Strategy-proof aggregation rules and single peakedness in bounded distributive lattices

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Abstract

It is shown that, under a very comprehensive notion of single peakedness, an aggregation rule on a bounded distributive lattice is strategy-proof on any rich domain of single peaked total preorders if and only if it admits one of three distinct and mutually equivalent representations by lattice-polynomials, namely whenever it can be represented as a *generalized weak consensus rule*, a *generalized weak sponsorship rule*, or an *iterated median rule*.

The equivalence of individual and coalitional strategy-proofness that is known to hold for single peaked domains in bounded linearly ordered sets and in finite trees typically *fails* in such an extended setting. A related impossibility result concerning non-trivial anonymous and coalitionally strategy-proof aggregation rules is also obtained.

Keywords: Strategy-proofness, Single Peakedness, Bounded Distributive Lattice, Aggregation Rule, Polynomials, Median.

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1 Introduction

The present work considers *strategy-proof aggregation rules* on a *very general* class of single peaked preference domains for a *very comprehensive* class of outcome spaces, i.e. bounded distributive lattices, and shows that they all admit several equivalent *polynomial closed-form characterizations* (this is the content of Theorem 1 below).[1]

Many decision problems involving several agents can be represented as elicitation-and-aggregation tasks where the items to be elicited and aggregated are *private* information tokens of a suitably specified type. Usually, those tokens are alternative items in the relevant outcome space (e.g. scores,

[1] Polynomial closed-form representations of aggregation rules and their significance are discussed below (see notes 7 and 20).

grades, signals, preferences, judgments, databases, locations of a physical environment or nodes of an abstract network), to be submitted by the agents. If the agents are in fact stakeholders and entertain nonverifiable 'preferential attitudes' on the items of the outcome space, then a reliable and effective decision protocol should be reputedly *strategy-proof,* i.e. immune to advantageous *individual* manipulations through submission of *false* information. Furthermore, if the agents have access to cheap communication facilities allowing them to coordinate their choices, then a reliable decision protocol should also be reputedly *coalitionally strategy-proof*, namely immune to *jointly* advantageous manipulations on the part of *coalitions* of voters. Thus, the agents' preferences on the outcome space are clearly involved in the process, but most decision protocols typically require that agents only provide information on their most preferred outcomes. Our paper focuses precisely on that kind of protocol, namely aggregation rules as opposed to social choice functions.[2] The reasons for such a choice are further discussed in subsection 1.1 below.

As it happens, *dictatorial* and *constant* rules are two available families of (coalitionally) strategy-proof aggregation rules for any domain of topped preference preorders, but they are of course scarcely appealing due to their intrinsic systematic biases and the resulting massive loss of information they bring about. Unfortunately, the well-known Gibbard-Satterthwaite theorem implies that if the domain of admissible preferences includes *every possible linear order* on the outcome space and the range of a strategy-proof aggregation rule includes *at least three* alternatives, then that aggregation rule is a dictatorial rule.

However, it turns out that a much wider and interesting range of strategy-proof aggregation rules is available for *rich* -i.e. suitably large- domains of *single peaked*[3] preference preorders. *Single peaked preference preorders* are preorders with a *unique top element* or maximum that arise in a natural way whenever each agent's representation of the outcome space is endowed with some 'natural' ternary *betweenness* relation establishing for any two outcomes x, y whether an arbitrary outcome z lies between x and y or not. The most appropriate and 'natural' interpretation of that betweeness relation is the following: 'outcome z lies between outcomes x and y' if and only if z is (commonly regarded as) a 'genuine compromise' between x and y. Namely, the betweenness relation is meant to represent *a shared structure of compromises between outcomes.* Thus, single peaked preorders are aptly described as those preorders with a *unique best outcome* that *respect such betweenness relation* i.e. are consistent with the 'compromise-structure' it represents.[4]

[2] Notice, however, that once the relevant preference domain is unambiguously specified aggregation rules can also be regarded as 'top-only' social choice functions (see subsection 1.1 below for more details).

[3] 'Single peakedness' will be used as a general non-technical term that admits of several specifications to be discussed below.

[4] Thus, concerning the betweenness relation the present work follows the 'classic' tradition that can be traced back at least to Black (1948) and was largely taken for granted in the early social choice theoretic literature: *the relevant betweenness relation is required to be invariant across agents hence unique,* modeling a representation of the outcome space structure that is entirely *shared* by all the involved parties. Remarkable contributions which are on the contrary consistent with an *agent-dependent betweenness* relation include Border and Jordan (1983), Bandelt and Barthélemy

However, several specifications of betweenness and single peakedness are in fact available. The specific notion of single peakedness we use in the present work -a quite general one, indeed- is described and justified in subsection 1.2 below.

1.1 Aggregation Rules and Social Choice Functions

It should be emphasized that *aggregation rules* are quite often encountered in the literature on collective decision protocols under several labels (including 'voting schemes', 'voting rules', 'social choice rules', 'aggregators', 'consensus functions'), and in fact encompass Arrowian 'social welfare functions' as a special case, but are certainly *not* the most commonly used model in those works that focus on *strategy-proofness properties* of decision mechanisms: *social choice functions* are. Thus, a few clarifying comments on our modeling choice are in order here.

While social choice functions map *profiles of preference relations defined on the outcome-space* into outcomes, aggregation rules map *outcome-profiles* into outcomes. Hence, when the outcome-profile is finite-dimensional (which is indeed the case in the present setting), the latter amount to algebraic operations on the outcome space. Aggregation rules have been extensively studied in a few remarkable and seminal attempts to extend the classic Arrowian framework (see e.g. Wilson (1975), Rubinstein and Fishburn (1986), Monjardet (1990)), and are a focal model in the now burgeoning field of *judgment aggregation* (see Endriss (2017) for a recent comprehensive survey). Moreover, outcome-profiles to be aggregated can be naturally regarded as lists of best options according to the agents' preferences. Hence, aggregation rules qualify as a 'natural' model for most voting mechanisms that, in order to save on information communication and processing costs, typically ask participants to communicate a *unique choice* among outcomes rather than a *preference ranking* of all outcomes. Of course, the study of strategy-proofness properties of aggregation rules requires a previous specification of a preference domain: in that connection, single-peaked preferences provide -as mentioned above- a very natural domain if agents have unique optima and share a betweenness relation describing compromises between outcomes.

Once the relevant preference domain has been specified, an obvious one-to-one correspondence between aggregation rules and 'top-only' social choice functions (namely, those social choice functions whose values only depend on profiles of top outcomes) is established. As a result, it becomes tempting to conflate aggregation rules and 'top-only' social choice functions since after all an aggregation rule and the corresponding 'top-only' social choice function compute the same function: this can be done, *hence the present work may also be regarded as a contribution to the study of 'top-only' social choice functions*.

Nevertheless, it should be emphasized again that from the perspective of communication and information-processing costs an aggregation rule is a protocol that provides an *efficient* implementa-

(1984), Peters, van der Stel and Storcken (1992), Peremans, Peters, van der Stel and Storcken (1997), Chichilnisky and Heal (1997), Bordes, Laffond and Le Breton (2011).

tion of the corresponding 'top-only' social choice function.[5] That is the main reason why the current work resists the foregoing conflation, and its results are presented in terms of aggregation rules (rather than of their 'top-only' social choice functions counterparts).[6]

1.2 Betweenness and single-peakedness in bounded distributive lattices

Generally speaking, single peaked preorders are those preorders with a *unique best outcome* that *respect* -are consistent with- *such betweenness relation*. It should be emphasized that such a broad description of single peakedness is in fact compatible with *several distinct* specifications of the domain of single peaked preference relations.

At least two salient issues require further preliminary clarification, namely:

(a) what is to be meant by a 'natural' ternary betweenness relation?

(b) what is precisely meant by 'consistency of preferences with the relevant betweenness relations'?

Concerning the first or 'betweenness-specification' issue, several choices are available here: the earliest works start from *bounded linearly ordered sets* (e.g. Black (1948), Moulin (1980)), while some subsequent contributions move to *finite products of bounded linearly ordered sets* (e.g. Barberà, Gul and Stacchetti (1993)) and to *finite trees* (Danilov (1994). A somewhat more general approach is proposed by Nehring and Puppe (2007 (a),(b)) who start from a certain *'property space'* on a finite outcome set. Most recently that approach has been furtherly extended to *infinite* outcome sets by Anno (2014). An even more general approach starts from an *arbitrary interval space* (see e.g. Vannucci (2016 (a)).

The present paper covers a sort of middle ground on 'betweenness-specification', focussing on the *betweenness relation induced by a bounded distributive lattice.*[7] Indeed, distributive lattices are a very common structure which generalize linearly ordered sets and share with trees the important property of admitting a well-defined ternary *median operation.* Relevant examples of aggregation problems

[5] To be sure, one might perhaps object to that statement on the comparative informational efficiency of aggregation rules along the following lines. While 'top-only' social choice functions impose a much heavier burden than the corresponding aggregation rules in terms of communication complexity and information-processing costs, they also elicit -if strategy-proof- a much larger amount of private information. Hence, the alleged superiority of aggregation rules with respect to 'top-only' social choice functions on informational efficiency grounds is in fact disputable. This objection, however, is moot. Indeed, strategy-proof 'top-only' social choice functions, by definition, do *not* use such supplementary amount of information to compute the 'socially best choice'. Now, if the entire preference ranking of each agent is socially relevant for reasons *other than* computation of the 'socially best choice' then presumably the corresponding aggregate 'social ranking' of alternatives is also socially relevant. Hence, arguably, the appropriate outcome space should consist of the possible *rankings* of alternatives. But then, Arrowian social welfare functions (i.e. a special class of aggregation rules) seem to be the most suitable model for that situation.

[6] Clearly enough, focussing on (strategy-proof) aggregation rules or 'top-only' social choice functions amounts to ignoring the class of (strategy-proof) *non* 'top-only' social choice functions, and consequently raises the following question: how significant is that restriction? We shall have something to say on that issue in Section 4 below.

[7] A distributive lattice is a partially ordered set such that any two elements admit a least upper bound and a greatest lower bound that mutually 'distribute' on each other, i.e. interact much like set-theoretic union and intersection.

having that structure include *multipart grading in education and competitions*, *panel selection of multidimensional poverty thresholds*, *behavioral preference aggregation*.[8] Take, for instance, multipart grading processes: a particularly interesting example of that is *judgement aggregation* in a political context when the assessment of candidates consists in the allocation of grades expressed in a common language. In this case, we have a (bounded) linearly ordered set of grades Λ=$(L, \leq)$, a (finite) population X of candidates to be evaluated, and a (finite) population N of evaluators. Therefore, the set of all possible gradings of X is L^X, which admits the point-wise partial order $\leqslant$ induced by $\leq$. The resulting partially ordered set of complete gradings of candidates is then a product of linearly ordered sets, i.e. $\mathcal{X}$ =$(L^X, \leqslant)$, which is a typical instance of a bounded distributive lattice. Hence, the natural betweenness relation is the one induced by the lattice of possible gradings.

This is indeed the formal setting recently proposed by Balinski and Laraki (2010) in order to advance their case for *majority judgment.* Alternative applications of the very same set-up could consist of aggregating grades achieved by a population of students in different subjects, assessments of wines according to several alternative graded criteria, or the graded performances of participants in a multi-trial competition.

In this particular setting, a proper definition of single-peakedness should arguably require (i) respect for betweenness whenever top-outcomes are concerned, while (ii) allowing for indifference among distinct non top-outcomes even if one of them is located between the other one and the top-outcome.

This kind of single-peakedness is indeed covered in the present paper *but it is typically not* in the extant literature.

Outcome spaces consisting of products of bounded linearly ordered sets have been widely studied in the literature on strategy-proof aggregation rules and social choice functions for single-peaked domains (see e.g. Barberà, Gul and Stacchetti (1993) and Nehring and Puppe (2007(a))). The present paper provides a characterization in algebraic or polynomial closed form (to be shortly discussed below) of all the aggregation rules for single-peaked preference domains on that kind of outcome space, under a very general notion of single-peakedness.[9]

As explained below, that choice is largely dictated by the principal aim of the present paper, namely generalizing or extending some characterizations of the strategy-proof aggregation rules *on 'rich' single peaked preference domains* that also provide a *closed-form (* actually, an *algebraic* or *polynomial)* description of such rules, along the lines of Moulin (1980), Danilov (1994) and a few

[8] A rather comprehensive list of possible applications is detailed in Annex 2.

[9] Previous contributions (e.g. Barbera, Gul and Stacchetti (1993) and Nehring and Puppe (2007a) provide higher-order characterizations that are not in algebraic closed form and with reference to a specialized version of single-peakedness, that is indeed a special case of our own notion. Earlier classic characterizations of strategic-proof rules in algebraic closed form, such as Moulin (1980) and Danilov (1994), concern other, more specialized, outcome spaces as discussed below.

other scholars.[10]

Concerning the second or 'consistency' issue we adopt a very broad notion denoted here as *local unimodality*, namely the requirement that individual preference relations have a unique maximum or top outcome and be such that an outcome located between the maximum and another distinct outcome is invariably regarded as *not worse* than the latter. Indeed, local unimodality encompasses the most widely used versions of single peakedness (see Remark 1 below).

1.3 Main results

As a result of the foregoing three major modelling choices, the present work focusses on the aggregation rules that are strategy-proof on an arbitrary *rich* -i.e. suitably 'large'- domain of locally unimodal total preorders[11] on a bounded distributive lattice: the aim of the paper is to characterize the entire class of such aggregation rules and represent them in *algebraic* closed form, along the lines of the seminal works by Moulin (1980) and Danilov (1994) concerning respectively bounded linearly ordered sets and finite trees.

Several characterizations of the entire class of strategy-proof aggregation rules on any rich locally unimodal domain are provided, generalizing or extending virtually all previously known results of that kind. In particular, *three distinct algebraic closed-form descriptions* of such aggregation rules are given in terms of certain lattice polynomials in both disjunctive and conjunctive normal form, and of certain iterated lattice-median polynomials. As mentioned above, polynomials amount to efficient algorithms, so the outputs of polynomial rules are by definition 'easily' computed. Moreover, distinct if equivalent polynomial representations of the same strategy-proof aggregation rule allow a wider perspective on the actually available strategy-proof protocols for any given decision problem. For instance, disjunctive normal form polynomials include meets of the outcomes proposed by members of a fixed coalition, and conjunctive normal form polynomials include joins of the outcomes proposed by members of a fixed coalition. So we have *two* versions of *collegial* rules, and Theorem 1 entails that are both strategy-proof. Similarly, the simple majority rule turns out to have three equivalent representations as (i) the join of the meets or consensus-outcomes of all majority coalitions, (ii) the meet of all the outcomes that are sponsored by members of some majority coalition, (iii) a suitably iterated median between outcome-triples. Conceivably, each one of those equivalent representations might be regarded as best suited for different decision problems. Quite remarkably, the *simple ma-*

[10] A *closed-form* expression admits finitary operations and -if real numbers are involved-, a restricted class of functions including logarithm, exponent, trigonometric functions: on the contrary, intrinsically infinitary operators such as limits and integrals are excluded. An *algebraic* (or, equivalently, *polynomial*) closed-form expression only admits finitary operations and is therefore highly convenient. Indeed, the values of an aggregation rule that has an algebraic closed-form representation are easily computable (for any argument or profile) by executing a well-defined and uniform finite sequence of steps.

[11] Strictly speaking, our results hold for any rich domain of locally unimodal preorders that are complete on the range of the aggregation rules taken into consideration.

jority rule (that is well-known to be strategy-proof and coalitionally strategy-proof on both unimodal and locally strictly unimodal domains in bounded linearly ordered sets) *is confirmed to belong to the strategy-proof class* even in the present wider setting. On the other hand, *it will also be shown that in a very large class of bounded distributive lattices that are not linearly ordered sets, and under minimal neutrality*[12] *requirements, no non-trivial anonymous aggregation rule is coalitionally strategy-proof on the foregoing class of single-peaked domains.* Our characterization results unify (generalizing or extending, and bringing together) several notions, approaches and results from the extant literature, namely:

- The characterizations contributed by the present paper generalize Moulin's original closed-form characterization of strategy-proof aggregation rules on the *full unimodal*[13] domain of total preorders in *bounded linearly ordered sets* to *any rich locally unimodal* domain in *any bounded distributive lattice*. Thus, we also obtain both a *disjunctive normal form* and a *conjunctive normal form lattice-polynomial*[14] representation of such rules. The latter is a generalized version of the min-max representation produced by Moulin (1980) for the special case of bounded linearly ordered sets.

- The disjunctive normal form lattice-polynomial representation mentioned above is in turn a generalization of 'latticial federation consensus functions' or, equivalently, of 'generalized committee aggregation rules' as introduced respectively, and independently, by Monjardet (1990) in his path-breaking contribution to (non-strategic) aggregation problems in (semi-)latticial structures, and by Barberà, Sonnenschein and Zhou (1991) and Barberà, Gul and Stacchetti (1993) in their study of *'top-only' strategy-proof social choice functions* on a specialized rich single-peaked (namely full locally strictly unimodal) domain in finite Boolean distributive lattices and in products of bounded linearly ordered sets, respectively (both of those outcome-structures being of course special cases of bounded distributive lattices).

- As mentioned above, our characterization also highlights the equivalence of that lattice-polynomial representation to another representation of strategy-proof aggregation rules on rich locally unimodal domains as *iterated median polynomials*, namely as *finite compositions of median, projection and constant operations.* Our proof relies on the fact that such iterated median polynomials can be conveniently represented as the *behavior maps* of certain *median tree-automata* acting on suitably labelled trees by computing a sequence of values of a *latticial median* ternary operation. Indeed, that tree-automata-theoretic representation essentially amounts to a streamlining and

[12]An aggregation rule is neutral with respect to a certain pair of outcomes when it treats them in an unbiased manner.

[13]A single peaked domain of a certain type is said to be *full* if it includes *all* the single peaked preferences of that type.

[14]A disjunctive normal form lattice-polynomial is a finite sequence of disjunctions (i.e. joins) of finite meets. Dually, a conjunctive normal form lattice-polynomial is a finite sequence of conjunctions (i.e. meets) of finite joins.

extension of the approach pioneered by Danilov (1994) in his remarkable representation of the strategy-proof aggregation rules on unimodal domains of linear orders in finite trees precisely as finite superpositions of median, projection and constant operations.[15]

- Finally, it is also proved that the equivalence between strategy-proofness and coalitional strategy-proofness - that is known to hold for rich locally unimodal domains in bounded linearly ordered sets and for unimodal domains of linear orders in finite trees- *fails* for both unimodal and locally strictly unimodal domains in bounded distributive lattices that are *not* linearly ordered sets,[16] hence *even in outcome spaces with a well-defined median operation* (Theorem 2). An impossibility theorem concerning coalitional strategy-proofness on rich unimodal or strict locally unimodal domains for anonymous voting rules satisfying very weak local sovereignty and neutrality requirements (Theorem 3) is also provided.[17] Thus, in particular, Theorem 3 implies that the equivalence between strategy-proofness and coalitional strategy-proofness established by Moulin (1980), Danilov (1994) for *other* structures endowed with a ternary median operation such as bounded linearly ordered sets and trees cannot be extended to arbitrary bounded distributive lattices *even for non-sovereign aggregation rules.*[18]

The remainder of the paper is organized as follows. The next section introduces notation and definitions. The main results of the paper are collected in Section 3. Section 4 is devoted to a detailed discussion of some related literature and offers some concluding remarks. Appendix 1 collects all the proofs. Annex 1 includes a detailed presentation of the basic notions on tree automata alluded to in the paper. Annex 2 provides a list of interesting examples of outcome spaces having the structure of a bounded distributive lattice. Annex 3 illustrates the results with an example concerning the Boolean square. Annex 4 is devoted to an extensive discussion of the related literature.

[15] As a matter of fact, such use of medians was foreshadowed in the representation of *anonymous* strategy-proof rules via generalized extended medians due to Moulin (1980).

[16] Vannucci (2016) provides a general incidence-geometric argument to explain that equivalence-failure.

[17] An aggregation rule is said to be sovereign if it is onto.

[18] It is indeed well-known that a multidimensional structure of the outcome space tends to undermine coalitional strategy-proofness of sovereign aggregation rules even for single peaked domains. However, Danilov's result as mentioned above implies that sovereign coalitionally strategy-proof aggregation rules do exist for locally unimodal domains in finite trees. Now, the order dimension of a finite tree may be 2 or even 3 (see e.g. Trotter and Moore (1977); recall that the order dimension of a partially ordered set $(A, \leqslant)$ is the minimum number of linear orders on A whose intersection is $\leqslant$) . So, multidimensionality cannot be the whole story underlying failure of coalitional strategy-proofness for sovereign aggregation rules. In that connection, Theorem 2 provides some useful information because it implies that coalitional strategy-proofness on a locally unimodal domain may fail for sovereign strategy-proof rules even in the Boolean square $\mathbf{2}^2$ (with $\mathbf{2} = (\{0,1\}, \leq)$), that has order dimension 2. Hence, Theorem 2 also entails that the key point here is not just (multi)dimensionality of the outcome space: subtler, more specific details of its order-theoretic structure also play a significant role here as implied by Theorem 3 itself (see also Remark 5 in Section 3 for further elaboration on that point).

2 The model: notation and definitions

The focus of the present work is the class of *strategy-proof aggregation rules* on certain *singled peaked preference domains* as defined on *outcome spaces with a bounded distributive latticial structure*. Accordingly, let us introduce one by one the main components of our model, and their basic structures.

I. **The outcome space: bounded distributive lattices.**

The outcome space is a **distributive lattice** $\mathcal{X} = (X, \leqslant)$, namely X is a set and $\leqslant$ is a partial order i.e. a reflexive, transitive and antisymmetric binary relation on X such that the following conditions hold:[19]

(i) (General lattice properties) for any $x, y \in X$ both the *least-upper-bound* (*l.u.b.*) or *join* $\vee$ and the *greatest-lower-bound (g.l.b.)* or *meet* $\wedge$ of x and y are well-defined binary operations on X. In particular, notice that $x \vee y = y$ and $x \wedge y = x$ hold if and only if $x \leqslant y$;[20]

(ii) (Distributivity) for all $x, y, z \in X$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (or, equivalently, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$) hold.[21]

A (distributive) lattice $\mathcal{X}$ is said to be **upper (lower) bounded** if there exists $\top \in X$ ($\bot \in X$) such that $x \leqslant \top$ ($\bot \leqslant x$) for all $x \in X$, and **bounded** -written $\mathcal{X}$ =$(X, \leqslant, \top, \bot)$ if it is both upper bounded and lower bounded.[22]

Order filters and several classes of filters of distributive lattices will also be employed in the proof of Claim 1 below and in the definition of some aggregation rules, and are therefore to be introduced here. An **order filter** of a distributive lattice $\mathcal{X}$ =$(X, \leqslant)$ is a set $Y \subseteq X$ such that for each $x \in X$ if there exists $y \in Y$ with $y \leqslant x$ then $x \in Y$. An order filter Y of $\mathcal{X}$ is a **filter** if $y \wedge z \in Y$ for all $y, z \in Y$. Moreover, a filter Y of $\mathcal{X}$ is **prime** if for any $y, z \in X$, if $y \vee z \in Y$ then either $y \in Y$ or $z \in Y$.[23]

Bounded distributive lattices can be represented in a few equivalent ways in terms of some alternative sets of algebraic operations. Any such set of algebraic operations characterizes distributive lattices and specifies a corresponding class of *polynomials*.[24] Since the focus of the present work is

[19] For a thorough introduction to lattice theory see Davey and Priestley (1990).

[20] It is well-known and easily checked that the joint and the meet thus defined satisfy associativity, commutativity, idempotency and absorption, (see definitions in the text below). Notice that thanks to associativity of $\vee$ and $\wedge$ the l.u.b and the g.l.b of *any finite* $Y \subseteq X$ are also well-defined and denoted by $\vee Y$ and $\wedge Y$, respectively. If Y is infinite, $\vee Y$ and $\wedge Y$ may or may not be well-defined.

[21] A linear order is a partial order $\leqslant$ such that $[x \leqslant y$ or $y \leqslant x]$ holds for all $x, y \in X$. Notice that a linearly ordered set, i.e. a partially ordered set $(X, \leqslant)$ such that $\leqslant$ is a linear order, does indeed satisfy the distributive identity above, and is therefore a special instance of a distributive lattice.

[22] A bounded distributive lattice $(X, \leqslant, \bot, \top)$ is *Boolean* if for each $x \in X$ there exists a *complement* namely an $x' \in X$ such that $x \vee x' = \top$ and $x \wedge x' = \bot$.

[23] Dually, an *order ideal* of $\mathcal{X}$ is a subset $Y \subseteq X$ such that for each $x \in X$ if there exists $y \in Y$ with $x \leqslant y$ then $x \in Y$. An order ideal Y of $\mathcal{X}$ is an *ideal* if and $y \vee z \in Y$ for all $y, z \in Y$, and an ideal Y of $\mathcal{X}$ is *prime* if for any $y, z \in X$, if $y \wedge z$ then either $y \in Y$ or $z \in Y$.

[24] Recall that the class of *polynomial functions* or polynomials of a certain algebra (or set endowed with a list of

precisely on *polynomial representations of the strategy-proof aggregation rules in bounded distributive lattices*, an explicit formulation of the relevant algebraic representations of distributive lattices is absolutely required.

The most common algebraic formulation of *all* distributive lattices (either bounded or not) is by far the one based on the *join* and *meet* operations: thus, a distributive lattice is a structure $\mathcal{X}' = (X, \vee, \wedge)$ where $\vee : X^2 \to X$ and $\wedge : X^2 \to X$ are *two binary operations* on X with the following properties:

(Associativity) : $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ for all $x, y, z \in X$;

(Commutativity): $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$ for all $x, y \in X$;

(Idempotency): $x \vee x = x$ and $x \wedge x = x$ for all $x \in X$;

(Absorption): $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$ for all $x, y \in X$;

(Distributivity): $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in X$.[25]

It is a remarkable fact that a ternary operation $\mu : X^3 \to X$ called *median* can be defined on any *bounded* distributive lattice $\mathcal{X}' = (X, \vee, \wedge)$ by the following rule:[26] for all $x, y, z \in X$,

$$\mu(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z).$$

The median μ of a bounded distributive lattice $\mathcal{X}'$ is indeed characterized by the following properties:

(μ_1): $\mu(x, x, y) = x$ for all $x, y \in X$;

(μ_2): $\mu(\mu(x, y, v), \mu(x, y, w), z) = \mu(\mu(v, w, z), x, y)$;

(μ_3): there exist $0, 1 \in X$ such that $\mu(0, x, 1) = x$ for all $x \in X$ (clearly, if $\mathcal{X}' = (X, \vee, \wedge, \top, \bot)$, then $1 = \top$ and $0 = \bot$ and, as it is easily checked, both $x \vee y = \mu(x, y, \top)$ and $x \wedge y = \mu(x, y, \bot)$ hold for all $x, y \in X$).

Therefore, a bounded distributive lattice can also be represented as an algebraic structure $\mathcal{X}'' = (X, \mu)$ with *one ternary operation* -the *median* μ- that satisfies the three properties (μ_1),(μ_2),(μ_3) listed above.

A ternary (latticial) **betweenness** relation $B_{\mathcal{X}} = \{(x, z, y) \in X^3 : x \wedge y \leqslant z \leqslant x \vee y\}$ is defined on a distributive lattice $\mathcal{X} = (X, \leqslant)$, and for any $x, y \in X$, $[x, y] = \{z \in X : x \wedge y \leqslant z \leqslant x \vee y\}$ is the

operations defined over some *finite* list of elements), is the smallest class of functions which includes the projections (selections of a single argument out of any lists of arguments of a given operation) and is closed with respect to every finite composition of operations of the algebra. Hence, the values of a polynomial function are consistently computed by performing a finite and uniform sequence of elementary operations.

[25] The partial order structure on $(X, \leqslant)$ induced by a distributive lattice $(X, \vee, \wedge)$ is defined as follows: for any $x, y \in X$, $x \leqslant y$ if and only if $y = x \vee y$ or equivalenty $x = x \wedge y$.

[26] The median of a bounded distributive lattice was first introduced by Birkhoff and Kiss (1947) who also provided a general characterization of that operation. The Birkhoff-Kiss axiom system for μ was subsequently simplified by Sholander who established the characterization mentioned in the text, based upon μ_1, μ_2 and μ_3 (Sholander (1952)), and provided a further characterization of the median of an arbitrary distributive lattice along the following lines (Sholander (1954)).

interval induced by x and y : therefore, *for any* $x, y, z \in X$, $z \in [x, y]$ *if and only if* $(x, z, y) \in B_{\mathcal{X}}$ (*also written* $B_{\mathcal{X}}(x, z, y)$), i.e. $[x, y] = \{z \in X : B_{\mathcal{X}}(x, z, y)\} = B_{\mathcal{X}}(x, ., y)$. In particular, a subset $Y \subseteq X$ is $B_{\mathcal{X}}$-***convex*** if for any $x, y, z \in X$, $B_{\mathcal{X}}(x, y, z)$ and $\{x, z\} \subseteq Y$ entail $y \in Y$.[27]

A few remarkable basic properties of $B_{\mathcal{X}}$ are listed under Claim 1 in Section 3 below.

It should be emphasized again that examples of interesting and relevant outcome spaces with a bounded distributive latticial structure abound, ranging from locations on a bounded grid or discrete ordered characteristic space to admissible gradings in learning or sport competitions, and feasible committees or poverty thresholds.[28]

II**. Preference domains: single peakedness.**

Now, consider the set T_X of all *topped* preorders on X (i.e. reflexive and transitive binary relations having a unique maximum in X). For any $\succcurlyeq \in T_X$, $top(\succcurlyeq)$ denotes the unique maximum of $\succcurlyeq$.[29] For any $Y \subseteq X$, $\succcurlyeq$ is Y**-complete** if for each $y, y' \in Y$ either $y \succcurlyeq y'$ or $y' \succcurlyeq y$ (or both), and **total** if it is X-complete. For any $x \in X$, $UC(\succcurlyeq, x) := \{y \in X : y \succcurlyeq x\}$ denotes the upper contour of $\succcurlyeq$ at x. *Single peaked* (total) preorders are those topped total preorders that 'respect' -i.e. are consistent with- the betweenness relation $B_{\mathcal{X}}$. The relevant notion of $B_{\mathcal{X}}$-consistency, however, is amenable to several distinct specifications: we shall use a very general one as defined below.

Definition 1. Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice. A topped preorder $\succcurlyeq \in T_X$ -with top outcome x^*- is **locally unimodal** (with respect to $B_{\mathcal{X}}$) if and only if, for each $y, z \in X$, $z \in B_{\mathcal{X}}(x^*, ., y)$ implies that $z \succcurlyeq y$. We denote by $\widehat{U}_{\mathcal{X}}$ the set of all locally unimodal preorders, and by $U_{\mathcal{X}} \subseteq T_X$ set of all locally unimodal *total preorders* (with respect to $B_{\mathcal{X}}$).

As mentioned above, the rationale underlying single peakedness as *local unimodality* may be plainly described as follows: a locally unimodal preference preorder *respects* betweenness $B_{\mathcal{X}}$ in that it never regards an intermediate or compromise outcome between the top outcome and another outcome as strictly worse than the latter.

Remark 1. It is worth noticing here that in the extant literature on single-peakedness two alternative specialized versions of local unimodality are most typically used. One of them, that we shall denote as **unimodality**, is a strengthening of local unimodality which consists in enlarging its scope to an arbitrary triple $x, y, z \in X$ (no requirement that x be the top outcome): the classic Moulin (1980) employs precisely that notion[30]. The second one, that we shall denote as **local strict**

[27] The following analysis could be pursued by replacing entirely betweenness relations with intervals (see Vannucci (2016) for such an approach in a more general setting).

[28] A more detailed list of examples is provided in Annex 2.

[29] We denote with $\succ$ and $\sim$ the asymmetric and symmetric components of $\succcurlyeq$, respectively.

[30] To be sure, in bounded linear orders local unimodality and unimodality are equivalent. Thus, Moulin's definition of single peakedness in Moulin (1980) as reported below in the following note can also be taken as an early advocacy of local unimodality. In any case, local unimodality qualifies as *the* natural extension of Moulin's definition of single peakedness to an arbitrary bounded distributive lattice.

unimodality (but is often labelled -somewhat misleadingly- as 'generalized single peakedness'), is an alternative strengthening of local unimodality that consists in requiring that the intermediate z be *strictly* preferred to y (i.e. that $z \succ y$ rather than just $z \succcurlyeq y$ hold). We shall denote by $U^*_{\mathcal{X}}$, $S_{\mathcal{X}}$ the *set of all unimodal (locally strictly unimodal, respectively) total preorders* with respect to $B_{\mathcal{X}}$. It is also worth mentioning here that unimodal and locally strictly unimodal preferences are not always firmly distinguished as they should be. For instance, in a very interesting and widely cited paper Nehring and Puppe (2007b, p.135) quote Moulin (1980) as a contribution on 'generalized single peaked' (i.e. locally strictly unimodal) preferences in the case of a line (but see also Barberà, Gul and Stacchetti (1993) who identify single peakedness and locally strict unimodality, suggesting that this is precisely the notion underlying Moulin's work, and provide a characterization of strategy-proof rules on the locally strictly unimodal domain in products of bounded intervals).[31]

We shall mostly focus on locally unimodal domains of preorders that satisfy a suitable richness condition, as made precise by the following:

Definition 4. A set $D_{\mathcal{X}} \subseteq U_{\mathcal{X}}$ of locally unimodal total preorders (with respect to $B_{\mathcal{X}}$) is **minimally rich** if for any $x \in X$ there exists $\succcurlyeq \in D_{\mathcal{X}}$ such that $top(\succcurlyeq) = x$ and **rich** if, for any $x, y \in X$, there exists $\succcurlyeq \in D_{\mathcal{X}}$ such that $top(\succcurlyeq) = x$ and $UC(\succcurlyeq, y) = B_{\mathcal{X}}(x, ., y)$. Moreover, a rich set $D_{\mathcal{X}} \subseteq U_{\mathcal{X}}$ is **strictly rich** if there exists a rich set $D'_{\mathcal{X}} \subseteq D_{\mathcal{X}}$ consisting of locally strictly unimodal preorders.[32]

Thus, a locally unimodal domain $D_{\mathcal{X}}$ is rich whenever for each pair of outcomes x, y there exists a preference relation in $D_{\mathcal{X}}$ having x as its top outcome and such that the subset of outcomes (weakly) preferred to y is precisely the interval between x and y. It should be noticed here that for each

[31] Indeed, Moulin's definition, once reformulated in terms of preferences (as opposed to utilities, as in the original Moulin (1980), p. 439) amounts to the following requirement: 'If a is the top outcome or peak on the line $(X, \leqslant)$ then $a \succ x \succcurlyeq y$ if $a < x \leqslant y$ or $y \leqslant x < a$.' Notice however that this condition is only consistent with *unimodality* as opposed to locally strict unimodality. To see this just consider $X = \{a, x, y\}$ with $a < x < y$, and total preorder $\succcurlyeq$ such that $a \succ x \sim y$: by construction, $\succcurlyeq$ is certainly consistent with Moulin's condition, and it is in fact unimodal but **not** at all locally strictly unimodal (or 'generalized single peaked'). Indeed, under the common 'single peakedness' label, Moulin (1980) and Danilov (1994) focus on unimodal preferences, while Barberà, Gul and Stacchetti (1993) and Nehring and Puppe (2007) consider locally strictly unimodal preferences.

To be sure, Nehring and Puppe's approach to single-peakedness via betweenness relations of finite property spaces (see Nehring and Puppe 2007(a), (b)) does provide a generalization of most notions of single-peakedness in the previous literature. But such a generalization amounts to the definition of single-peakedness with respect to betweenness relations *other than* the usual betweenness relations of linear orders, trees or distributive lattices. Nevertheless, when applied to (bounded) distributive lattices, Nehring and Puppe's notion of single-peakedness is incomparable to Moulin's, hence it is *not* a generalization of the latter.

[32] In words, 'richness' requires that for any pair of outcomes (x, y), there exists a locally unimodal preference such that x is its top element and its upper contour at y consists precisely of the outcomes that lie between x and y, that is essentially the standard richness condition used in the literature concerning single-peakedness (see e.g. Nehring and Puppe (2007 (a), (b)). Notice that a set D of preference relations on X is sometimes said to be 'minimally rich' whenever it only satisfies property (i).

$x, y \in X$ one such rich locally unimodal preorder $\succcurlyeq^*_{x,y}$ with three indifference classes is easily defined as follows: take $\{x\}$, $B_{\mathcal{X}}(x,.,y)\setminus\{x\}$, and a subset of $X\setminus B_{\mathcal{X}}(x,.,y)$ to be the top, middle, and bottom indifference classes of $\succcurlyeq^*_{x,y}$, respectively.

III. **Aggregation rules and strategy-proofness.**

Let $N = \{1, ..., n\}$ denote the finite population of agents and $\mathcal{P}(N)$ the set of all subsets of N or coalitions: an *aggregation rule* for (N, X) is a function $f : X^N \to X$. Hence, $U^N_{\mathcal{X}}$ denotes the corresponding set of all N-*profiles* $(\succcurlyeq_i)_{i\in N}$ of locally unimodal total preorders or *full locally unimodal domain.* Moreover, for any $(\succcurlyeq_i)_{i\in N} \in U^N_{\mathcal{X}}$, we denote by *top*$((\succcurlyeq_i)_{i\in N})$ the corresponding profile of maxima *top*$(\succcurlyeq_i)$. Similarly, $U^{*N}_{\mathcal{X}}$ and $S^N_{\mathcal{X}}$ denote the corresponding *full* unimodal and *full* locally strictly unimodal domains.

The following strategy-proofness properties of an aggregation rule will play a pivotal role in the present analysis:

Definition 5. For any $i \in N$, let $D_i \subseteq U_{\mathcal{X}}$ be minimally rich. Then, an aggregation rule $f : X^N \to X$ is (individually) **strategy-proof** on $\Pi_{i\in N} D_i \subseteq U^N_{\mathcal{X}}$ if and only if, for all $x_N \in X^N$, $i \in N$ and $x'_i \in X$, and for all $\succcurlyeq_i \in D_i$, $f(top(\succcurlyeq_i), x_{N\setminus\{i\}}) \succcurlyeq_i f(x'_i, x_{N\setminus\{i\}})$.

Definition 6. For any $i \in N$, let $D_i \subseteq U_{\mathcal{X}}$ be minimally rich. Then, an aggregation rule $f : X^N \to X$ is **coalitionally strategy-proof** on $\Pi_{i\in N} D_i \subseteq U^N_{\mathcal{X}}$ if and only if for all $x_N \in X^N$, $C \subseteq N$ and $x'_C \in X^C$, and for all $\succcurlyeq = (\succcurlyeq_j)_{j\in N} \in \Pi_{i\in N} D_i$ such that *top*$((\succcurlyeq_i)_{i\in N}) = x_N$, there exists $i \in C$ such that $f(x_N) \succcurlyeq_i f(x'_C, x_{N\setminus C})$.

Clearly, not all strategy-proof aggregation rule are otherwise appealing. As mentioned in the Introduction, two notable classes of strategy-proof voting rules are the *projections* (or *dictatorial rules)* $\pi_i : X^N \to X$, $i \in N$ where for all $x_N \in x^N$, $\pi_i(x_N) = x_i$, and the *constant* rules $f_x : X^N \to X$, $x \in X$ where for all $x_N \in X^N$, $f_x(x_N) = x$. In order to single out 'nice' strategy-proof rules we shall mainly rely on the following benchmark properties.

An aggregation rule $f : X^N \to X$ is **anonymous** if $f((x_j)_{j\in N})) = f((x_{\sigma(j)})_{j\in N}))$ for all $x_N \in X^N$ and any permutation σ of N; **idempotent** or **unanimity-respecting** if $f(x_N) = x$ for each $x_N \in X^N$ such that $x_i = x$ for all $i \in N$; and **sovereign** (or **onto**) if for all $x \in X$ there exists $x_N \in X^N$ such that $f(x_N) = x$ (it is well-known and easily checked that idempotence implies sovereignty by definition, and is in turn implied by sovereignty under strategy-proofness).

As mentioned in the Introduction, the main objective of the present work is to characterize the *entire class* of strategy-proof aggregation rules on single peaked domains, and to represent them in a suitable *algebraic closed form* along the lines of previous results on (more specialized) single peaked domains in linear orders due to Moulin (1980) or on (related but incomparable) single peaked domains in finite trees due to Danilov (1994).

The following condition on aggregation rules for distributive lattices requiring consistence of the aggregation rule and the latticial betweenness relation will play a crucial role in our main characterization result.

Definition 7. Let $\mathcal{X}$ =$(X, \leqslant)$ be a distributive lattice. An aggregation rule $f : X^N \to X$ is $B_{\mathcal{X}}$**-monotonic** if and only if for all $x_N = (x_j)_{j\in N} \in X^N$, $i \in N$ and $x'_i \in X$: $f(x_N) \in [x_i, f(x'_i, x_{N\setminus\{i\}})]$, i.e. $B_{\mathcal{X}}(x_i, f(x_N), f(x'_i, x_{N\setminus\{i\}}))$.

Thus, an aggregation rule is $B_{\mathcal{X}}$**-monotonic** whenever the outcome that an agent obtains by submitting a certain outcome x lies between x itself and the outcome that the agent would obtain by submitting another outcome (for any fixed profile of proposals/submissions on the part of the other agents). In other terms, f 'respects' the betweenness relation $B_{\mathcal{X}}$.

Remark 2. As noticed above, two classes of strategy-proof aggregation rules are the *projections* (or *dictatorial rules)* $\pi_i : X^N \to X$, $i \in N$ where for all $x_N \in x^N$, $\pi_i(x_N) = x_i$, and the *constant* rules $f_x : X^N \to X$, $x \in X$ where for all $x_N \in X^N$, $f_x(x_N) = x$. It is also easily checked that *both dictatorial and constant rules are* $B_{\mathcal{X}}$*-monotonic by Closure of* $B_{\mathcal{X}}$ as defined and proved in Claim 1 of Section 3 below.

Three classes of polynomial aggregation rules corresponding to the main algebraic presentations -$(X, \vee, \wedge)$ and (X, μ)- of *bounded* distributive lattices will be considered, namely **generalized weak committee rules**, **generalized weak sponsorship rules** and **iterated median rules**. It will be shown that those three classes are indeed *equivalent* and provide *alternative algebraic closed-form descriptions of the entire class of aggregation rules which are strategy-proof on rich single peaked domains in an arbitrary bounded distributive lattice.*

Definition 8. A **generalized committee** in N is a set of coalitions $\mathcal{C} \subseteq \mathcal{P}(N)$ such that $T \in \mathcal{C}$ if $T \subseteq N$ and $S \subseteq T$ for some $S \in \mathcal{C}$ (a *committee* in N being a *non-empty* generalized committee in N which does *not* include the *empty* coalition).[33]

Definition 9. Let $\mathcal{X}$ =$(X, \leqslant)$ be a bounded distributive lattice. A **generalized weak consensus rule** is a function $f : X^N \to X$ such that, for some fixed family $\{z_S : z_S \in X\}_{S\in\mathcal{P}(N)}$, and for all $x_N \in X^N$, $f(x_N) = \vee_{S\in\mathcal{P}(N)}((\wedge_{i\in S}x_i) \wedge z_S)$.

A special case of generalized weak consensus rule is a **generalized consensus rule**, namely a function $f : X^N \to X$ such that, for some fixed generalized committee $\mathcal{C} \subseteq \mathcal{P}(N)$ and for all $x_N \in X^N$, $f(x_N) = \vee_{S\in\mathcal{C}}(\wedge_{i\in S}x_i)$.

Dually, we have:

Definition 10. Let $\mathcal{X}$ =$(X, \leqslant)$ be a bounded distributive lattice. A **generalized weak sponsorship rule** is a function $f : X^N \to X$ such that, for some fixed family $\{z_S : z_S \in X\}_{S\in\mathcal{P}(N)}$, and for all $x_N \in X^N$, $f(x_N) = \wedge_{S\in\mathcal{P}(N)}((\vee_{i\in S}x_i) \vee z_S)$.

A special case of a generalized weak sponsorship rule is a **generalized sponsorship rule**, namely

[33] Thus, a generalized committee is just an *order filter* of the partially ordered set $(\mathcal{P}(N), \subseteq)$ of coalitions of N.

a function $f : X^N \to X$ such that, for some fixed generalized committee $\mathcal{C} \subseteq \mathcal{P}(N)$ and for all $x_N \in X^N$, $f(x_N) = \wedge_{S \in \mathcal{C}}(\vee_{i \in S} x_i)$.

Thus, a *generalized consensus rule* is an aggregation rule that selects the join of the consensus-outcomes of the coalitions of a prefixed set of 'winning' coalitions.[34] Dually, a *generalized sponsorship rule* selects the meet of the joins of outcomes sponsored by some member of each coalition of a prefixed set of 'winning' coalitions. A *generalized weak consensus rule* (*generalized weak sponsorship rule)* works as a generalized consensus rule (generalized sponsorship rule) except for the fact that *every coalition* is involved and it also admits characteristic and possibly non-trivial *ceilings*[35] of its admissible consensus-outcomes (*floors* on the largest proposal sponsored by some its members). Generalized consensus rules and generalized sponsorship rules correspond to the special case in which the coalitions which do not belong to a certain fixed committee get the bottom element $\perp$ as their ceiling (the top element $\top$ as their floor). Notice that both generalized (weak) consensus rules and generalized (weak) sponsorship rules are instances of closed-form -indeed algebraic- aggregation rules since they are simple lattice polynomials involving the join and meet operations.[36]

A third class of polynomial rules that we shall denote as *iterated median rules* may be defined starting from the algebraic representation of a (bounded) distributive lattice as a structure $\mathcal{X}'' = (X, \mu, \top, \perp)$.

An *iterated median rule* $f : X^N \to X$ assigns to each outcome-profile $x_N \in X^N$ the output of a certain *nested sequence of medians* $\mu(...\mu(\mu(u, x_i, z), x_i, \mu(u', x_i, z'))...)$ starting with medians of projections x_i of x_N, $i = 1, ..., n$ and of the values of f at the 2^n extremal profiles $y_N \in \{\top, \perp\}^N$ following the instructions specified by the non-terminal nodes of a finite labelled tree. That is made more precise by the following:

Definition 11. (**Iterated median rules**) Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice. Then, an aggregation rule $f : X^N \longrightarrow X$ is an **iterated median rule** if for all $x_N \in X^N$,

$$f(x_N) = \mu[\overbrace{\mu(...\mu(}^{n-1-times} f(\perp, ..., \perp, \perp), x_1, f(\perp, ..., \perp, \top))...), x_n, \overbrace{\mu(...\mu(}^{n-1-times} f(\top, ..., \top, \perp), x_1, f(\top, ..., \top, \top)))...)].$$

Thus, an aggregation rule $f : X^N \longrightarrow X$ is an *iterated median* rule if at any profile x_N its value $f(x_N)$ is a *certain n-iterated median of the profile-projections x_i, $i \in N$, and of constants given by the values of f at extremal profiles $z_N \in \{\perp, \top\}^N$*. To put it in another equivalent terms, iterated median

[34] Indeed, any term $\wedge_{i \in S} x_i$ can be regarded as the consensus of coalition S at profile x_N in the following sense: if one identifies each outcome z with the set of the extensional properties $\mathbb{P}_y = \{u \in X : u \leqslant y\}$ with $y \in X$, then $\wedge_{i \in S} x_i$ is precisely the outcome which satisfies the largest possible set of properties that are satisfied by all the x_i with $i \in S$.

[35] A ceiling is non-trivial if it is *not* the top outcome $\top$ of $\mathcal{X}$.

[36] Notice that *generalized (weak) consensus rules* can also be regarded as *generalized max-min operators*, and *generalized weak sponsoring rules* as *generalized min-max operators* (as a matter of fact, the latter min-max representation was used in the original characterization of strategy-proof aggregation rules on full single peaked domains in bounded linearly ordered sets due to Moulin (1980): see also Monjardet (1990)).

rules are parameterized by their values at the extremal profiles. A detailed and general formulation of the sequence of medians involved is best explained by regarding iterated median rules as the behavior maps of certain tree automata (see Adámek and Trnková (1990) for a thorough treatment of tree automata, and Annex 1 for all the relevant details).

The example below provides a very simple illustration of all of the above with $n = 3$.

Example 1. Let $\mathcal{X} = (X, \leqslant, \top, \bot)$ be a bounded distributive lattice. *An iterated median rule* $f : X^3 \to X$ for $(\{1, 2, 3\}, X)$ is defined as follows: for any $x = (x_1, x_2, x_3) \in X^3$,

$$\begin{aligned} f(x_N) \quad = \quad & \mu(\mu(\mu(f(\bot,\bot,\bot), x_1, f(\bot,\bot,\top)), x_2, \mu(f(\bot,\top,\bot), x_1, f(\bot,\top,\top))), \\ & x_3, \mu(\mu(f(\top,\bot,\bot), x_1, f(\top,\bot,\top)), x_2, \mu(f(\top,\top,\bot), x_1, f(\top,\top,\top)))). \end{aligned}$$

The following diagram summarizes the workings of f:[37]

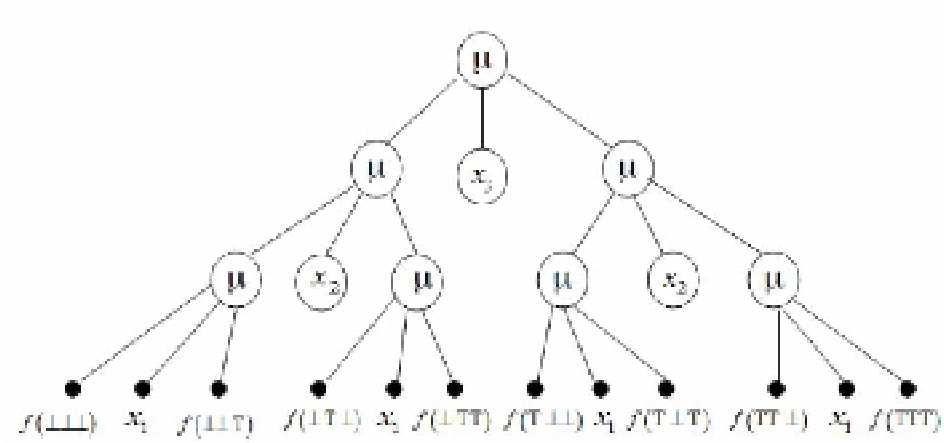

Figure 1

3 Results

We are now ready to state the main result of this paper establishing a few alternative characterizations of strategy-proof aggregation rules on rich locally unimodal profiles as: (a) the $B_{\mathcal{X}}$-monotonic rules, (b) the generalized weak consensus rules, (c) the generalized weak sponsorship rules, (d) the iterated median rules. To start with, we need to introduce a few basic properties of the betweenness relation $B_{\mathcal{X}}$ that will be widely used in the relevant proofs to follow. Such properties are collected under the following:

Claim 1. Let $\mathcal{X}$ =$(X, \leqslant)$ be a distributive lattice. Then, the latticial betweenness relation $B_{\mathcal{X}}$ satisfies the following conditions:

[37] We thank Mariateresa Ciommi for assistance in drawing the present figure.

(i) *Symmetry*: for all $x, y, z \in X$, if $B_{\mathcal{X}}(x, z, y)$ then $B_{\mathcal{X}}(y, z, x)$;

(ii) *Closure* (or Reflexivity): for all $x, y \in X$, $B_{\mathcal{X}}(x, x, y)$ and $B_{\mathcal{X}}(x, y, y)$;

(iii) *Idempotence*: for all $x, y \in X$, $B_{\mathcal{X}}(x, y, x)$ only if $y = x$;

(iv) *Convexity* (or Transitivity): for all $x, y, z, u, v \in X$, if $B_{\mathcal{X}}(x, u, y)$, $B_{\mathcal{X}}(x, v, y)$ and $B_{\mathcal{X}}(u, z, v)$ then $B_{\mathcal{X}}(x, z, y)$;

(v) *Antisymmetry*: for all $x, y, z \in X$, if $B_{\mathcal{X}}(x, y, z)$ and $B_{\mathcal{X}}(y, x, z)$ then $x = y$.

(vi) *Separation*: for all $x, y, z \in X$, **not** $B_{\mathcal{X}}(x, y, z)$ implies that $y \notin H \supseteq \{x, z\}$ for some *half-space* H of $\mathcal{X}$, namely for some non-empty and $B_{\mathcal{X}}$-convex $H \subseteq X$ such that $X \setminus H$ is also non-empty and $B_{\mathcal{X}}$-convex.

(vii) *Median-Equivalence*: for all $x, y, z \in X$, $B_{\mathcal{X}}(x, z, y)$ if and only if $\mu(x, z, y) = z$.

Our characterization result relies on the following three lemmas.

The first lemma simply establishes the equivalence between *$B_{\mathcal{X}}$-monotonicity* with respect to an arbitrary distributive lattice $\mathcal{X}$ and *strategy-proofness* of an aggregation rule $f : X^N \to X$ on any rich locally unimodal domain $\Pi_{i \in N} D_i \subseteq \widehat{U}_{\mathcal{X}}^N$ of $f\left[X^N\right]$-complete preorders, where $f\left[X^N\right] = \left\{x \in X : \text{ there exists } x_N \in X^N \text{ such that } x = f\left(x_N\right)\right\}$.

Lemma 1. Let $\mathcal{X} = (X, \leqslant)$ be a distributive lattice, $f : X^N \to X$ an aggregation rule for (N, X), and $\Pi_{i \in N} D_i \subseteq U_{\mathcal{X}}^N$ a rich locally unimodal domain of $f\left[X^N\right]$-total preorders. Then, the following statements are equivalent:

(i) f is *$B_{\mathcal{X}}$-monotonic;*

(ii) f is strategy-proof on $\Pi_{i \in N} D_i$.

Remark 3. Lemma 1 above extends Lemma 1 of Danilov (1994) (concerning linear orders in a finite tree that are unimodal with respect to tree-betweenness since, as it is easily checked, the latter satisfies the properties of $B_{\mathcal{X}}$ required by our proof, namely Symmetry, Closure, Idempotence and Convexity).

Observe that a restricted aggregation rule may be strategy-proof on its restricted unimodal domain while being not monotonic (i.e. the implication from (ii) to (i) of the previous lemma does not hold in general for restricted aggregation rules).[38] It should also be noticed that Lemma 1 can be extended to any rich locally unimodal domain of $f\left[X^N\right]$-complete preorders.

The next lemma ensures that in an arbitrary distributive lattice the median operation as applied to aggregation rules does preserve $B_{\mathcal{X}}$-monotonicity.

Lemma 2. Let $\mathcal{X} = (X, \leqslant)$ be a distributive lattice, and $f : X^N \to X$, $g : X^N \to X$, $h : X^N \to X$ aggregation rules that are $B_{\mathcal{X}}$-monotonic. Then $\mu(f, g, h) : X^N \to X$ (where $\mu(f, g, h)(x_N) = \mu(f(x_N), g(x_N), h(x_N))$ for all $x_N \in X^N$) is also $B_{\mathcal{X}}$-monotonic.

Finally, the next lemma - that only concerns *bounded* distributive lattices - provides an *iterated median representation* of all $B_{\mathcal{X}}$-monotonic aggregation rules hence - in view of Lemma 1 above - of

[38] A simple example is available from the authors upon request.

all strategy-proof voting rules on the corresponding full unimodal domain.

Lemma 3. Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice and $f : X^N \to X$ a $B_{\mathcal{X}}$*-monotonic aggregation rule*. Then, f is an iterated median rule .

The main implications of the foregoing lemmas are indeed summarized by the following:

Theorem 1. Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice, $B_{\mathcal{X}}$ its latticial betweenness relation, $f : X^N \to X$ an aggregation rule for (N, X), and $D_i \subseteq U_{\mathcal{X}}$ a rich domain of locally unimodal $f\left[X^N\right]$-complete preorders for each $i \in N$. Then, the following statements are equivalent:

(i) f is $B_{\mathcal{X}}$-monotonic;

(ii) f is strategy-proof on $\Pi_{i \in N} D_i$*;*

(iii) f is an iterated median rule;

(iv) f is a generalized weak consensus rule;

(v) f is a generalized weak sponsorship rule**.**

Remark 4. It should be emphasized again that Theorem 1 generalizes Moulin's characterization of strategy-proof aggregation rules on (full) unimodal domains in bounded chains to rich locally unimodal domains in arbitrary bounded distributive lattices. Thus, it also offers a direct extension to all bounded distributive lattices of Moulin's original min-max representation of strategy-proof aggregation rules. Moreover, the generalized weak consensus rules provide an explicit and slightly generalized polynomial version of the alternative characterization via families of 'left-coalition systems' on (full) locally strictly unimodal domains in products of bounded chains due to Barberà, Gul and Stacchetti (1993), which relies heavily on the product-structure of the underlying lattices. In particular, Theorem 1 implies strategy-proofness of the simple majority rule on rich locally unimodal domains (with an odd population of voters), since it can be quite easily shown that the former is $B_{\mathcal{X}}$-monotonic (see e.g. Monjardet (1990) for a formal definition and study of the simple majority -or extended median-rule in a semi-latticial framework). It follows that in an arbitrary bounded distributive lattice there exist many nicely 'inclusive' aggregation rules, including of course the simple majority rule which jointly satisfies anonymity (i.e. symmetric treatment of voters), neutrality (i.e. symmetric treatment of outcomes), unanimity (i.e. faithful respect of unanimity of votes) and strategy-proofness on any rich locally unimodal domain. Notice that the class of single peaked domains used in our characterizations is larger than the class of domains considered by both Nehring and Puppe (2007 (a),(b)) and Anno (2014). Thus, *when it comes to bounded distributive lattices,* our *polynomial* characterizations of strategy-proof aggregation rules have a *strictly larger scope* than the Nehring-Puppe-Anno non-polynomial combinatorial (and higher-order) characterizations.

It can also be established, however, that strategy-proofness and coalitional strategy-proofness of an aggregation rule are *not* equivalent on arbitrary rich locally unimodal domains in bounded distributive lattices. This is made precise by the following:

Theorem 2. Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice. Then the following holds: if $|X| \geq 4$ and $\mathcal{X}$ is not a linearly ordered set, then there exists a bounded sublattice $\mathcal{Y} = (Y, \leqslant_Y)$ of $\mathcal{X}$ (with $|Y| \geq 4$) and an aggregation rule $f' : Y^N \to Y$ that is strategy-proof on $U_{\mathcal{Y}}^N$ and on $S_{\mathcal{Y}}^N$ but *not* coalitionally strategy-proof on $U_{\mathcal{Y}}^N$ or on $S_{\mathcal{Y}}^N$.

Notice that if $f : X^N \to X$ is strategy-proof on $U_{\mathcal{X}}^N$ and $|X| \leq 3$ then f is also coalitionally strategy-proof on $U_{\mathcal{X}}^N$: that implication follows from a straightforward adaptation of the proof of Theorem 1 of Barberà, Berga and Moreno (2010) to aggregation rules as combined with Proposition 1 of the same paper.

Moreover, as a further straightforward consequence of Theorem 2 (and of previously known results), we have the following:

Corollary 1. Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice. Then the following statements are equivalent:

(i) for each sublattice $\mathcal{Y} = (Y, \leqslant_{|Y})$ of $\mathcal{X}$ and each aggregation rule $f : Y^N \to Y$, f is strategy-proof on $U_{\mathcal{Y}}^N$ (on $S_{\mathcal{Y}}^N$, respectively) if and only if it is also coalitionally strategy-proof on $U_{\mathcal{Y}}^N$ (on $S_{\mathcal{Y}}^N$, respectively);

(ii) $\mathcal{X} = (X, \leqslant)$ is a linearly ordered set.

Thus, we have here a remarkable characterization of *bounded linearly ordered sets* as *the only bounded distributive lattices where equivalence of individual and coalitional strategy-proofness of aggregation rules on any rich unimodal or strict locally unimodal domains holds.*

Indeed, the failure of equivalence between simple and coalitional strategy-proofness pointed out by Theorem 2 is readily extended to an impossibility result concerning availability of non-trivial (i.e. non-constant) anonymous and coalitionally strategy-proof aggregation rules for rich unimodal domains and locally strictly unimodal domains in a very general class of bounded distributive lattices, even if (full) sovereignty is dropped. In order to make it precise a few more definitions are needed.

A *join irreducible* element of a distributive lattice $\mathcal{X} = (X, \leqslant)$ is any $j \in X$ such that $j \neq \wedge X$ and for any $Y \subseteq X$ if $j = \vee Y$ then $j \in Y$. The set of all join irreducible elements of $\mathcal{X}$ is denoted $J_{\mathcal{X}}$. An *atom* of a lower bounded $\mathcal{X}$ is any $\leqslant$-minimal $x \in X \setminus \{\perp\}$, (i.e. $x \in X$, $x \neq \perp$, and there is no $y \in X \setminus \{\perp\}$ such that $y \leqslant x$ and $y \neq x$). The set of all atoms of X is denoted $A_{\mathcal{X}}$: clearly, $A_{\mathcal{X}} \subseteq J_{\mathcal{X}}$.

An aggregation rule $f : X^N \to X$ is (weakly) **efficient** on preference domain $D_{\mathcal{X}}^N \subseteq \widehat{U}_{\mathcal{X}}^N$ if and only if for all $(\succcurlyeq_j)_{j \in N} \in D_{\mathcal{X}}^N$ and $y \in X$, $y \notin f((top(\succcurlyeq_j)_{j \in N}))$ if there exists $x \in X$ such that $x \succ_j y$ for all $j \in N$; **locally join-irreducible-neutral** on $Y \subseteq X$ if $f((\tau_{jk}(x_i))_{i \in N}) = \tau_{jk}(f((x_i)_{i \in N}))$ for all $x_N \in Y^N$ and every pair of distinct joint irriducible elements $j, k \in J_{\mathcal{X}} \cap Y$ (where $\tau_{jk} : Y \to Y$ is the elementary permutation of Y such that $\tau_{jk}(j) = k$, $\tau_{jk}(k) = j$ and $\tau_{jk}(x) = x$ for any $x \neq j, k$); **locally unanimity-respecting** on $Y \subseteq X$ if $f(y_N) = z$ for each $y_N \in Y^N$ such that $y_i = z$ for all $i \in N$; **locally sovereign** on $Y \subseteq X$ if for all $z \in Y$ there exists $y_N \in Y^N$ such that $f(y_N) = z$.

Theorem 3. Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice with at least two distinct atoms $x, z \in X$ and $\mathcal{Y} = (Y, \leqslant_Y)$ the sublattice of $\mathcal{X}$ induced by the restriction of $\leqslant$ to $Y = \{0, x, z, x \vee z\}$. Then, there is no anonymous aggregation rule $f : X^N \to X$ that is locally sovereign and locally join-irreducible-neutral on Y, and coalitionally strategy-proof on $U_{\mathcal{X}}^N$, or on $S_{\mathcal{X}}^N$.

Thus, in sharp contrast to what happens in bounded linearly ordered sets, no minimally 'nice' anonymous and coalitionally strategy-proof aggregation rules are available on standard full unimodal or locally strictly unimodal domains *in bounded distributive lattices with at least two atoms,* including Boolean k-hypercubes with $k > 1$,[39] even if the simple majority aggregation rule is well-defined, and (weak) efficiency or even (full) sovereignty are *not* required at all.

Remark 5. Clearly, Theorem 3 also holds for the Boolean square $\mathbf{2}^2$ (see note 33) whose order-dimension is precisely 2.[40] Notice, however, that Danilov (1994) (as discussed at length in the next Section) implies the existence of anonymous and sovereign strategy-proof aggregation rules on full unimodal domains in arbitrary finite trees. Since there exist (planar) finite trees of dimension 2 and 3 (see e.g. Trotter and Moore (1977)), it follows that multidimensionality cannot be the whole story underlying failures of non-trivial coalitional strategy-proofness for aggregation rules. Theorem 3 suggests that what is key here is the existence of at least *two* atoms (hence of cycles of the covering graph of the lattice, i.e. of the graph having the elements of the lattice as nodes, and connecting two distinct elements if and only if one of them is larger than the other and no further element lies between the two of them). To put it in other terms, Theorem 3 forcefully suggests that the source of the problem that prevents the existence of non-trivial anonymous and coalitionally strategy-proof aggregation rules on the relevant domains is (not multidimensionality as such, but) the incidence geometry of the underlying ordered outcome space. That is in fact the case, as shown in detail by Vannucci (2016).

4 Related literature and concluding remarks

The main result of the present work, namely Theorem 1, is an addition to a quite extensive and rich literature on characterizations of strategy-proof mechanisms on single peaked domains.

In particular, Theorem 1 extends and unifies several earlier results concerning characterizations of strategy-proof decision protocols on different sorts of 'large' single-peaked domains. The latter include Moulin (1980), Barberà, Sonnenschein and Zhou (1991), Barberà, Gul and Stacchetti (1993), Danilov (1994), Ching (1997), Nehring and Puppe (2007 (a) and (b)), Weymark (2011) and Anno

[39] A Boolean k-hypercube is a bounded distributive lattice $\mathbf{2}^k$ for some positive integer $k > 1$, where $\mathbf{2}$ is the linearly ordered set $(\{0,1\}, \leqslant)$, with $0 \leqslant 1$ and $0 \neq 1$ (see also annex 3 for an detailed discussian of the Boolean 2-hypercube or the Boolean square.

[40] Recall that the order-dimension of a partially ordered set $(X, \leqslant)$ is the minimum number of linear orders on X having $\leqslant$ as their common intersection.

(2014).

In order to compare the scope of Theorem 1 of the present work to characterizations provided in earlier contributions, we should distinguish between the respective *outcome sets*, *preference domains*, and *characterization-type* for strategy-proof decision protocols on single peaked domains.

Outcome sets. To begin with, the set-containment relationship between the class of outcome spaces in our paper and in the other contributions (or *outcome-set containment*, denoted *osc* in the table below) is to be considered. It should be noticed that most of the previous contributions concern a class of outcome spaces that is a *subclass* (denoted $(-)$ in the table below) of the class of bounded distributive lattices. But there are a few exceptions covering a class of outcome spaces which is *containment-incomparable to* (denoted $(\shortparallel)$ in the table below) or even -in a single case- *wider than* (denoted $(+)$ in the table below) the class of bounded distributive lattices (denoted respectively $(\shortparallel)$ and $(+)$ in the table below) .

Specifically, the classes of ordered outcome sets typically considered in the relevant literature include: *bounded linearly ordered sets, finite products of bounded linearly ordered sets, finite distributive lattices, bounded distributive lattices* (denoted respectively BO, $FPBO$, FDL, BDL in the table below).

Preference domains. As repeatedly mentioned above, several distinct single peaked domains consisting of profiles of total preference preorders on the relevant outcome space have been considered in the literature, including: the *full unimodal linear domain* of i.e. the set of *all* profiles of unimodal linear (namely antisymmetric) total preference preorders (e.g. Danilov (1994)), the *full unimodal domain* of i.e. the set of *all* profiles of unimodal total preference preorders (e.g. Moulin (1980)), the *full locally strictly unimodal linear domain* i.e. the set of *all* profiles of locally strictly unimodal linear total preference preorders (e.g. Barberà, Gul and Stacchetti (1993)), the *full locally strictly unimodal domain* of i.e. the set of *all* profiles of locally strictly unimodal total preference preorders (e.g. Ching (1997), Weymark (2011)), or more generally *any rich locally strictly unimodal linear domain* i.e. *any rich* set of profiles of locally strictly unimodal linear total preference preorders (see e.g. Nehring and Puppe (2007 (a),(b))), *any rich locally strictly unimodal domain* i.e. *any rich* set of profiles of locally strictly unimodal total preference preorders (see e.g. Anno (2014)), *any rich locally unimodal domain* i.e. *any rich* set of profiles of locally unimodal total preference preorders (this paper). The last preference domain, consisting of an arbitrary *rich locally unimodal domain* (and denoted $RLUD$ in the table below), obviously encompasses all the previous ones (see Section 2.II above for the relevant definitions).

Characterization-type. The emphasis of the present paper is on *polynomial* characterizations of strategy-proof decision protocols on single peaked domains.Therefore, first and foremost, we distinguish between *non-polynomial characterizations* (denoted NPC in the table below) and polynomial ones. The latter are partitioned into three subclasses:

(i) conjunctive normal forms or generalized sponsorship rules (denoted GmM -for *Generalized* min max- in the table below);

(ii) disjunctive normal forms or generalized consensus rules (denoted GMm -for *Generalized* max min- in the table below);

(iii) iterated medians or extended median rules (denoted $ItMed$ in the table below).

The modifier (A) (for *anonymous)* is appended to characterizations of the restricted class of anonymous strategy-proof aggregation rules.

The following table summarizes the marginal contribution of our Theorem 1 with respect to such previous characterization results as discussed above.

Papers/Features	OSC	BO	FPBO	FDL	BDL	RLUD	NPCh	GmM	GMm	$ItMed$
Moulin (1980)	$-$	*yes*	$\ominus$	$\ominus$	$\ominus$	*no*	*no*	*yes*	*no*	$yes(A)$
Ching (1997)	$-$	*yes*	$\ominus$	$\ominus$	$\ominus$	*no*	*yes*	*yes*	*yes*	*yes*
Weymark (2011)	$-$	*yes*	$\ominus$	$\ominus$	$\ominus$	*no*	*yes*	*yes*	*no*	$yes(A)$
Danilov (1994)	$\vert\vert$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	*no*	*yes*	no^*	no^*	*yes*
BGS (1993)	$-$	*yes*	*yes*	$\ominus$	$\ominus$	*no*	*yes*	*no*	*no*	*no*
NP (2007)	$\vert\vert$	$\ominus$	$\ominus$	*yes*	$\ominus$	*no*	*yes*	no^*	no^*	no^*
Anno (2014)	$+$	*yes*	*yes*	*yes*	*yes*	*no*	*yes*	no^*	no^*	no^*
This Paper	$=$	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*	*yes*

Legenda: $\ominus$: *class of outcomes spaces not covered by the paper assigned to this row* ;
no^* : *the characterization is not provided in this paper* ;

Concerning the equivalence of strategy-proofness and coalitional strategy-proofness on single peaked domains, Theorems 2 and 3 contribute to a somewhat sparse but not negligible body of literature which includes Moulin (1980), Barberà, Sonnenschein and Zhou (1991), Danilov (1994), Nehring and Puppe (2007 (b)), Le Breton and Zaporozhets (2009), Barberà, Berga and Moreno (2010), Vannucci (2016).

Indeed, Moulin (1980) establishes the equivalence of strategy-proofness and coalitional strategy-proofness for *all* aggregation rules -and 'top-only' social choice functions- on full unimodal domains.

In a similar vein, Danilov (1994) shows that strategy-proofness and coalitional strategy-proofness of aggregation rules on the foregoing domain are *equivalent* properties for aggregation rules on the full domain of all unimodal *linear orders* (i.e. *antisymmetric* total preorders) when the outcome set is the vertex set of an undirected (finite) *tree*. Extending a similar result obtained by Barberà, Sonnenschein and Zhou (1991) for outcomes sets consisting of finite Boolean distributive lattices, Nehring and Puppe (2007 (b)) also prove that the only efficient and strategy-proof 'top-only' social choice functions on rich domains of locally strictly unimodal profiles of linear orders in finite Boolean m-hypercubes with $m \geq 3$ are almost dictatorial.[41] It should also be mentioned that the main result in Nehring

[41] Namely, entrust a single player with veto power against any alternative of the profile of top outcomes, whenever the latter includes just two distinct outcomes. Notice, however, that coalitional strategy-proofness on single-peaked total

and Puppe (2007 (b)) *does entail* equivalence-failure for simple and coalitional strategy-proofness in Boolean k-hypercubes for $k \geq 3$. Of course, in view of the characterization results summarized in Table 1, such result implies lack of equivalence between individual and coalitional strategy-proofness for arbitrary rich single peaked domains in bounded distributive lattices. But Nehring and Puppe's result concerns a domain of locally strictly unimodal linear orders that is distinct -and in fact *disjoint* in the case of the Boolean square- from all rich domains of unimodal total preorders, which are on the contrary covered by the present work. Moreover, it does not apply to rich locally strictly unimodal domains on the Boolean square $\mathbf{2}^2$. By contrast, our Theorem 2 covers *every* rich locally unimodal domain in both infinite bounded distributive lattices and arbitrary Boolean hypercubes, including of course $\mathbf{2}^2$.[42]

Barberá, Berga and Moreno (2010) address the general issue of equivalence between simple and coalitional strategy-proofness and extends to locally strictly unimodal domains Moulin's equivalence between individual and coalitional strategy-proofness on unimodal domains in bounded linearly ordered sets (see also Le Breton and Zaporozhets (2009) on that equivalence-issue). Notice, incidentally, that the Barberá-Berga-Moreno argument for such an equivalence result *cannot be extended to arbitrary rich domains of locally unimodal total preorders even in bounded linear orders.* An extensive discussion of those issues is provided in a companion paper to the present work (see Vannucci (2016)).

Concerning the significance of the exclusion of possible strategy-proof non 'top-only' social choice functions, it is well-known that any sovereign social choice function that is strategy-proof on appropriately rich locally strictly unimodal domains satisfies the 'top-only' property (see e.g. Nehring and Puppe (2007 (a) and Anno (2014)). Thus, nothing substantial is to be gained anyway by considering social choice functions without the 'top-only' property, at least on a large class of rich single peaked domains, including the full domain of all locally unimodal preferences. For arbitrary rich locally unimodal domains, however, this is still an open issue (Chatterji and Sen (2011) offers some potentially relevant ideas to address it).

Finally, it should also be remarked that some results of the present paper can be reproduced in a more general setting, e.g. in any median algebra (see Isbell (1980), Bandelt and Hedlíková (1983)). But, to the best of the authors' knowledge, polynomial characterizations of strategy-proof aggregation rules on single peaked domains in arbitrary median algebras are not yet available. This intriguing open issue is however best left as a topic for future research.

5 Appendix 1: Proofs

Proof of Claim 1. (i) If $B_{\mathcal{X}}(x,z,y)$ then $x \wedge y \leqslant z \leqslant x \vee y$. Since by definition $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ it obviously follows $y \wedge x \leqslant z \leqslant y \vee x$ hence $B_{\mathcal{X}}(y,z,x)$ also holds.

preorders implies *weak efficiency* (as opposed to efficiency), and only *provided that the aggregation rule is sovereign.*

[42] We refer the interested reader to Annex 3 for further details.

(ii) Since by definition $x \wedge y \leqslant x \leqslant x \vee y$ and $x \wedge y \leqslant y \leqslant x \vee y$ hold for any $x, y \in X$, both $B_{\mathcal{X}}(x,x,y)$ and $B_{\mathcal{X}}(x,y,y)$ hold.

(iii) If $B_{\mathcal{X}}(x,y,x)$ then $x = x \wedge x \leqslant y \leqslant x \vee x = x$ hence $y = x$.

(iv) If $B_{\mathcal{X}}(x,u,y)$, $B_{\mathcal{X}}(x,v,y)$ and $B_{\mathcal{X}}(u,z,v)$ then $x \wedge y \leqslant u \leqslant x \vee y$, $x \wedge y \leqslant v \leqslant x \vee y$ and $u \wedge v \leqslant z \leqslant u \vee v$. Thus, by definition of $\wedge$ and $\vee$, $x \wedge y \leqslant u \wedge v \leqslant x \vee y$ (that implies $x \wedge y \leqslant z$) and $x \wedge y \leqslant u \vee v \leqslant x \vee y$ (that implies $z \leqslant x \vee y$). It follows that $B_{\mathcal{X}}(x,z,y)$ as required;

(v) If $B_{\mathcal{X}}(x,y,z)$ and $B_{\mathcal{X}}(y,x,z)$ then $x \wedge z \leqslant y \leqslant x \vee z$ and $y \wedge z \leqslant x \leqslant y \vee z$ hence $x = x \vee (y \wedge z) = (x \wedge (y \vee z)) \vee (y \wedge z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) =$

$(x \wedge y) \vee (y \wedge z) \vee (x \wedge z) = (y \vee (x \wedge z)) \vee (x \wedge z) = y \vee (x \wedge z) = y$.

(vi) Let $x, y, z \in X$ be such that *not* $B_{\mathcal{X}}(x,y,z)$ i.e. *either* $y \nleqslant x \wedge z$ *or* $y \nleqslant x \vee z$ (or both).

Then, it is well-known and easily checked that there exist a filter F and an ideal I of $\mathcal{X}$ -hence, by the Prime Ideal Theorem (see Davey, Priestley (1990)) a *prime filter* F' and a *prime ideal* I' of $\mathcal{X}$ with $F \subseteq F'$, $I \subseteq I'$ and $F' \cap I' = \varnothing$- such that either $y \in I'$ and $\{x,z\} \subseteq F'$ or $y \in F'$ and $\{x,z\} \subseteq I'$. Then, the thesis follows from the well-known fact that the half spaces of a lattice $\mathcal{X}$ with respect to $B_{\mathcal{X}}$ are precisely its prime filters and prime ideals (see e.g. van de Vel (1993)).

(vii) (see Birkhoff, Kiss (1947)). We give here an explicit proof just for the sake of completeness. From distributivity of $\mathcal{X}$ it follows that $(x \wedge y) \vee (y \wedge z) \vee (x \wedge z) = ((x \wedge y) \vee y) \wedge ((x \wedge y) \vee z)) \vee (x \wedge z) = (y \wedge ((x \wedge y) \vee z))) \vee (x \wedge z) =$

$= (y \vee (x \wedge z)) \wedge (((x \wedge y) \vee z))) \vee (x \wedge z)) =$

$= ((x \wedge z) \vee y) \wedge ((x \wedge y) \vee z)) = (x \vee y) \wedge (z \vee y) \wedge (x \vee z) \wedge (y \vee z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$.

Thus, $x \wedge y \leqslant z \leqslant x \vee y$ entails that $(x \wedge y) \vee (y \wedge z) \vee (x \wedge z) \leqslant z \leqslant (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$, whence $z = \mu(x,y,z) = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$. Conversely, if $z = \mu(x,y,z) = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$ then clearly $x \wedge y \leqslant z \leqslant x \vee y$. $\square$

Proof of Lemma 1. $(i) \Rightarrow (ii)$ Let us assume that $f : X^N \to X$ is *not* $B_{\mathcal{X}}$-monotonic: thus, there exist $i \in N$, $x'_i \in X$ and $x_N = (x_i)_{i \in N} \in X^N$ such that $(x_i, f(x_N), f(x'_i, x_{N \smallsetminus \{i\}})) \notin B_{\mathcal{X}}$. Thus, by Closure of $B_{\mathcal{X}}$, $x_i \neq f(x_N) \neq f(x'_i, x_{N \smallsetminus \{i\}})$. Moreover, by Idempotence of $B_{\mathcal{X}}$ (see Claim 1), for each $x, y \in X : y \in B_{\mathcal{X}}(x,.,x)$ if and only if $y = x$.

Next, consider an $f[X^N]$-complete preorder $\succcurlyeq^* \in D_i$ such that $x_i = top(\succcurlyeq^*)$ and $UC(\succcurlyeq, f(x'_i, x_{N \smallsetminus \{i\}})) = B_{\mathcal{X}}(x_i, ., f(x'_i, x_{N \smallsetminus \{i\}}))$. Such a preorder exists since D_i is rich.

Now, by assumption $f(x_N) \in X \backslash B_{\mathcal{X}}(x_i, ., f(x'_i, x_{N \smallsetminus \{i\}}))$ while $f(x'_i, x_{N \backslash \{i\}}) \in B_{\mathcal{X}}(x_i, ., f(x'_i, x_{N \backslash \{i\}}))$ by Closure of $B_{\mathcal{X}}$, hence by construction and $f[X^N]$-completeness $f(x'_i, x_{N \smallsetminus \{i\}}) \succ^* f(x_N)$. Finally, posit $(\succcurlyeq_j)_{j \in N} \in \Pi_{i \in N} D_i$ such that $x_j = top(\succcurlyeq_j)$ for all $j \in N$ and $\succcurlyeq_i = \succcurlyeq^*$: then, f is *not* strategy-proof on $\Pi_{i \in N} D_i$.

$(ii) \Rightarrow (i)$ Conversely, let f be $B_{\mathcal{X}}$-monotonic. Next, consider any locally unimodal profile $\succcurlyeq = (\succcurlyeq_j)_{j \in N} \in \Pi_{i \in N} D_i$ and any $i \in N$. By definition of $B_{\mathcal{X}}$-monotonicity $f(top(\succcurlyeq_i), x_{N \backslash \{i\}}) \in B_{\mathcal{X}}(top(\succcurlyeq_i), ., f(x_i, x_{N \backslash \{i\}}))$ for all $x_{N \backslash \{i\}} \in X^{N \backslash \{i\}}$ and $x_i \in X$. But then, since clearly by definition $top(\succcurlyeq_i) \succcurlyeq_i f(top(\succcurlyeq_i), x_{N \backslash \{i\}})$, either $f(top(\succcurlyeq_i), x_{N \backslash \{i\}}) = top(\succcurlyeq_i)$ or $f(top(\succcurlyeq_i), x_{N \backslash \{i\}}) \succcurlyeq_i f(x_i, x_{N \backslash \{i\}})$ by local unimodality of $\succcurlyeq_i$. Hence, $f(top(\succcurlyeq_i), x_{N\setminus\{i\}}) \succcurlyeq_i f(x_i, x_{N\setminus\{i\}})$ in any case. It follows that f is indeed strategy-proof on $\Pi_{i\in N} D_i$. $\square$

Proof of Lemma 2. Take any $x_N \in X^N$. By definition of $B_{\mathcal{X}}$-monotonicity, it suffices to show that for any $i \in N$ and $x'_i \in X$, $\mu(f,g,h)(x_N) \in [x_i, \mu(f,g,h)(x'_i, x_{N\setminus\{i\}})]$. Indeed, by monotonicity of f, g, h with respect to $\mathcal{X}$, $f(x_N) \in [x_i, f(x'_i, x_{N\setminus\{i\}})]$, $g(x_N) \in [x_i, g(x'_i, x_{N\setminus\{i\}})]$, and $h(x_N) \in [x_i, h(x'_i, x_{N\setminus\{i\}})]$. A change of variables is in order here for the sake of convenience, namely $x_f = f(x_N)$, $x'_f = f(x'_i, x_{N\setminus\{i\}})$, $x_g = g(x_N)$, $x'_g = g(x'_i, x_{N\setminus\{i\}})$, $x_h = h(x_N)$, $x'_h = h(x'_i, x_{N\setminus\{i\}})$, whence $\mu(f,g,h)(x_N) = \mu(x_f, x_g, x_h)$, and $\mu(f,g,h)(x'_i, x_{N\setminus\{i\}}) = \mu(x'_f, x'_g, x'_h)$. Thus, $x_i \wedge x'_l \leqslant x_l \leqslant x_i \vee x'_l$, $l = f, g, h$, by hypothesis, while the thesis amounts to $x_i \wedge \mu(x'_f, x'_g, x'_h) \leqslant \mu(x_f, x_g, x_h) \leqslant x_i \vee \mu(x'_f, x'_g, x'_h)$. Now, $\mu(x'_f, x'_g, x'_h) = (x'_f \wedge x'_g) \vee (x'_g \wedge x'_h) \vee (x'_f \wedge x'_h)$ hence by distributivity and the basic latticial identities we get:

$$
\begin{aligned}
& x_i \wedge ((x'_f \wedge x'_g) \vee (x'_g \wedge x'_h) \vee (x'_f \wedge x'_h)) = \\
= \quad & (x_i \wedge (x'_f \wedge x'_g)) \vee (x_i \wedge (x'_g \wedge x'_h)) \vee (x_i \wedge (x'_f \wedge x'_h)) = \\
= \quad & ((x_i \wedge x'_f) \wedge (x_i \wedge x'_g)) \vee ((x_i \wedge x'_g) \wedge (x_i \wedge x'_h)) \vee ((x_i \wedge x'_f) \wedge (x_i \wedge x'_h)).
\end{aligned}
$$

However, by hypothesis, distributivity and the basic latticial identities again:

$$
\begin{aligned}
& ((x_i \wedge x'_f) \wedge (x_i \wedge x'_g)) \vee ((x_i \wedge x'_g) \wedge (x_i \wedge x'_h)) \vee ((x_i \wedge x'_f) \wedge (x_i \wedge x'_h)) \leqslant \\
\leqslant \quad & (x_f \wedge x_g) \vee (x_g \wedge x_h) \vee (x_f \wedge x_h) = \mu(x_f, x_g, x_h) \leqslant \\
\leqslant \quad & ((x_i \vee x'_f) \wedge (x_i \vee x'_g)) \vee ((x_i \vee x'_g) \wedge (x_i \vee x'_h)) \vee ((x_i \vee x'_f) \wedge (x_i \vee x'_h)) = \\
= \quad & (x_i \vee (x'_f \wedge x'_g)) \vee (x_i \vee (x'_g \wedge x'_h)) \vee (x_i \vee (x'_f \wedge x'_h)) = \\
= \quad & x_i \vee ((x'_f \wedge x'_g) \vee (x'_g \wedge x'_h) \vee (x'_f \wedge x'_h)) = x_i \vee \mu(x'_f, x'_g, x'_h)
\end{aligned}
$$

as required. $\square$

Proof of Lemma 3. The proof is by induction on n, the cardinality of N. Let us consider first the case $n = 1$, to establish that if $f : X \to X$ is $B_{\mathcal{X}}$-monotonic, then $f(x) = \mu(f(\bot), x, f(\top))$, for any $x \in X$. To begin with, observe that $B_{\mathcal{X}}$-monotonicity of f entails that $f(\bot) \leqslant f(x) \leqslant f(\top)$ for each $x \in X$, whence $\mu(f(\bot), f(x), f(\top)) = f(\bot) \vee f(x) = f(x)$ for all $x \in X$. Indeed, by $B_{\mathcal{X}}$-monotonicity, $B_{\mathcal{X}}(\bot, f(\bot), f(x))$ and $B_{\mathcal{X}}(\top, f(\top), f(x))$ for all $x \in X$ i.e. -by Median-Equivalence of $B_{\mathcal{X}}$- $f(\bot) = \mu(\bot, f(\bot), f(x))$ and $f(\top) = \mu(\top, f(\top), f(x))$. Now, $\mu(\bot, f(\bot), f(x)) = (\bot \wedge f(\bot)) \vee (f(\bot) \wedge f(x)) \vee (\bot \wedge f(x)) = f(\bot) \wedge f(x)$, hence $f(\bot) \wedge f(x) = f(\bot)$, namely $f(\bot) \leqslant f(x)$. Moreover, $\mu(\top, f(\top), f(x)) = (\top \wedge f(\top)) \vee (f(\top) \wedge f(x)) \vee (\top \wedge f(x)) = f(\top) \vee f(x)$, hence $f(\top) \vee f(x) = f(\top)$, namely $f(x) \leqslant f(\top)$. Next, observe that from $B_{\mathcal{X}}$-monotonicity of f and Median-Equivalence of $B_{\mathcal{X}}$ it follows that $\mu(x, f(x), f(\bot)) = f(x) = \mu(x, f(x), f(\top))$. Thus $f(x) = \mu(x, f(x), f(\bot)) =$

$= (x \wedge f(x)) \vee (f(x) \wedge f(\bot)) \vee (x \wedge f(\bot)) =$

$= (x \wedge f(x)) \vee f(\bot) = (x \vee f(\bot)) \wedge (f(x) \vee f(\bot)) = (x \vee f(\bot)) \wedge f(x)$ whence $f(x) \leqslant (x \vee f(\bot))$.

It follows that $f(x) = f(\top) \wedge f(x) \leqslant f(\top) \wedge (x \vee f(\bot))$. Moreover, $f(x) = \mu(x, f(x), f(\top)) =$

$(x \wedge f(x)) \vee (f(x) \wedge f(\top)) \vee (x \wedge f(\top)) = (x \wedge f(x)) \vee f(x) \vee (x \wedge f(\top)) = f(x) \vee (x \wedge f(\top))$ whence $(x \wedge f(\top)) \leqslant f(x)$. It follows that $f(\bot) \vee (x \wedge f(\top)) \leqslant f(\bot) \vee f(x) = f(x)$.

Now, notice that $\mu(f(\bot), x, f(\top)) = (f(\bot) \wedge x) \vee (x \wedge f(\top)) \vee (f(\bot) \wedge f(\top)) = f(\bot) \vee (x \wedge f(\top))$.

Furthermore, $f(\top) \wedge (x \vee f(\bot)) = (f(\top) \wedge x) \vee (f(\top) \wedge f(\bot)) = f(\bot) \vee (x \wedge f(\top)) = \mu(f(\bot), x, f(\top))$.

It follows that, $\mu(f(\bot), x, f(\top)) = f(\bot) \vee (x \wedge f(\top)) \leqslant f(x) \leqslant f(\top) \wedge (x \vee f(\bot)) = \mu(f(\bot), x, f(\top))$ whence $f(x) = \mu(f(\bot), x, f(\top))$ as required. Let us then assume that if $f : X^K \to X$ is $B_{\mathcal{X}}$-monotonic and $K = \{1, ..., n-1\}$, then for all $x_K \in X^K$:

$$f(x_K) = \mu[\overbrace{\mu(...\mu(}^{n-2-times} f(\bot, ..., \bot, \bot), x_1, f(\bot, ..., \bot, \top))...), x_{n-1}, \overbrace{\mu(...\mu(}^{n-2-times} f(\bot, ..., \top, \bot), x_1, f(\bot, ..., \top, \top)))...)].$$

Next, posit $N = \{1, ..., n\} = K \cup \{n\}$, consider $f : X^N \to X$, and observe that the following fact holds, and is easily checked:

(*) for all $x_N = (x_i)_{i \in N} \in X^N$ and $i \in N$, two functions $f_{x_{N \setminus \{i\}}} : X \to X$ and $f_{x_i} : X^{N \setminus \{i\}} \to X$ can be defined by the rules $f_{x_{N \setminus \{i\}}}(z) = f((x_{N \setminus \{i\}}, z))$ for each $z \in Z$, $f_{x_i}(z_{N \setminus \{i\}}) = f(x_i, z_{N \setminus \{i\}})$ for each $z_{N \setminus \{i\}} \in X^{N \setminus \{i\}}$, and $B_{\mathcal{X}}$-monotonicity of f clearly entails that any such $f_{x_{N \setminus \{i\}}}$ and f_{x_i} are also $B_{\mathcal{X}}$-monotonic. Then, by the basic-inductive step $f_{x_{N \setminus \{n\}}}(x_n) = \mu(f_{x_{N \setminus \{n\}}}(\bot), x_n, f_{x_{N \setminus \{n\}}}(\top)) =$

$= \mu(f(x_{N \setminus \{n\}}, \bot), x_n, f(x_{N \setminus \{n\}}, \top)) = \mu(f_\bot(x_{N \setminus \{n\}}), x_n, f_\top(x_{N \setminus \{n\}})$. Moreover, from the inductive step as applied to $f_\bot : X^K \to X$ and $f_\top : X^K \to X$ it follows that:

$$f_\bot(x_{N \setminus \{n\}}) = \mu[\overbrace{\mu(...\mu(}^{n-2-times} f_\bot(\bot, ..., \bot, \bot), x_1, f_\bot(\bot, ..., \bot, \top))...),$$
$$x_{n-1}, \overbrace{\mu(...\mu(}^{n-2-times} f_\bot(\bot, ..., \top, \bot), x_1, f_\bot(\bot, ..., \top, \top)))...)]$$

and

$$f_\top(x_{N \setminus \{n\}}) = \mu[\overbrace{\mu(...\mu(}^{n-2-times} f_\top(\bot, ..., \bot, \bot), x_1, f_\top(\bot, ..., \bot, \top))...),$$
$$x_{n-1}, \overbrace{\mu(...\mu(}^{n-2-times} f_\top(\bot, ..., \top, \bot), x_1, f_\top(\bot, ..., \top, \top)))...)].$$

Therefore, for all $x_N \in X^N$,

$$f(x_N) = f_{x_{N \setminus \{n\}}}(x_n) = \mu(f_{x_{N \setminus \{n\}}}(\bot), x_n, f_{x_{N \setminus \{n\}}}(\top)) = \mu(f_\bot(x_{N \setminus \{n\}}), x_n, f_\top(x_{N \setminus \{n\}})) =$$
$$= \mu[\overbrace{\mu(...\mu(}^{n-1-times} f_\bot(\bot, ..., \bot, \bot), x_1, f_\bot(\bot, ..., \bot, \top))...), x_{n-1}, \overbrace{\mu(...\mu(}^{n-1-times} f_\bot(\bot, ..., \top, \bot), x_1, f_\bot(\bot, ..., \top, \top)))...)] =$$
$$= \mu[\overbrace{\mu(...\mu(}^{n-1-times} f(\bot, ..., \bot, \bot), x_1, f(\bot, ..., \bot, \top))...), x_{n-1}, \overbrace{\mu(...\mu(}^{n-1-times} f(\bot, ..., \top, \bot), x_1, f(\bot, ..., \top, \top)))...)]$$

and the thesis is established. □

Proof of Theorem 1. $(i) \Longleftrightarrow (ii)$ It follows from Lemma 1.

$(i) \Longrightarrow (iii)$ Immediate from Lemma 3.

$(iii) \Longrightarrow (i)$ It follows immediately from the definition of iterated median rule, from the observation that projections and constants induce $B_{\mathcal{X}}$ -monotonic aggregation rules, and from Lemma 2.

$(iii) \Longrightarrow (iv)$ Suppose f is an iterated median rule. Then, by definition, for all $x_N \in X^N$

$$f(x_N)=\mu[\overbrace{\mu(...\mu(}^{n-1-times}\quad f(\bot,...,\bot,\bot),x_1,f(\bot,...,\bot,\top))...),x_{n-1},\overbrace{\mu(...\mu(}^{n-1-times}\quad f(\bot,...,\top,\bot),x_1,f(\bot,...,\top,\top)))...)].$$

Let us know proceed to a sequential elimination of occurrences of μ in the right hand side of the previous formula by substituting for each of them the corresponding μ-value, in accordance with the following clauses:

(a) each x_i is marked with a suffix $'v'$ (for 'variable') and rewritten as $x_i^{(v)}$, and each $f(z_N)$ (with $z_N \in \{\bot,\top\}^N$) is marked with a suffix $'c'$(for 'constant') and rewritten as $f(z_N)^{(c)}$, in order to track their 'provenance';

(b) no simplification of composite terms involving terms marked with *distinct* suffixes is allowed (e.g. even if $x_i^{(v)}=f(z_N)^{(c)}$, $x_i^{(v)}\wedge f(z_N)^{(c)}$ or $x_i^{(v)}\vee f(z_N)^{(c)}$ are *not* allowed to be rewritten as $x_i^{(v)}$or $f(z_N)^{(c)}$).

In view of the identity $\mu(x,y,z)=(x\wedge y)\vee(y\wedge z)\vee(x\wedge z)$ which holds for each $x,y,z\in X$, where $x\vee y=\mu(x,y,\top)$, $x\wedge y=\mu(x,y,\bot)$, it is easily checked that such a sequential elimination of μ's occurrences results in a disjunctive normal-form lattice-polynomial equation $f(x_N)=\vee_{j=1}^{m}((\wedge_{i\in S_j\subseteq N}x_i^{(c)})\wedge y_j)$ where $y_j=\wedge_{k=1}^{h_j}f(z_N^k)^{(c)}$, with $h_j\leq 2^n$.

Notice that, by construction, $y_j\leqslant y_{j'}$ whenever $S_{j'}\subseteq S_j$. Thus, collecting terms with the same $S_j=S\subseteq N$, taking the join y_S^* of the corresponding y_j's for each S, and proceeding to some trivial rearrangements of terms, one obtains $f(x_N)=\vee_{S\subseteq N}((\wedge_{i\in S\subseteq N}x_i)\wedge y_S^*)$ hence f is indeed a generalized weak consensus aggregation rule.

$(iv)\Longleftrightarrow(v)$ It follows immediately from the definitions and the identity $(x\wedge y)\vee(y\wedge z)\vee(x\wedge z)=(x\vee y)\wedge(y\vee z)\wedge(x\vee z)$.

$(iv)\Longrightarrow(i)$ Let $f:X^N\rightarrow X$ be a generalized weak consensus aggregation rule i.e. there exists an order filter $\mathcal{F}$ of $(\mathcal{P}(N),\subseteq)$ such that $f(x_N)=\vee_{S\in\mathcal{F}}((\wedge_{i\in S}x_i)\wedge y_S^*)$ for all $x_N\in X^N$. Then, observe that -for any $x,y\in X$- $x\wedge y=\mu(x,y,\bot)$ and $x\vee y=\mu(x,y,\top)$. Hence by repeated application of Lemma 2 it follows that f is $B_{\mathcal{X}}$ -monotonic. □

Proof of Theorem 2. Let us assume without loss of generality that $|X|=4$ and let $X=\{a,b,c,d\}$ and $\Delta_X=\{(x,x):x\in X\}$. Next, define $\leqslant^{**}=\{(a,b),(a,c),(a,d),(b,d),(c,d)\}\cup\Delta_X$. It is easily checked that $\mathcal{X}^{**}=(X,\leqslant^{**})$ is the Boolean lattice 2^2 with $a=\top$, $d=\bot$. Now, define the family $\{f(x^*)\}_{x^*\in\{\bot,\top\}^N}$ as follows: for all $x_{N\smallsetminus\{1,2\}}\in\{\bot,\top\}^{N\setminus\{1,2\}}$

$$f(a,a,x_{N\smallsetminus\{1,2\}})=a,\ f(d,d,x_{N\setminus\{1,2\}})=d,\ f(a,d,x_{N\setminus\{1,2\}})=b,\ f(d,a,x_{N\setminus\{1,2\}})=c.$$

Then, consider the nested sequence of medians that provides the run of the median tree-automaton $\mathcal{A}_\mu^{I,\lambda}$ as initialized with ballot profile x_N and applied to the finite (Σ^μ,I)-tree $T=T(x_N,\{f(x^*)\}_{x^*\in\{\bot,\top\}^N})$ with terminal nodes suitably labelled by projections of x_N and elements of $\{f(x^*)\}_{x^*\in\{\bot,\top\}^N}$ as defined above. A few simple if tedious calculations immediately establish that for all $x_{N\smallsetminus\{1,2\}}\in$

$X^{N\smallsetminus\{1,2\}}$:

$$\begin{aligned}
f(a,c,x_{N\setminus\{1,2\}}) &= f(b,a,x_{N\setminus\{1,2\}})=f(b,c,x_{N\setminus\{1,2\}})=a,\\
f(b,b,x_{N\setminus\{1,2\}}) &= f(a,b,x_{N\setminus\{1,2\}})=f(b,d,x_{N\setminus\{1,2\}})=b,\\
f(c,c,x_{N\setminus\{1,2\}}) &= f(c,a,x_{N\setminus\{1,2\}})=f(d,c,x_{N\setminus\{1,2\}})=c,\\
f(c,d,x_{N\setminus\{1,2\}}) &= f(d,c,x_{N\setminus\{1,2\}})=f(c,b,x_{N\setminus\{1,2\}})=d.
\end{aligned}$$

By construction, and in view of Lemma 3 above, f is $B_{\mathcal{X}^{**}}$- monotonic. Therefore, by Lemma 1, f is also strategy-proof on $U^N_{\mathcal{X}^{**}}$.

Now, take

$$\begin{aligned}
\succcurlyeq\ &=\{(a,b),(a,c),(a,d),(b,c),(b,d),(c,d),(d,c)\}\cup\Delta_X,\\
\succcurlyeq\ '&=\{(d,b),(d,c),(d,a),(b,c),(b,a),(c,a),(a,c)\}\cup\Delta_X.
\end{aligned}$$

First, observe that both $\succcurlyeq$ and $\succcurlyeq'$are in $U^N_{\mathcal{X}^{**}}$, i.e. are unimodal with respect to $\mathcal{X}^{**}$: indeed, $top(\succcurlyeq)=a$, $top(\succcurlyeq')=d$ and it is immediately seen that

$$B_{\mathcal{X}}(X,\quad\leqslant\quad^{**})=\left\{\begin{array}{c}(a,b,d),(a,c,d),(b,a,c),(b,d,c),(d,b,a),\\(d,c,a),(c,a,b),(c,d,b)\end{array}\right\}\cup$$
$$\cup\left\{(x,y,z)\in X^3:x=y\text{ or }z=y\right\}.$$

But then, since $\{(b,d),(c,d),(a,b),(d,c)\}\cup\Delta_X$ is a subrelation of $\succcurlyeq$ and $\{(b,a),(c,a),(a,c),(d,c)\}\cup\Delta_X$ is a subrelation of $\succcurlyeq'$, it follows that $\succcurlyeq$ and $\succcurlyeq'$are also unimodal with respect to $\mathcal{X}^{**}$. Now, take any preference profile $(\succcurlyeq_i)_{i\in N}$ such that $\succcurlyeq_1=\succcurlyeq'$ and $\succcurlyeq_2=\succcurlyeq$, hence $top(\succcurlyeq_1)=d$, $top(\succcurlyeq_2)=a$. Then, for any $x_{N\smallsetminus\{1,2\}}\in X^{N\smallsetminus\{1,2\}}$, both $f(a,d,x_{N\smallsetminus\{1,2\}})\succ_1 f(top(\succcurlyeq_1),top(\succcurlyeq_2),x_{N\smallsetminus\{1,2\}})$ and $f(a,d,x_{N\smallsetminus\{1,2\}})\succ_2 f(top(\succcurlyeq_1),top(\succcurlyeq_2),x_{N\setminus\{1,2\}})$: it follows that, again, coalition $\{1,2\}$ can manipulate the outcome at $(\succcurlyeq_i)_{i\in N}$ namely f is *not* coalitionally strategy-proof. Again, strategy-proofness and failure of coalitional strategy-proofness of f on $S^N_{\mathcal{X}^{**}}$ follows from the very same argument, by positing $\succcurlyeq_1=\succcurlyeq''$ and $\succcurlyeq_2=\succcurlyeq'''$. □

Proof of Corollary 1. $(i)\Longrightarrow(ii)$ It follows immediately from Theorem 2 (ii) above;

$(ii)\Longrightarrow(i)$ For the case concerning $U^N_{\mathcal{Y}}$, the statement follows from a straightforward extension and adaptation of the proof of Proposition 4 of Danilov (1994) concerning aggregation rules on unimodal domains of *linear orders* in *undirected finite trees* (details available from the authors upon request), and is indeed already stated with a sketch of the proof in Moulin (1980). As far as $S^N_{\mathcal{Y}}$ is concerned, the statement follows e.g. from Theorem 2 and Proposition 3 of Barberá, Berga and Moreno (2010). □

Proof of Theorem 3. Let us assume that on the contrary there exists an aggregation rule $f:X^N\to X$ which is anonymous, locally JI-neutral on Y, locally sovereign on Y, and coalitionally strategy-proof on $U^N_{\mathcal{X}}$ (on $S^N_{\mathcal{X}}$, respectively). By Theorem 1, it follows that there exists an order filter

$\mathcal{F}$ of $(\mathcal{P}(N), \subseteq)$ such that $f(x_N) = \vee_{S\in\mathcal{F}}((\wedge_{i\in S}x_i) \wedge y_S^*)$ for all $x_N \in X^N$. To begin with, observe that coalitional strategy-proofness and local sovereignty on Y jointly imply local idempotence on Y (indeed, suppose there exists $u \in Y$, $u \neq f(u^N)$; of course, by local sovereignty there exists $x_N \in X^N$ such that $f(x_N) = u$. But then f is coalitionally manipulable at any preference profile $(\succcurlyeq_i)_{i\in N} \in U_{\mathcal{X}}^N$ ($(\succcurlyeq_i)_{i\in N} \in S_{\mathcal{X}}^N$, respectively) such that $top(\succcurlyeq_i) = u$ for all $i \in N$, a contradiction). Next, for any $u \in Y$ denote by S_u the set of all minimal coalitions $T \in \mathcal{F}$ such that $u \leqslant f(u^T, w^{N\smallsetminus T})$ for all $w^{N\smallsetminus T} \in X^{N\smallsetminus T}$. By local idempotence of f on Y , $S_u \neq \varnothing$. By anonymity of f, $|T| = |T'| = n_u$ for all $T, T' \in S_u$, and $y_S^* = y_{S'}^* = y_s^*$ for any $S, S' \in \mathcal{F}$ such that $|S| = |S'| = s$. Moreover, since by Theorem 1 coalitional strategy-proofness entails in particular $B_{\mathcal{X}}$-monotonicity, it also follows -by definition of $B_{\mathcal{X}}$-monotonicity- that for any $i \in N \smallsetminus T$ $u = u \wedge f(u^T, w^{N\smallsetminus T}) \leqslant f((u^{T\cup\{i\}}, w^{N\smallsetminus(T\cup\{i\})}) \leqslant u \vee f(u^T, w^{N\smallsetminus T})$ whence, by repeated application of that argument $u \leqslant f(u^{T'}, w^{N\smallsetminus T'})$ for any $T' \subseteq N$ such that $|T'| \geq n_u$. Also, by local JI-neutrality on Y of f, $n_x = n_z = q$. Four cases are to be distinguished according to the sign of $(q - n/2)$ and the parity of n.

(α): Let us first suppose that $q \leq n/2$. Then, in order to address the unimodal case consider the following triple of preference relations:

$\succcurlyeq^* := [x \succ^* 0 \succ^* x \vee z \sim^* z \sim^* w$ for all $w \in X \smallsetminus Y]$,

$\succcurlyeq^{**} := [z \succ^{**} 0 \succ^{**} x \vee z \sim^{**} x \sim^{**} w$ for all $w \in X \smallsetminus Y]$,

$\succcurlyeq^{***} := [0 \succ^{***} x \sim^{***} z \sim^{***} x \vee z \sim^{***} w$ for all $w \in X \smallsetminus Y]$.

Notice that by construction such preferences are unimodal with respect to $\mathcal{X}$, i.e. $\{\succcurlyeq^*, \succcurlyeq^{**}, \succcurlyeq^{***}\} \subseteq U_{\mathcal{X}}^N$. Two subcases are distinguished according to the parity of n, namely

(i) $n = 2k + 1$ for some positive integer k, and (ii) $n = 2k$ for some positive integer k.

If $(\alpha(i))$ obtains then take preference profile $\succcurlyeq_{[O]} = ((\succcurlyeq_i^*)_{i\in\{1,...,k\}}, (\succcurlyeq_i^{**})_{i\in\{k+1,...,2k\}}, \succcurlyeq_{2k+1}^{***})$ and compute $f(y_N) = \vee_{S\in\mathcal{F}}((\wedge_{i\in S}y_i) \wedge y_s^*)$ where $y_N = top(\succcurlyeq_{[O]})$ i.e. $y_i = x$ for all $i \in \{1, ..., k\}$, $y_i = z$ for all $i \in \{k + 1, ..., 2n\}$, and $y_{2k+1} = 0$. By construction, $f(y_N)$ is the l.u.b. of a nonempty family $\mathcal{T}$ of terms belonging to some of the following jointly exhaustive, partially overlapping classes:

$T_1 = \{\wedge_{j\in J}v_j : J$ is a finite set J and there exists $j \in J$ such that $v_j = 0\}$,

$T_2 = \{\wedge_{j\in J}v_j : J$ is a finite set J and there exist $j, h \in J$ such that $v_j = x$ and $v_h = z\}$,

$T_3 = \{\wedge_{j\in J}v_j :$ J is a finite set and there exists $J' \subseteq J$ such that $|J'| \geq q$ and $v_j = x$ for all $j \in J'\}$,

$T_4 = \{\wedge_{j\in J}v_j :$ J is a finite set and there exists $J' \subseteq J$ such that $|J'| \geq q$ and $v_j = z$ for all $j \in J'\}$.

Moreover, $t = \wedge_{j\in J}v_j = 0$ for all $t \in T_1 \cup T_2$ hence, by construction, $T_3 \cap \mathcal{T} \neq \varnothing \neq T_4 \cap \mathcal{T}$. On the other hand, $t_3 \geqslant x$ and $t_4 \geqslant z$ for any $t_3 \in T_3$ and $t_4 \in T_4$.

It follows that $f(y_N) \geqslant x \vee z$. If $(\alpha(ii))$ obtains then take preference profile $\succcurlyeq_{[E]} = ((\succcurlyeq_i^*)_{i\in\{1,...,k\}}, (\succcurlyeq_i^{**})_{i\in\{k+1,...,2k\}})$, and compute $f(y'_N) = \vee_{S\in\mathcal{F}}((\wedge_{i\in S}y'_i) \wedge y_s^*)$, where $y'_N = top(\succcurlyeq_{[E]})$ i.e. $y'_i = x$ for all $i \in \{1, ..., k\}$, and $y'_i = z$ for all $i \in \{k + 1, ..., 2n\}$. Again, $f(y'_N)$ is the l.u.b. of a nonempty family $\mathcal{T}$ of terms belonging to some of the following jointly exhaustive, partially overlapping classes:

$T'_1 = \{\wedge_{j\in J}v_j : J$ is a finite set J and there exist $j, h \in J$ such that $v_j = x$ and $v_h = z\}$,

$T_2' = \{\wedge_{j\in J} v_j :\ J$ is a finite set and there exists $J' \subseteq J$ such that $|J'| \geq q$ and $v_j = x$ for all $j \in J'\}$,

$T_3' = \{\wedge_{j\in J} v_j :\ J$ is a finite set and there exists $J' \subseteq J$ such that $|J'| \geq q$ and $v_j = z$ for all $j \in J'\}$.

Moreover, $t = \wedge_{j\in J} v_j = 0$ for all $t \in T_1'$ hence, by construction, $T_2' \cap \mathcal{T} \neq \varnothing \neq T_3' \cap \mathcal{T}$. On the other hand, $t_2 \geqslant x$ and $t_3 \geqslant z$ for any $t_2 \in T_2'$ and $t_3 \in T_3'$. It follows, again, that $f(y_N') \geqslant x \vee z$. Now, take $u_N \in X^N$ with $u_i = 0$ for all $i \in N$: by local idempotence, $f(u_N) = 0$. Thus, if $n = 2k+1$, $f((u_i = 0)_{i\in N\smallsetminus\{2k+1\}}, y_{2k+1} = 0) = f(u_N) \succ_i f(y_N)$ for all $i \in N \smallsetminus \{2k+1\}$. Similarly, if $n = 2k$, then $f(u_N) \succ_i f(y_N)$ for all $i \in N$. Hence, f is coalitionally manipulable at unimodal preference profile $\succcurlyeq^{[O]}$ (at unimodal preference profile $\succcurlyeq^{[E]}$, respectively), a contradiction.

The locally strictly unimodal case can be addressed precisely by the same argument, provided preference profile $(\succcurlyeq^*, \succcurlyeq^{**}, \succcurlyeq^{***})$ is replaced by any locally strictly unimodal preference profile $(\succcurlyeq'$ $, \succcurlyeq'', \succcurlyeq''')$ such that

$\succcurlyeq' := [x \succ' 0 \succ' x \vee z \succ' z \succ w$ for all $w \in X \smallsetminus Y]$,

$\succcurlyeq'' := [z \succ'' 0 \succ'' x \vee z \succ'' x \succ'' w$ for all $w \in X \smallsetminus Y]$,

$\succcurlyeq''' := [0 \succ''' x \succ''' z \succ''' x \vee z \succ''' w$ for all $w \in X \smallsetminus Y]$.

(β) Let us now assume that, on the contrary, $q > (n/2)$. Then, consider the following triple of preference relations:

$\succcurlyeq^\circ := [x \succ^\circ x \vee z \succ^\circ 0 \sim^\circ z \sim^\circ w$ for all $w \in X \smallsetminus Y]$,

$\succcurlyeq^{\circ\circ} := [z \succ^{\circ\circ} x \vee z \succ^{\circ\circ} 0 \sim^{\circ\circ} x \sim^{\circ\circ} w$ for all $w \in X \smallsetminus Y]$,

$\succcurlyeq^{\circ\circ\circ} := [0 \succ^{\circ\circ\circ} x \sim^{\circ\circ\circ} z \sim^{\circ\circ\circ} x \vee z \sim^{\circ\circ\circ} w$ for all $w \in X \smallsetminus Y]$.

Notice that by construction such preferences are unimodal with respect to $\mathcal{X}$, i.e. $\{\succcurlyeq^\circ, \succcurlyeq^{\circ\circ}, \succcurlyeq'\} \subseteq U_{\mathcal{X}}$. Two subcases are distinguished again according to the parity of n, namely

(i) $n = 2k+1$ for some positive integer k, and (ii) $n = 2k$ for some positive integer k. If $(\beta(i))$ obtains, then take preference profile $\succcurlyeq^\circ_{[O]} = ((\succcurlyeq^\circ_i)_{i\in\{1,...,k\}}, (\succcurlyeq^{\circ\circ}_i)_{i\in\{k+1,...,2k\}}, \succcurlyeq^{\circ\circ\circ}_n)$ and compute $f(w_N) = \vee_{S\in\mathcal{F}}((\wedge_{i\in S} w_i) \wedge y_s^*)$ where $w_N = top(\succcurlyeq^\circ_{[O]})$ i.e. $w_i = x$ for all $i \in \{1, ..., k\}$, $w_i = z$ for all $i \in \{k+1, ..., 2k\}$, and $w_n = 0$. By construction, $f(w_N)$ is the l.u.b. of a nonempty family $\mathcal{T}$ of terms belonging to some of the following jointly exhaustive, partially overlapping classes:

$T_1 = \{\wedge_{j\in J} v_j : J$ is a finite set J and there exists $j \in J$ such that $v_j = 0\}$,

$T_2 = \{\wedge_{j\in J} v_j : J$ is a finite set J and there exist $j, h \in J$ such that $v_j = x$ and $v_h = z\}$,

$$T_3 = \left\{ \begin{array}{c} \wedge_{j\in J} v_j :\ J \text{ is a finite set and there exists} \\ \text{a nonempty } J' \subseteq J \text{ such that } |J'| \leq k < q \text{ and } v_j = x \text{ for all } j \in J' \end{array} \right\},$$

$$T_4 = \left\{ \begin{array}{c} \wedge_{j\in J} v_j :\ J \text{ is a finite set and there exists} \\ \text{a nonempty } J' \subseteq J \text{ such that } |J'| \leq k < q \text{ and } v_j = z \text{ for all } j \in J' \end{array} \right\}.$$

Notice that, again, $t = \wedge_{j\in J} v_j = 0$ for all $t \in T_1 \cup T_2$. Moreover, by construction, $t = \wedge_{j\in J} v_j < x$ for all $t \in T_3$ and $t = \wedge_{j\in J} v_j < z$ for all $t \in T_4$. Since both x and y are atoms of $\mathcal{X}$, it follows that $t = \wedge_{j\in J} v_j = 0$ for all $t \in T_3 \cup T_4$ whence $f(w_N) = 0$.

If $(\beta(ii))$ obtains then take preference profile $\succcurlyeq^\circ_{[E]} = ((\succcurlyeq^\circ_i)_{i\in\{1,...,k\}}, (\succcurlyeq^{\circ\circ}_i)_{i\in\{k+1,...,2k-1\}}, \succcurlyeq^{\circ\circ\circ}_n)$, and compute $f(w_N') = \vee_{S\in\mathcal{F}}((\wedge_{i\in S} w_i') \wedge y_s^*)$, where $w_N' = top(\succcurlyeq'_{[E]})$ i.e. $w_i' = x$ for all $i \in \{1, ..., k\}$,

$w'_i = z$ for all $i \in \{k+1, ..., 2k-1\}$, and $w'_n = 0$. Again, $f(w'_N)$ is the l.u.b. of a nonempty family $\mathcal{T}$ of terms belonging to some of the following jointly exhaustive, partially overlapping classes:

$T'_1 = \{\wedge_{j \in J} v_j : J$ is a finite set J and there exists $j \in J$ such that $v_j = 0\}$,

$$T'_2 = \left\{ \begin{array}{c} \wedge_{j \in J} v_j : J \text{ is a finite set } J \text{ and there exist} \\ j, h \in J \text{ such that } v_j = x \text{ and } v_h = z \end{array} \right\},$$

$$T'_3 = \left\{ \begin{array}{c} \wedge_{j \in J} v_j : J \text{ is a finite set and there exists} \\ \text{a nonempty } J' \subseteq J \text{ such that } |J'| < q \text{ and } v_j = x \text{ for all } j \in J' \end{array} \right\},$$

$$T'_4 = \left\{ \begin{array}{c} \wedge_{j \in J} v_j : J \text{ is a finite set and there exists} \\ \text{a nonempty } J' \subseteq J \text{ such that } |J'| < q \text{ and } v_j = z \text{ for all } j \in J' \end{array} \right\}.$$

Notice that $t = \wedge_{j \in J} v_j = 0$ for all $t \in T'_1$, and for all $t \in T'_2$ as well since $x \wedge z = 0$. Moreover, since $f(w'_N) = \vee_{S \in \mathcal{F}}((\wedge_{i \in S} w'_i) \wedge y^*_s)$, it also follows that $t = \wedge_{j \in J} v_j < x$ for all $t \in T'_3 \cap \mathcal{T}$ and $t = \wedge_{j \in J} v_j < z$ for all $t \in T'_4 \cap \mathcal{T}$. On the other hand, $t_2 \geqslant x$ and $t_3 \geqslant z$ for any $t_2 \in T'_2$ and $t_3 \in T'_3$. It follows, again, that $f(w'_N) = 0$. Now, take $u'_N \in X^N$ with $u'_i = x \vee z$ for all $i \in \{1, ..., n-1\} = N \smallsetminus \{n\}$, and $u'_n = 0$. By construction, $f(u'_N) = \vee_{S \in \mathcal{F}}((\wedge_{i \in S} u'_i) \wedge y^*_s)$ is the l.u.b. of a nonempty family $\mathcal{T}$ of terms belonging to some of the following jointly exhaustive, partially overlapping classes:

$T''_1 = \{\wedge_{j \in J} v_j : J$ is a finite set J and there exists $j \in J$ such that $v_j = 0\}$,

$$T''_2 = \left\{ \begin{array}{c} \wedge_{j \in J} v_j : J \text{ is a finite set and there exists} \\ \text{a nonempty } J' \subseteq J \text{ such that } |J'| < q \text{ and } v_j = x \vee z \text{ for all } j \in J' \end{array} \right\},$$

$$T''_3 = \left\{ \begin{array}{c} \wedge_{j \in J} v_j : J \text{ is a finite set and there exists} \\ J' \subseteq J \text{ such that } |J'| \geq q \text{ and } v_j = x \vee z \text{ for all } j \in J' \end{array} \right\}.$$

Observe that $t = \wedge_{j \in J} v_j = 0$ for all $t \in T''_1$. Moreover, by definition of f and q, both $y^*_{s'} < x$ and $y^*_{s'} < z$ for all $s' < q$, hence $t = \wedge_{j \in J} v_j = 0$ for all $t \in T''_2$ as well. Furthermore, $T''_3 \cap \mathcal{T} \neq \varnothing$ and , by definition of f and q, it must be the case that for all $s \geq q$, both $x \leqslant y^*_s$ and $z \leqslant y^*_s$ hold. Therefore, $x \vee z \leqslant y^*_s$. It follows that $f(u'_N) = x \vee z$. Thus, if $n = 2k+1$, $f(u'_N) \succ_i f(w_N)$ for all $i \in N \smallsetminus \{n\}$. Similarly, if $n = 2k$, then $f(u'_N) \succ_i f(w'_N)$ for all $i \in N \smallsetminus \{n\}$. Hence, f is coalitionally manipulable at unimodal preference profile $\succcurlyeq^\circ_{[O]} \in U^N_{\mathcal{X}}$ (at unimodal preference profile $\succcurlyeq^\circ_{[E]} \in U^N_{\mathcal{X}}$, respectively), a contradiction again, and the proof is complete. The locally strictly unimodal case can be addressed precisely by the same argument, provided preference profile $(\succcurlyeq^*, \succcurlyeq^{**}, \succcurlyeq^{***})$ is replaced by any locally strictly unimodal preference profile $(\succcurlyeq^+, \succcurlyeq^{++}, \succcurlyeq^{+++})$ such that

$\succcurlyeq^+ := [x \succ^+ x \vee z \succ^+ 0 \succ^+ z \succ^+ w$ for all $w \in X \smallsetminus Y]$,

$\succcurlyeq^{++} := [z \succ^{++} x \vee z \succ^{++} 0 \succ^{++} x \succ^{++} w$ for all $w \in X \smallsetminus Y]$,

$\succcurlyeq^{+++} := [0 \succ^{+++} x \succ^{+++} z \succ^{+++} x \vee z \succ^{+++} w$ for all $w \in X \smallsetminus Y]$. □

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