta C & A^T PB - \frac{C^T}{2} + C^T \Delta D \\ B^T PA - \frac{C}{2} + D^T \Delta C & B^T PB - \frac{D+D^T}{2} + D^T \Delta D + \Phi \end{array} \right] < 0. \end{array} \tag{2}$$

In order to address our problems, we find it convenient to look for an alternative formulation of the above result in which some of the quantities involved, in particular matrices $B$, $D$, and $\Delta^{-1}$, appear linearly in the matrix inequalities that define the passivity condition (2). This can be accomplished via a Schur complement argument, as the following result shows.

**Lemma 2.** *Let $\Delta > 0$. System $\Sigma_d$ is passive ($\Delta-$OSP, $\Phi-$ISP) if and only if there exist a symmetric matrix $Q \in \mathbb{R}^n$ and a matrix $R \in \mathbb{R}^{(n+2m)\times n}$ satisfying the constraints*

$$\begin{array}{ll} \text{(a)} & Q > 0 \\ \text{(b)} & \left[ \begin{array}{cc} Q & R^T \\ R & S \end{array} \right] > 0 \\ \text{(c)} & R = \left[ \begin{array}{c} \frac{C}{2} \\ A \\ C \end{array} \right] Q \end{array} \tag{3}$$

*where*

$$S = \left[ \begin{array}{ccc} \frac{D^T+D}{2} + \Phi & B^T & D^T \\ B & Q & 0 \\ D & 0 & \Delta^{-1} \end{array} \right].$$

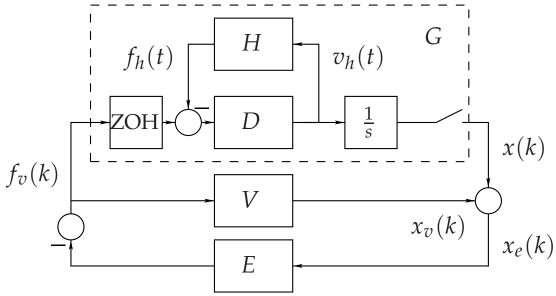


Fig. 1. Haptic loop

**Proof.** By pre- and post- multiplication of (3)(b) by the positive definite nonsingular matrix $\begin{bmatrix} Q^{-1} & 0 \\ 0 & I_{n+2m} \end{bmatrix}$ we get that (3)(b)-(3)(c) are equivalent to

$$\begin{bmatrix} Q^{-1} & \frac{C^T}{2} & A^T & C^T \\ \frac{C}{2} & \frac{D+D^T}{2}+\Phi & B^T & D^T \\ A & B & Q & 0 \\ C & D & 0 & \Delta^{-1} \end{bmatrix} > 0. \tag{4}$$

Taking the Schur complement with respect to the submatrix $\begin{bmatrix} Q & 0 \\ 0 & \Delta^{-1} \end{bmatrix} > 0$, (4) is in turn equivalent to

$$\begin{bmatrix} Q^{-1} & \frac{C^T}{2} \\ \frac{C}{2} & \frac{D+D^T}{2}+\Phi \end{bmatrix} - \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} > 0$$

which finally is equivalent to (2) once $P = Q^{-1} > 0$.

### 2.2 Modeling and passivity analysis of multi-contact haptic systems

We characterize multi-contact haptic systems starting from the well-established framework of Colgate & Schenkel (1997), Miller et al. (1999)-Miller et al. (2004). In that framework, a haptic system is modeled as a sampled-data system (with sampling period $T$) resulting from the interconnection of four main components described by suitable I/O mappings (see Fig. 1): a human operator block $H$, a haptic device block $D$, a computer-simulated virtual environment $E$, and a virtual coupling $V$, whose role is to act as a controller in order to ensure the stability of the closed-loop. The mappings $H$ and $D$ are continuous-time, while $E$ and $V$ are described by discrete-time dynamical systems.

In this chapter, we fit the above framework to the case of $N$ haptic devices $D_i$, $i = 1, \ldots, N$, where each device $D_i$ is assumed to have $m_i$ degrees of freedom. One or several human operators are assumed to interact with each device and the $N$ devices are coupled through a $m-$input, $m-$output (with $m = \sum_{i=1}^{N} m_i$) virtual environment $E$ and through a virtual coupling $V$, which is described by a $m-$input, $m-$output dynamical system as well. In order to simplify our exposition, we assume the absence of delay in the computations and consider

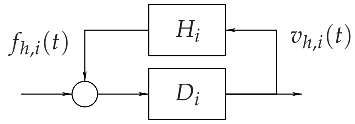


Fig. 2. Human-device interconnection

only the impedance causality representation of the haptic system (see Miller (2000)), although the proposed results are believed to be easily extendable to cover both the delayed case and admittance causality.
The interaction of each device $D_i$ with the respective human operator(s) $H_i$ can be described by the feedback loop in Fig. 2, in which $f_{h,i}(t) \in \mathbb{R}^{m_i}$ represents the generalized force vector and $v_{h,i}(t) \in \mathbb{R}^{m_i}$ is the corresponding generalized velocity vector.
It turns out that the overall system is described by the interconnection in Fig. 1, where

$$\begin{aligned} H &= \text{blockdiag}(H_1, \dots, H_N) \in \mathcal{BD}(m; m_1, \dots, m_N) \\ D &= \text{blockdiag}(D_1, \dots, D_N) \in \mathcal{BD}(m; m_1, \dots, m_N) \\ f_h(t) &= [f_{h,1}^T(t) \ \dots \ f_{h,N}^T(t)]^T \\ v_h(t) &= [v_{h,1}^T(t) \ \dots \ v_{h,N}^T(t)]^T \end{aligned} \tag{5}$$

and where $x(k) \in \mathbb{R}^m$ and $f_v(k) \in \mathbb{R}^m$ are the sampled generalized device displacement vector and sampled generalized force feedback vector, respectively.

**Remark 2.** Note that no peculiar structure is enforced a-priori on $V$ and $E$. However, due to decentralized computation requirements, it is often the case that the haptic device and virtual coupling are lumped together. This requirement can be easily taken into account by assuming that $V$ has a suitable block-diagonal structure as well. Clearly, additional requirements arising from decentralized computation and communication restrictions may enforce different contraints on $V$. For the sake of simplicity, in the sequel we will assume that $V$ may only be constrained to be block-diagonal, even though the proposed approach is simply generalizable to a wide range of different structures.

Passivity-based stability analysis of haptic systems is typically based on the assumption that both the human and the device can be seen as passive operators; in particular, when an impedance causality model is employed, the device is assumed to be OSP to some extent (Miller et al. (2000)). The OSP level pertaining to a given device can be related to the amount of damping introduced into the system by the device itself. The problem of its computation has been addressed in Miller et al. (1999) for linear devices via a robust stability argument, while in Miller et al. (2004) it is shown, using a standard Lagrangian description of the device, that the OSP level can be related to dissipation in joint space. The latter results are easily generalizable to the OSP notion in Definition 1.
Motivated by the above observations, the following assumption is made.

**Assumption 1.** (a) Each device block $D_i$ is $\Delta_{D_i}$−OSP, and (b) each human block $H_i$ is a passive continuous-time $m_i$-input, $m_i$-output operator.

The following well-known result on passivity of general interconnected systems holds also in view of the OSP notion in Definition 1.

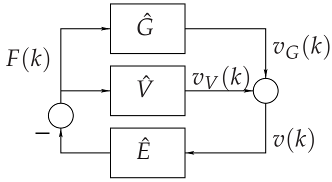


Fig. 3. Transformed haptic loop

**Lemma 3.** *Let Assumption 1 hold. Then, the feedback interconnection of $D_i$ with $H_i$ is also $\Delta_{D_i}-OSP$.*

Given the block-diagonal structure of the operators $D$ and $H$ and in view of Assumption 1 and Lemma 3, it is almost predictable that their feedback interconnection gives rise to a $\Delta_D-$OSP system where $\Delta_D$ has precisely the same structure. Indeed, the following result holds.

**Theorem 1.** *Let Assumption 1 hold. Then, the feedback interconnection of $D$ and $H$ is $\Delta_D-OSP$ where*

$$\Delta_D = \text{blockdiag}(\Delta_{D_1}, \ldots, \Delta_{D_N}). \tag{6}$$

**Proof.** For each $i$, a positive definite storage function $V(\psi_i)$ exists such that $\dot{V}_i(\psi_i(t)) < f_{h,i}^T(t)v_{h,i}(t) - v_{h,i}^T(t)\Delta_{D_i}v_{h,i}(t)$, where $\psi_i$ is the state vector of the interconnection of $H_i$ and $D_i$. Therefore, taking $V(\psi_1, \ldots, \psi_N) = \sum_{i=1}^N V_i(\psi_i)$ one gets that $V(\psi_1, \ldots, \psi_N)$ is positive definite and that

$$\begin{aligned} &\dot{V}(\psi_1(t), \ldots \psi_N(t)) \\ &< \textstyle\sum_{i=1}^N [f_{h,i}^T(t)v_{h,i}(t) - v_{h,i}^T(t)\Delta_{D_i}v_{h,i}(t)] \\ &= f_h^T(t)v_h(t) - v_h^T(t)\Delta_D v_h(t) \end{aligned}$$

with $\Delta_D$ as in (6).

Let $G$ denote the discrete-time mapping describing the ZOH-equivalent of the interconnection of $H$ and $D$ (see again Fig. 1), and consider a loop transformation which places the system into the form of Fig. 3, where

$$\hat{G} = \frac{z-1}{Tz}[G+K],\ \hat{V} = \frac{z-1}{Tz}[V-K],\ \hat{E} = \frac{Tz}{z-1}E \tag{7}$$

being $K$ a constant $m \times m$ matrix. Clearly, the loop transformation in (7) is a MIMO version of the one employed in Miller et al. (1999; 2000; 2004). The resulting interconnection of $\hat{G}$, $\hat{V}$ and $\hat{E}$ is a purely discrete-time system in which the transformed components can be characterized in terms of their OSP or ISP passivity levels according to Definition 2, leading to the following result.

**Theorem 2.** *Consider the haptic loop in Fig. 1 and its transformation in Fig. 3, with $\hat{G}$, $\hat{V}$, $\hat{E}$ as in (7). Suppose that Assumption 1 holds and let*

$$K = \frac{T}{2}\Delta_D^{-1} \tag{8}$$

*with $\Delta_D$ as in (6). Then, $\hat{G}$ is (discrete-time) $\Delta_{\hat{G}}-OSP$ with*

$$\Delta_{\hat{G}} = \Delta_D. \tag{9}$$

**Proof.** This result extends Lemma 2 in Miller et al. (2004) quite straightforwardly, and the proof is omitted.

**Remark 3.** It is worth noting that both $\Delta_{\hat{G}}$ and $\hat{G}$ itself have the block-diagonal structure of $H$ and $D$. This is a simple consequence of (9), (7),(8), and the fact that matrix inversion preserves the block-diagonal structure.

## 3. LMI stability condition

A generally accepted notion of stability in the haptic context is that the velocity $v_h(t)$ of the device must converge to zero in steady state: in turn, this condition ensures that the system is oscillation free and that all states such as the device position remain bounded.
In Miller et al. (2000) it is shown that if the interconnection of transformed blocks $\hat{G}$, $\hat{V}$, and $\hat{E}$ is asymptotically stable, then the generalized velocity vector $v_h(t)$ presented to the user(s) goes to zero in steady state. Moreover, based on standard passivity results for parallel and feedback interconnected systems, suitable bounds are provided on the (scalar) OSP/ISP levels of the transformed blocks in order to guarantee stability. This way, also the case of non-passive virtual environments (i.e., of virtual environments lacking a certain amount of ISP) can be accounted for.
The following result provides a stability criterion for the haptic system under investigation that generalizes the one in Miller et al. (2000) by exploiting the OSP and ISP notions in Definition 2. The resulting stability conditions are expressed in matrix inequality form.

**Theorem 3.** *Suppose that there exist symmetric matrices $\Delta_{\hat{V}}$ and $\Phi_{\hat{E}}$ such that*

1. *$\hat{V}$ is $\Delta_{\hat{V}}-OSP$,*
2. *$\hat{E}+\Phi_{\hat{E}}$ is passive, i.e., $\hat{E}$ is $(-\Phi_{\hat{E}})-ISP$.*
3. *the following matrix inequality holds:*

$$\begin{bmatrix} \Delta_D - \Phi_{\hat{E}} & -\Phi_{\hat{E}} \\ -\Phi_{\hat{E}} & \Delta_{\hat{V}} - \Phi_{\hat{E}} \end{bmatrix} > 0 \tag{10}$$

*Then, the signal $v_h(t)$ presented to the human operator goes to zero in steady state.*

**Proof.** Let $\psi_{\hat{G}}, \psi_{\hat{V}}, \psi_{\hat{E}}$ be the state vectors of $\hat{G}, \hat{V}, \hat{E}$, respectively. By conditions 1. and 2., there exist positive definite storage functions $V_{\hat{G}}(\psi_{\hat{G}})$, $V_{\hat{V}}(\psi_{\hat{V}})$, and $V_{\hat{E}}(\psi_{\hat{E}})$ such that

$$\begin{aligned} \Delta V_G(\psi_{\hat{G}}(k)) &\leq F^T(k)v_G(k) - v_G^T(k)\Delta_D v_G(k) \\ \Delta V_V(\psi_{\hat{V}}(k)) &\leq F^T(k)v_V(k) - v_V^T(k)\Delta_{\hat{V}} v_V(k) \\ \Delta V_E(\psi_{\hat{E}}(k)) &\leq -F^T(k)v(k) + v^T(k)\Phi_{\hat{E}} v(k) \end{aligned}$$

Taking $V_{\hat{G}+\hat{V}}(\psi_{\hat{G}}, \psi_{\hat{V}}) = V_{\hat{G}}(\psi_{\hat{G}}) + V_{\hat{V}}(\psi_{\hat{V}})$ we get

$$\begin{aligned} \Delta V_{\hat{G}+\hat{V}}(\psi_{\hat{G}}(k), \psi_{\hat{V}}(k)) \leq F^T(k)v(k) - v^T(k)\Phi_{\hat{E}} v(k) - \\ \begin{bmatrix} v_G^T(k) & v_V^T(k) \end{bmatrix} \begin{bmatrix} \Delta_{\hat{G}} - \Phi_{\hat{E}} & -\Phi_{\hat{E}} \\ -\Phi_{\hat{E}} & \Delta_{\hat{V}} - \Phi_{\hat{E}} \end{bmatrix} \begin{bmatrix} v_G(k) \\ v_V(k) \end{bmatrix} \end{aligned} \tag{11}$$

and hence, by (10)

$$\Delta V_{\hat{G}+\hat{V}}(\psi_{\hat{G}}(k), \psi_{\hat{V}}(k)) \leq F^T(k)v(k) - v^T(k)\Phi_{\hat{E}} v(k)$$

i.e., the parallel interconnection of $\hat{G}$ and $\hat{V}$ is $\Phi_{\hat{E}}-$OSP.
Finally, taking $V(\psi_{\hat{G}}, \psi_{\hat{V}}, \psi_{\hat{E}}) = V_{\hat{G}+\hat{V}}(\psi_{\hat{G}}, \psi_{\hat{V}}) + V_{\hat{E}}(\psi_{\hat{E}})$ it holds that $V(\psi_{\hat{G}}, \psi_{\hat{V}}, \psi_{\hat{E}})$ is positive definite and moreover

$$\Delta V(\psi_{\hat{G}}(k), \psi_{\hat{V}}(k), \psi_{\hat{E}}(k)) < 0$$

i.e., $V(\psi_{\hat{G}}, \psi_{\hat{V}}, \psi_{\hat{E}})$ is a Lyapunov function that proves asymptotic stability of the transformed closed-loop system, and therefore that the velocity vector $v_h(t)$ goes to zero in steady state.

## 4. Stabilizing structured virtual coupling parameterization

We are now interested in providing a computationally viable parameterization of a class of stabilizing virtual coupling systems that have a given structure. More specifically, we seek the parameterization of a set $\mathcal{V}$ of linear virtual coupling systems $V$ of given order $n$ which share the following properties:

(R1) $V$ stabilizes the haptic loop,

(R2) $V$ has an arbitrarily assigned block-diagonal structure, i.e., $V = \text{blockdiag}(V_1, \ldots, V_{\bar{N}}) \in \mathcal{BD}(m; \bar{m}_1, \ldots, \bar{m}_{\bar{N}})$, where each block $V_i$ is a linear system with state space dimension $\bar{n}_i$, $\sum_{i=1}^{\bar{N}} \bar{n}_i = n$, for given $\bar{N}, \bar{m}_i, \bar{n}_i$, $i = 1, \ldots, \bar{N}$.

**Remark 4.** The block-diagonal structure of $V$ need not necessarily be related to the structure of the device block $D$, although it may be convenient to enforce a virtual coupling structure that reflects actual implementation constraints such as controller decentralization.

Let $(A_V, B_V, C_V, D_V)$ denote a state space representation of $V$. Requirement (R2) is equivalent to the condition that $A_V \in \mathcal{BD}(n; \bar{n}_1, \ldots, \bar{n}_{\bar{N}})$, $B_V \in \overline{\mathcal{BD}}(\bar{n}_1 \times \bar{m}_1, \ldots \bar{n}_{\bar{N}} \times \bar{m}_{\bar{N}})$, $C_V \in \overline{\mathcal{BD}}(\bar{m}_1 \times \bar{n}_1, \ldots \bar{m}_{\bar{N}} \times \bar{n}_{\bar{N}})$ and $D_V \in \mathcal{BD}(m; \bar{m}_1, \ldots, \bar{m}_{\bar{N}})$.

A straightforward computation yields the following state space representation of the transformed virtual coupling $\hat{V}$:

$$\begin{aligned} A_{\hat{V}} &= \begin{bmatrix} A_V & -\frac{1}{T} B_V \\ 0 & 0 \end{bmatrix}, \quad B_{\hat{V}} = \begin{bmatrix} \frac{1}{T} B_V \\ I_m \end{bmatrix}, \\ C_{\hat{V}} &= \begin{bmatrix} C_V & -\frac{1}{T} D_V + \frac{1}{T} K \end{bmatrix}, \quad D_{\hat{V}} = \frac{1}{T} D_V - \frac{1}{T} K. \end{aligned} \tag{12}$$

We are now ready to state the main design result, which provides the parameterization of a set $\mathcal{V}$ of controllers that satisfy requirements (R1) and (R2).

**Theorem 4.** *Consider the haptic loop L, let $\Delta_D$ be the device OSP level as in (6) and $(A_{\hat{E}}, B_{\hat{E}}, C_{\hat{E}}, D_{\hat{E}})$ be a state space realization of $E(z)\frac{Tz}{z-1}$. Let $P, \Delta_{\hat{V}}, \Phi_{\hat{E}}, \Sigma_{\hat{V}}, Q, Y, A_V, B_V, C_V, D_V$ be any solution of the LMI problem*

$$\begin{aligned} &A_V \in \mathcal{BD}(n; \bar{n}_1, \ldots, \bar{n}_{\bar{N}}) \\ &C_V \in \overline{\mathcal{BD}}(\bar{m}_1 \times \bar{n}_1, \ldots \bar{m}_{\bar{N}} \times \bar{n}_{\bar{N}}) \\ &B_V \in \overline{\mathcal{BD}}(\bar{n}_1 \times \bar{m}_1, \ldots \bar{n}_{\bar{N}} \times \bar{m}_{\bar{N}}) \\ &D_V \in \mathcal{BD}(m; \bar{m}_1, \ldots, \bar{m}_{\bar{N}}) \end{aligned} \tag{13}$$

$$
\begin{array}{l}
P > 0 \\
\Delta_{\hat{V}} > 0 \\
\left[ \begin{array}{cc} A_{\hat{E}}^T P A_{\hat{E}} - P & A_{\hat{E}}^T P B_{\hat{E}} - \frac{C_{\hat{E}}^T}{2} \\ B_{\hat{E}}^T P A_{\hat{E}} - \frac{C_{\hat{E}}}{2} & B_{\hat{E}}^T P B_{\hat{E}} - \frac{D_{\hat{E}}}{2} - \frac{D_{\hat{E}}^T}{2} - \Phi_{\hat{E}} \end{array} \right] < 0 \\
\left[ \begin{array}{cc} \Delta_D - \Phi_{\hat{E}} & -\Phi_{\hat{E}} \\ -\Phi_{\hat{E}} & \Delta_{\hat{V}} - \Phi_{\hat{E}} \end{array} \right] > 0
\end{array} \tag{14}
$$

$$
\begin{array}{l}
Q > 0 \\
\left[ \begin{array}{cc} Y & R^T \\ R & S \end{array} \right] > 0
\end{array} \tag{15}
$$

*where*

$$
R = \left[ \begin{array}{c} \frac{C_{\hat{V}}}{2} \\ A_{\hat{V}} \\ C_{\hat{V}} \end{array} \right] \qquad
S = \left[ \begin{array}{ccc} \frac{D_{\hat{V}} + D_{\hat{V}}^T}{2} & B_{\hat{V}}^T & D_{\hat{V}}^T \\ B_{\hat{V}} & Q & 0 \\ D_{\hat{V}} & 0 & \Sigma_{\hat{V}} \end{array} \right] \tag{16}
$$

*being $A_{\hat{V}}, B_{\hat{V}}, C_{\hat{V}}, D_{\hat{V}}$ as in (12), with the further non-convex constraint*

$$
\Sigma_{\hat{V}} = \Delta_{\hat{V}}^{-1}, \;\; Y = Q^{-1}. \tag{17}
$$

*Then, the virtual coupling $V$ defined by the state space representation $(A_V, B_V, C_V, D_V)$ has the structure $\mathcal{BD}(m; \bar{m}_1, \ldots, \bar{m}_{\bar{N}})$ and stabilizes the haptic loop $L$.*

**Proof.** Constraints (13) enforce that the virtual coupling has the required structure. Moreover, (14) and (15),(16),(17) ensure, respectively, that the OSP level $\Delta_{\hat{V}}$ satisfies the loop stability condition of Theorem 3 for some $\Phi_{\hat{E}}$ and that it can be achieved by a virtual coupling $V$ of the given structure; the latter property follows from Lemma 2.

**Remark 5.** The set $\mathcal{V}$ parameterized via Theorem 4 clearly does not encompass all linear controllers of the given structure satisfying (R1) and (R2). Nevertheless, the proposed parameterization has the advantage of being linear in the design parameters $(A_V, B_V, C_V' D_V)$. This is a key property that allows for exploiting the proposed parameterization to address performance problems, as it will be further explained in Section 5 below.

### 4.1 Tackling the non-convex constraint (17)

The cone complementarity linearization algorithm in ElGhaoui et al. (1997) can be employed successfully to reformulate the LMI problem (13)-(16) with the additional non-convex condition (17) as a sequence of LMI optimization problems. We illustrate this reformulation for the basic problem of checking feasibility of (13)-(17), i.e., of checking the existence of a stabilizing controller of the proposed class, although this method can be seamlessly incorporated in several possible performance problems involving the controller parameterization in Theorem 4.

1. Fix a tolerance $\varepsilon$ and set $j = 1$.

2. Find a feasible solution $W, Z$ of the relaxed problem (13)-(16) with the convex constraint

$$\begin{bmatrix} W & I \\ I & Z \end{bmatrix} > 0 \tag{18}$$

where

$$W = \begin{bmatrix} \Sigma_{\hat{V}} & 0 \\ 0 & Q \end{bmatrix}, \quad Z = \begin{bmatrix} \Delta_{\hat{V}} & 0 \\ 0 & Y \end{bmatrix}, \tag{19}$$

if no solution exists, then exit and declare the problem infeasible.

3. Fix $W^j = W^{j-1}$, $Z^j = Z^{j-1}$, and find a solution $W^{j+1}$, $Z^{j+1}$ of the LMI optimization problem

$$\begin{array}{c} \min \ \ \mathrm{trace}(W^j Z + Z^j W) \\ \text{subject to (13)-(16),(18),(19)} \end{array}$$

4. If $\|W^{j+1}Z^{j+1} - I\| < \varepsilon$ then output the solution, otherwise set $j = j + 1$ and go to 3) unless a maximum number of iterations has been reached.

### 4.2 Spring-damper virtual coupling design

In order to provide a qualitative evaluation of the impact of the virtual coupling on the the sensation felt by the users during haptic interaction, it is an established practice to design the virtual coupling as a discretized spring-damper system Miller (2000); Miller et al. (2000).
In order to generalize this notion to the multi-dimensional case, let us consider a virtual coupling $V$ of the form

$$V = V(z) = \left[\mathbf{K} + \mathbf{B}\frac{z-1}{Tz}\right]^{-1} \tag{20}$$

where $\mathbf{K} \in \mathbb{R}^{m \times m}$ and $\mathbf{B} \in \mathbb{R}^{m \times m}$ will be referred to as the stiffness and damping matrix, respectively. The transfer matrix $V(z)$ in (20) models a network of interconnected virtual springs and dampers that connect the contact endpoints to each other according to the structure of matrices $\mathbf{K}$ and $\mathbf{B}$. Note that enforcing $\mathbf{K}$ and $\mathbf{B}$ to have some block diagonal structure results in the controller $V$ having the same structure.
For this special case, a simple computation shows that a state-space realization of $V$ is of order $m$ and has the form $(A_V,\ B_V,\ C_V,\ D_V)$ where

$$\begin{aligned} B_V &= TA_V \\ D_V &= TC_V \end{aligned} \tag{21}$$

and the pair $(A_V, C_V)$ is related one-to-one to the pair $(\mathbf{K}, \mathbf{B})$ as follows:

$$\mathbf{K} = \frac{1}{T}(I_m - A_V)C_V^{-1}, \quad \mathbf{B} = \ A_V C_V^{-1}. \tag{22}$$

In (22), note that $(\mathbf{K}, \mathbf{B}) \in \mathcal{BD}(m; \bar{m}_1, \dots, \bar{m}_{\bar{N}})$ if and only if $(A_V, C_V) \in \mathcal{BD}(m; \bar{m}_1, \dots, \bar{m}_{\bar{N}})$. Therefore, the parameterization of a set of stabilizing spring-damper controllers having the structure $\mathcal{BD}(m; \bar{m}_1, \dots, \bar{m}_{\bar{N}})$ is readily obtained from Theorem 4 by taking $A_V, B_V, C_V$ and $D_V$ to be matrices of $\mathcal{BD}(m; \bar{m}_1, \dots, \bar{m}_{\bar{N}})$ satisfying the linear equations (21). Obviously, the design parameters reduce to $A_V$ and $C_V$ only.

## 5. A performance problem: virtual coupling transparency

An important performance objective in a haptic system is the transparency of the virtual coupling. When a controller is required in order to guarantee stability, it is desirable that its effect on the dynamics of the virtual environment, as it is seen by the human-device block, be minimized. This amounts to the requirement that the transfer function (admittance) of the $E - V$ loop between the sampled displacement $x$ and the virtual force $f_v$ (see Fig. 1) be as close as possibile to the virtual environment admittance according to some criterion.
We have that

$$f_v = -\mathbf{F}(E, V)x$$

where $\mathbf{F}(E, V)$ is given by

$$\mathbf{F}(E, V) = (I_m + EV)^{-1}E.$$

The problem of designing a stabilizing virtual coupling $V$ ensuring the transparency condition, i.e., $\mathbf{F}(E, V) \approx E$, is most effectively formulated in the context of $H_\infty$ control theory according to two different criteria.
*Closed-loop $H_\infty$ design.* The transparency problem can be cast as the problem of finding a stabilizing $V$ such that the weighted $\infty-$norm of $(\mathbf{F}(E, V) - E)$ be less than a given (possibly minimum) $\mu$, i.e.,

$$\|\mathbf{W}_1(\mathbf{F}(E, V) - E)\mathbf{W}_2\|_\infty < \mu \tag{23}$$

where $\mathbf{W}_1$ and $\mathbf{W}_2$ are transfer functions that allow for weighing the admittance error differently at different frequencies. A similar formulation was used in Naghshtabrizi & Hespanha (2006).

*$H_\infty$ loop shaping.* The condition $\mathbf{F}(E, V) \approx E$ can be achieved by "shaping" the loop gain $EV$ so that it is small at given frequencies. This amounts to the requirement

$$\|\mathbf{L}(E, V, \mathbf{W}_1, \mathbf{W}_2)\|_\infty < \gamma \tag{24}$$

where

$$\mathbf{L}(E, V, \mathbf{W}_1, \mathbf{W}_2) = \mathbf{W}_1 EV \mathbf{W}_2 = \mathbf{W}_1 EV \mathbf{W}_2 \tag{25}$$

for given $\gamma$ and weighing functions $\mathbf{W}_1$ and $\mathbf{W}_2$.
These problems fall into the class of standard $H_\infty$ control problems, and can be solved by means of the combination of LMI-based procedures. The computational details are out of the scope of this chapter, but it is absolutely important to note that the key requirement in these schemes is that a controller parameterization which is linear in $V$ be available, as it is indeed the case with Theorem 4.

## 6. Examples

**Example 1.** *We consider a haptic system in which the interaction with a $3-$DOF linear virtual environment $E$ is performed through three $1-$DOF haptic devices $d_x$, $d_y$ and $d_z$.*
*According to our characterization, each block of the haptic loop L is described by a $m-$input, $m-$output system with $m = 3$.*
*Let the virtual environment $E$ be the backward Euler discretized version with sample period $T = 0.001$s of the mechanical system in Fig. 4, where $x_e = [x_{e,1}\ x_{e_2}\ x_{e_3}]^T$ and $f_v = [f_{v,1}\ f_{v_2}\ f_{v_3}]^T$. Assume the parameter values $B_1 = 0.1,\ B_2 = 0.2,\ B_3 = 0.3,\ B = 0.5,\ k_1 = 3800,\ k_2 = 3500,\ k_3 = 3300,\ k = 2600,\ M = 3$, in standard measurement units.*
*We assume that the three haptic devices are characterized by the OSP levels $\Delta_{d_x} = \Delta_{d_y} = \Delta_{d_z} = 1.37$.*

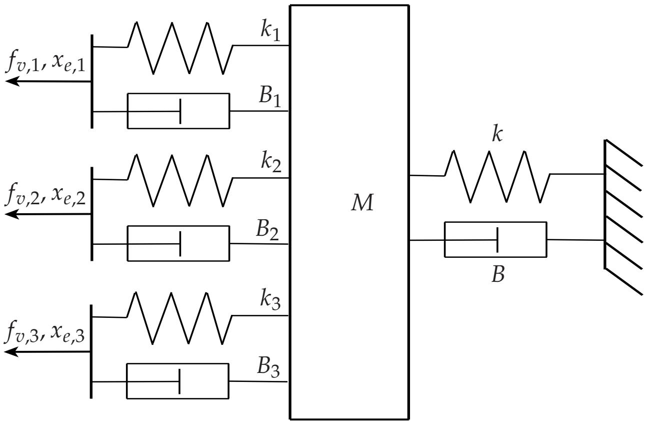


Fig. 4. Example 1: Virtual environment model

*These values are computed according to the results in Miller et al. (2004) from the identified dynamics along one axis of Force Dimension's Omega*™ *device.*
*In the absence of virtual coupling, the haptic system is not stable for all passive human operator blocks as shown by the simulation in Figure 5, in which the human operator is modeled by the following passive continuous-time linear system*

$$H(s) = \begin{bmatrix} \frac{0.02(s^2+60s+1000)}{s^2+10.3s+100} & 0.1 & 0.1 \\ 0.1 & \frac{0.02(s^2+60s+1000)}{s^2+10.3s+100} & 0.1 \\ 0.1 & 0.1 & \frac{0.02(s^2+60s+1000)}{s^2+10.3s+100} \end{bmatrix}.$$

*We look for a decentralized spring-damper virtual coupling as described in Section 4.2.*
*A feasible solution is given by*

$$\mathbf{K} = \begin{bmatrix} 704.9058 & 0 & 0 \\ 0 & 704.9336 & 0 \\ 0 & 0 & 704.8754 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.0372 & 0 & 0 \\ 0 & 0.0372 & 0 \\ 0 & 0 & 0.0372 \end{bmatrix}. \tag{26}$$

*Without enforcing the decentralization constraint on $V$, we get the following feasible solution.*

$$\mathbf{K} = \begin{bmatrix} 660.4367 & -0.2614 & -0.3282 \\ -0.2614 & 660.4181 & -0.3429 \\ -0.3282 & -0.3429 & 660.4602 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0.1674 & -0.0003 & -0.0005 \\ -0.0003 & 0.1685 & -0.0006 \\ -0.0005 & -0.0006 & 0.1675 \end{bmatrix}. \tag{27}$$

*Figures 6 and 7 show simulations of the haptic system with decentralized and centralized virtual coupling as in (26) and (27).*

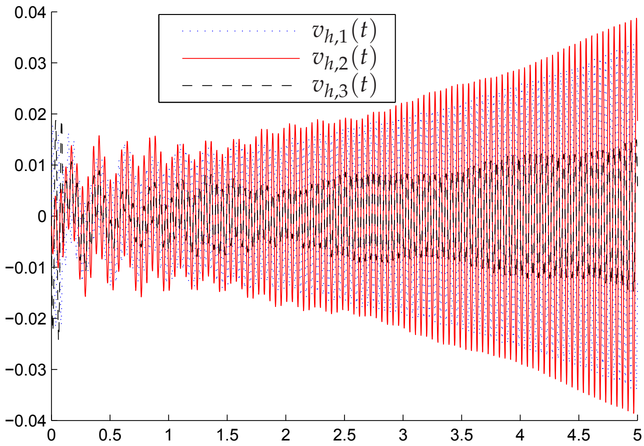


Fig. 5. Example 1: simulation of haptic loop dynamics (velocity vector) without virtual coupling.

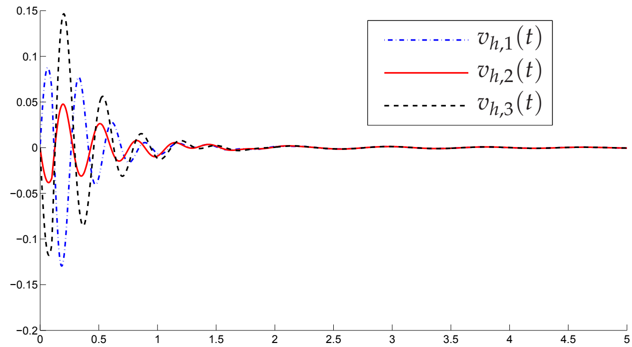


Fig. 6. Example 1, case (i): Simulation of haptic loop dynamics (velocity vector) with decentralized virtual coupling (26).

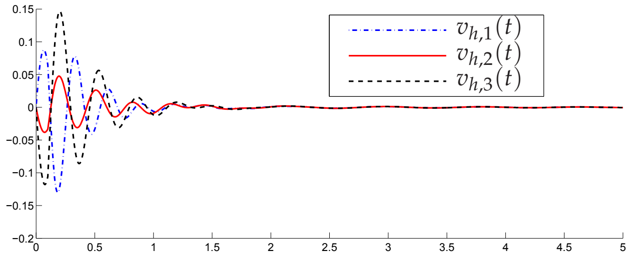


Fig. 7. Example 1, case (i): simulation of haptic loop dynamics (velocity vector) with centralized virtual coupling (27).

**Example 2.** *Let us consider case (i) in the previous example. For this case, we are interested in computing a (decentralized) spring-damper controller that stabilizes the haptic loop while optimizing controller transparency. To this purpose, we solve the loop shaping problem in (24),(25). In particular, we try to minimize $\gamma$ such that (24),(25) hold subject to the proposed controller parameterization. As weighing functions we choose*

$$\mathbf{W}_1 = w(z)I_3, \quad \mathbf{W}_2 = I_3$$

*where $w(z)$ is the backward Euler discretization of the first-order filter*

$$w(s) = \frac{1}{1 + s/\omega_0}, \quad \omega_0 = 125$$

*so that the virtual coupling transparency requirement is emphasized in the frequency range $0-20$ Hz. By proceeding by bisection on $\gamma$ and iteratively solving the loop shaping problem, solutions are found down to the value $\gamma = 12.6$, and the corresponding virtual coupling is given by:*

$$\begin{array}{c} \mathbf{K} = \begin{bmatrix} 2.7355 & 0 & 0 \\ 0 & 2.7346 & 0 \\ 0 & 0 & 2.7355 \end{bmatrix} 10^3, \\ \mathbf{B} = \begin{bmatrix} -0.2648 & 0 & 0 \\ 0 & -0.2798 & 0 \\ 0 & 0 & -0.2838 \end{bmatrix} 10^{-3} \end{array} . \tag{28}$$

*A simulation is shown in Fig. 8.*

## 7. Conclusion

In this chapter, the problem of stability assessment and virtual coupling design for haptic interaction systems involving multiple devices and human operators has been addressed in a passivity-based LMI framework. A class of stabilizing virtual coupling controllers which can be parameterized via a sequence of LMI problems has been introduced. Such a class is quite flexible, since it allows for taking into account decentralization constraints imposed on the control system. Finally, the solution of the controller transparency problem within this class has been outlined.

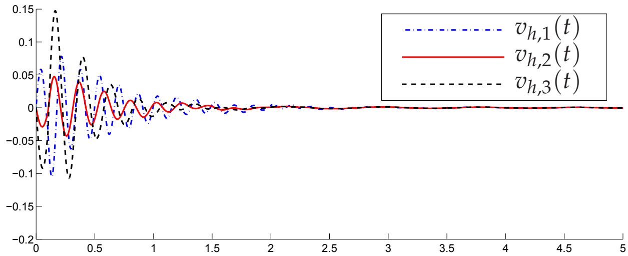


Fig. 8. Example 2: simulation of haptic loop dynamics (velocity vector) with the virtual coupling in (28).

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Haptic interfaces are divided into two main categories: force feedback and tactile. Force feedback interfaces are used to explore and modify remote/virtual objects in three physical dimensions in applications including computer-aided design, computer-assisted surgery, and computer-aided assembly. Tactile interfaces deal with surface properties such as roughness, smoothness, and temperature. Haptic research is intrinsically multi-disciplinary, incorporating computer science/engineering, control, robotics, psychophysics, and human motor control. By extending the scope of research in haptics, advances can be achieved in existing applications such as computer-aided design (CAD), tele-surgery, rehabilitation, scientific visualization, robot-assisted surgery, authentication, and graphical user interfaces (GUI), to name a few. Advances in Haptics presents a number of recent contributions to the field of haptics. Authors from around the world present the results of their research on various issues in the field of haptics.