Majority judgment and strategy-proofness: a characterization

MAJORITY JUDGMENT AND STRATEGY-PROOFNESS: A CHARACTERIZATION

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ABSTRACT. Majority judgment as recently formulated and advocated by Balinski and Laraki in their influential monograph (*Majority Judgment* (2010)) is a method to aggregate profiles of judgments which are expressed in a common language consisting of a linearly ordered, and typically bounded, set of grades. It is shown that majority judgment thus defined is strategy-proof but *not* coalitionally strategy-proof on a very comprehensive class of rich single peaked preference domains. The proof relies on the key observation that a common bounded linear order of grades makes the set of gradings a product of bounded chains, which is a special instance of a bounded distributive lattice.

Relying on the foregoing result, this paper also provides a *simple characterization of majority judgment with an odd number of agents by anonymity, bi-idempotence and strategy-proofness on rich single peaked domains.*

Key words: Strategy-proofness, bounded distributive lattice, single peakedness, majority rule, majority judgment

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1. INTRODUCTION

Majority judgment is a family of collective decision mechanisms based upon a common language of grades, and relying on the following basic protocol (MJ): fix a *(bounded) linear order* Λ as a *common language of grades* and ask the n agents/voters to submit a *grading* i.e. an assignment of Λ-grades to alternatives, then assign to each alternative an aggregate grade given by the median of the grades received if n is odd, or the lower middlemost of the grades received if n is even[1].

[1]Hence, majority judgment methods are defined by their *input* consisting of grading profiles expressed in a common language of ordered grades. On the contrary, the output-type or outcome space may vary, depending on the relevant decision problem. Indeed, the basic protocol MJ is what Balinski and Laraki (2010) refer to as the Method of Grading (or more specifically the Social Grading Function) underlying Majority Judgment methods. But the output of MJ is a grading of the alternatives that may be used to select one or more alternatives (e.g. one of the

Thus, if N denotes the n-sized finite set of agents and Y the set of alternatives, MJ is an aggregation rule on $\Lambda^{Y\times N}$: a function $f : \Lambda^{Y\times N} \to \Lambda^{Y}$ i.e. an n-ary aggregation operation that maps grading profiles into gradings, thus producing gradings of all alternatives as outcomes.

The range of possible applications of majority judgment methods is vast, including mass elections, elections of variable-size committees, selections of research projects in funding decisions, score allocations in sport competitions and wine contests.

In their remarkable monograph, Balinski and Laraki (2010) strongly advocate majority judgement showing -inter alia- that majority judgment methods tend to be remarkably resistant to certain types of manipulation. In particular, they notice that MJ is strategy-proof for single-alternative grades on a certain single peaked domain of preferences, and mention that it is also 'group strategy-proof' (i.e. coalitionally strategy-proof) on such preference domain. What is missing in Balinski-Laraki's monograph is a full-fledged analysis of the strategy-proofness properties of MJ on single peaked domains of preferences on *outcomes* i.e. *gradings of all alternatives,* as opposed to grades for any *single* alternative i.e. *outcome-components.*

In order to address that issue, the present paper focusses on the special -but most typical and 'natural'- case where the ordered set of grades is *bounded.* Then, it relies on the key observation that the use of *a common language consisting of a bounded linear order of grades amounts to making the set of possible gradings a product of bounded chains, hence a special instance of a bounded distributive lattice.* Accordingly, strategy-proofness properties of MJ are studied within the more general setting of aggregation rules on *bounded distributive lattices*, which allows a considerable extension and unification of several scattered results already available in the extant literature. In particular, it is shown that *MJ is indeed strategy-proof on any rich locally unimodal domain of preferences on gradings* (a considerably generalized single peaked domain that encompasses virtually all the several distinct notions of single peakedness typically used in the current literature). That result in turn makes it possible to obtain *a simple characterization of majority judgment in terms of strategy-proofness on those rich single peaked odd domains* (i.e. when the number of agents is odd). It is also shown that, on the contrary, MJ is *coalitionally manipulable* on the same rich single peaked domains, and on the smaller ones which

alternatives with a maximum majority/median grade). See Section 4 for further details and elaborations on this point.

are most commonly used in related works. The rest of the paper is organized as follows. Section 2 is devoted to the introduction of the basic definitions and preliminaries. Section 3 includes the main results of the paper. In Section 4 some related literature is discussed, while Section 5 provides some concluding remarks. The proofs are collected in the final Appendix.

2. Majority judgment, single peakedness, and strategy-proofness: definitions and preliminaries

Let $N = \{1, ..., n\}$ denote the *finite* population of agents, and

$\mathcal{X}$ =$(X, \leqslant)$ the partially ordered set of alternative outcomes (i.e. $\leqslant$ is a reflexive, transitive and antisymmetric binary relation on X). We denote as $x||y$ any pair x, y of $\leqslant$-incomparable outcomes, and assume $|N| \geq 3$ in order to avoid tedious qualifications, where $|\cdot|$ denotes the cardinality of a set.

We also assume that $\mathcal{X} = (X, \leqslant)$ is a **distributive lattice** namely both the *least-upper-bound* (or *join*) $\vee$ and the *greatest-lower-bound* (or *meet*) $\wedge$ of any $x, y \in X$ -as induced by $\leqslant$- are well-defined binary operations on X, and for all $x, y, z \in X$,

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ (or, equivalently, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$)

i.e. the two equivalent *distributive identities* hold.

Finally, we assume that $\mathcal{X}$ is **bounded** i.e. there exist $\bot, \top \in X$ such that $\bot \leqslant x \ \leqslant \top$ for all $x \in X$.

An **order filter** of $\mathcal{X}$ is a set $F \subseteq X$ such that $z \in F$ if and only if $z \in X$ and $y \leqslant z$ for some $y \in F$: it is said to be *non-trivial* if $F \neq \varnothing$ and *proper* if $F \neq X$. An order filter F of $\mathcal{X} = (X, \leqslant)$ is a **(latticial) filter** if $x \wedge y \in F$ for any $x, y \in F$. A (latticial) filter F of $\mathcal{X} = (X, \leqslant)$ is an **ultrafilter** if it is proper and *maximal* (i.e. there is no proper filter $F'$of $\mathcal{X}$ such that $F \subset F'$) and is **principal** if there exists an $x \in X$ such that $F = [x) := \{y \in X : x \leqslant y\}$: clearly, as it is easily checked any proper principal filter is an ultrafilter. Moreover, a filter F of $\mathcal{X} = (X, \leqslant)$ is **prime** if $x \vee y \in F$ implies that $x \in F$ or $y \in F$. The set of all *non-trivial and proper* prime filters of $\mathcal{X}$ is denoted by $\mathcal{F}_P$.

It should be recalled here the following important and well-known fact to be used below, namely:

Fact (i): there is a bijection between the elements of a bounded distributive lattice $\mathcal{X}$ and the sets of prime filters of $\mathcal{X}$ they belong to, namely the function $\phi : X \rightarrow 2^{\mathcal{F}_P}$ defined by the rule $\phi(x) =$

$\{F \in \mathcal{F}_P : x \in F\}$ is both injective and surjective (see e.g. Davey, Priestley (1990), chpt. 10).

Moreover, it is easily established that the following holds:

Fact (ii): *every ultrafilter (hence in particular every (proper) principal filter) of a bounded distributive lattice is a (proper) prime filter* (see Davey, Priestley (1990), Theorem 9.7).

In particular, $\mathcal{X} = (X, \leqslant)$ is a **linear order** or **chain** if $[x \leqslant y$ or $y \leqslant x]$ holds for all $x, y \in X$ (recall that, as it is easily checked, a chain does indeed satisfy the basic latticial properties and the distributive identities above, hence a linear order is an instance of a distributive lattice). It is also easily checked that *a product of bounded linear orders is a bounded distributive lattice*. Moreover, a (bounded) distributive lattice is **discrete** if it does not include any infinite chain.

A ternary **betweenness** relation

$B_{\mathcal{X}} = \{(x, z, y) \in X^3 : x \wedge y \leqslant z \leqslant x \vee y\}$ is defined on $\mathcal{X}$,

and for any $x, y \in X$,

$B_{\mathcal{X}}(x, ., y) = \{z \in X : x \wedge y \leqslant z \leqslant x \vee y\}$ is the *interval* induced by x and y: therefore, *for any* $x, y, z \in X$, $z \in B_{\mathcal{X}}(x, ., y)$ *if and only if* $(x, z, y) \in B_{\mathcal{X}}$ (*also written* $B_{\mathcal{X}}(x, z, y)$).

Moreover, a ternary **median** operation $\mu : X^3 \to X$ is defined on $\mathcal{X}$ =$(X, \leqslant)$ by the following rule: for any $x, y, z \in X$,

$$\mu(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (x \wedge z) = (x \vee y) \wedge (y \vee z) \wedge (x \vee z)$$

(the latter identity is of course a consequence of distributivity).

A few remarkable basic properties of $B_{\mathcal{X}}$ are listed below, and easily checked:

Claim 1. The latticial betweenness relation $B_{\mathcal{X}}$ of a distributive lattice $\mathcal{X}$ =$(X, \leqslant)$ satisfies the following conditions:

(i) *Symmetry*: for all $x, y, z \in X$, if $B_{\mathcal{X}}(x, z, y)$ then $B_{\mathcal{X}}(y, z, x)$;

(ii) *Closure* (or *Reflexivity*): for all $x, y \in X$, $B_{\mathcal{X}}(x, x, y)$ and $B_{\mathcal{X}}(x, y, y)$;

(iii) *Idempotence*: for all $x, y \in X$, $B_{\mathcal{X}}(x, y, x)$ only if $y = x$;

(iv) *Convexity* (or *Transitivity*): for all $x, y, z, u, v \in X$, if $B_{\mathcal{X}}(x, u, y)$, $B_{\mathcal{X}}(x, v, y)$ and $B_{\mathcal{X}}(u, z, v)$ then $B_{\mathcal{X}}(x, z, y)$;

(v) *Antisymmetry*: for all $x, y, z \in X$, if $B_{\mathcal{X}}(x, y, z)$ and $B_{\mathcal{X}}(y, x, z)$ then $x = y$.;

(vi) *Median-Equivalence*: for all $x, y, z \in X$, $B_{\mathcal{X}}(x, y, z)$ if and only if $\mu(x, y, z) = y$.

Now, consider the set T_X of all *topped* preorders on X (i.e. reflexive and transitive binary relations having a unique maximum in X). For any $\succcurlyeq \in T_X$, $top(\succcurlyeq)$ denotes the unique maximum of $\succcurlyeq$ (while $\succ$

and $\sim$ denote the asymmetric and symmetric components of $\succcurlyeq$, respectively, and - for any $x \in X$ - $UC(\succeq, x) := \{y \in X : y \succcurlyeq x\}$ denotes the upper contour of $\succcurlyeq$ at x). *Single peaked* (total) preorders are those topped (total) preorders that 'respect' -i.e. are consistent with- the betweenness relation $B_{\mathcal{X}}$. The relevant notion of $B_{\mathcal{X}}$-consistency, however, is amenable to several distinct specifications including *unimodality, metricity, local strict unimodality,* and *local unimodality.*

Definition 1. *A topped preorder $\succcurlyeq \in T_X$ is* ***unimodal*** *(with respect to $B_{\mathcal{X}}$) if and only if, for each $x, y, z \in X$, $z \in B_{\mathcal{X}}(x, ., y)$ implies that either $z \succcurlyeq x$ or $z \succcurlyeq y$ (or both).*

As mentioned above, the rationale underlying single peakedness as *unimodality* may be plainly described as follows: an unimodal total preference preorder *respects* betweenness $B_{\mathcal{X}}$ in that it never regards an intermediate or compromise outcome as strictly worse than both of its 'extreme'-counterparts.

An alternative notion of single peakedness has also been widely adopted in the literature under several labels including 'generalized single peakedness' (see e.g. Nehring and Puppe (2007 (a,b)) among others). It will be relabeled here '*locally strict unimodality*' for the sake of convenience, and may be formulated as follows in the present setting:

Definition 2. *A topped preorder $\succcurlyeq \in T_X$ -with top outcome x^*- is* ***locally strictly unimodal*** *(with respect to $B_{\mathcal{X}}$) if and only if, for each $y, z \in X$, $z \in B_{\mathcal{X}}(x^*, ., y) \smallsetminus \{y\}$ implies $z \succ y$.*

Remark 1. *It is worth noticing here that in the extant literature unimodality and locally strict unimodality are not always firmly distinguished as they should be. For instance, in a very interesting and widely cited paper (Nehring and Puppe (2007(a))) Moulin (1980) is quoted as a contribution on 'generalized single peaked' (i.e. locally strictly unimodal) preferences on an ordered line (but see also Barberà, Gul and Stacchetti (1993) who identify single peakedness and locally strict unimodality, suggesting that this is precisely the notion underlying Moulin's work, and provide a characterization of strategy-proof rules on the locally strictly unimodal domain in products of bounded intervals). Now, Moulin's definition, once reformulated in terms of preferences (as opposed to utilities, as in the original Moulin (1980), p. 439) amounts to the following requirement: 'If a is the top outcome or peak on the line $(X, \leqslant)$ then $a \succ x \succcurlyeq y$ if $a < x \leqslant y$ or $y \leqslant x < a$'. Notice however that this condition is only consistent with unimodality as opposed to local strict unimodality. To see this just consider $X = \{a, x, y\}$ with*

*$a < x < y$, and total preorder $\succcurlyeq$ such that $a \succ x \sim y$: by construction, $\succcurlyeq$ is certainly consistent with Moulin's condition, and it is in fact unimodal but **not** at all locally strictly unimodal (or 'generalized single peaked' in Nehring and Puppe's parlance). Indeed, under the common 'single peakedness' label, Moulin (1980) and Danilov (1994) focus on unimodal preferences, while Barberà, Gul and Stacchetti (1993) and Nehring and Puppe (2007 (a),(b)) consider locally strictly unimodal preferences.*

In view of the foregoing observations, we shall rely on a more general notion of single peakedness that encompasses both unimodality and locally strict unimodality, as made precise by the following definition

Definition 3. *A topped preorder $\succcurlyeq \in T_X$ -with top outcome x^*-is **locally unimodal** (with respect to $B_{\mathcal{X}}$) if and only if, for all $y, z \in X$, $z \in B_{\mathcal{X}}(x^*, ., y)$ implies that $z \succcurlyeq y$; moreover, for any $Y \subseteq X$, $\succcurlyeq$ is Y-**complete** if for each $y, y' \in Y$ either $y \succcurlyeq y'$ or $y' \succcurlyeq y$ (or both), and **total** if it is X-complete.*

Notice that -as it is easily verified, and left to the reader to check- if $\mathcal{X}$ is a (bounded) linear order then a topped total preorder is locally unimodal if and only if it is also unimodal (with respect to $B_{\mathcal{X}}$). Thus, local unimodality (as opposed to local strict unimodality) may be arguably regarded as *the* natural extension of the 'classic' notion of single-peakedness used by Moulin (1980) to arbitrary (bounded) distributive lattices.

Remark 2. *It should also be mentioned that a related class of preferences, namely **metric** topped preorders, are also occasionally if implicitly used in the literature (see e.g. Bandelt, Barthélemy (1984)). In the setting of the present paper, if $\mathcal{X} = (X, \leqslant)$ is discrete, a topped preorder $\succcurlyeq \in T_X$ -with top outcome x^*- is **metric** with respect to $B_{\mathcal{X}}$ (written $\succcurlyeq \in M_{\mathcal{X}}$) whenever, for each $y, z \in X$ with $y \in B_{\mathcal{X}}(x^*, ., z) \smallsetminus \{z\}$, $y \succ z$ if and only if $d(x^*, y) < d(x^*, z)$ (for some metric d, usually the geodesic i.e. shortest-path distance on the Hasse diagram or **covering graph**[2] of $\mathcal{X}$). By definition, a metric topped preorder is a special instance of a locally strictly unimodal total preorder **which is entirely determined by its top outcome.***

Let $U_{\mathcal{X}} \subseteq T_X$ denote the *set of all locally unimodal total preorders* (with respect to $B_{\mathcal{X}}$), and $U_{\mathcal{X}}^N$ the corresponding set of all N-*profiles*

[2]The *covering graph* of $\mathcal{X} = (X, \leqslant)$ is the graph $G(\mathcal{X})$ having X as its set of vertices, and edges connecting precisely those pairs $\{x, y\} \subseteq X$ such that $x \neq y$ and $B_{\mathcal{X}}(x, ., y) = \{x, y\}$.

of locally unimodal total preorders or *full locally unimodal domain*. Similarly, $M_{\mathcal{X}}, U^{*}_{\mathcal{X}}, S_{\mathcal{X}} \subseteq T_X$ denote respectively the *set of all metric, unimodal, locally strictly unimodal total preorders* (with respect to $B_{\mathcal{X}}$), and $M^N_{\mathcal{X}}$,$U^{*N}_{\mathcal{X}}$, $S^N_{\mathcal{X}}$ the corresponding *full* metric, unimodal and locally strictly unimodal domains. Notice that, by definition, $M_{\mathcal{X}} \subseteq S_{\mathcal{X}}$, and $S_{\mathcal{X}} \cup U^{*}_{\mathcal{X}} \subseteq U_{\mathcal{X}}$. We shall mostly focus on locally unimodal domains of preorders that *need not be total but satisfy a suitable richness condition*, as made precise by the following definition:

Definition 4. *A set $D_{\mathcal{X}}$ of locally unimodal preorders (with respect to $B_{\mathcal{X}}$) is **rich** if for all $x, y \in X$ there exists $\succcurlyeq \in D_{\mathcal{X}}$ such that $top(\succcurlyeq) = x$ and $UC(\succeq, y) = B_{\mathcal{X}}(x, ., y)$.*

It should be noticed here that for each $x, y \in X$ one such rich locally unimodal preorder $\succcurlyeq^{*}_{x,y} \in U_{\mathcal{X}}$ with three indifference classes is easily defined as follows: take $\{x\}$, $B(x, ., y) \smallsetminus \{x\}$, and any subset of $X \smallsetminus B_{\mathcal{X}}(x, ., y)$ to be the top, middle, and bottom indifference classes of $\succcurlyeq^{*}$, respectively. A similar construction is available for $U^{*N}_{\mathcal{X}}$ and $S^N_{\mathcal{X}}$. It should also be emphasized that on the contrary $M_{\mathcal{X}}$ is in general *not rich* (because an outcome z may well be at the same positive distance from top outcome x as another outcome y -hence in the same metric indifference class of y- while not staying between x and y).

An **aggregation rule** for (N, X) is a function $f : X^N \to X$. Moreover, for any $x_N \in X^N$ and prime filter $F \in \mathcal{F}_P$ we posit $N_F(x_N) := \{i \in N : x_i \in F\}$. The following properties of an aggregation rule will play a crucial role in the ensuing analysis:

Definition 5. *An aggregation rule $f : X^N \to X$ is $B_{\mathcal{X}}$-**monotonic** if and only if for all $x_N = (x_j)_{j \in N} \in X^N$, $i \in N$ and $x'_i \in X$, $f(x_N) \in B_{\mathcal{X}}(x_i, ., f(x'_i, x_{N \smallsetminus \{i\}})$.*

Definition 6. *An aggregation rule $f : X^N \to X$ is **independent** if and only if for all $x_N, y_N \in X^N$ and all $F \in \mathcal{F}_P$: if $N_F(x_N) = N_F(y_N)$ then $f(x_N) \in F$ if and only if $f(y_N) \in F$.*

Definition 7. *An aggregation rule $f : X^N \to X$ is **monotonically independent** if and only if for all $x_N, y_N \in X^N$ and all $F \in \mathcal{F}_P$: if $N_F(x_N) \subseteq N_F(y_N)$ then $f(x_N) \in F$ implies $f(y_N) \in F$.*

Observe that the aforementioned properties rely on the order-theoretic structure of the outcome set, with somewhat different emphases. Indeed, $B_{\mathcal{X}}$-*monotonicity* requires consistency of f with the betweenness relation of the underlying outcome set. The *independence* condition requires that whether or not the outcome of f at a certain profile satisfies a certain property depends *solely* on the characteristic pattern of *that* property at the given profile (and *not at all* on the characteristic patterns of *other* properties at the same profile)[3]. Clearly, the *monotonic independence* property is in turn a strenghtening of independence.

A **generalized committee** in N is a set of coalitions $\mathcal{W} \subseteq \mathcal{P}(N)$ such that $T \in \mathcal{W}$ if and only if $T \subseteq N$ and $S \subseteq T$ for some $S \in \mathcal{C}$ (a *committee* in N being a *non-empty* generalized committee in N which does *not* include the *empty* coalition)[4]. The *basis* of a generalized committee $\mathcal{W} \subseteq \mathcal{P}(N)$ is the set $\mathcal{W}_m \subseteq \mathcal{W}$ of its *minimal* coalitions[5]. A generalized committee $\mathcal{W} \subseteq \mathcal{P}(N)$ is **transversal** (or a **quorum system**) if $S \cap T \neq \varnothing$ for any $S, T \in \mathcal{W}$, **collegial** if $\bigcap \mathcal{W} \neq \varnothing$, **proper** if -for any $S \subseteq N$- $S \in \mathcal{W}$ entails $N \smallsetminus S \notin \mathcal{W}$, **strong** if -for any $S \subseteq N$- $S \notin \mathcal{W}$ entails $N \smallsetminus S \in \mathcal{W}$, **self-dual** if -for any $S \subseteq N$- $S \in \mathcal{W}$ if and only if $N \smallsetminus S \notin \mathcal{W}$ (i.e. if it is both proper and strong), **inclusive** if $\bigcup \mathcal{W}_m = N$, and **anonymous** if for any $S, T \subseteq N$ such that $|S| = |T|$, $S \in \mathcal{W}$ if and only if $T \in \mathcal{W}$.

A **generalized committee aggregation rule** is a function $f : X^N \rightarrow X$ such that, for some fixed generalized committee $\mathcal{W} \subseteq \mathcal{P}(N)$ and for all $x_N \in X^N$, $f(x_N) = \vee_{S \in \mathcal{W}} (\wedge_{i \in S} x_i)$. Such an aggregation rule is also denoted as $f = f_{\mathcal{W}}$. A prominent example of a generalized committee voting rule is of course the ***majority rule*** f^{maj} defined as follows: for all $x_N \in X^N$, $f^{maj}(x_N) = \vee_{S \in \mathcal{W}^{maj}} (\wedge_{i \in S} x_i)$ where $\mathcal{W}^{maj} = \left\{ S \subseteq N : |S| \geq \lfloor \frac{|N|+2}{2} \rfloor \right\}$ (i.e. $f^{maj} = f_{\mathcal{W}^{maj}}$).

The **majority judgment rule** - henceforth denoted f^{maj^*}- is precisely the majority rule as applied to $\mathbf{\Lambda}^Y = (\Lambda^Y, \leqslant)$ where Y is the set of alternative candidates, $\mathbf{\Lambda} = (\Lambda, \leq)$ is a bounded linear order that denotes a common grading language: 0 and 1 denote the bottom and

[3]Independence as defined above may be regarded as a suitably generalized and adapted version of the Arrowian 'independence of irrelevant alternatives' as made clear by its counterparts in Wilson (1975) and Monjardet (1990) under the labels 'simplicity' and 'J-decisivity', respectively.

[4]Thus, a generalized committee is just an *order filter* of the partially ordered set $(\mathcal{P}(N), \subseteq)$ of coalitions of N, while a committee is a *non-trivial* and *proper order filter* of $(\mathcal{P}(N), \subseteq)$ namely an order filter other than $\varnothing$ or $\mathcal{P}(N)$. Notice that in common game-theoretic parlance the pair $(N, \mathcal{W})$ -where $\mathcal{W}$ is an order filter of $(\mathcal{P}(N), \subseteq)$- is precisely a *simple game* on N, and $\mathcal{W}$ denotes its set of *winning coalitions*. A simple game $(N, \mathcal{W})$ is *proper* (or *superadditive*) if $\mathcal{W}$ is *proper*, and *strong* (or *subadditive*) if $\mathcal{W}$ is *strong* (the relevant definitions for $\mathcal{W}$ are provided immediately below in the text).

[5]It should be noticed that $\mathcal{W}_m$ is by construction a *clutter* (or *antichain*) of $(\mathcal{P}(N), \subseteq)$ i.e. for any $S, T \in \mathcal{W}_m$ if $S \neq T$ then both $S \nsubseteq T$ and $T \nsubseteq S$ hold. Thus, $\mathcal{W}_m$ is not a generalized committee i.e. an order filter, except for the trivial cases where $\mathcal{W}_m = \mathcal{W} = \{N\}$ or $\mathcal{W}_m = \mathcal{W} = \varnothing$.

top elements of $\mathbf{\Lambda}$, respectively, and $\leqslant$ is the component-wise order induced by $\mathbf{\Lambda}$ on Λ^Y. If in particular $\mathbf{\Lambda} = (\{0,1\}, \leq)$ the majority judgment rule amounts to *majority approval.*

Since aggregation rules only mention outcomes (as opposed to preferences on outcomes) their strategy-proofness properties require of course an explicit specification of the relevant preference domains. The ensuing analysis is mainly focussed on rich domains of locally unimodal preorders as made precise by the following definitions:

Definition 8. *Let $f : X^N \to X$ be an aggregation rule and $D_{\mathcal{X}} \subseteq U_{\mathcal{X}}$ be a rich domain of locally unimodal and $f[X^N]$-complete*[6] *preorders (with respect to $B_{\mathcal{X}}$). Then, f is (individually)* ***strategy-proof*** *on $D_{\mathcal{X}}^N$ if and only if, for all $x_N \in X^N$, $i \in N$ and $x' \in X$, and for all $\succcurlyeq_N = (\succcurlyeq_j)_{j\in N} \in D_{\mathcal{X}}^N$ such that $x_N = (top(\succcurlyeq_i))_{i\in N}$, not $f(x', x_{N\smallsetminus\{i\}}) \succ_i f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}})$.*

Definition 9. *Let $f : X^N \to X$ be an aggregation rule and $D_{\mathcal{X}} \subseteq U_{\mathcal{X}}$ be a rich domain of locally unimodal $f[X^N]$-complete preorders with respect to $B_{\mathcal{X}}$. Then, f is* ***coalitionally strategy-proof*** [7] *on $D_{\mathcal{X}}^N$ if and only if for all $x_N \in X^N$, $C \subseteq N$ and $x'_C \in X^C$, and for all $\succcurlyeq_N = (\succcurlyeq_j)_{j\in N} \in D_{\mathcal{X}}^N$ such that $x_N = (top(\succcurlyeq_i))_{i\in N}$, there exists $i \in C$ such that $f(x_N) \succcurlyeq_i f(x'_C, x_{N\smallsetminus C})$.*

3. Main results

The following Lemma and Proposition extend some previous results concerning strategy-proofness of aggregation rules on trees and on bounded distributive lattices (see Danilov (1994), Vannucci (2016)).

Lemma 1. *Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice, $f : X^N \to X$ an aggregation rule for (N, X), and $D_{\mathcal{X}}$ a rich domain of locally unimodal $f[X^N]$-complete preorders (with respect to $B_{\mathcal{X}}$). Then, the following statements are equivalent:*
(i) f is strategy-proof on $D_{\mathcal{X}}^N$;
(ii) f is $B_{\mathcal{X}}$-monotonic;
(iii) f is monotonically independent.

Remark 3. *Notice that richness of the preference domain is only required to prove that strategy-proofness implies $B_{\mathcal{X}}$-monotonicity, while the reverse implication holds anyway. Without the richness restriction strategy-proofness is a weaker condition than $B_{\mathcal{X}}$-monotonicity.*

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[6] Here $f[X^N]$ denotes of course the range of f.

[7] 'Group-strategy-proof' is an equivalent label also frequently met in the extant literature.

Proposition 1. *Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice and $B_{\mathcal{X}}$ its latticial betweenness relation. Then, a generalized committee aggregation rule $f : X^N \to X$ is $B_{\mathcal{X}}$-monotonic.*

As a corollary, it immediately follows that the majority rule on an arbitrary bounded distributive lattice -including majority judgment- is strategy-proof on any rich locally unimodal domain, namely

Corollary 1. *Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice, $B_{\mathcal{X}}$ its latticial betweenness relation, and $D_{\mathcal{X}} \subseteq U_{\mathcal{X}}$ a rich domain of locally unimodal total preorders (with respect to $B_{\mathcal{X}}$). Then, $f^{maj} : X^N \to X$ is strategy-proof on $D_{\mathcal{X}}^N$. In particular, the majority judgment rule $f^{maj^*} : \Lambda^{Y \times N} \to \Lambda^Y$ is strategy-proof on $D_{\mathbf{\Lambda}^Y}^N$.*

Observe that the common grading language Λ may be such that $|\Lambda| = |Y|$: in that case, the grades correspond to the possible rank-values of a ranking without ties of all the alternatives in Y . Thus, the final grading of any $y \in Y$ may well change if an alternative $z \in Y \smallsetminus \{y\}$ is removed from the set of available alternatives. If this kind of 'agenda manipulation' is to be avoided, then the common grading language should indeed be *fixed* across distinct agendas, to ensure independence of aggregate grades from the set of available alternatives. However, Corollary 1 establishes that for any fixed set of alternatives to be assessed, a *common* language of bounded grades is enough to enforce strategy-proofness of majority judgment on any rich domain of single peaked preferences.

Altogether, Lemma 1 and Proposition 1 can also be used to obtain a neat *characterization through strategy-proofness* of the majority rule, and of majority judgment as a special case, for an odd number of players.

To begin with, some further notions and properties of aggregation rules are to be introduced.

Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice, $F \in \mathcal{F}_P$ a (non-trivial, proper) prime filter of $\mathcal{X}$, and $f : X^N \to X$ an aggregation rule. A coalition $S \subseteq N$ is F*-decisive for* f if there exists a profile $x_N \in X^N$ such that $N_F(x_N) = S$ and $f(x_N) \in F$, and *decisive for* f if there exists some $F \in \mathcal{F}_P$ such that S is F-decisive for f. We denote by $\mathcal{W}^f$ the set of all decisive coalitions for f, and by $\mathcal{W}_m^f \subseteq \mathcal{W}^f$ the set of all *minimal* decisive coalitions for f. Notice that if f is monotonically independent then $\mathcal{W}^f$ is indeed a generalized committee as defined above, namely an order filter of $(\mathcal{P}(N), \subseteq)$. The following properties will be of special interest for the ensuing analysis.

Definition 10. *An aggregation rule $f : X^N \rightarrow X$ is* ***inclusive*** *if and only if* $\bigcup_{S \in \mathcal{W}_m^f} S = N$.

Definition 11. *An aggregation rule $f : X^N \rightarrow X$ is* ***anonymous*** *if and only if for all $x_N = (x_i)_{i \in N} \in X^N$ and all permutations $\sigma : N \rightarrow N$ $i \in N$, $f(x_N) = f(x_{\sigma(N)})$ (where $(x_{\sigma(N)} = (x_{\sigma(i)})_{i \in N}$).*

Definition 12. *An aggregation rule $f : X^N \rightarrow X$ is* ***idempotent*** *if for any $x \in X$ and any profile $x_N = (x_i)_{i \in N} \in X^N$ with $x_i = x$ for each $i \in N$, $f(x_N) = x$.*

Inclusivity of an aggregation rule ensures that each agent is locally positively pivotal hence may positively determine the final outcome by participating. Anonymity is a standard requirement to ensure an equal treatment of agents. Idempotence is a condition requiring respect for agents' unanimity. Notice that inclusivity and anonymity are mutually independent conditions. Indeed, by definition the constant aggregation rules are *trivially anonymous* but *not inclusive* (because for any $x \in X$ the only minimal decisive coalition of the constant aggregation rule f_x is $\emptyset$, the empty coalition). Conversely, for any *collegial* $\mathcal{W} \subseteq \mathcal{P}(\mathcal{N})$ with $\bigcup \mathcal{W}_m = N$ (such as e.g. $\mathcal{W} = \{\{1, 2, ..., i\}, \{i, i+1, ..., n\}\}$) $f_{\mathcal{W}}$ is inclusive but *not anonymous* (but see Remark 5 below for a further example involving a non-collegial *quorum system*). However, it is easily checked that any *idempotent aggregation rule is anonymous only if it is also inclusive* (but not conversely).

Moreover, it should be recalled here that inclusivity and anonymity of a general committee aggregation rule have already been defined in terms of properties of the underlying general committee. But such specialized notions are in fact consistent with the newly introduced general ones, as made precise by the following Claim.

Claim 2. Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice. Then,

(i) an inclusive aggregation rule $f : X^N \rightarrow X$ is a generalized committee rule $f_{\mathcal{W}}$ if and only if $\mathcal{W}$ is inclusive;

(ii) an anonymous aggregation rule $f : X^N \rightarrow X$ is a generalized committee rule $f_{\mathcal{W}}$ if and only if $\mathcal{W}$ is anonymous.

Definition 13. *If $\mathcal{X} = (X, \leqslant)$ is a bounded distributive lattice an aggregation rule $f : X^N \to X$ is* ***neutral*** *if and only if for all $x_N \in X^N$, and all $F, F' \in \mathcal{F}_P$: if $N_F(x_N) = N_{F'}(x_N)$ then $f(x_N) \in F$ if and only if $f(x_N) \in F'$.*

Of course, neutrality is a version of the standard requirement of an equal treatment of outcomes.

Definition 14. *An aggregation rule $f : X^N \to X$ is* ***bi-idempotent*** *if and only if for all $x_N = (x_i)_{i \in N} \in X^N$ and all $y, z \in X$, if $x_i \in \{y, z\}$ for all $i \in N$ then $f(x_N) \in \{y, z\}$.*

Bi-idempotence is a very specific and strong *'decisiveness plus faithfulness'* condition: it requires both ability of a rule to decide by choosing *one of the only two* overtly advanced proposals in (perfectly) *polarized* situations (a local *'no guessing'* condition), and at the same time to respect *unanimity* i.e. to record and reproduce faithfully any (perfectly) *consensual* proposal (observe that bi-idempotence entails idempotence). Thus, bi-idempotence is a *local* condition in that it only puts restrictions on the behaviour of an aggregation rule under two sorts of very specific, and polarly opposed, circumstances: perfect polarization of revealed opinions[8], and perfect consensus of revealed opinions. And in both cases, bi-idempotence requires a behavioural pattern that is consistent with the behaviour of the majority rule and of the dictatorial rules, under such circumstances. It turns out that for any generalized committee aggregation rule $f_{\mathcal{W}}$ on a lattice bi-idempotence is equivalent to a very interesting restriction on the underlying generalized committee $\mathcal{W}$ of winning coalitions. This is made precise by the following Claim first established by Monjardet (1990):

Claim 3. (Monjardet (1990)).[9] Let $f_{\mathcal{W}} : X^N \to X$ be a generalized committee aggregation rule on a lattice $\mathcal{X} = (X, \leqslant)$. Then $f_{\mathcal{W}}$ is bi-idempotent if and only if $\mathcal{W}$ is self-dual.

[8] See the extended discussion of polarized situations in Balinski and Laraki (2016) who argue that the majority aggregation rule is especially suitable to cope with them.

[9] See Monjardet (1990), Proposition 1.3. To be sure, such a proposition only refers to arbitrary *finite* (meet-semi)lattices i.e. partially ordered sets endowed with a well-defined meet operation, but it is immediately checked that Monjardet's proof also applies verbatim to *arbitrary* lattices.

We are now ready to state the main characterization result of the present work.

Theorem 1. *Let $\mathcal{X} = (X, \leqslant)$ be a bounded distributive lattice with $|X| \geq 3$, $B_{\mathcal{X}}$ its latticial betweenness relation, $f : X^N \to X$ an aggregation rule for (N, X), and $D_{\mathcal{X}} \subseteq U_{\mathcal{X}}$ a rich domain of locally unimodal total preorders with respect to $B_{\mathcal{X}}$. Then, f is anonymous, bi-idempotent and strategy-proof on $D_{\mathcal{X}}^N$ if and only if there exists a positive integer k such that $|N| = 2k + 1$ and $f = f^{maj}$.*

Remark 4. *Notice that the general characterization of f^{maj} provided by Theorem 1 is tight . To see this, consider any constant rule $f_x : X^N \to X$ (the constant rule that selects $x \in X$ for each $x_N \in X^N$), any dictatorial rule $f_i : X^N \to X$ with $i \in N$ (the i-th projection such that $f_i(x_N) = x_i$ for all $x_N \in X^N$), and the minority rule $f^{\min} : X^N \to X$ (namely, for all $x_N \in X^N$, $f^{\min}(x_N) = \vee \{z \in X : 1 \leq |\{i \in N : x_i = z\}| \leq k\}$ with $|N| = 2k + 1$). Clearly, f_x is anonymous and strategy-proof on $D_{\mathcal{X}}^N$ but not bi-idempotent, f_i is bi-idempotent and strategy-proof on $D_{\mathcal{X}}^N$ but not anonymous, $f^{\min}$ is anonymous and bi-idempotent but not strategy-proof on $D_{\mathcal{X}}^N$. Observe that the required condition on the cardinality of the outcome space is satisfied by the majority judgment rule f^{maj^*} whenever $\min\{|Y|, |\Lambda|\} \geq 2$ or ($1 \leq \min\{|Y|, |\Lambda|\}$ and $\max\{|Y|, |\Lambda|\} \geq 3$).*

Remark 5. *It should also be noticed that f^{maj} is indeed inclusive, since it is both anonymous and neutral. But anonymity cannot be replaced with inclusivity, because it is easily checked that there exist aggregation rules other than f^{maj} which are inclusive, bi-idempotent and strategy-proof on $D_{\mathcal{X}}^N$: with $N = \{1, 2, 3, 4, 5\}$ consider for instance $f = f_{\mathcal{W}}$ with*

$$\mathcal{W}_m = \{\{2,4\}, \{1,2,3\}, \{1,2,5\}, \{1,3,4\}, \{1,4,5\}, \{2,3,5\}, \{3,4,5\}\}.$$

Observe that $\mathcal{W}$ is an inclusive and self-dual quorum system but is not anonymous since e.g. $\mathcal{W} \cap \{\{2,4\}, \{1,2\}\} = \{\{2,4\}\}$.

Thus, Theorem 1 cannot be ameliorated in that respect. Moreover, if $|X| = 2$ then bi-idempotence reduces to idempotence. Hence, it is easily checked that the unanimity rules f_u^x and f_u^y (where $f_u^x(z_N) = y$ unless $z_i = x$ for each $i \in N$, and $f_u^y(z_N) = x$ unless $z_i = y$ for each $i \in N$) are also (bi-)idempotent, anonymous and strategy-proof on $D_{\mathcal{X}}^N$.

As a by-product, we also immediately obtain the following characterization of the majority judgment rule.

Corollary 2. *Let* $|N|$ *be an odd integer, and* $\min\{|Y|,|\Lambda|\} \geq 2$. *Then the majority judgment rule* $f^{maj^*} : \Lambda^{Y\times N} \to \Lambda^Y$ *is the only anonymous and bi-idempotent aggregation rule on* Λ^Y *which is strategy-proof on* $D^N_{\mathbf{\Lambda}^Y}$.

As mentioned above, Balinski and Laraki (2010) present and discuss a MJ-version of the classic characterization of strategy-proof voting rules for single-peaked domains on (bounded) chains due to Moulin (1980) to the effect that the order functions are the only 'strategy-proof-in-grading' social grading functions on a single peaked domain. Balinski and Laraki observe that Moulin's theorem implies that majority aggregation of the grade of each single alternative $y \in Y$ -i.e. $f^{maj^*}_y : \Lambda^N \to \Lambda-$ is strategy-proof and coalitionally strategy proof (or 'group-strategy-proof') on single peaked domains (Balinski and Laraki (2010), p. 193). However, since gradings of *all* alternatives are the actual outcomes of the aggregation rule, the main issues concerning manipulability are precisely strategy-proofness and coalitional strategy-proofness of f^{maj} on single peaked domains of *preferences on gradings* (as opposed to preferences on single-alternative grades).

Indeed, we have the following

Theorem 2. *Let* $\min\{|Y|,|\Lambda|\} \geq 2$. *Then, the majority judgment rule* $f^{maj^*} : \Lambda^{Y\times N} \to \Lambda^Y$ *is not coalitionally strategy-proof on* $U^{*N}_{\mathbf{\Lambda}^Y}$, $S^N_{\mathbf{\Lambda}^Y}$ *or* $U^N_{\mathbf{\Lambda}^Y}$.

The following two-grade, two-alternative, three-agent example may aptly illustrate the content of Theorem 2.

Example 1. *Let* $N = \{1,2,3\}$ *and* $\Lambda = \{1,2\}$ *as endowed with the natural order, and suppose that the profile of top gradings of the agents is* $\alpha = ((2,1),(1,2),(2,2))$. *Then,* $f^{maj^*}(\alpha) = (2,2)$. *Now, suppose that the agents' preference profile* $\succcurlyeq_N$ *is such that* $(2,1) \succ_1 (1,1) \succ_1 (2,2) \sim_1 (1,2)$, $(1,2) \succ_2 (1,1) \succ_2 (2,2) \sim_2 (2,1)$, *and* $(2,2) \succ_3 (2,1) \sim_3 (1,2) \sim_3 (1,1)$ *(it can be easily shown that such a profile is unimodal with respect to* $B_{\mathcal{X}}$*). Similarly, it is easily checked that preference profile* $\succcurlyeq'_N$ *with* $(2,1) \succ'_1 (1,1) \succ'_1 (2,2) \succ'_1 (1,2)$, $(1,2) \succ'_2 (1,1) \succ'_2 (2,2) \succ'_2 (2,1)$, *and* $(2,2) \succ'_3 (2,1) \succ'_3 (1,2) \succ'_3 (1,1)$ *is indeed locally strictly unimodal with respect to* $B_{\mathcal{X}}$. *By construction,* $top(\succcurlyeq) = top(\succcurlyeq') = \alpha$, *and* $f^{maj^*}((1,1),(1,1),(2,2)) = (1,1) \succ^+_i (2,2) = f^{maj^*}(\alpha)$, $\succ^+_i \in \{\succcurlyeq_i, \succcurlyeq'_i\}$, $i = 1,2$, *hence* f^{maj^*} *is manipulable by coalition* $\{1,2\}$ *at both profiles* $\succcurlyeq_N$ *and* $\succcurlyeq'_N$.

Notice, however, that the only possible metric topped preference preorders on $\mathbf{\Lambda}^2 = (\{1,2\}^2, \leqslant)$ *(with respect to* $B_{\mathbf{\Lambda}^Y}$*) are*
$\succcurlyeq_{(1,1)}, \succcurlyeq_{(1,2)}, \succcurlyeq_{(2,1)}, \succcurlyeq_{(2,2)}$ *defined as follows:*
$(1,1) \succ_{(1,1)} (1,2) \sim_{(1,1)} (2,1) \succ_{(1,1)} (2,2)$
$(1,2) \succ_{(1,2)} (1,1) \sim_{(1,2)} (2,2) \succ_{(1,2)} (2,1)$
$(2,1) \succ_{(2,1)} (1,1) \sim_{(2,1)} (2,2) \succ_{(2,1)} (1,2)$
$(2,2) \succ_{(2,2)} (1,2) \sim_{(2,2)} (2,1) \succ_{(2,2)} (1,1)$.
It follows that none of the preferences of profiles $\succcurlyeq_N$ *and* $\succcurlyeq'_N$ *are metric topped preference preorders (it is easily checked that such statement also applies to the corresponding profiles occurring in the proof of Theorem 2).*

Thus, strategy-proofness of the majority judgment rule on rich single peaked domains does not entail at all it is also *coalitionally* strategy-proof on such domains. It can be shown that this is due to a very general fact involving the 'natural' incidence-geometric structure induced by the latticial betweenness of $\mathbf{\Lambda}^Y$ whenever both Λ and Y include at least two elements (see Section 2 above for a definition of the kind of betweenness relation involved, and Vannucci (2016) for the relevant details).

4. Related literature

The present work is mainly focussed on a characterization of majority judgment through its strategy-proofness properties, as suggested by its title. But such characterization is obtained by exploiting some general results concerning strategy-proofness properties of aggregation rules in bounded distributive lattices. Accordingly, related results are conveniently grouped under two distinct headings, namely: (a) strategy-proof aggregation rules in bounded distributive lattices, and (b) majority judgment and its strategy-proofness properties.

(a) *Strategy-proof aggregation rules in bounded distributive lattices.* Several results of disparate generality on strategy-proof aggregation rules in bounded distributive lattices are already available in the literature, but they are typically either more specialized than or just incomparable to Proposition 1 and Corollary 1 of the present work (note, however, that Proposition 1 is in fact implied by the main characterization result for strategy-proof aggregation rules of Savaglio, Vannucci (2018), forthcoming).

Indeed, Corollary 1 provides a generalization of the main results of Moulin (1980) concerning strategy-proof aggregation rules on the full unimodal domain over a (bounded) linear order. Moreover, Barberà, Gul and Stacchetti (1993) implies a particular version of Corollary 1 namely strategy-proofness of the majority judgment rule on the full locally strictly unimodal domain $S^N_{\mathbf{\Lambda}^Y}$ when $\mathbf{\Lambda}$ is a bounded integer interval (or even a bounded real interval) and $\beta_{\mathbf{\Lambda}^Y}$ is the betweenness relation canonically induced by the 'taxi-cab' metric. Nehring and Puppe (2007 (a)) also includes (or implies) some strictly related results on aggregation rules in *finite* median (interval) spaces -including *finite* distributive lattices- as induced by certain *property spaces,* and their strategy-proofness properties on rich locally *strictly* unimodal domains. Specifically, Nehring and Puppe prove that the characteristic set of decisive (or winning) coalitions of any sovereign and neutral social choice function that is strategy-proof on the aforementioned domains is self-dual (see Nehring and Puppe (2007 (a), Theorem 4). They also point out that, as a consequence, for any *finite* property-based (interval) space of outcomes and any rich *locally strictly unimodal* domain on that space there exists a strategy-proof social choice function defined on such domain that is sovereign, neutral and anonymous (namely, the top-only social choice function having the set of simple majorities as its characteristic set of winning coalitions)[10] if and only if the population of agents is *odd* and the underlying outcome space is *median* (ibidem, Corollary 5). Now, any distributive lattice is indeed a median space with respect to its own latticial betweenness relation, and any aggregation rule on a bounded distributive lattice (once a rich single peaked domain on such outcome space has been specified) may be regarded as a social choice function with the top-only property (i.e. a social choice function whose outcomes at each preference profile only depend on the corresponding profile of announced optima). Therefore, both results mentioned above overlap to a non-negligible extent with Proposition 1, Corollary 1 and Theorem 1 of the present work. Of course, neither of such Nehring and Puppe's results follows *entirely* from our own: they simply can't, because they concern *every* finite *median* space (thus including structures that are *not* distributive lattices, such as e.g. trees). Notice however that -when it comes to the majority aggregation rule on (bounded) *distributive lattices*- Proposition 1 is *considerably more general* than the results of Nehring and Puppe (2007 (a)) and all the other results mentioned above, since it covers *arbitrary* bounded distributive lattices (which may be neither finite nor products of bounded chains), and *arbitrary* rich domains of *locally unimodal* preferences (including

[10]But no uniqueness claim concerning that social choice function is made. Thus, strictly speaking, this is *not* a characterization of the majority aggregation rule (or, equivalently, of the corresponding top-only social choice function).

both unimodal and locally strictly unimodal preferences). Moreover, Theorem 1 provides a simple characterization of MJ with an odd number of agents (and no finiteness restriction on the common language of grades) in terms of strategy-proofness on such a large class of *very general* single peaked domains, and -most significantly- *as a special case of the majority rule* on a *specific* bounded distributive lattice.

Furthermore, Nehring, Puppe (2007 (b)) includes a result implying lack of Pareto efficiency for the majority rule on rich locally strictly unimodal domains of linear orders in finite non cube-free distributive lattices. Notice that such a result entails failure of the majority rule both to satisfy coalitional strategy-proofness and to select the Condorcet winner on every odd profile of that single-peaked domain as a 'sincere' strong equilibrium. Since that domain is strictly included in the domain of the present paper, such a failure also holds for the latter. It also follows that even the locally strictly unimodal domain of linear orders *is not included* in the domain consisting of profiles of linear orders that admit a strict Condorcet winner (the domain used for a characterization of the majority *social choice function* by *strategy-proofness*, *non-dictatorship* and *Condorcet-winner selection* due to Campbell, Kelly (2003)). It follows that even in terms of domains Theorem 1 is not comparable to Campbell and Kelly's characterization result.

On the other hand, Theorem 2 covers a *much richer* class of rich single peaked domains than locally strictly unimodal domains, as defined on the more general class of *all* bounded distributive lattices (including the trivially cube-free Boolean square).

Hence, as far as strategy-proofness properties of the majority aggregation rule on a bounded distributive lattice are concerned, and to the best of the author's knowledge, Theorem 1 and Theorem 2 above offer a definitely more general result than any other result previously available in the extant literature.

(b) *Majority judgment and its strategy-proofness properties.*

The extensive analysis of majority judgment methods provided by Balinski and Laraki (2010, 2014, 2016) makes use of a rich array of strictly related, but subtly different models and representations of majority judgment including (what they variously denote as) *methods of grading (MG), methods or ranking (MR)* induced by MG[11], and their

[11]Indeed, each grading $\alpha \in \Lambda^Y$ induces a total preorder $\succcurlyeq \in R_{Y(|\Lambda|)}$ where $R_{Y(|\Lambda|)}$ denotes the class of all total preorders on Y having at most $|Y|$ indifference classes (in particular, if $|\Lambda| \geq |Y|$ then every total preorder on Y is thus generated by some grading). It follows that any MG on Λ^Y induces in a natural way an MR i.e. an aggregation rule on $R_{Y(|\Lambda|)}$ (in fact an Arrowian social welfare function on Y when $|\Lambda| \geq |Y|$). Such an MR can be further refined (see e.g. the *majority gauge,*

respective well-behaved subclasses consisting of *social grading functions (SGF)* and *social ranking functions (SRF)* (the subclasses of anonymous, neutral, idempotent MG and MR, respectively, that are also suitably 'independent' and 'monotonic', and -possibly- continuous whenever the set of grades is explicitly endowed with a topology). Moreover, in Balinski and Laraki (2010, 2016) several distinct outcome spaces and respective preference domains are considered: to each specification of an outcome space and the relevant domain of preference relations defined on it, a distinct strategy-proofness issue is attached. Notice that the aggregation rule f^{maj^*} of the basic protocol MJ underlying majority judgment methods is a MG (and in particular a SGF), since it is in fact a (suitably 'regular') aggregation rule on Λ^Y i.e. $f^{maj^*} : \Lambda^{Y\times N} \rightarrow \Lambda^Y$. This fact by itself suggests the opportunity to consider the set Λ^Y as a possible outcome space in order to study the strategy-proofness properties of f^{maj^*} (as it is in fact done in the present work). However, this is hardly the option favored by Balinski and Laraki (2010) who tend rather to focus on the set Y of alternative candidates, or the set $Y \times \Lambda$ of (candidate, grade)-pairs[12]. As a consequence, they focus on strategy-proofness properties of the composite functions $\arg\max_{y\in Y} f_y^{maj^*} : \Lambda^{Y\times N} \rightarrow \mathcal{P}(Y)$ (or one its single-valued selections) or $\arg\max_{y\in Y} f_y^{maj^*} \times f^{maj^*}_{\arg\max_\Lambda} : \Lambda^{Y\times N} \rightarrow \mathcal{P}(Y\times\Lambda)$ i.e. the function selecting the pairs consisting of candidates with maximum median grade and their grade (or one its single-valued selections), and discuss their strategy-proofness properties with respect to single grades.[13] But then, f^{maj^*} is anyway in the picture as *the* aggregation rule of majority judgment methods. Thus, even if Y or $Y \times \Lambda$ -as opposed to Λ^Y- are eventually regarded as the most common or significant outcome spaces, a firm grasp of the strategy-proofness properties

a refined MJ-induced MR introduced by Balinski and Laraki (2010) mainly for use in mass elections).

[12]An outcome space consisting of candidate-grade pairs is the relevant one for the notion of Condorcet-Judgment-Winner introduced and briefly discussed by Balinski and Laraki in their monograph (see Balinski and Laraki (2010) chpt. 20.6).

[13]Indeed, such a move is not entirely unlike the early trend to shift the focus from social welfare functions to social choice functions that is detectable even in Arrow's classic monograph. Of course, with three or more alternatives, social choice functions face the Gibbard-Satterthwaite impossibility theorem. It is easily checked that if $|\Lambda| \geq 3$ the same holds for the majority judgment methods based on the composite functions mentioned above in the text. But while$f^{maj^*}$is a well-defined aggregation rule on Λ^Y there is no well-defined and majority-based Arrowian social welfare function.

of the underlying aggregation rule f^{maj^*} seems to be a crucial preliminary step toward a full understanding of the entire class of majority judgment methods and the source of their distinctive features.

5. Concluding remarks

'Majority judgment' has been repeatedly advertised by its advocates as a substitute for -and a major improvement on- 'voting', including 'majority voting' (see e.g. the catching motto 'Judge, don't vote!' which provides the title of Balinski and Laraki (2014)). Of course, that statement is to some extent disputable in that 'majority judgment' itself can also be regarded as a majority voting protocol of sorts, as plainly confirmed by Theorem 1 and Corollary 2. Nevertheless, even leaving aside labels, Balinski and Laraki (2010, 2014, 2016) make quite clear what is the main difference between (what they choose to call) 'judgment' as opposed to 'voting' protocols (or, for that matter, any standard social choice protocol): comparisons between alternative outcomes/candidates should rely on *a common grading language comprising a (typically bounded, or even finite) linearly ordered set of grades.* But in fact they add -somewhat implicitly- a further requirement to the effect *that the size of the set of available grades should not depend on the size of the set of available alternatives.* The latter requirement is indeed mandatory if 'structural' outcome manipulation through agenda manipulation is to be prevented, and most reasonable anyway. Nevertheless, *commonality* and *fixed-size* of the grading language are two *independent* requirements to be firmly distinguished. In fact, it is precisely their *combination* that lends support to the 'judge-don't vote' proposition. But then, what about *the role of a common grading language as such*? What are its *specific* effects, if any?

The present work singles out a further, structural reason that helps to explain why *reliance on a common grading language is indeed momentous by itself, within a fixed-outcome-space setting.* To be sure, several standard social choice methods -e.g. social choice functions or correspondences - are *not aggregation operations* on ordered structures, hence are generally not covered by the present model. Indeed, social welfare functions are typically formulated as aggregation operations on ordered structures such as total preorders or linear orders. But such ordered domains -namely the poset of total preorders with respect to set inclusion, or the lattice of linear orders viewed as permutations of a reference order- are *not at all distributive lattices* or even median meet-semilattices, hence do not support a well-defined *and* well-behaved majority rule [14]. In that setting, introducing a common bounded linearly ordered set of grades ensures that for any set of alternative options the resulting aggregation rule of gradings *is* an aggregation operation on a product of bounded chains, which is an instance of a bounded distributive lattice. And in that environment, majority judgment as a nicely working version of the majority aggregation rule is well-defined and *strategy-proof* (though unfortunately *not* coalitionally strategy-proof) on a very comprehensive class of rich single peaked domains.

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[14] To be sure, the poset of total preorders (or weak orders) on a set ordered by set-inclusion is an upper distributive join-semilattice with the co-coronation (or meet-Helly) property i.e. a median join-semilattice, but of course not a distributive lattice (see Janowitz (1984)). It follows that such a domain only supports a co-majority aggregation rule that -even with an odd number of agents- is clearly *not* equivalent to the majority rule precisely because the underlying semilattice is not a distributive lattice: see e.g. Monjardet (1990).

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6. Appendix: Proofs

Proof of Lemma 1.

Proof. (i)⇒(ii) Let us assume that $f : X^N \to X$ is *not* $B_{\mathcal{X}}$-monotonic: hence, there exist $i \in N$, $x'_i \in X$ and $x_N = (x_i)_{i\in N} \in X^N$ such that $(x_i, f(x_N), f(x'_i, x_{N\smallsetminus\{i\}})) \notin B_{\mathcal{X}}$. Thus, by Closure of $B_{\mathcal{X}}$, $x_i \neq f(x_N) \neq f(x'_i, x_{N\smallsetminus\{i\}})$. To begin with, observe that for any $x \in X$, if $\succcurlyeq \in D_{\mathcal{X}}$ is such that $x = top(\succcurlyeq)$ then $B_{\mathcal{X}}(x, y, x)$ if and only if $y = x$. Indeed, suppose that there exists $y \neq x$ such that $(x, y, x) \in B_{\mathcal{X}}$: then by definition $\succcurlyeq$ is not locally unimodal hence $\succcurlyeq \notin D_{\mathcal{X}}$, a contradiction.

Next, consider an $f[X^N]$-complete preorder $\succcurlyeq^* \in D_{\mathcal{X}}$ such that $x_i = top(\succcurlyeq^*)$ and $UC(\succcurlyeq^*, f(x'_i, x_{N\smallsetminus\{i\}})) = B_{\mathcal{X}}(x_i, ., f(x'_i, x_{N\smallsetminus\{i\}}))$. Such a preorder exists since $D_{\mathcal{X}}$ is rich.

Now, by assumption $f(x_N) \in X \smallsetminus B_{\mathcal{X}}(x_i, ., f(x'_i, x_{N\smallsetminus\{i\}}))$

while $f(x'_i, x_{N\smallsetminus\{i\}}) \in B_{\mathcal{X}}(x_i, ., f(x'_i, x_{N\smallsetminus\{i\}}))$ by Closure of $B_{\mathcal{X}}$, hence by construction and $f[X^N]$-completeness $f(x'_i, x_{N\smallsetminus\{i\}}) \succ^* f(x_N)$.

Finally, posit $(\succcurlyeq_j)_{j\in N} \in D^N_{\mathcal{X}}$ such that $x_j = top(\succcurlyeq_j)$ for all $j \in N$ and $\succcurlyeq_i = \succcurlyeq^*$: then, f is *not* strategy-proof on $D^N_{\mathcal{X}}$.

(ii) ⇒(i) Conversely, let f be $B_{\mathcal{X}}$-monotonic. Next, consider any locally unimodal profile $\succcurlyeq_N = (\succcurlyeq_j)_{j\in N} \in D^N_{\mathcal{X}}$ and any $i \in N$. By definition of $B_{\mathcal{X}}$-monotonicity $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) \in B_{\mathcal{X}}(top(\succcurlyeq_i), ., f(x_i, x_{N\smallsetminus\{i\}}))$ for all $x_{N\smallsetminus\{i\}} \in X^{N\smallsetminus\{i\}}$ and $x_i \in X$. But then, since clearly by definition $top(\succcurlyeq_i) \succcurlyeq_i f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}})$, either $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) = top(\succcurlyeq_i)$ or $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) \succcurlyeq_i f(x_i, x_{N\smallsetminus\{i\}})$ by local unimodality of $\succcurlyeq_i$.

Hence, $f(top(\succcurlyeq_i), x_{N\smallsetminus\{i\}}) \succcurlyeq_i f(x_i, x_{N\smallsetminus\{i\}})$ in any case. It follows that f is indeed strategy-proof on $D^N_{\mathcal{X}}$.

(ii)⇒(iii) Suppose that f is $B_{\mathcal{X}}$-monotonic**.** Hence, for all $i \in N$, $y_i \in X$, and $x_N \in X^N$, $B_{\mathcal{X}}(x_i, f(x_N), f(y_i, x_{N\smallsetminus\{i\}}))$ i.e. by Median-Equivalence of $B_{\mathcal{X}}$, $f(x_N) = \mu(x_i, f(x_N), f(y_i, x_{N\smallsetminus\{i\}}))$. Therefore, for any prime filter $F \in \mathcal{F}_P$,

$f(x_N) \in F$ if and only if

$\mu(x_i, f(x_N), f(y_i, x_{N\smallsetminus\{i\}})) =$
$= (x_i \wedge f(x_N)) \vee (f(x_N) \wedge f(y_i, x_{N\smallsetminus\{i\}})) \vee (x_i \wedge f(y_i, x_{N\smallsetminus\{i\}})) \in F.$

It follows that for any $F \in \mathcal{F}_P$, if $f(x_N) \in F$ then $(x_i \wedge f(x_N)) \vee (f(x_N) \wedge f(y_i, x_{N\smallsetminus\{i\}})) \vee (x_i \wedge f(y_i, x_{N\smallsetminus\{i\}})) \in F$. Hence, if $f(x_N) \in F$ but $x_i \notin F$ and $f(y_i, x_{N\smallsetminus\{i\}}) \notin F$ it follows that -by definition of filter- $F \cap \left\{x_i \wedge f(x_N),\ f(x_N) \wedge f(y_i, x_{N\smallsetminus\{i\}}),\ x_i \wedge f(y_i, x_{N\smallsetminus\{i\}})\right\} = \varnothing$. But then, primality of F entails $(x_i \wedge f(x_N)) \vee (f(x_N) \wedge f(y_i, x_{N\smallsetminus\{i\}})) \vee (x_i \wedge f(y_i, x_{N\smallsetminus\{i\}})) \notin F$, a contradiction. Therefore,

(*) for any prime filter $F \in \mathcal{F}_P$, $f(x_N) \in F$ implies that either $x_i \in F$ or $f(y_i, x_{N\smallsetminus\{i\}}) \in F$.

Moreover, for any prime filter $F \in \mathcal{F}_P$, if $(x_i \wedge f(x_N)) \vee (f(x_N) \wedge f(y_i, x_{N\smallsetminus\{i\}})) \vee (x_i \wedge f(y_i, x_{N\smallsetminus\{i\}})) \in F$ then $f(x_N) \in F$. Thus, if $x_i \in F$ and $f(y_i, x_{N\smallsetminus\{i\}}) \in F$ then $x_i \wedge f(y_i, x_{N\smallsetminus\{i\}}) \in F$ (by definition of latticial filter). It follows that, by definition of (order) filter, $(x_i \wedge f(x_N)) \vee (f(x_N) \wedge f(y_i, x_{N\smallsetminus\{i\}})) \vee (x_i \wedge f(y_i, x_{N\smallsetminus\{i\}})) \in F$ whence $f(x_N) \in F$.

Therefore,

(**) for any prime filter $F \in \mathcal{F}_P$, if $x_i \in F$ and $f(y_i, x_{N\smallsetminus\{i\}}) \in F$ then $f(x_N) \in F$.

Now, let us consider $x_N, y_N \in X^N$ and a prime filter $F \in \mathcal{F}_P$ such that $f(x_N) \in F$ and $N_F(x_N) \subseteq N_F(y_N)$.

Hence, by (*), either $x_1 \in F$ or $f(y_1, x_{N\smallsetminus\{1\}}) \in F$. It follows that $f(y_1, x_{N\smallsetminus\{1\}}) \in F$ holds anyway. Indeed, if $x_1 \in F$ then $y_1 \in F$ since $N_F(x_N) \subseteq N_F(y_N)$. Therefore, by (**), $f(y_1, x_{N\smallsetminus\{1\}}) \in F$ holds.

Next, consider $(y_1, x_{N\smallsetminus\{1\}})$ and $(y_1, y_2, x_{N\smallsetminus\{1,2\}})$. Since $f(y_1, x_{N\smallsetminus\{1\}}) \in F$ holds, and $N_F(y_1, x_{N\smallsetminus\{1\}}) \subseteq N_F(y_1, y_2, x_{N\smallsetminus\{1,2\}})$ by construction, we may repeat the previous argument to conclude that $f(y_1, y_2, x_{N\smallsetminus\{1,2\}}) \in F$. Thus, by a suitable iteration of that argument, we may conclude that $f(y_N) \in F$. It follows that f is monotonically independent.

(iii) $\Longrightarrow$(ii) Suppose that f is monotonically independent but *not* $B_{\mathcal{X}}$-monotonic. Thus, there exist $i \in N$, $(x_h)_{h\in N} \in X^N$, $y_i \in X$ such that

$f((x_j)_{j\in N}) \neq \mu(x_i, f((x_j)_{j\in N}), f(y_i, (x_j)_{j\in N\smallsetminus\{i\}}))$. Therefore, from the bijection between the elements of a bounded distributive lattice and the sets of prime filters mentioned above (see e.g. Davey, Priestley (1990), Theorem 10.18) it follows that either

(α) there exists a prime filter $F \in \mathcal{F}_P$ such that $f(x_N) \in F$ but $(x_i \wedge f((x_j)_{j\in N})) \vee (f((x_j)_{j\in N}) \wedge f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})) \vee (x_i \wedge f(y_i, (x_j)_{j\in N\smallsetminus\{i\}})) \notin F$

or

(β) there exists a prime filter $F' \in \mathcal{F}_P$ such that

$(x_i \wedge f(x_N)) \vee (f(x_N) \wedge f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \vee (x_i \wedge f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \in F'$ and $f(x_N) \notin F'$.

Thus, let us first assume that (α) holds. Now, if $x_i \in F$ then - by definition of (latticial) filter- $x_i \wedge f(x_N) \in F$, and similarly if $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \in F$ then $(f(x_N) \wedge f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \in F$.

In either case, by definition of (order) filter,

$(x_i \wedge f(x_N)) \vee (f(x_N) \wedge f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \vee (x_i \wedge f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \in F$, a contradiction.

Therefore, $f(x_N) \in F$ implies $F \cap \{x_i, f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})\} = \varnothing$.

However, $x_i \notin F$ clearly implies $N_F(x_N) \subseteq N_F((y_i,(x_j)_{j\in N\smallsetminus\{i\}}))$, by definition. Therefore, since $f(x_N) \in F$ and f is monotonically independent, it follows that $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \in F$, a contradiction.

Let us then assume that (β) holds.

By primality of F' applied twice,

$(x_i \wedge f(x_N)) \vee (f(x_N) \wedge f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \vee (x_i \wedge f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \in F'$ implies that

$F' \cap \{x_i \wedge f(x_N), f(x_N) \wedge f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}), x_i \wedge f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}))\} \neq \varnothing$.

But, by definition of (order) filter, $(x_i \wedge f(x_N)) \in F'$ implies $f(x_N) \in F'$, and $(f(x_N) \wedge f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \in F'$ also implies $f(x_N) \in F'$, a contradiction.

Thus, it must be the case that $(x_i \wedge f(y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \in F'$. Hence, by definition of (order) filter, both $x_i \in F'$ and $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \in F'$.

But then, $N_{F'}((y_i,(x_j)_{j\in N\smallsetminus\{i\}})) \subseteq N_{F'}(x_N)$, and since f is monotonically independent, $f(y_i,(x_j)_{j\in N\smallsetminus\{i\}}) \in F'$ implies $f(x_N) \in F'$, which contradicts (β).

It follows that if f is monotonically independent then it is also $B_{\mathcal{X}}$-monotonic. □

Proof of Proposition 1.

Proof. By definition, there exists an order filter $\mathcal{W}$ of $(\mathcal{P}(N),\subseteq)$ such that for each $x_N = (x_j)_{j\in N} \in X^N$

$f(x_N) = \vee_{S\in\mathcal{W}}(\wedge_{j\in S}x_j)$.

Moreover, for any $i \in N$ and $x'_i \in X$,

$f(x'_i, x_{N\smallsetminus\{i\}}) = (\vee_{i\notin S\in\mathcal{W}}(\wedge_{j\in S}x_j)) \vee (\vee_{i\in S\in\mathcal{W}}(x'_i \wedge (\wedge_{j\in S\smallsetminus\{i\}}x_j))$.

Then,

$$\begin{aligned}
&x_i \wedge f(x'_i, x_{N\smallsetminus\{i\}}) = x_i \wedge [(\vee_{i\notin S\in\mathcal{W}}(\wedge_{j\in S}x_j)) \vee (\vee_{i\in S\in\mathcal{W}}(x'_i \wedge (\wedge_{j\in S\smallsetminus\{i\}}x_j)))] = \\
&= (x_i \wedge (\vee_{i\notin S\in\mathcal{W}}(\wedge_{j\in S}x_j))) \vee (x_i \wedge (\vee_{i\in S\in\mathcal{W}}(x'_i \wedge (\wedge_{j\in S\smallsetminus\{i\}}x_j)))] \leqslant \\
&\leqslant (\vee_{i\notin S\in\mathcal{W}}(\wedge_{j\in S}x_j)) \vee (\vee_{i\in S\in\mathcal{W}}(x_i \wedge (\wedge_{j\in S\smallsetminus\{i\}}x_j))) = f(x_N) \leqslant \\
&\leqslant (\vee_{i\notin S\in\mathcal{W}}(\wedge_{j\in S}x_j)) \vee (x_i \vee (\vee_{i\in S\in\mathcal{W}}(x'_i \wedge (\wedge_{j\in S\smallsetminus\{i\}}x_j))) = \\
&= x_i \vee [(\vee_{i\notin S\in\mathcal{W}}(\wedge_{j\in S}x_j)) \vee (\vee_{i\in S\in\mathcal{W}}(x'_i \wedge_{j\in S\smallsetminus\{i\}}x_j))] = x_i \vee f(x'_i, x_{N\smallsetminus\{i\}})
\end{aligned}$$

hence
$f(x_N) \in B_{\mathcal{X}}(x_i, ., f(x'_i, x_{N \smallsetminus \{i\}})$ and f is indeed $B_{\mathcal{X}}$-monotonic. □

Proof of Corollary 1.

Proof. Immediate from Lemma 1 and Proposition 1, and from the observation that Λ^Y is a product of bounded chains and is therefore a bounded distributive lattice with respect to the component-wise order. □

Proof of Claim 2.

Proof. First, observe that by Lemma 1 and Proposition 1 a generalized committee aggregation rule $f = f_{\mathcal{W}}$ on a bounded distributive lattice is monotonically independent. Suppose then that $f = f_{\mathcal{W}}$ i.e. $f(x_N) = \vee_{T \in \mathcal{W}}(\wedge_{i \in T} x_i)$ for any $x_N \in X^N$. Therefore, $\mathcal{W} \subseteq \mathcal{W}^f$, by definition of $\mathcal{W}^f$. Conversely, let $S \in \mathcal{W}^f$ i.e. there exists $x_N \in X^N$ and a non-trivial proper prime filter $F \in \mathcal{F}_P$ such that $N_F(x_N) = S$ and $f(x_N) \in F$. Now, suppose $S \notin \mathcal{W}$. Then, $T \nsubseteq S$ for each $T \in \mathcal{W}$ i.e. for any $T \in \mathcal{W}$ there exists $i_T \in T \setminus S$. Next, consider profile $y_N \in X^N$ defined as follows: $y_i = x_i$ for each $i \in S$, and $y_i = \perp$ for each $i \in N \setminus S$. Since F is proper i.e. $F \neq X$, $\perp \notin F$ whence $N_F(y_N) = N_F(x_N) = S$. Therefore, since $f = f_{\mathcal{W}}$ is monotonically independent and $f(x_N) \in F$, it follows that $f(y_N) \in F$. However, $f(y_N) = f_{\mathcal{W}}(y_N) = \vee_{T \in \mathcal{W}}(\wedge_{i \in T} x_i) = \perp$ since by construction for each $T \in \mathcal{W}$ there exists $i_T \in T$ with $y_{i_T} = \perp$: a contradiction. Hence $\mathcal{W}^f \subseteq \mathcal{W}$ as well i.e. $\mathcal{W}^f = \mathcal{W}$ and consequently $\mathcal{W}^f_m = \mathcal{W}_m$. Since f is inclusive if and only if $\bigcup_{S \in \mathcal{W}^f_m} S = N$ and $\mathcal{W}$ is inclusive if and only if $\bigcup_{S \in \mathcal{W}_m} S = N$, part (i) of the Claim follows immediately. Suppose now that $f = f_{\mathcal{W}}$ is anonymous, and $\mathcal{W}$ is not anonymous i.e. there exist $S, T \subseteq N$ such that $|S| = |T|$ with $\mathcal{W} \cap \{S, T\} = \{S\}$. Therefore, there exist $F \in \mathcal{F}_P$ and $x_N \in X^N$ such that $N_F(x_N) = S$ and $f(x_N) \in F$, yet there are no $F \in \mathcal{F}_P$ and $y_N \in X^N$ such that $N_F(y_N) = T$ and $f(y_N) \in F$.

But that contradicts anonymity of f since any permutation $\sigma : N \to N$ with $\sigma(T) = S$ yields $f(x_{\sigma(N)}) = f(x_N) \in F$ with $N_F(x_{\sigma(N)}) = T$. Therefore, $\mathcal{W}$ is anonymous. Conversely, suppose that $f = f_{\mathcal{W}}$ with $\mathcal{W}$ anonymous. Clearly, if $f(x_N) = \perp$ for each $x_N \in X^N$ i.e. if $\mathcal{W} = \varnothing$, then f is a constant function hence it is trivially anonymous. Otherwise, there exist $x_N \in X^N$ and prime filter $F \in \mathcal{F}_P$ such that

$N_F(x_N) = S$ and $f(x_N) \in F$ i.e. $S \in \mathcal{W}^f = \mathcal{W}$. But then, for each permutation $\sigma : N \to N$, $\sigma(S) \in \mathcal{W} = \mathcal{W}^f$ since $|\sigma(S)| = |S|$.

It follows that, for any $x_N \in X^N$, $f_{\mathcal{W}}(x_N) = \vee_{T \in \mathcal{W}}(\wedge_{i \in T} x_i) = \vee_{\sigma(T) \in \mathcal{W}}(\wedge_{i \in \sigma(T)} x_{\sigma(i)}) = f_{\mathcal{W}}(x_{\sigma(N)})$, namely $f = f_{\mathcal{W}}$ is anonymous. □

Proof of Claim 3. (the following proof is due to Monjardet (1990), who develops it in a framework that only considers *arbitrary finite meet-semilattices*, including of course *finite* lattices: it is reproduced here in a very detailed style for the sake of completeness, and in order to make it possible for the reader to appreciate that such a proof also applies to *arbitrary lattices*)

Proof. $\Rightarrow$ The proof consists of four steps.

Step 1. To begin with, observe that for any $x, y \in X$ such that $x||y$ (i.e. x and y are $\leqslant$-incomparable) it must be the case that the following relationships hold: $x > x \wedge y$, $y > x \wedge y$, $x < x \vee y$, $y < x \vee y$. That is so because by definition $x \geqslant x \wedge y$ and $y \geqslant x \wedge y$ but neither $x = x \wedge y$ (which implies $x \leqslant y$) nor $y = x \wedge y$ (which implies $y \leqslant x$) hold, since $x||y$. Similarly, $x \leqslant x \vee y$ and $y \leqslant x \vee y$ by definition, but neither $x = x \vee y$ (which implies $x \geqslant y$) nor $y = x \vee y$ (which implies $y \geqslant x$) hold, since $x||y$.

Step 2. We prove the following claim: if $f : X^N \to X$ is a generalized committee aggregation rule with $f = f_{\mathcal{W}}$ then $\mathcal{W}$ is *proper* (i.e. for all $S \subseteq N$, $S \in \mathcal{W}$ entails $N \smallsetminus S \notin \mathcal{W}$) if and only if [for every $x, y \in X$ with $x||y$ and $S \subseteq N$, and for any $x_N \in X^N$ with $x_i = x$ for each $i \in S$, and $x_i = y$ for each $i \in N \smallsetminus S$: $f_{\mathcal{W}}(x_N) < x \vee y$].

Indeed, suppose $\mathcal{W}$ is *proper*. For any $S \subseteq N$, if $S \notin \mathcal{W}$ then for each $T \in \mathcal{W}$ it must be the case that $T \cap (N \smallsetminus S) \neq \varnothing$ (because otherwise $T \subseteq S \notin \mathcal{W}$, a contradiction since by hypothesis $\mathcal{W}$ is an order filter). Thus, for any $x, y \in X$ with $x||y$, and for any $x_N \in X^N$ with $x_i = x$ for each $i \in S$, and $x_i = y$ for each $i \in N \smallsetminus S$: $f_{\mathcal{W}}(x_N) = \vee_{T \in \mathcal{W}}(\wedge_{i \in T} x_i) \leqslant y < x \vee y$. If on the contrary $S \in \mathcal{W}$, and therefore by properness $N \smallsetminus S \notin \mathcal{W}$, then for each $T \in \mathcal{W}$ it must be the case that $T \cap S \neq \varnothing$ (because otherwise $T \subseteq (N \smallsetminus S) \notin \mathcal{W}$, a contradiction again since by hypothesis $\mathcal{W}$ is an order filter). Hence, for any $x, y \in X$ with $x||y$, and for any $x_N \in X^N$ with $x_i = x$ for each $i \in S$, and $x_i = y$ for each $i \in N \smallsetminus S$: $f_{\mathcal{W}}(x_N) = \vee_{T \in \mathcal{W}}(\wedge_{i \in T} x_i) \leqslant x < x \vee y$. It follows that $f_{\mathcal{W}}(x_N) < x \vee y$ as required.

To establish the converse statement, let us proceed by contraposition. Thus, suppose that $\mathcal{W}$ is *not* proper i.e. there exists $S \in \mathcal{W}$ such that $(N \smallsetminus S) \in \mathcal{W}$. Then, consider any $x, y \in X$ with $x||y$, and any $x_N \in X^N$

with $x_i = x$ for each $i \in S$, and $x_i = y$ for each $i \in (N \smallsetminus S)$. It follows, by construction, that $f_{\mathcal{W}}(x_N) = \vee_{T \in \mathcal{W}}(\wedge_{i \in T} x_i) = x \vee y$.

Step 3. We prove the following claim: if $f : X^N \rightarrow X$ is a generalized committee aggregation rule with $f = f_{\mathcal{W}}$ then $\mathcal{W}$ is *strong* (i.e. for all $S \subseteq N$, $S \notin \mathcal{W}$ entails $(N \smallsetminus S) \in \mathcal{W}$) if and only if [for every $x, y \in X$ with $x||y$ and $S \subseteq N$, and for any $x_N \in X^N$ with $x_i = x$ for each $i \in S$, and $x_i = y$ for each $i \in N \smallsetminus S$: $f_{\mathcal{W}}(x_N) > x \wedge y$].

Indeed, suppose that $f = f_{\mathcal{W}}$ such that $\mathcal{W}$ is *strong* and consider $x, y \in X$ with $x||y$ and $S \subseteq N$, and $x_N \in X^N$ with $x_i = x$ for each $i \in S$, and $x_i = y$ for each $i \in N \smallsetminus S$. If $S \in \mathcal{W}$ then by construction $f_{\mathcal{W}}(x_N) \geqslant x > x \wedge y$. If on the contrary $S \notin \mathcal{W}$ then $(N \smallsetminus S) \in \mathcal{W}$ (by strongness of $\mathcal{W}$), hence $f_{\mathcal{W}}(x_N) \geqslant y > x \wedge y$. It follows that $f_{\mathcal{W}}(x_N) > x \wedge y$ as required.

To establish the converse statement, let us proceed again by contraposition. Suppose there exists $S \subseteq N$ such that both $S \notin \mathcal{W}$ and $(N \smallsetminus S) \notin \mathcal{W}$. Therefore, it must be the case that for each $T \in \mathcal{W}$ both $T \cap S \neq \varnothing$ and $T \cap (N \smallsetminus S) \neq \varnothing$ (because otherwise either $T \subseteq (N \smallsetminus S) \notin \mathcal{W}$ or $T \subseteq S \notin \mathcal{W}$, a contradiction in both cases since $\mathcal{W}$ is by hypothesis an order filter). But then, for any $x, y \in X$ with $x||y$, $S \subseteq N$, and $x_N \in X^N$ with $x_i = x$ for each $i \in S$, and $x_i = y$ for each $i \in N \smallsetminus S$, it follows that by construction $f_{\mathcal{W}}(x_N) = x \wedge y$ whence indeed *not* $f_{\mathcal{W}}(x_N) > x \wedge y$.

Step 4. Now, consider a *bi-idempotent* generalized committee aggregation rule $f_{\mathcal{W}} : X^N \rightarrow X$, i.e. $f_{\mathcal{W}}$ is such that for every $x, y \in X$, $S \subseteq N$ and $x_N \in X^N$ with $x_i = x$ if $i \in S$ and $x_i = y$ if $i \in N \smallsetminus S$, $f_{\mathcal{W}}(x_N) = \vee_{T \in \mathcal{W}}(\wedge_{i \in T} x_i) \in \{x, y\}$. Thus, in particular, if $x||y$ then $x \wedge y < f_{\mathcal{W}}(x_N) < x \vee y$ (by Step 1 above). But then, from $f_{\mathcal{W}}(x_N) < x \vee y$ and Step 2 above it follows that $\mathcal{W}$ is proper, and from $x \wedge y < f_{\mathcal{W}}(x_N)$ and Step 3 above it follows that $\mathcal{W}$ is strong. Therefore, $\mathcal{W}$ is indeed *self-dual* as required.

$\Longleftarrow$ Straightforward. Suppose that $\mathcal{W}$ is self-dual i.e. both proper and strong, consider any $x, y \in X$, and any $x_N \in X^N$ such that $x_i \in \{x, y\}$ for all $i \in N$, and posit $S = \{i \in N : x_i = x\}$ (whence $N \smallsetminus S = \{i \in N : x_i = y\}$). Then, by self-duality of $\mathcal{W}$, either (i) $S \in \mathcal{W}$ and $(N \smallsetminus S) \notin \mathcal{W}$ or (ii) $(N \smallsetminus S) \in \mathcal{W}$ and $S \notin \mathcal{W}$. If (i) holds then by definition $f_{\mathcal{W}}(x_N) = x$, whereas if (ii) holds then $f_{\mathcal{W}}(x_N) = y$. It follows that $f_{\mathcal{W}}$ is actually bi-idempotent. $\square$

Proof of Theorem 1.

Proof. Clearly, f^{maj} is anonymous by definition, and strategy-proof on $D_{\mathcal{X}}^N$ by Corollary 1. Moreover, if $|N|$ is odd then f^{maj} is bi-idempotent, by definition.

Conversely, suppose that $f : X^N \to X$ is anonymous, bi-idempotent and strategy-proof on $D^N_{\mathcal{X}}$. Since f is strategy-proof on $D^N_{\mathcal{X}}$, it follows by Lemma 1 that f is *monotonically independent* (hence in particular it is also *independent*).

Now it can be shown that

(a) *if f is independent and bi-idempotent then it is also neutral.*

Indeed, suppose that f is independent and bi-idempotent, and let $F, F' \in \mathcal{F}_p$ and $x_N \in X^N$ such that $N_F(x_N) = N_{F'}(x_N) = A$, and $f(x_N) \in F$. If $A = \varnothing$ then $N_F(x_N) = A = N_F((\perp, ..., \perp))$ (because $F \in \mathcal{F}_p$ hence it is a *proper* filter and thus $\perp \notin F$): it follows that $f((\perp, ..., \perp)) \in F$, by independence of f. But bi-idempotence implies idempotence, hence $\perp = f((\perp, ..., \perp)) \in F$, a contradiction. Therefore, $A \neq \varnothing$ i.e. there exists $j \in N$ such that $x_j \in F \cap F'$ (clearly enough, $x_j \neq \perp$). Now, define $x'_N \in X^N$: $x'_i = x_j$ for every $i \in A$, and $x'_i = \perp$ otherwise. But then it follows from bi-idempotence that $f(x'_N) = x_j \in F \cap F'$ hence $f(x'_N) \in F'$. Moreover, $A = N_{F'}(x_N) = N_{F'}(x'_N)$ by construction hence by independence $f(x_N) \in F'$, and neutrality of f is thus established.

(b) *if f is monotonically independent and neutral then f is a generalized committee aggregation rule* i.e. there exists an order filter $\mathcal{W}$ of $(\mathcal{P}(N), \subseteq)$ such that for all $x_N \in X^N$

$f(x_N) = \vee_{A \in \mathcal{W}} \wedge_{i \in A} x_i$.

To check the validity of that claim, for any $F \in \mathcal{F}_p$ define

$$\mathcal{W}^f_F := \left\{ \begin{array}{c} A \subseteq N : \text{there exists } x_N \in X^N \\ \text{such that } N_F(x_N) = A \text{ and } f(x_N) \in F \end{array} \right\}$$

and notice that, by *monotonic independence* of f, $A \in \mathcal{W}^f_F$ if and only if $f(y_N) \in F$ for every $y_N \in X^N$ such that $N_F(y_N) \supseteq A$, i.e. $\mathcal{W}^f_F$ is an order filter of $(\mathcal{P}(N), \subseteq)$.

Next, observe that the following principle holds:

$(*)$ for any $x, y \in X$, $x \leqslant y$ if and only if $y \in F$ for all $F \in \mathcal{F}_p$ such that $x \in F$.

To check that, recall that any proper principal filter is also prime, hence it belongs to $\mathcal{F}_p$. Thus, for any $x \neq \perp$, $[x) := \{z \in X : x \leqslant z\} \in \mathcal{F}_p$. But then, if $y \in F$ for all $F \in \mathcal{F}_p$ such that $x \in F$ it follows that $y \in [x)$ (since $x \in [x) \in \mathcal{F}_p$), whence $x \leqslant y$ as claimed. The converse implication follows immediately from the order-filter-property of (latticial) filters.

Now, *neutrality* of f implies indeed that, by construction, for any $F, F' \in \mathcal{F}_p$, $\mathcal{W}^f_F = \mathcal{W}^f_{F'}$ i.e. there exists an order filter $\mathcal{W}^f$ of $(\mathcal{P}(N), \subseteq)$ such that $\mathcal{W}^f_F = \mathcal{W}^f$ for every $F \in \mathcal{F}_p$.

Let us first establish that, for any $x_N \in X^N$, $f(x_N) \leqslant \vee_{A\in\mathcal{W}^f} \wedge_{i\in A} x_i$. If $f(x_N) = \perp$ there is nothing to prove, thus we can assume without loss of generality that $f(x_N) \neq \perp$ (which implies that $f(x_N) \in F$ for some $F \in \mathcal{F}_p$, including of course the principal ultrafilter $[f(x_N)))$. Then, take any such F, namely $f(x_N) \in F \in \mathcal{F}_p$ and posit $A^* := N_F(x_N)$. By definition, A^* is F-decisive for f whence $A^* \in \mathcal{W}^f$. By construction, $x_i \in F$ for all $i \in A^*$: thus, $\wedge_{i\in A^*} x_i \in F$ (by $\wedge$-closedness of F , finiteness of $A^* \subseteq N$, and associativity of $\wedge$). But then $\vee_{A\in\mathcal{W}^f} \wedge_{i\in A} x_i \in F$ as well, since clearly $\wedge_{i\in A^*} x_i \leqslant \vee_{A\in\mathcal{W}^f} \wedge_{i\in A} x_i$, and F is in particular an order filter. It follows from $(*)$ that $f(x_N) \leqslant \vee_{A\in\mathcal{W}^f} \wedge_{i\in A} x_i$ as required.

To establish that $\vee_{A\in\mathcal{W}^f} \wedge_{i\in A} x_i \leqslant f(x_N)$ also holds for any $x_N \in X^N$, let us consider any $F \in \mathcal{F}_p$ such that $(\vee_{A\in\mathcal{W}^f} \wedge_{i\in A} x_i) \in F$. By primality of F, finiteness of $\mathcal{W}^f \subseteq \mathcal{P}(N)$, and associativity of $\vee$, there exists $A \in \mathcal{W}^f$ such that $(\wedge_{i\in A} x_i) \in F$ whence $x_j \in F$ for each $j \in A$ since clearly $\wedge_{i\in A} x_i \leqslant x_j$ for any $j \in A$. But then, $N_F(x_N) \supseteq A \in \mathcal{W}^f$, whence $N_F(x_N) \in \mathcal{W}^f$ since as observed above $\mathcal{W}^f$ is an order filter of $(\mathcal{P}(N), \subseteq)$ (because f is monotonically independent) and consequently $f(x_N) \in F$ by definition of $\mathcal{W}^f$. Thus, $\vee_{A\in\mathcal{W}^f} \wedge_{i\in A} x_i \leqslant f(x_N)$ holds (by $(*)$, again). As a result, $f(x_N) = \vee_{A\in\mathcal{W}^f} \wedge_{i\in A} x_i$ holds, thus establishing claim (b) above.

Next, notice that anonymity of f implies that there exists a positive integer $k \leq |N|$ such that $\mathcal{W}^f = \{A \subseteq N : |A| \geq k\}$. Furthermore, bi-idempotence of f implies that $|N| - k = k - 1$. That is so because $|A| = k$ implies $A \in \mathcal{W}^f$, and by Claim 3 above bi-idempotence of f implies that $\mathcal{W}^f$ is self-dual. Thus, $A \in \mathcal{W}^f$ implies that $N \smallsetminus A \notin \mathcal{W}^f$ i.e. $|N \smallsetminus A| = n-k < k$, namely $n-k \leq k-1$. But if $n-k < k-1$, then $\mathcal{W}^f \cap \{A \smallsetminus \{i\}, (N \smallsetminus A) \cup \{i\}\} = \varnothing$ for every $i \in A$, contradicting self-duality of $\mathcal{W}^f$. It follows that $n - k = k - 1$ whence $|N| = n = 2k - 1$ and $k = \frac{n+1}{2}$. Therefore, $\mathcal{W}^f = \left\{A \subseteq N : |A| \geq \frac{n+1}{2}\right\} = \mathcal{W}^{maj}$, namely $f = f_{\mathcal{W}^f} = f^{maj}$. □

Proof of Corollary 2.

Proof. Straightforward, by Theorem 1 as combined with the observation that Λ^Y is a product of bounded linear orders and thus it is indeed a bounded distributive lattice. □

Proof of Theorem 2.

Proof. To check that f^{maj^*} is *not* coalitionally-strategy-proof, consider the following h-grade, m-alternative situation with $\Lambda = \{1, 2, .., h\}$ as endowed with the natural order, and suppose that the profile of top gradings of the agents is $\alpha = (\alpha_i)_{i\in N}$ with

$\alpha_i = (h, h-1, 1, ..., 1)$, $i = 1, .., k^*$

$\alpha_i = (h-1, h, 1, ..., 1)$, $i = k^*+1, ..., n-1$

$\alpha_n = (h, h, 1, ..., 1)$

with $k^* = k$ if $n = 2k+1$, and $k^* = k-1$ if $n = 2k$.

Then, by definition, $f^{maj^*}(\alpha) = (h, h, 1, ..., 1)$.

Now, denote by $[\lambda]_{\sim i}$ the indifference class of $\lambda \in \Lambda^Y$ according to preference relation $\succcurlyeq_i$ and suppose that the agents' preference profile $\succcurlyeq_N$ is such that

$(h, h-1, 1..., 1)) \succ_i (h-1, h-1, 1..., 1) \succ_i [\beta]_{\sim i}$, $i = 1, ..., k^*$

$(h-1, h, 1, ..., 1)) \succ_j (h-1, h-1, 1, ..., 1) \succ_j [\gamma]_{\sim j}$, $j = k^*+1, ..., n-1$

$(h, h, 1, ..., 1)) \succ_n [\delta]_{\sim n}$,

with $\beta \in \Lambda^Y \smallsetminus \{(h, h-1, 1, ..., 1), (h-1, h-1, 1, ..., 1)\}$,

and $[\beta]_{\sim i} = \Lambda^Y \smallsetminus \{(h, h-1, 1, ..., 1), (h-1, h, 1, ..., 1)\}$,

$\gamma \in \Lambda^Y \smallsetminus \{(h-1, h, 1, ..., 1), (h-1, h-1, 1, ..., 1)\}$,

and $[\gamma]_{\sim j} = \Lambda^Y \smallsetminus \{(h-1, h, 1, ..., 1), (h-1, h-1, 1, ..., 1)\}$,

$\delta \in \Lambda^Y \smallsetminus \{(h, h, 1, ..., 1)\}$

and $[\delta]_{\sim n} = \Lambda^Y \smallsetminus \{(h, h, 1, ..., 1)\}$.

(it can be easily checked that such a profile is unimodal hence belongs to $U^{*N}_{\mathcal{X}}$).

Similarly, it is easily checked that a preference profile $\succcurlyeq'_N$of linear orders on Λ^Y such that

$(h, h-1, 1, ..., 1) \succ'_i (h-1, h-1, 1, ..., 1) \succ'_i \beta$ for all $\beta \in \Lambda^Y \smallsetminus \{(h, h-1, 1, ..., 1), (h-1, h-1, 1, ..., 1)\}$, $i = 1, ..., k^*$

$(h-1, h, 1, ..., 1) \succ'_j (h-1, h-1, 1, ..., 1) \succ'_i \gamma$ for all $\gamma \in \Lambda^Y \smallsetminus \{(h-1, h, 1, ..., 1), (h-1, h-1, 1, ..., 1)\}$, $j = k^*+1, ..., n-1$

(and k^* as defined above),

$(h, h, 1, ..., 1)) \succ_n \delta$ for all $\delta \in \Lambda^Y \smallsetminus \{(h, h, 1, ..., 1)\}$,

is indeed a locally strictly unimodal profile hence belongs to $S^N_{\mathcal{X}}$.

Notice that $top(\succcurlyeq) = top(\succcurlyeq') = \alpha$.

Now, observe that

$f^{maj^*}((h-1, h-1, 1, ..., 1), ..., (h-1, h-1, 1, ..., 1), (h, h, 1, ..., 1)) = (h-1, h-1, 1, ..., 1) \succ^+_i$

$\succ^+_i (h, h, 1, ..., 1) = f^{maj^*}(\alpha)$, $\succ^+_i \in \{\succcurlyeq_i, \succcurlyeq'_i\}$, $i = 1, ..., n-1$. Thus, f^{maj} is manipulable by coalition $\{1, ..., n-1\}$ at both profiles $\succcurlyeq_N$ and $\succcurlyeq'_N$.

It follows that f^{maj^*} is not coalitionally strategy-proof on $U^{*N}_{\mathcal{X}}$ or $S^N_{\mathcal{X}}$, hence a fortiori it is not coalitionally strategy-proof on $U^N_{\mathcal{X}} \supseteq U^{*N}_{\mathcal{X}} \cup S^N_{\mathcal{X}}$. □

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