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Abstract. The magnetic dressing phenomenon occurs when spins precessing in a static field (holding field) are subjected to an additional strong alternating field. It is usually studied when such extra field is homogeneous and oscillates in one direction. We study the dynamics of spins under dressing condition in two unusual configurations. In the first instance, an inhomogeneous dressing field produces a space-dependent dressing phenomenon, which helps to operate the magnetometer in a strongly inhomogeneous static field. In the second instance, besides the usual configuration with a static and a strong orthogonal oscillating magnetic fields, we add a secondary oscillating field, which is perpendicular to both. The system shows novel and interesting features that are accurately explained and modelled theoretically. Possible applications of these novel features are briefly discussed.

1. Introduction

The physics of a precessing magnetization in a static magnetic field is well-known since the studies performed by Joseph Larmor at the end of the 19th century. The phenomenology of precessing systems is enriched when additional time-dependent fields are introduced. In many applications like magnetic resonance (MR) experiments, an oscillating transverse field (which is usually much weaker than the static one) is applied resonantly to the Larmor one $\Omega_L/2\pi$. In this case the resonant condition helps in describing the system dynamics in the rotating wave approximation: the oscillating field is seen as a superposition of two counter-rotating components, one of which appears to be static in the reference frame rotating at Ω_L around the holding field: it causes a slow precession in that rotating frame.

The presence of a transverse (off-resonant) alternating field arouses interest also in the complementary regime, when its strength exceeds the static one. This condition is technically unfeasible in conventional NMR at tesla level, while it is more easily accessible in atomic physics experiments, where the MR is commonly studied in much weaker fields, e.g. at a microtesla level.

The seminal work studying precessing spins subjected to a strong, off-resonant, transverse field (*dressing field*) dates back to the late nineteen sixties, when S. Haroche and co-workers [1] introduced the concept of magnetic dressing. That contribution started a vivid activity around the subject [2-5].

Recently, the magnetic dressing was studied also with cold atoms and condensates [6, 7]. The magnetic dressing of atoms offers a powerful tool in quantum control experiments [6, 8] and high-resolution magnetometry [9].

Our group devoted research activity to characterizing the magnetic dressing phenomenon occurring when an atomic sample is subject to an anharmonic dressing field [10]. More recently, the potential of applying a non-uniform dressing field to counteract the effects of static field inhomogeneities was demonstrated [11]. This has important implications in NMR imaging (MRI) in the ultra-low field regime [12].

As a general feature, the mentioned works studied the magnetic dressing in configurations where the precession around the holding field is modified by one time dependent field oriented along one direction in the plane perpendicular to the holding one (the precession plane).

In a more recent work [13], we studied theoretically and experimentally new features that emerge when the design of the dressing field uses both dimensions of the precession plane. We addressed the case when such time-dependent two-dimensional field is periodic in time. The periodicity is obtained by superposing two perpendicular field components, which oscillate harmonically with various amplitudes and with frequencies that are equal or in (small) integer ratio *p*.

The system is studied by means of a model that produces analytical results thanks to a perturbative approach. This approach requires that one of the time-dependent components is much larger than both the static field and the other oscillating term. Despite its weakness, the latter (denominated *tuning* field) plays an important role.

This paper is organized as follows: in Sec.2, we describe the model and derive the expression for the effective precession frequency in terms of the dressing parameters (amplitudes, frequencies and relative phase of the dressing and tuning fields); in Sec.3, we describe a practical application of magnetic dressing, which makes a Bell and Bloom atomic magnetometer suited to operate also in the presence of a strong field gradient. In Sec.4, we report experimental results obtained in the tuning-dressing configuration and consider possible applications where our findings may be of interests. An appendix deepens some technical aspects of the calculations on which the model is based.

2. Model

The evolution of atomic magnetization in an external field is commonly described using the Bloch equations. In this work, the considered atomic sample is a buffered Cs cell at the core of a Bell & Bloom magnetometer [14]. The presence of a *weak* pump radiation synchronously modulated at the effective Larmor frequency compensates the *weak* relaxation phenomena and enables the use of a model that determines the steady-state solution on the basis of the simpler Larmor equation.

Consider the Larmor equation $\dot{M} = \gamma B \times M$ for the atomic magnetization $\boldsymbol{M}$ in presence of a magnetic field of the form

$$\boldsymbol{B} = B_1 \cos(\Omega t)\, \hat{\boldsymbol{x}} + B_2 \cos(p\Omega t + \phi)\, \hat{\boldsymbol{y}} + B_3 \hat{\boldsymbol{z}} \tag{1}$$

with p an integer. We refer to B_3 as *static* field, to B_1 as *dressing* field and to B_2 as *tuning* field, respectively. Using the adimensional time $\tau = \Omega t$ one obtains explicitly ($\omega_i = \gamma B_i$)

$$\frac{d\,\boldsymbol{M}}{d\,\tau} = \left[\frac{\omega_1}{\Omega}\cos(\tau)\ A_1 + \frac{\omega_2}{\Omega}\cos(p\tau + \phi)\ A_2 + \frac{\omega_3}{\Omega} A_3\right] \boldsymbol{M}, \tag{2}$$

where the three-dimensional matrices A_i are antisymmetric with only two elements different from zero i.e. $(A_1)_{2,3} = -(A_1)_{3,2} = -1$. A_2 and A_3 are obtained from cyclic permutations of the indices: see equations (A.1) in the appendix.

Assuming that $\xi = \omega_1/\Omega \gg \omega_2/\Omega,\ \omega_3/\Omega$ the perturbation theory let factorize the time evolution operator $U(\tau)$ ($M(\tau) \equiv U(\tau)M(0)$) in the interaction representation as

$$U(\tau) = \exp[\xi \, \sin\tau \, A_1] \, U_I(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\varphi) & -sin(\varphi) \\ 0 & cos(\varphi) & cos(\varphi) \end{pmatrix} U_I(\tau), \tag{3}$$

where $\varphi = \xi \, sin \, \tau$, obtaining a dynamical equation for $U_I(\tau)$: dU_I/d is given by
$$\left[\left(\frac{\omega_3}{\Omega}\sin\varphi + \frac{\omega_2}{\Omega}\cos\varphi \, \cos(p\tau + \phi)\right) A_2 + \left(\frac{\omega_3}{\Omega}\cos\varphi - \frac{\omega_2}{\Omega}\sin\varphi \, \cos(p\tau + \phi)\right) A_3\right] U_I \equiv \epsilon A(\tau) U_I,$$

where the parameter ϵ is used to identify the various orders of perturbation theory and to label the corresponding terms.

The matrix $A(\tau)$ is periodic $A(\tau + 2\pi) = A(\tau)$ so we can use the Floquet theorem and write

$$U_I(\tau) = e^{\Lambda(\tau)} \, e^{\tau F} \tag{4}$$

with $\Lambda(0) = 0$ and $\Lambda(\tau + 2\pi) = \Lambda(\tau)$, and the Floquet matrix F is time independent. Next, we expand U_I following the Floquet-Magnus [15] expansion

$$\Lambda = \epsilon\Lambda_1 + \epsilon^2\Lambda_2 + \cdots, \quad F = \epsilon F_1 + \epsilon^2 F_2 + \cdots \tag{5}$$

where the first terms are obtained from the relations

$$F_1 = \frac{1}{2\pi}\int_0^{2\pi} A(\tau) \mathrm{d}\tau, \quad \Lambda_1(\tau) = \int_0^{\tau} A(\tau') \mathrm{d}\tau' - \tau F_1. \tag{6}$$

Using the relation involving the Bessel functions J_n

$$e^{iz\sin\theta} = \sum_{n=-\infty}^{+\infty} J_n(z) \, e^{in\theta} \tag{7}$$

one finds

$$\int_0^{\tau} \cos\big(\varphi(\tau')\big) \, \mathrm{d}\tau' = J_0(\xi)\tau + f_1(\tau) \tag{8a}$$

$$\int_0^{\tau} \sin\big(\varphi(\tau')\big) \, \mathrm{d}\tau' = f_2(\tau) \tag{8b}$$

$$\int_0^{\tau} \cos\big(\varphi(\tau')\big) \cos(p\tau' + \phi) \, \mathrm{d}\tau' = \frac{1 + (-1)^p}{2} \tau J_p(\xi) \cos\phi + f_3(\tau) \tag{8c}$$

$$\int_0^{\tau} \sin\big(\varphi(\tau')\big) \cos(p\tau' + \phi) \, \mathrm{d}\tau' = \frac{-1 + (-1)^p}{2} \tau J_p(\xi) \sin\phi + f_4(\tau). \tag{8d}$$

The functions $f_i(\tau)$ are reported in the appendix (see equations (A.4)-(A.7)). Using the equation (8) one finds

$$F_1 = \begin{cases} \frac{\omega_3}{\Omega} J_0(\xi) \, A_3 + \frac{\omega_2}{\Omega} J_p(\xi) \cos\phi \, \mathrm{A}_2 & p \text{ even} \\ \left[\frac{\omega_3}{\Omega} J_0(\xi) + \frac{\omega_2}{\Omega} J_p(\xi) \sin\phi\right] A_3 & p \text{ odd} \end{cases} \tag{9}$$

and

$$\Lambda_1 = \left(\frac{\omega_3}{\Omega} f_2(\tau) + \frac{\omega_2}{\Omega} f_3(\tau)\right) A_2 + \left(\frac{\omega_2}{\Omega} f_1(\tau) - \frac{\omega_2}{\Omega} f_4(\tau)\right) A_3 \tag{10}$$

Using the equation (9) (see appendix for details) we can calculate the dressed Larmor frequency measured in the experiment for the component of the magnetization parallel to the dressing field:

$$\Omega_\mathrm{L} = \begin{cases} \sqrt{\omega_3^2 J_0^2(\xi) + \omega_2^2 J_p^2(\xi)\ \cos^2\phi} & p \text{ even} \\ \left|\omega_3 J_0(\xi) + \omega_2 J_p(\xi)\ \sin\phi\right| & p \text{ odd} \end{cases} \tag{11}$$

The equation (11) is at the focus of this paper. It includes also the well known and simpler case of a single dressing field: for zero values of the tuning field (i.e. $\omega_2 = 0$) it gives $\Omega_\mathrm{L} = |J_0(\xi)\omega_3|$. The latter is the instance discussed in Sec.3, where only the dressing field is applied, with the peculiarity of a position-dependent parameter: $\xi = \xi(x)$. The more general case, with the presence of both the dressing and the tuning fields, is discussed in Sec.4.

3. Atomic resonance enhancement with inhomogeneous dressing

An interesting magnetometric application of the dressing phenomenon is based on applying a position-dependent dressing (and/or tuning) field to compensate atomic MR broadening caused by inhomogeneities of the static field. Unless counteracted, the broadening induced by strongly inhomogeneous static field would severely deteriorate the performance of the magnetometer.

Details of the magnetometric experimental apparatus can be found elsewhere [16]. Beside the field control system described in ref. [16], here several additional coils enable the application of the time dependent (dressing and tuning) fields. In particular, the stronger dressing field is applied with a solenoidal coil surrounding the sensor and the weaker tuning field is applied by means of a Helmholtz pair. Alternatively, when an inhomogeneous dressing field is required, it is produced by a small source (a coil wound on a hollow-cylinder ferrite, with the laser beams passing across the hole) placed in the proximity of the atomic cell.

As discussed in Refs. [11, 12], the application of inhomogeneous dressing field paves the way to detecting NMR imaging signals *in loco*, that is with an atomic sensor operating in the same place where the NMR sample is. In fact, the frequency encoding technique used in MRI requires the application of static gradients which would destroy the atomic resonance.

The basic principle of operation can be summarized as follows: let the NMR sample and the sensor atom be in a static field (B_3 (x)) with an intensity dependent on the x co-ordinate , thus $\omega_3 = \omega_3(x) = \gamma B_3(0) + Gx$, where $G = \partial B_3 / \partial x$ is the gradient applied to the frequency encoding purpose, which in our case is up to about 150 nT/cm. Over the centimetric size of the atomic sample, such a gradient would broaden the atomic MR from its original few-Hz width up to kHz level, smashing it completely and preventing any magnetometric measurement.

The narrow width of the atomic magnetic resonance can be restored by applying tailored tuning and/or dressing fields. We will consider here the case of single field dressing, where the gradient G makes Ω_L position-dependent according to

$$\Omega_\mathrm{L}(x) = \left|J_0\big(\xi(x)\big)\omega_3(x)\right|, \tag{12}$$

where we have considered a position dependent dressing field, making the dressing parameter ξ dependent on x, as well.

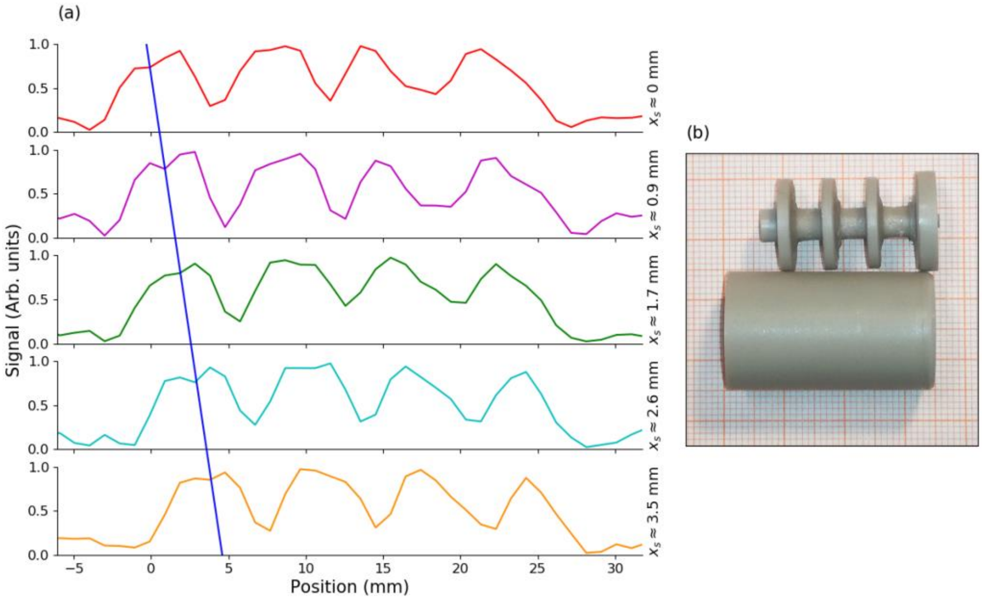

Figure 1. 1-D imaging of a structured (four disks) water sample contained in the plastic cartridge shown in the picture (b). The plots (a) represent MRI profiles, as obtained when the sample occupies four different positions x_s. Both the structure details and the positions (actual x_s values are reported the right-side y-axes) are determined with submillimetric precision.

An appropriate inhomogeneity of ξ makes possible to achieve the condition $\partial\Omega_L/\partial x = 0$, under which the atomic resonance width is restored. It is worth noting that the nuclear precession is unaffected by the dressing field. The much smaller gyromagnetic factor of nuclei makes their dressing parameter vanishing and the effects of B_1 negligible: the MRI frequency encoding based on G is preserved.

In this configuration, B_1 is inhomogeneous, and, in a dipole approximation, at a distance x from the cell centre, the dressing field B_x is

$$B_x(x,t) = \frac{\mu_0}{2\pi}\frac{m(t)}{(x_0 + x)^3} = B_1(x)\cos(\Omega t), \tag{13}$$

where μ_0 is the vacuum permittivity, $m(t) = m_0\cos(\Omega t)$ is the oscillating dipole momentum, and x_0 is the distance of the dipole from the cell centre. Taking now into account the dependence on x of both the static and the dressing fields, accordingly with the equation (11), the dressed angular frequency results

$$\Omega_L(x) = \frac{\gamma_{\mathrm{Cs}}}{2\pi}(B_0 + Gx)J_0\left(\frac{\gamma_{\mathrm{Cs}}B_1(x)}{\Omega}\right), \tag{14}$$

or, in a first-order Taylor approximation,

$$\Omega_{\mathrm{L}}(x) = \Omega_{\mathrm{L}}(0) + \Omega_{\mathrm{L}}'(0)\,x + O(x^2) \approx \frac{\gamma_{\mathrm{Cs}}}{2\pi}\left(B_0J_0(\alpha) + \left[\frac{3B_0\alpha J_1(\alpha)}{x_0} + GJ_0(\alpha)\right]x\right),$$

where $\alpha = (\mu_0/2\pi)(\gamma_{\mathrm{Cs}}m_0)/(\Omega x_0^3)$. In conclusion, the condition for compensating the effect of the gradient G is found to be:

$$-3\frac{B_0}{x_0}\frac{\alpha J_1(\alpha)}{J_0(\alpha)} = G. \tag{15}$$

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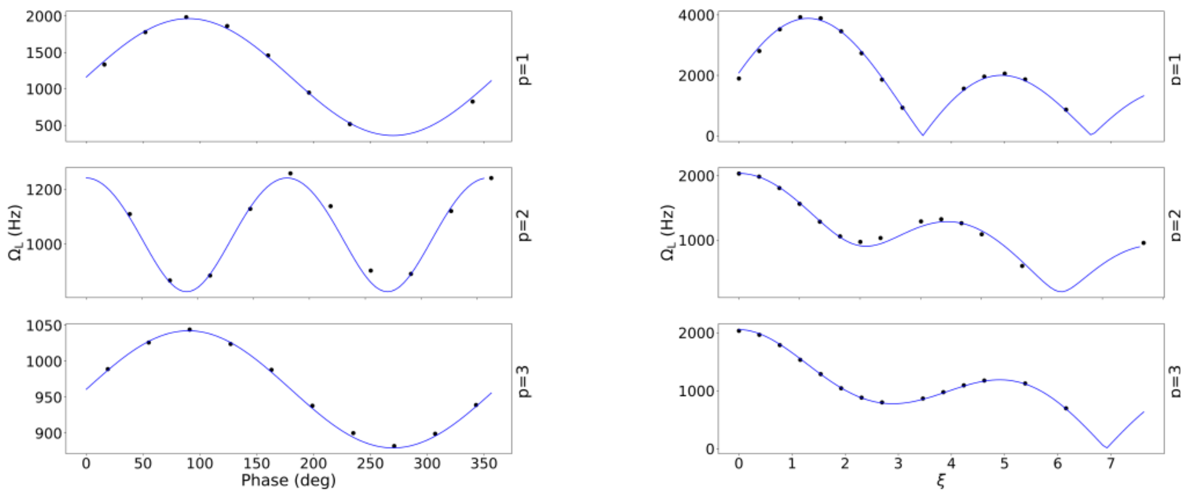
This shows that with an appropriate choice of α, the width of the atomic magnetic resonance is substantially restored, as to recover the magnetometer performance to a level making it suited to detect weak MRI signals.

Figure 1 shows MRI profiles obtained with this methodology. The frequency encoding is here obtained with a field gradient $G \approx 100$ nT/cm. The presence of G broadens the atomic MR from its original 25 Hz up to about 800 Hz. Its width is then restored down to 35 Hz by inhomogeneous dressing.

4. Tuning-dressing experiment

The equation (11) expresses the dependence of dressed angular frequency Ω_L on the experimental parameters (strengths, relative phase and frequencies of the tuning and dressing fields). The predicted behavior is verified using the mentioned Bell and Bloom magnetometer.

Measurements are made with $p = 1,2,3$, and Ω_L is recorded as a function of ω_1 and ϕ.



(a) (b)

Figure 2. (a) Ω_L as a function of the relative phase ϕ between the fields B_1 and B_2. In black dots the experimental measurements, in blue the theoretical predictions. Upper, middle and bottom panels refer respectively to the cases with $p = 1, 2, 3$, in which ξ has been fixed to 1.38,3.83 ,1.54.
(b) Ω_L as a function of the Bessel functions argument ξ. In black dots the experimental measurements, in blue the theoretical predictions. Upper, middle and bottom panels refer respectively to the cases with $p = 1, 2, 3$. In the cases with an odd p the phase between B_1 and B_2 is fixed to $\phi = \pi/2$, when p is even the phase ϕ is set to zero. In each plot, ξ changes varying the value of B_1, which ranges from zero to ≈ 15 μT.

Figure 2 shows comparisons of measured and calculated values of Ω_L as a function of ϕ for given ξ and as a function of ξ for given ϕ, respectively. The data shown in figure 2 (b) are measured with ϕ values that maximize the tuning effect. i.e. is $\phi = \pi/2$ for the odd p values and $\phi = 0$ for the even one.

There is an excellent accordance between the experimental data and the theoretical prediction. In particular, both the dependence of the relative phase ϕ and on the pertinent J_p (with $p = 1,2,3$) Bessel functions are perfectly verified by the experiment. Minor deviations can be attributed to experimental imperfections, mainly to not-exact perpendicularity of the applied fields B_1 and B_2.

The developed model and its excellent correspondence with the experimental results demonstrates the possibility to enhance the control of spins evolution by means of the described tuning-dressing field arrangement.

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This enhancement opens up to a variety of new experimental configurations in which the new set of parameters (B_2, p, and ϕ) add to B_1 and Ω to make new *handles* available to finely control the atomic magnetization dynamics. Compared to the known cases of harmonic [1] or anharmonic [10] dressing field oscillating along one direction, noticeably here the dressed frequency Ω_L (equation (11)) may exceed ω_3.

The tuning-dressing scheme makes possible to choose a parameter set to achieve an arbitrary $\mathrm{d}\Omega_L/d\xi$ in conjunction with an arbitrary (to some extent) value of Ω_L. For instance, it is possible to produce a condition of critical dressing (equalization of precession frequencies of different species) [17] with no first-order dependence on ξ, so to attenuate the detrimental effects caused by B_1inhomogeneities, which constitute a severe limiting problem in high-resolution experiments [9].

Similarly, it is possible to fulfil the condition of a large $\mathrm{d}\Omega_L/d\xi$ avoiding the constraint of a strong Ω_L reduction. The latter may help in applications like that described in Sec.3, where Ω_L is made deliberately position-dependent by means of a spatially inhomogeneous ξ. In addition, for that kind of application, it is worth noting that the presented scheme makes it possible to render Ω_L space dependent by means of an inhomogeneity of the field B_2, which is of easier implementation and control, being $B_2 \ll B_1$.

Other applications where the tuning-dressing scheme may offer important potentials is suggested by the dependence on ϕ. As recently reported [18], an emerging application of highly sensitive magnetometers concerns the detection of targets made of weakly conductive materials. In that case, the typical setup is based on a radio-frequency magnetometer, where the target modifies the amplitude or the phase of a (resonant) radio-frequency field driving the magnetometer. Alternative setups could be developed, where the target modifies the field B_2, whose frequency is not required to match the atomic resonance. In this case, provided that $J_p(\xi)$ is large (e.g. $\xi \approx 1.84$ in the case of $p = 1$), the system would have a large response to any variation of either the amplitude (if $\phi = \pm\pi/2$) or the phase (if $\phi = 0, \pi$) of B_2 caused by eddy currents induced in the target.

Appendix

The explicit expressions for the matrices in the main text are

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.1}$$

Defining $L_j = iA_j$ and taking the base that diagonalizes L_3 one obtains the familiar form for the angular momentum operators acting on the $|L = 1, M_L\rangle$ states. This demonstrates that the algebra generated by the A_i matrices is the same of the quantum angular momentum operators. Moreover, it is straightforward to demonstrate that the quantum mean value $\langle \gamma \boldsymbol{L} \rangle$ satisfies the same classical Larmor equation.

The action of a general antisymmetric matrix $W = a_1 A_1 + a_2 A_2 + a_3 A_3$ on a given vector $\boldsymbol{v}$ is the cross-product

$$W\boldsymbol{v} = \boldsymbol{a} \times \boldsymbol{v} \qquad \boldsymbol{a} \equiv (a_1, a_2, a_3). \tag{A.2}$$

Using the Cayley-Hamilton theorem, the needed matrix exponentials can be analytically evaluated. Let's write $\boldsymbol{a} = \theta\hat{a}$, with $\hat{a} \cdot \hat{a} = 1$ then

$$e^W \boldsymbol{v} = \sum_{n=0}^{+\infty} \frac{W^n}{n!} \boldsymbol{v} = \boldsymbol{v} + \sin\theta \; \hat{a} \times \boldsymbol{v} + (1 - \cos\theta) \, \hat{a} \times (\hat{a} \times \boldsymbol{v}). \tag{A.3}$$

The auxiliary f_i functions introduced in the main text are defined as

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$$f_1(\tau) = \sum_{n=1}^{\infty} \frac{J_{2n}(\xi)}{n} \sin(2n\tau) \tag{A.4}$$

$$f_2(\tau) = 4 \sum_{n=0}^{\infty} \frac{J_{2n+1}(\xi)}{2n+1} \sin^2\big((n+1/2)\tau\big) \tag{A.5}$$

$$f_3(\tau) = \mathrm{Re}\big(g(\tau)\big) \tag{A.6}$$

$$f_4(\tau) = \mathrm{Im}\big(g(\tau)\big) \tag{A.7}$$

where

$$g(\tau) = e^{i\phi} \sum_{n\neq -p} \frac{J_n(\xi)}{i(n+p)} \left(e^{i(n+p)\tau} - 1\right) + e^{-i\phi} \sum_{n\neq p} \frac{J_n(\xi)}{i(n-p)} \left(e^{i(n-p)\tau} - 1\right) \tag{A.8}$$

These functions have a limited and oscillating behaviour and are needed in the evaluation of $e^{\Lambda_1(\tau)}$. One can see by inspection that $e^{\Lambda_1} \approx \mathbb{1}$ is a good approximation.

The initial condition appropriate for the experiment is $\boldsymbol{M}(0) \propto (1,0,0)$ and the quantity monitored is $M_x(\tau)$. So, applying equation (A.3) to evaluate $e^{\tau F_1}$, and then equation (3), one finds

$$M_x(t) \propto \cos(\Omega_\mathrm{L}\, t). \tag{A.9}$$

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