UNIVERSITÀ DI SIENA
1240

PhD: Information Engineering and Science

Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche

Hilbert functions and symmetric tensors identifiability

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XXXIII Cycle - 2017/2018

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Chapter 0

Introduction

The thesis considers the study of (Waring) decompositions of homogeneous polynomials (forms), mainly under the point of view of methods which can determine the uniqueness and minimality of a given decomposition.

Forms are natural algebraic objects that become multilinear objects because they can be associated to symmetric tensors. The growing interest on tensors that rises from different branches of Mathematics, both theoretical and applied, is determined by the growing number of applications that tensors find in Statistics, Signal Theory, Artificial Intelligence, Algebraic Geometry, Numerical Analysis, and other fields. We will not mention the wide literature on the subject, which should involve many titles, and remain nevertheless rather partial and incomplete.

The focus of the thesis is on methods of Algebraic Geometry that can guarantee the identifiability of a symmetric tensor. The word *identifiability* has a precise, general meaning in applied Mathematics. From the point of view of Algebraic Statistics, it is associated to parametric models, and indicates that an element uniquely determines the parameters it comes from. In the case of forms, the parametric description that we consider is the so called *Waring decomposition*, that presents a form as a sum of powers of linear forms. Indeed, since we will work over an algebraically closed field $\mathbb{C}$, every form F has finite representations of type

$$F = L_1^d + \cdots + L_r^d,$$

where each L_i is a linear form. In these expressions, that we call *Waring decompositions*, we are particularly interested in minimizing the number r, that we call the *length* of the decomposition.

Waring decompositions owe their importance to the fact that, in the natural association between forms and symmetric tensors, powers of linear forms correspond to tensors of rank 1. So, if one accepts that tensors of rank 1 are the simplest objects in the category, then r is a good measure for the complexity of a decomposition, and the minimal r for which the decomposition exists is a good measure of the complexity of F itself. The minimal value is now called the *Waring rank* of F.

The name of *Waring* decomposition is historically connected with the English mathematician Edward Waring, that in his work Meditationes Algebraicae (See [48]) presented the problem of finding the minimal expression of a natural number as a sum of powers. The original conjecture made by Waring is that any natural number can be written as a sum of 4 squares, of 9 cubes, of 19 fourth-powers. The conjecture for squares, which indeed goes back to Diophantus, was then proved by the Italian mathematician G.L. Lagrange at the end of the XVIII century. Generalizing the problem and translating it in the setting of polynomials, we obtain the starting point of the theory mentioned above.

Though the Waring rank of a general form is known, in terms of the degree and the number of variables, due to the basic paper by Alexander and Hirschowitz [1], the problem of finding a minimal decomposition of a specific form is not easy to solve, except for forms of degree 2, which correspond to symmetric matrices. In fact Hillar and Lim in 2013 proved that the problem of finding the rank of a tensor is NP-hard (see Theorem 8.2 of [35]).

In the thesis, we assume to know a decomposition of a form F, and our problem concerns its minimality and uniqueness. Indeed, since it is easy to see that uniqueness implies minimality, we will mainly focus on the latter problem. The assumption that we already know some decomposition is restrictive, but widely satisfied in many settings. A lot of heuristic methods are available for the computation, mainly approximate but also, in some cases, algebraically precise, of some decomposition of a tensor. Besides, there are situations in which a decomposition is naturally known (Strassen's problem, Comon's problem, etc.), and the problem is the certification of minimality and uniqueness.

We will take the problem from the point of view of projective geometry. In almost all cases in which are used, tensors can be considered modulo rescaling, i.e. modulo the multiplication by non-zero scalars. This suggests to look at projective spaces of tensors, and consequently identify a decomposition with a finite set A of points in a projective space $\mathbb{P}^n$ (for forms in $n+1$ variables), given in homogeneous coordinates by the coefficients of the L_i's. The problem naturally translates to the question on the minimal r such that the point corresponding to F belongs to the r-th secant variety of the d-th Veronese embedding of $\mathbb{P}^n$, d being the degree of F.

The aim of the thesis is to illustrate how geometric methods for the study of finite sets in projective spaces can determine effective algorithms to detect the identifiability of forms. The methods that we will use are based on classical and modern results on the Hilbert function of the finite set A.

The definition of Hilbert function of a finite set A is quite natural. Given a set of coordinates for the points, one can consider the evaluation of forms at the r points of A. If R_i denotes the linear space of forms of degree i in the polynomial ring $R = \mathbb{C}[x_0, \dots, x_n]$, then the Hilbert function h_A sends i to the dimension of the cokernel of the evaluation map $R_i \to \mathbb{C}^r$, which is independent on the choice of the coordinates. Several important properties of the set A are encoded in the Hilbert function h_A and in its first difference Dh_A.

In our setting, we start with a decomposition A of F, of length r, and we want to exclude the existence of a second decomposition B with length $r' \leq r$. The main tool we will use is the analysis of the Hilbert function of $Z = A \cup B$. Indeed, by induction on r', we show that we can restrict ourselves to the case in which A, B are disjoint. If $A \cap B = \emptyset$, we get from classical and modern projective geometry a long list of restrictions on the difference Dh_Z, that eventually imply that B cannot exist.

If we compare our method with classical methods universally employed to guarantee the identifiability of symmetric tensors (the celebrated Kruskal criterion [38], and its reshaped version [20]), we see that we can determine the uniqueness of a decomposition A in a range for the length r which cannot be explored (for theoretical reasons!) with the aforementioned procedures.

The analysis follows the guidelines of several recent papers on the subject, starting with [20], [7], to the more recent [6], [40], and [5], [4]. The results that we obtain are mainly focused on forms in three variables (ternary forms). For degree $d \leq 6$, Kruskal criterion provides a complete analysis for reasonably general forms. We improve the situation by completing the analysis in degree 7, where even the reshaped Kruskal method fails for high value of the length r. Kruskal method also often fails for low r, when the position of A is not sufficiently general. We show how a deep study of the Hilbert function allows to handle similar cases. In particular, we determine bounds for the identifiability of the form F, when the given decomposition A sits in a cubic curve.

We want to stress that if the decomposition A turns out to be non-unique or non-minimal, yet our methods, via the analysis of the Hilbert-Burch matrix of A and the liaison process, can construct from A a minimal decomposition of F. In many cases, e.g. for decompositions lying in a cubic curve, our method provides an algorithm for the identifiability whose complexity (i.e. the number of needed elementary operations) is lower than the Kruskal algorithm, even when this last is applicable.

In the last part of the thesis we deal with forms in 4 variables (quaternary forms). The first situation in which Kruskal-based methods fail to determine the identifiability is the case of forms of degree 5 with a decomposition of length 13. For quaternary forms the analysis is more complicated, because the resolution of the ideal of A involves two matrices, instead of a unique Hilbert-Burch matrix. In more geometric terms, in order to determine the identifiability we are forced to analyze not only hypersurfaces containing A, but also curves of $\mathbb{P}^3$ containing the set of points, and the latter task is much more demanding. For decomposition of length 13 of a quintic quaternary form, we can provide an algorithm which tests the minimality, except for a final gap, related to determine the smoothness of a general cubic surface containing the union $Z = A \cup B$ defined above. This is the reason why we present the final analysis in the form of work in progress.

The structure of the thesis is the following.

In Chapter 1 we recall some basic definitions and results about multi-projective varieties.

In Chapter 2 we describe the main aspect regarding symmetric tensors. In particular we state the identifiability problem and we present the main results known in literature that describe when a given symmetric tensor T is identifiable or not.

In Chapter 3 we define the Hilbert function of a given graded module and the Cayley-Bacharach property. In particular we explain how to use these two objects to study the identifiability problem for symmetric tensors.

In Chapter 4 we give a more geometrical proof of the Kruskal Theorem for symmetric tensors.

In Chapter 5 we present some new results about the identifiability problem for symmetric tensors. In particular, we describe the results originally presented in [6] and [40].

In Chapter 6 we study the properties of symmetric tensor T of type $4 \times \cdots \times 4$ (5-times) with a given decomposition A of length at most 13.

Chapter 1

Algebraic Geometry of Multiprojective Varieties and Secant Varieties

In this section we collect, for reference, some results and definitions about projective and multiprojective varieties that will be useful in our investigation. In particular, we are interested in defining what is the dimension of a variety and what are secant varieties of a given projective variety.

Recall that the *projective space* over a field $\mathbb{K}$, $\mathbb{P}(\mathbb{K}^{n+1})$, is the set of equivalence classes of $(n+1)$-tuples $(a_0, a_1, \dots, a_n)$ of elements of $\mathbb{K}$, not all zero, under the equivalent relation given by

$$(a_0, a_1, \dots, a_n) \sim \lambda(a_0, a_1, \dots, a_n)$$

with $0 \neq \lambda \in \mathbb{K}$. For a point $P = (a_0, \dots, a_n) \in \mathbb{P}(\mathbb{K}^{n+1})$ we will also write $P = [a_0 : a_1 : \dots : a_n]$ and we will call the a_i's the *(projective) coordinates of* P.

If there is no confusion about the field we will write $\mathbb{P}^n$ instead of $\mathbb{P}(\mathbb{K}^{n+1})$. As usual, the *projective dimension* of $\mathbb{P}^n$ is n, equal to the dimension of $\mathbb{K}^{n+1}$ minus one.

In the next definition we recall some basic concepts of linear algebra, just to fix the notation that we will use in the rest of the thesis.

Definition 1.0.1. Given a non-empty finite set $A \subset \mathbb{P}^n$ we denote by $\ell(A)$ the cardinality of A. Moreover, we say that A is *linearly independent* if any set of coordinates for the points of A consists of linearly independent vectors. Finally, we denote with $< A >$ the linear span of A, i.e. the set of points whose coordinates are linearly dependent from A.

Remark 1.0.2. One of the main advantages of working in projective space is that given a set of points $X \subseteq \mathbb{P}^n$ it is easy to compute $< X >$. In fact, differently from the affine case, the linear projective space generated by X is exactly the projectivization of the linear space generated by coordinates of the given points (seen as affine vectors). Thus, finding the linear space generated by r projective points $P_0, \dots, P_r \in \mathbb{P}^n$ is equivalent to finding the linear space generated by r lines in $\mathbb{K}^{n+1}$, passing through the origin, of directions equal to $(P_0),\dots,(P_r)$ where $(P_0),\dots,(P_r)$ are coordinates of the projective points $P_0,\dots,P_r$. So, we have that the linear projective space $< P_0,\dots,P_r >$ is equal to $\mathbb{P}(L((P_0),\dots,(P_r)))$ where $L((P_0),\dots,(P_r))$ is the linear space spanned by $(P_0),\dots,(P_r)$.

Next example shows some differences between the affine and the projective case.

Example 1.0.3. Let $P_0 = (a_0, a_1, a_2), P_1 = (b_0, b_1, b_2)$ be two points of the affine space $\mathbb{C}^3$. We want to find the affine line L passing through P_0, P_1.

In order to do that we need first to find the direction of the line $P_1 - P_0$ and then we need to impose the passage through P_1. So a parametric expression of the line will be of the form $L : \alpha(b_0 - a_0, b_1 - a_1, b_2 - a_2) + (a_0, a_1, a_2)$. Notice in particular that the linear space spanned by P_0 and P_1 is different from the line passing through these two points.

On the other hand, let $\overline{P}_0 = [a_0 : a_1 : a_2]$ and $\overline{P}_1 = [b_0 : b_1 : b_2]$ be two points of $\mathbb{P}^2$. The projective line L passing through $\overline{P}_0$ and $\overline{P}_1$ can be expressed in parametric form just as $L : \alpha P_0 + \beta P_1$. Moreover the space which defines L coincides with the linear projective space $< P_0, P_1 >$.

Remark 1.0.4. Notice that given a polynomial $F \in \mathbb{K}[x_0, x_1, \dots, x_n]$ we cannot interpret f as a function to $\mathbb{P}(\mathbb{K}^{n+1})$ to $\mathbb{K}$. In fact, given f a non constant polynomial in $n+1$ variables and $[a_0, a_1 : \cdots : a_n] = [\lambda a_0 : \lambda a_1 : \cdots : \lambda a_n]$ a point in $\mathbb{P}^n$, we have that in general $f(a_0, a_1, \dots, a_n) \neq f(\lambda \cdot a_0, \lambda \cdot a_1, \dots, \lambda \cdot a_n)$. However, if we work with homogeneous polynomials, we can still define correctly what is the zero locus. In fact if $f \in \mathbb{K}[x_0, x_1, \dots, x_n]$ is homogeneous of degree d, then $f(a_0, a_1, \dots, a_n) = 0$ implies $f(\lambda \cdot a_0, \lambda \cdot a_1, \dots, \lambda \cdot a_n) = \lambda^d f(a_0, a_1, \dots, a_n) = 0$ for all $\lambda \neq 0$ and viceversa. As a consequence, we can give a definition of projective variety.

Definition 1.0.5. Given $T \subseteq \mathbb{K}[x_0, x_1, \dots x_n]$ we call the *variety generated* by T the set $V(T) \subseteq \mathbb{P}(\mathbb{K}^{n+1})$ such that:

$$V(T) = \left\{ x \in \mathbb{P}(\mathbb{K}^{n+1}) \text{ s.t. } f(x) = 0 \text{ for all homogeneous polynomials } f \in T \right\}.$$

The results we need about projective varieties are quite standard, so we avoid to recall it directly (for some reference see Chapter 1 of [34] or [23]).

Instead, we will focus mainly on the properties of the so called *multiprojective varieties*, a generalization of varieties for a product of projective spaces. Indeed, since projective varieties can be considered as particular multiprojective varieties, all the properties and definitions we give for multiprojective varieties applies also in general for any projective variety. As the theory of multiprojective variety is not completely standard, we recall it in details.

First of all, in order to give the definition of multiprojective variety, we recall that a point $P \in \mathbb{P}(\mathbb{K}^{a_1+1}) \times \cdots \times \mathbb{P}(\mathbb{K}^{a_r+1})$ is an r-tuple of (a_i+1)-tuples $(p_{i,0}, \dots, p_{i,a_i})$ modulo the equivalence relation given by

$$\begin{aligned}((p_{1,0}, \dots, p_{1,a_1}), \dots, (p_{r,0} : \cdots : p_{r,a_r})) &= \\ &= (\lambda_1(p_{1,0}, \dots, p_{1,a_1}), \dots, \lambda_r(p_{r,0}, \dots, p_{r,a_r}))\end{aligned}$$

with $0 \neq \lambda_i \in \mathbb{K}$. As before, for a point $P \in \mathbb{P}(\mathbb{K}^{a_1+1}) \times \cdots \times (\mathbb{K}^{a_r+1})$ we will also write:

$$P = ([p_{1,0} : \cdots : p_{1,a_1}], \ldots, [p_{r,0} : \cdots : p_{r,a_r}]).$$

It is not trivial to show that multiprojective spaces are isomorphic to particular projective varieties. Indeed, we will show in the next section that a multiprojective variety can be identified with a projective variety via the *Segre embedding*.

Notice in particular that all the components of P can be scaled independently, so we cannot use homogeneous polynomials to define a multiprojective variety. Instead we will use *multiprojective* homogeneous polynomials.

Definition 1.0.6. Let $A = \{x_0, \ldots x_a\}$ be a set of $a+1 = (a_1+1)+\cdots+(a_r+1)$ variables ordered with the lexicographic order and let $\mathbb{K}[x_0, \ldots, x_a] = \mathbb{K}[A]$ be the corresponding ring of polynomials. Fix a partition of the set of variables $A = A_0 \cup \cdots \cup A_r$ with A_i of cardinality $a_i + 1$. We say that a polynomial $f \in \mathbb{K}[A]$ is *multihomogeneous of multidegree* $(d_1, \ldots, d_r)$ if, for all $i = 0, \ldots, r$, f considered as a polynomial in the variables A_i is a homogeneous polynomial of degree d_i.

Example 1.0.7. Let $A = \{x_0, x_1, x_2, x_3\}$ be our set of variables and let $f = x_0x_1x_2^4 + x_0^2x_2^3x_3 + x_1^2x_2x_3^3$ be a polynomial in $\mathbb{K}[x_0, x_1, x_2, x_3]$. If we fix $A_1 = \{x_0, x_1\}$ and $A_2 = \{x_2, x_3\}$ we have that f is multihomogeneous of degree $(2, 4)$. On the other hand, if we fix $A_1 = \{x_0\}$ and $A_2 = \{x_1, x_2, x_3\}$ then f is not multihomogeneous.

Remark 1.0.8. Let $R = \mathbb{K}[x_0, \ldots, x_a]$ be a ring of polynomials with $a + 1 = (a_0 + 1) + (a_1 + 1) + \cdots + (a_r + 1)$. In order to simplify the notation, instead of giving explicitly a partition for the set of variables of R, we will write R as $\mathbb{K}[x_{1,0}, \ldots, x_{1,a_1}, \ldots, x_{r,0}, \ldots x_{r,a_r}]$ or, more shortly, $\mathbb{K}[x_{1,0}, \ldots, x_{r,a_r}]$. In this case we will always take as a partition of the set of variables of R the one made by the sets $A_i = \{x_{i,0}, \ldots, x_{i,a_i}\}$ for $i = 1, \ldots, r$.

Remark 1.0.9. As for the projective case, if we work with multihomogeneous polynomials, we can still define correctly what is the zero locus of a polynomial. In fact if $f \in \mathbb{K}[x_{1,0}, \ldots, x_{r,a_r}]$ is multihomogeneous of multidegree $(d_1, \ldots, d_r)$, then $f(b_{1,0}, b_{1,1}, \ldots, b_{r,a_r}) = 0$ implies

$$\begin{aligned} f(\lambda_1 b_{1,0}, \ldots, \lambda_1 b_{1,a_1}, \ldots, \lambda_r b_{r,0}, \ldots, \lambda_r b_{r,a_r}) = \\ = \lambda_1^{d_1} \cdots \lambda_r^{d_r} \cdot f(b_{1,0}, b_{1,1}, \ldots, b_{r,a_r}) = 0 \end{aligned}$$

for all $\lambda_i \neq 0$ and viceversa. As a consequence, we can give a definition of what is a multiprojective variety.

Definition 1.0.10. A *multiprojective variety* is a subset $X \subset \mathbb{P}(\mathbb{K}^{a_1+1}) \times \cdots \times \mathbb{P}(\mathbb{K}^{a_r+1})$ defined as the vanishing locus of a set $T \subset \mathbb{K}[x_{1,0}, \ldots, x_{r,a_r}]$ of multihomogeneous polynomials.

Given $S \subset \mathbb{K}[x_{1,0}, \ldots, x_{r,a_r}]$ we indicate with $V(S) \subset \mathbb{P}(\mathbb{K}^{a_1+1}) \times \cdots \times (\mathbb{K}^{a_r+1})$ the multiprojective variety defined by the vanishing locus of the multihomogeneous polynomials in S.

Remark 1.0.11. Let $\mathbb{K}[x_0, \ldots, x_r]$ be a ring of polynomials. If we take as a partition of the set of variables $\{x_0, \ldots, x_r\}$ the trivial one, the notion of projective and multiprojective varieties coincide.

Example 1.0.12. The variety generated by a single multihomogeneous polynomial f, $V(f) \subset \mathbb{P}(\mathbb{K}^{a_1+1}) \times \cdots \times (\mathbb{K}^{a_r+1})$, is called a hypersurface.

Example 1.0.13. The projective variety $X \subset \mathbb{P}(\mathbb{K}^{n+1})$ generated by linear homogeneous polynomials $\{f_1, \ldots, f_r\}$ is a variety which can be identified with the linear projective subspace generated by the solution of the set of polynomials.

Example 1.0.14. The *twisted cubic curve* in $\mathbb{P}(\mathbb{C}^4)$ is a projective variety defined by the following polynomials in $\mathbb{K}[x_0, x_1, x_2, x_3]$:

$$f_0 = x_0x_3 - x_1^2; \quad f_1 = x_1x_3 - x_2^2; \quad f_2 = x_0x_3 - x_1x_2.$$

Notice that $f_0, f_1, f_2 \in \mathbb{C}[x_0, x_1, x_2, x_3]$ form a multiprojective variety only if we take as a partition of the variables the trivial one.

Remark 1.0.15. Not all multiprojective varieties are a (Cartesian) product of projective varieties. As an example take $\mathbb{K} = \mathbb{C}$ and consider the variety V of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $V = V(x_0y_1 - x_1y_0)$. We have that $V \neq \mathbb{P}^1 \times \mathbb{P}^1$ since for example $Q = ([1 : 2], [3 : 5])$ does not belong to V. Moreover, given a point $P \in \mathbb{P}^1$ we have that all the points of the form (P, P) belongs to V. As a consequence, we have that V cannot be written as the product of two subvarieties of $\mathbb{P}^1$, one of which needs to be different from the whole space $\mathbb{P}^1$.

Definition 1.0.16. Let R be a ring and let $f_1, \ldots, f_r$ be elements of R. The set

$$I = \{a_1f_1 + \cdots + a_rf_r \text{ s.t. } a_i \in R\}$$

is an ideal called *the ideal generated by* $f_1, \ldots, f_r$. We indicate the ideal generated by $f_1, \ldots, f_r$ as $I(f_1, \ldots, f_r)$.

The multiprojective variety generated by a set of multihomogeneous polynomials T is equal to the variety $V(I(T))$ where $I(T)$ is the ideal generated by T. Moreover, given a variety V we can define what is the ideal generated by the variety.

Definition 1.0.17. Given $V \subset \mathbb{P}(\mathbb{K}^{a_1+1}) \times \cdots \times \mathbb{P}(\mathbb{K}^{a_r+1})$ a multiprojective variety, we define the *ideal generated by* V, $I(V)$ as follows.

$$I(V) = I(\{f \in \mathbb{K}[x_{1,0}, \ldots, x_{r,a_r}] \text{ s.t. } \\ f \text{ is multihomogeneous and } \; f(x) = 0 \;\forall x \in V\}).$$

Another property of the ideals defined by varieties is the following.

Definition 1.0.18. Let $R = \mathbb{K}[x_1, \ldots, x_r]$ be a ring of polynomials. Any $f \in R$ can be written uniquely as $f = \sum f_i$ where each f_i is the sum of multihomogeneous monomials of fixed multidegree in f. The polynomials f_i are called the *multihomogeneous components of* f.

We say that an ideal $I \subseteq R$ is *multihomogeneous* if for all $F \in I$ all the multihomogeneous components of F belongs to I.

It is easy to prove that $I(V)$ is a multihomogeneous ideal. Moreover the maps I and V define an inclusion reversing correspondence between the class of multihomogeneous ideals and the class of multiprojective varieties. In fact we have the following proposition.

Proposition 1.0.19. *Given $T_1, T_2 \subseteq \mathbb{K}[x_{1,0}, \ldots, x_{r,a_r}]$ and $Y, Y_1, Y_2 \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$ we have:*

- *If $T_1 \subseteq T_2$ then $V(T_1) \supseteq V(T_2)$*
- *If $Y_1 \subseteq Y_2 \subset \mathbb{P}^n$ then $I(Y_1) \supseteq I(Y_2)$*
- $V(I(Y)) = Y$
- *for all multihomogeneous ideal $\mathfrak{a} \subseteq \mathbb{K}[x_{1,0}, \ldots, x_{r,a_r}]$ we have that $I(V(\mathfrak{a})) \supseteq \mathfrak{a}$*

Proof. We can find a proof of this proposition for projective varieties in Proposition 1.2 in Chapter 1 of [34]. The same proof can be easily adapted for the multiprojective case. □

The union of two multiprojective varieties is a variety, the intersection of any family of multiprojective varieties is a variety and the whole space and the empty sets are multiprojective varieties (a proof of this fact in Proposition 1.1 of [34]). So, multiprojective varieties can be considered as the collection of closed sets of a topology.

Definition 1.0.20. We define the *Zariski topology* on $\mathbb{P}^n$ as the topology such that the open sets are complements of multiprojective varieties.

Since not all multiprojective varieties are generated by products of projective varieties, we have that the Zariski topology of a multiprojective space $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$ is not the product topology of the Zariski topology of the $\mathbb{P}^{a_i}$. Indeed the following example shows that the Zariski topology of a multiprojective space is finer then the product topology.

Example 1.0.21. Suppose $\mathbb{K} = \mathbb{C}$. In $\mathbb{P}^1$ all the closed sets are finite unions of points, except $\mathbb{P}^1$. In fact, we have that a point $[a_0, a_1]$ can be seen as the zero locus of the linear polynomial $a_1x_0 - a_0x_1$ so it is a Zariski closed set. Conversely, take a homogeneous polynomial $f \in \mathbb{C}[x_0, x_1]$. Setting $x_0 = 1$ we obtain a polynomial $(\overline{f}) \in \mathbb{C}[x_1]$ that decomposes as: $(\overline{f}) = \gamma(x_1 - \alpha_1)^{n_1} \cdots (x_1 - \alpha_k)^{n_k}$ where $\alpha_1 \ldots \alpha_k$ with $\gamma \in \mathbb{C}$ and $n_1, \ldots, n_k \in \mathbb{N}$. Thus, going back to f, there exists a power x_0^α with $\alpha \geq 0$ such that f can be written as $f = kx_0^\alpha(x_1 - \alpha_1 x_0)^{n_1} \cdots (x_1 - \alpha_k x_0)^{n_k}$. So f has only a finite number of solutions i.e. the variety generated by f is a finite union of points.

Notice in particular that the Zariski topology satisfies the separation axiom T_1 but it is not Hausdorff.

Another important fact is that the ideal generated by a multiprojective variety is finitely generated. This follows directly from the Hilbert Basis Theorem that we cite below. First of all we give a definition of *Noetherian ring.*

Definition 1.0.22. A ring A is *Noetherian* if for every ideal $I \subseteq A$ and for any set S of generators for I there exist elements $f_1, f_2 \ldots, f_n \in S$ such that I is the ideal generated by $f_1, f_2 \ldots, f_n$.

Equivalently, we can say that a ring A is Noetherian if it satisfies the ascending chain condition i.e. given a chain of ideals:

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \ldots$$

there exist an index i such that for every $k \geq i$ $I_k = I_{k+1}$.

Example 1.0.23. Any field $\mathbb{K}$ is Noetherian. In fact $\mathbb{K}$ contains only two finitely generated ideals $I_1 = I(0)$ and $I_2 = I(1)$.

Now we can state the Hilbert Basis Theorem. From this theorem follows directly that every projective variety can be defined by the zero locus of a finite set of polynomials.

Theorem 1.0.24 (Hilbert Basis theorem)**.** *Let A be a ring. If A is a Noetherian ring then also its ring of polynomials $A[x_1, \ldots x_n]$ is Noetherian. In particular, every polynomial ring over a field is Noetherian.*

Proof. See Theorem 4 in Chapter 5 of [23]. □

A direct consequence of the Hilbert Basis Theorem is that every multiprojective variety is the intersection of a finite number of hypersurfaces (even though we do not know a priori the number).

There is a strong connection between multiprojective varieties and multihomogeneous ideals. In particular, we can prove that there is a one to one correspondence between a particular class of homogeneous ideals and multiprojective varieties. In order to show this relation we need first to understand which are the ideals that generate the empty variety of a projective space. This class of ideals is described by the *weak (multi)homogeneous Hilbert Nullstellensatz Theorem.*

Theorem 1.0.25 (weak (multi)homogeneous Hilbert Nullstellensatz Theorem)**.** *Assume $\mathbb{K}$ algebraically closed. Let $R = \mathbb{K}[x_{1,0}, x_{2,0}, \ldots x_{r,a_r}]$ be the polynomial ring in the field $\mathbb{K}$ with the usual partition of the variables $A_1, \ldots, A_r$ with A_i of cardinality $a_1 + 1$ and let I be a multihomogeneous ideal. Then the following are equivalent:*

1) *the variety generated by I is empty;*

2) *there exists an i with $0 \leq i \leq r$ such that for $0 \leq j \leq a_j$ there exist an integer $m_j \geq 1$ such that $x_{i,j}^{m_j} \in I$;*

3) *there exists an i with $0 \leq i \leq r$ and an integer $r \geq 1$ such that $x_{i,0}^r, \ldots, x_{i,a_i}^r \in I$.*

We call an ideal I satisfying one between 1) 2) and 3) an irrelevant ideal.

Proof. A proof of this theorem for homogeneous ideals can be found in Theorem 8 in Chapter 8 of [23]. The same proof can be adapted for the multiprojective case. □

Next example shows how to use the weak Hilbert Nullstellensatz Theorem to prove that the product of closed subsets $X_1 \subset \mathbb{P}^{a_1}, \dots, X_r \subset \mathbb{P}^{a_r}$ is again a multiprojective variety of $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$.

Example 1.0.26. Let $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$ be a multiprojective variety and let X_i be a closed set of $\mathbb{P}^{a_i}$ for $i = 1, \dots, r$. The product $X_1 \times \cdots \times X_r$ is a closed set of $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$ (in the Zariski topology).

Indeed, each X_i is the vanishing locus of an homogeneous set of polynomials $T_i \subset \mathbb{K}[x_{i,0}, \dots, x_{i,a_i}]$. We extend each T_i to a finite set J_i such that J_i defines the empty set in $\mathbb{P}^{a_i}$. By the weak Hilbert Nullstellensatz Theorem 1.0.25 we can do this simply adding to T_i the variables $x_{i,0}, \dots, x_{i,a_i}$. We claim that $X_1 \times \cdots \times X_r$ is defined by the set of multihomogeneous polynomials

$$J = \{f_1 \cdots f_r \text{ s.t.} \forall\ i\ f_i \in J_i \text{ and } \exists j : f_j \in T_j\}.$$

Indeed, if $P \in X_1 \times \cdots \times X_r$ then P annihilates all the elements in J. Conversely suppose that $P = (P_1, \dots, P_r) \notin X_1 \times \cdots \times X_r$ so that there exists a j such that $P_j \notin X_j$. We construct an element of J which does not vanish at P as follows. For every $i \neq j$ take g_i any homogeneous polynomial in J_i such that $g_i(P_i) \neq 0$, and take $g_j \in T_j$ such that $g_j(P_j) \neq 0$. The product $g_1 \dots g_r$ belongs to J and it does not vanish at P.

The weak Hilbert Nullstellensatz theorem is used to prove the so-called *strong (multi)homogeneous Hilbert Nullstellensatz Theorem*. This result clarifies the correspondence between homogeneous ideals and projective varieties. In order to state the theorem, we recall the definition of *radical ideal*.

Definition 1.0.27. Given an ideal $I \subseteq \mathbb{K}[x_0, x_1, \dots, x_n]$ we define its *radical* as:

$$\sqrt{I} = \{f \in \mathbb{K}[x_0, x_1, \dots, x_n] \text{ s.t. } f^n \in I \text{ for some } \ n \in \mathbb{Z}_+\}.$$

We say that an ideal I is *radical* if $I = \sqrt{I}$.

Example 1.0.28. Let $A = \{x_0, x_1, x_2, x_3\}$ be a set of variables and let $A_0 = \{x_0, x_1\}$, $A_1 = \{x_2, x_3\}$ be the usual partition of A. The multihomogeneous ideal $I = I(x_0^2 x_1^2) \subset \mathbb{C}[x_0, x_1, x_2, x_3]$ is not a radical ideal since $x_0 x_1$ does not belongs to the ideal. The radical of I, $\sqrt{I}$ is the ideal generated by $x_0 x_1$. Notice in particular that both these ideals define the same variety in $\mathbb{P}^1 \times \mathbb{P}^1$.

Now we can state the strong (multi)homogeneous Hilbert Nullstellensatz Theorem.

Theorem 1.0.29 (Strong (multi)homogeneous Hilbert Nullstellensatz Theorem)**.** *Assume $\mathbb{K}$ algebraically closed. Let $I \subset R = \mathbb{K}[x_0, x_2, \dots x_n]$ be a multihomogeneous ideal and assume $V(I) \neq \emptyset$. Then $I(V(I)) = \sqrt{I}$.*

Proof. For a proof of the homogeneous case see Theorem 9 of [23]. Once again the same proof can be adapted for the multiprojective case. □

Remark 1.0.30. Both the weak and strong Hilbert Nullstellensatz Theorem can be stated for the multiprojective case. Once again the proof of this results in the more general setting is similar to the projective one.

As a consequence, we have the following corollary that describes the correspondence between varieties and multihomogeneous ideals.

Corollary 1.0.31. *There is a one to one correspondence between non empty multiprojective varieties and radical multihomogeneous ideals. In particular, given a non-empty variety V we can find an unique multihomogeneous radical ideal I such that $I = I(V)$. Conversely, we can associate to every radical ideal I which does not contain any irrelevant ideal an unique variety $V(I)$.*

Proof. The proof is a direct consequence of Hilbert Nullstellensatz Theorem 1.0.29 and the fact that the maps I and V are inclusion reversing. A complete proof for the projective case can be found in Chapter 8 of [23] (see Theorem 10). Once again this proof can be adapted easily for the multiprojective case. □

An important class of varieties is the class of *irreducible varieties*. We recall the definition of irreducible subset of a topological space, and then we define an irreducible variety as an irreducible closed subset of the Zariski topology.

Definition 1.0.32. A nonempty subset V of a topological space X is *irreducible* if it cannot be expressed as the union $V = Y_1 \cup Y_2$ of two proper closed subsets of X. The empty set is not considered as irreducible. A multiprojective variety V is *irreducible* if it is an irreducible topological space in the Zariski topology.

Irreducible varieties have a nice interpretation in terms of ideals, whenever we work in a *unique factorization domain*.

Definition 1.0.33. Let R be a ring. We recall also that an element $f \in R$ is *irreducible* if it is not a unit and if $f = a \cdot b$ with $a, b \in R$ implies either a or b is a unit. We say that R is a *unique factorization domain* if it is an integral domain and all the elements of R can be factored uniquely into irreducible elements up to factors which are units.

Example 1.0.34. A polynomial ring $\mathbb{K}[x_0, \dots, x_r]$ is a unique factorization domain where the irreducible elements are the irreducible polynomials.

A well known fact about unique factorization domain is that all irreducible elements are primes i.e. irreducible elements generate *prime ideals* where a prime ideal is defined as follows.

Definition 1.0.35. An ideal $I \subseteq \mathbb{K}[x_0, \dots, x_n]$ is *prime* if given two polynomials $f, g \in \mathbb{K}[x_0, \dots, x_n]$ we have that $fg \in I$ if and only if $f \in I$ or $g \in I$.

Remark 1.0.36. It is easy to see that nonempty irreducible multiprojective varieties correspond, via the map defined in Corollary 1.0.31, to prime ideals. Moreover, since we will work with unique factorization domains, we have that prime ideals are generated by irreducible polynomials.

Example 1.0.37. The twisted cubic curve $V(x_0x_3 - x_1^2, x_1x_3 - x_2^2, x_0x_3 - x_1x_2) \subseteq \mathbb{P}(\mathbb{C}^4)$ is an irreducible variety. Indeed the ideal $I(x_0x_3 - x_1^2, x_1x_3 - x_2^2, x_0x_3 - x_1x_2) = I$ is a prime ideal i.e. if there are two polynomial f, g such that $f \cdot g \in I$ then either $f \in I$ or $g \in I$.

We can prove this fact just using the division algorithm to express f and g in function of $x_0x_3 - x_1^2, x_1x_3 - x_2^2$ and $x_0x_3 - x_1x_2$ and using the fact that the twisted cubic curve is the image of the map $\phi : \mathbb{P}^1 \to \mathbb{P}^3$ which sends a point $p = [x : y] \in \mathbb{P}^1$ to the point $[x^3 : x^2y : xy^2 : y^3] \in \mathbb{P}^3$.

We can describe a variety in terms of its irreducible components. In particular we have the following theorem.

Theorem 1.0.38. *Let $X \subset \mathbb{P}^n$ be a multiprojective variety. Then there exist a unique finite collection of subvarieties $X_1, \dots, X_r \subset X$ such that every X_i is an irreducible variety, $X_i \subsetneq X_j$ for $i \neq j$ and*

$$X = X_1 \cup \cdots \cup X_r.$$

We call $X_1, \dots, X_r$ the irreducible components *of X.*

Proof. See Theorem 6 in Chapter 8 of [23]. □

Example 1.0.39. Let $V \subset \mathbb{P}^n$ be a hypersurface defined by a polynomial f. Since the ring of polynomials of a field is an unique factorization domain, we can write f as the product of its irreducible components i.e. $f = f_1 \cdot f_2 \cdot \ldots f_r$. Thus we have that the irreducible components of V are the varieties $V(f_1), \dots, V(f_r)$.

1.1 Multiprojective maps

In this section we want to describe good maps between multiprojective varieties. First we want to define what is a *regular map* and in particular the object called the *sheaf of regular maps* from a projective variety to a field $\mathbb{K}$. After that, we will define two classes of objects strictly related to the problem of decomposability of a tensor, the Veronese and Segre maps.

Since we are interested mainly to work over the complex field in this section we will take $\mathbb{K} = \mathbb{C}$. Thus, instead of $\mathbb{P}(\mathbb{C}^{n+1})$ we will write $\mathbb{P}^n$.

We start by describing the class of maps from a projective variety to the field $\mathbb{C}$ which can be written locally as a quotient of polynomials of the same degree.

Definition 1.1.1. Let X be a multiprojective variety in $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ and let $U \subset X$ be an open set. A function $f : U \to \mathbb{C}$ is *regular at a point* P if there exists an open neighbourhood U of P in X and multihomogeneous polynomials g, h, of the same multidegree, such that g is nowhere 0 in U and for all $Q \in U$ we have $f(Q) = \frac{h(Q)}{g(Q)}$. We say that f is *regular in* $J \subseteq X$ if it is regular at every point of J.

We indicate with $\mathcal{O}(J)$ the *ring of all regular functions* on J and with $\mathcal{O}_P$ the *ring of germs of regular functions* at $P \in X$ i.e. the ring of regular function in a neighbourhood of P modulo the following equivalent relation: given two regular maps f, g at P we have that $f \simeq g$ if and only if there is an open neighbourhood of P in which $f = g$. Indeed it is easy to see that the sum and the product of two regular function at P is still regular at P.

In the next proposition we show that the ring $\mathcal{O}_P$ is a local ring i.e. a ring with only one maximal ideal.

Proposition 1.1.2. *Let $X \subset \mathbb{P}^n$ be a projective variety and let P be a point in X. The ring of regular function at P, $\mathcal{O}_P$, is a regular local ring. In particular $\mathcal{O}_P/m_p$ is a field isomorphic to $\mathbb{C}$.*

Proof. Let m_p be the ideal of $\mathcal{O}_P$ generated by quotients of polynomials defined in an open neighbourhood of P which vanish at P. m_p is maximal. Indeed if we add another element $f \in \mathcal{O}_P$ we have that there is a neighbourhood of P in which f does not vanish. Thus $1/f$ is regular at P and is an unit i.e. an invertible element. From Proposition 1.6 of [9] we have that $\mathcal{O}_P$ is a local ring. Moreover since m_p is maximal we have that $\mathcal{O}_P/m_p \simeq \mathbb{C}$ (a proof of this fact can be found in Chapter 1 of [9]).

□

Remark 1.1.3. Let $f, g \in \mathbb{C}[x_{1,0}, \dots x_{r,a_r}]$ be two multihomogeneous polynomials of the same multidegree $(d_1, \dots, d_r)$ and let U be a Zariski open set in which g does not vanish. Then, the map $f/g : U \to \mathbb{C}$ which sends a point $([P_1], \dots, [P_r]) \in \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$ to $f((P_1), \dots, (P_r))/g((P_1), \dots, (P_r))$, where $(P_1), \dots, (P_r)$ are sets of coordinates for the point $([P_1], \dots [P_r])$, is a well defined function.

Indeed for every point $([P_1], \dots [P_r]) \in \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$ we have that

$$\frac{f(\lambda_1(P_1), \dots, \lambda_r(P_r))}{g(\lambda_1(P_1), \dots, \lambda_r(P_r))} = \frac{\lambda_1^{d_1} \cdots \lambda_r^{d_r}}{\lambda_1^{d_1} \cdots \lambda_r^{d_r}} \cdot \frac{f((P_1), \dots, (P_r))}{g((P_1), \dots, (P_r))} = \frac{f((P_1), \dots, (P_r))}{g((P_1), \dots, (P_r))}$$

for all λ_i different from 0.

Proposition 1.1.4. *Let $B = \{P_1, \dots, P_r\} \subset \mathbb{P}^n$ be a finite set of length $\ell(B) = r$. Then, the ring of regular functions $\mathcal{O}(B)$ is isomorphic to $\mathbb{C}^r$ as a linear space over $\mathbb{C}$.*

Proof. To show this, we define a map $\phi : \mathcal{O}(B) \to \mathbb{C}^r$ as follows. For every $f \in \mathcal{O}(B)$ and for every $P_i \in B$ there exist an open neighbourhood $U_i \subseteq \mathbb{P}^n$ of P_i such that $f = h_i/g_i$ on U_i with h_i, g_i of the same degree. Thus, we define $\phi(f)$ as:

$$\phi(f) = (h_0/g_0(P_1)), \dots, h_r/g_r(P_r)).$$

Let f, g be two regular function in B such that $\phi(f) = \phi(g)$. This means that, there exists h_i, h_i' and g_i, g_i' polynomials and $U_i \dots U_i'$ neighbourhoods such that $h_i/g_i(p_i) = h_i'/g_i'(p_i)$. Thus, f and g are equal as maps from B to $\mathbb{C}$ so ϕ is injective.

Moreover B is a finite set, so we can always find a polynomial g which vanish in no points of B and another polynomial h of the same degree of g vanishing in all the point of B but one. So, we have that h/g, up to scaling, is an element of the canonical base of $\mathbb{C}^r$. As a consequence ϕ is surjective.

□

Now we give a definition of *multiprojective maps* i.e. maps between multiprojective spaces.

Definition 1.1.5. Let $X \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ and $Y \subset \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_j}$ two multiprojective varieties. We say that a map $f : X \to Y$ is *multiprojective* if for any point $P \in X$ there exists an open set U of X containing P and polynomials $f_{1,0}, \dots, f_{j,m_j} \in \mathbb{K}[x_{1,0}, \dots, x_{r,n_r}]$ multihomogeneous of the same multidegree such that for all $Q \in U$:

$$f(Q) = [f_{1,0}(Q) : \cdots : f_{j,m_j}(Q)].$$

Remark 1.1.6. Let X be a multiprojective variety and let $P \in X$. Notice that every regular map can be also considered as a multiprojective map from an open neighbourhood $U \subset X$ to $\mathbb{P}^1$.

Example 1.1.7. Given two projective spaces $\mathbb{P}^n, \mathbb{P}^m$ with $n \leq m$ we have that the map $\phi : \mathbb{P}^n \to \mathbb{P}^m$ defined as:

$$\phi([x_0 : \cdots : x_n]) = [x_0 : \cdots : x_n : 0 : \cdots : 0]$$

is an injective projective map.

An important property of projective maps is the following.

Proposition 1.1.8. *Projective maps are continuous in the Zariski topology.*

Proof. See Proposition 9.3.5 of [15]. $\square$

As for the projective case, it is easy to prove that the composition of two multiprojective maps is a multiprojective map, and that multiprojective maps are continuous in the Zariski topology.

An interesting class of projective maps are the maps from the whole space $\mathbb{P}^n$ to a variety $Y \subseteq \mathbb{P}^m$ defined starting from a linear map $f : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$. The following example shows the construction of such projective maps.

Example 1.1.9. Given a set of linear injective maps $f_i : \mathbb{C}^{a_i+1} \to \mathbb{C}^{b_i+1}$ $(i = 1, \ldots, r)$, we can define a projective map $\overline{f} : \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r} \to \mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_r}$ as follows:

$$\begin{aligned}\overline{f}([x_{1,0} : \cdots : x_{1,a_1}], \ldots, [x_{r,0} : \cdots : x_{r,a_r}]) = \\ = ([f_1(x_{1,0}, \ldots, x_{1,a_1})], \ldots, [f_r(x_{r,0}, \ldots, x_{r,a_r})]).\end{aligned}$$

Notice that, from the fact that f is linear and injective, $\overline{f}$ defined as before is a well defined multiprojective map. We call such a map a *linear multiprojective map*.

Notice that, in the previous example, if one of the linear maps $f_i : \mathbb{C}^{a_i+1} \to \mathbb{C}^{b_1+1}$ is not injective, we cannot define the multiprojective map $\overline{f}$. In fact, since f_i is not injective, we have that the kernel of f_i is different from $\{(0, \ldots, 0)\}$, so there will be a point $[x_{i,0} : \cdots : x_{i,a_i}]$ such that $f_i(x_{i,0}, \ldots, x_{i,a_i})$ is equal to 0. So, we have that $[f_i(x_{i,0}, \ldots, x_{i,a_i})]$ is not defined.

However, also in this situation, we can avoid this problem restricting the domain of the projective map in order to avoid the kernel of the map $\overline{f}$. This justify the following definition.

Definition 1.1.10. Given a linear map $f : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$ and a projective variety $X \subseteq \mathbb{P}^n$ such that $X \cap \mathbb{P}(\ker(f)) = \emptyset$ we have that if we restrict the domain of f to X, the restricted map

$$f_{|X} : X \to \mathbb{P}^n$$

which send a point $P \in X$ of coordinates $[a_0, \ldots, a_n]$ to the point $[f(a_0, \ldots, a_n)]$ is a well defined projective map.

We call such a map the *projection of* V *from* $\mathbb{P}(\ker(f))$. The subspace $\ker(f)$ is called the center of the *projection* and we call $\mathbb{P}(\ker(f))$ the *projective kernel of* f.

Remark 1.1.11. Any linear multiprojective map $\phi : \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n} \to \mathbb{P}^m$ is defined, up to isomorphism and change of coordinates, by multihomogeneous polynomials of multidegree $(1, \ldots, 1)$. In other words, for every point $P \in \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$, ϕ can be written as a quotient of polynomials taking as a neighbourhood U of P the whole space. The proof of this fact follows almost directly from the irreducibility of projective spaces and can be found generalizing the proof of Proposition 9.3.2 of [15]. The same observation applies also for projective maps.

Remark 1.1.12. We can repeat the argument used to define projections also to find maps from a multiprojective variety to a multiprojective space induced by linear maps. In particular, a set of linear maps $\phi_i : \mathbb{C}^{a_i+1} \to \mathbb{C}^{b_i+1}$ $(i = 1, \ldots, r)$ induces a multiprojective map from $X \subset \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$ to $\mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_r}$ when $(\mathbb{P}(\ker(\phi_1) \times \cdots \times \mathbb{P}(\ker(\phi_r)) \cap X = \emptyset$.

An important property of projections is the following.

Proposition 1.1.13. *Let f_i be an injective linear map $f_i : \mathbb{C}^{a_i+1} \to \mathbb{C}^{b_i+1}$ with $i = 1, \ldots r$ and $b_i \geq a_i$. Then, given a projective variety $X \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$, we have that f induces a projective map $\overline{f} : X \to \mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_r}$ which is a closed map in the Zariski topology i.e the image of a closed set is still closed.*

Proof. We prove it by induction of the number of factors of the multiprojective space $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$.

Suppose $r = 1$. If f is injective then, the proof is easy and it is based on the fact that f can be written as a composition of two maps $\psi \circ f'$ where ψ is a change of variables and f' is the map which send a point $P = (a_0, \ldots, a_n)$ to the point $(a_0, \ldots, a_n, 0, \ldots, 0)$. A complete proof of this case can be found in Proposition 10.1.4 of [15].

If f is not injective then, we can write f as $f = \psi \circ f''$ where ψ is the natural surjection from $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n+1}/\ker(f)$ and f'' is the map from $\mathbb{C}^{n+1}/\ker(f)$ to $\mathbb{C}^m$ which send a point $P + \ker(f) \in \mathbb{C}^{n+1}/\ker(f)$ to $f(P)$. The complete proof of this case can be found in Proposition 10.4.2. of [15].

The proof for $r > 1$ follows directly from the general fact that the product of closed maps is closed. $\square$

We will use this kind of maps in the next session to define formally what is the dimension of a variety.

Proposition 1.1.14. *The projection $\pi_i : \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r} \to \mathbb{P}^{a_i}$, which send a point $P = ([a_{1,0} : \cdots : a_{1,a_1}], \ldots, [a_{r,0} : \cdots : a_{r,a_r}])$ to the point $[a_{i,0} : \cdots : a_{i,a_i}]$, is a closed projective map.*

Proof. It is easy to see that the map π_i is a multiprojective map. As a matter of fact, it follows directly from the definition that π_i is defined by multihomogeneous polynomials of multidegree $(0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is the degree of the i-th set of variables. Indeed, the multihomogeneous polynomials are just the variables $x_{i,j}$ for $j = 0, \ldots, a_i$.

To prove that π_i is closed we have to show that given a closed set $X \subset \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$ the image of X through π_i is still closed.

Now we can proceed by induction. If $\pi_i(X) = \mathbb{P}^{a_i}$ then there is nothing to prove since $\mathbb{P}^{a_i}$ is closed. In the same way, if $a_i = 0$ for some i then $\mathbb{P}^{a_i}$ is a point so the image of every non-empty closed subset of $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$ is closed.

Now we proceed by induction. Since $\pi_{|X}$ is the map induced by the natural map $f : \mathbb{C}^{a_1+1} \times \cdots \times \mathbb{C}^{a_r+1} \to \mathbb{C}^{a_i+1}$ the conclusion follows. □

We introduce now two projective maps that will be involved in problems related to tensors. This maps are the *Veronese and Segre maps*.

We can define *Veronese maps* as follows.

Definition 1.1.15. The *Veronese map* $\overline{v}_{d,n} : \mathbb{P}^n \to \mathbb{P}^N$ with $N = \binom{n+d}{d} - 1$ is the (non-linear) map such that:

$$\overline{v}_{d,n}(P) = [m_0(P) : m_1(P) : \cdots : m_n(P)]$$

where the m_i are the monomials of degree d (ordered in the lexicographic order).

Remark 1.1.16. In the literature, *Veronese maps* are defined also in other ways. In particular we can consider the Veronese maps to be defined as

$$\overline{v}_{d,n}(P) = [\alpha_0 m_0(P) : \alpha_1 m_1(P) : \cdots : \alpha_n m_n(P)] \tag{1.1}$$

where the α_i are some coefficients in $\mathbb{C}$. It is easy to see that all those definitions are equal up to a change of coordinates. Thus, all the properties that hold for the maps defined as in 1.1.15 also hold for the maps defined in 1.1.

There is a way to choose the scalars α_i in Equation 1.1 that is particularly useful when we are dealing with symmetric tensors. This choice of the coefficients is strictly related to the concept of *dual of a vector space* and can be described as follows.

Fix a base of $\mathbb{P}^n$ $x_0, x_1, \ldots, x_n$ and write $P = [a_0 : a_1 : \cdots : a_n] \in \mathbb{P}^n$ as:

$$[a_0 : a_1 : \cdots : a_n] = a_0 x_0 + a_1 x_1 + \cdots + a_n x_n.$$

Definition 1.1.17. The *(dual) Veronese map* $v_{d,n} : \mathbb{P}^n \to \mathbb{P}^N$ (with $N = \binom{n+d}{d} - 1$) is the (non-linear) map that associate to every point $[a_0 : a_1 : \cdots : a_n]$ the coefficients of $[(a_0 x_0 + a_1 x_1 + \cdots + a_n x_n)^d]$.

Equivalently, $v_{n,d}$ is the map obtained by choosing the coefficient α_i in 1.1 equal to the Newton binomials, i.e. the i-th coordinate of $v_{d,n}([a_0 : a_1 : \cdots : a_n])$ is the evaluation in $(a_0, a_1, \ldots, a_n)$ of the i-th monomial obtained from the expansion of $(a_0 x_0 + a_1 x_1 + \cdots + a_n x_n)^d$ ordered lexicographically.

The following example shows that Definition 1.1.17 and Definition 1.1.15 differ only for some coefficients.

Example 1.1.18. Fix as a base for $\mathbb{P}^5$ $\{x_0, x_1, x_2, x_3, x_4, x_5\}$. The standard Veronese map $\overline{v}_{2,2} : \mathbb{P}^2 \to \mathbb{P}^5$ is the map that sends at each point $P = [a_0 : a_1 : a_2]$ to the point

$$\overline{v}_{2,2}(P) = [a_0^2 : a_0 a_1 : a_0 a_2 : a_1^2 : a_1 a_2 : a_2^2].$$

On the other hand we have that

$$v_{2,2}([a_0 : a_1 : a_2]) = [a_0^2 : 2a_0 a_1 : 2a_0 a_2 : a_1^2 : 2a_1 a_2 : a_2^2].$$

An important property of Veronese maps is that $v_{n,d}$ is injective for all $n, d \in \mathbb{Z}^+$.

Lemma 1.1.19. *For all $n, d \in \mathbb{N}$ the Veronese map $v_{n,d}$ is injective.*

Proof. Suppose that there are two points $P = (p_0, \dots, p_n)$, $Q = (q_0, \dots, q_n)$ in $\mathbb{P}^n$ such that $v_{n,d}(P) = v_{n,d}(Q)$. Thus we have that, up to a scalar multiplication, $p_i^d = q_i^d$ and then $p_i = e_i q_i$ with e_i a d-th root of unity. If the e_i's are not all equal to 1, then there exists a monomial m_j such that $m_j(e_0, \dots, e_n) \neq 1$. Thus, we have that the monomial $\alpha_j m_j(p_0, \dots, p_n)$ is not equal to $\alpha_j m_j(q_0, \dots, q_n)$ where α_j is the coefficient that comes from the expansion of $(p_0, \dots, p_n)^d$. Thus, since α_j is a positive natural number different from zero for all monomials m_j, we have a contradiction.

□

Another important property of Veronese maps is that the image is a variety, so there exists a set of equations that describes the image of $\overline{v}_{n,d}$.

Proposition 1.1.20. *The image of a Veronese map is a projective subvariety. In particular it is defined by a set of homogeneous polynomials of degree 2. Moreover, every Veronese map is closed in the Zariski topology.*

Proof. Since $\overline{v}_{n,d}$ and $v_{n,d}$ are equal up to a change of coordinates, we sketch the proof of this theorem for $v_{n,d}$. A complete proof can be found in Theorem 10.5.4 and Theorem 10.5.7 of [15].

The Veronese variety can be defined as the vanishing locus of quadratic equations as follows.

Fix $(a_0, \dots, a_n)$ and $(b_0, \dots, b_n)$ two $(n+1)$-tuples of non-negative integers such that $\sum a_i = \sum b_i = d$ and let $(c_1, \dots, c_n)$ another $(n+1)$-tuple such that $\sum c_i = d$ and such that for all i we have $a_i + b_i \geq c_i$. We define $D = (d_o, \dots, d_n)$ as $d_i = a_i + b_i - c_i$. Then we have that $\sum d_i = d$ and that the Veronese variety V is the zero locus of the quadratic polynomial of the form:

$$m^A m^B - m^C m^D$$

where each m^i is a monomial of multidegree i.

To prove that the Veronese map is closed we have to show that the image of every projective subvariety of $\mathbb{P}^n$ through $v_{n,d}$ is a projective subvariety of $\mathbb{P}^N$.

The proof is based on the fact that every polynomial $F \in \mathbb{C}[x_0, \dots, x_n]$ of degree kd can be interpreted as a form of degree k in the coordinates $M_0, \dots, M_N$ of $\mathbb{P}^N$ together with the fact that for every variety $X \subset \mathbb{P}^n$ the ideal $I(X)$ is equal to the ideal IM^d where M is the maximal ideal of $\mathbb{C}[x_1, \dots, x_n]$. □

Given a Veronese map $v_{n,d}$, we call its image a *Veronese variety*. Since a Veronese map is injective, sometimes we will refer to a Veronese map also with the name of Veronese embedding.

Another important class of maps between multiprojective spaces, is the class of Segre maps.

Definition 1.1.21. Fix $a_1, \dots, a_n \in \mathbb{N}$ and set $N = (a_1 + 1) \cdot (a_2 + 1) \cdot \dots \cdot (a_n + 1) - 1$.

The *Segre map* of $a_1, \dots, a_n$ is the map $s_{a_1,\dots,a_n} : \mathbb{P}^{a_1} \times \dots \times \mathbb{P}^{a_n} \to \mathbb{P}^N$ that sends a point $P = ([p_{1,0} \dots p_{1,a_1}], \dots, [p_{n,0} : \dots : p_{n,a_n}])$ to $[M_0(P) : \dots : M_N(P)]$ where each M_i is a monic monomial of multidegree $(1, \dots, 1)$ in the variables $x_{1,0} \dots x_{1,a_1}, \dots, x_{n,0} \dots x_{n,a_n}$ (ordered in lexicographic order).

As for the Veronese map, we can prove that all the Segre maps are injective.

Lemma 1.1.22. *Every Segre map is injective.*

Proof. Let $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ be a multiprojective space with n factors. We prove our statement by induction on n.

If $n = 1$ then the Segre map is the identity map, so this case is trivial.

Let $P, Q \in \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ be two points of the form

$$P = ([p_{1,0} : \cdots : p_{1,a_1}], \ldots, [p_{n,0} \ldots p_{n,a_n}])$$

$$Q = ([q_{1,0} : \cdots : q_{1,a_1}], \ldots, [q_{n,0} \ldots q_{n,a_n}])$$

and assume that $s_{a_1,\ldots,a_n}(P) = s_{a_1,\ldots,a_n}(Q)$.

Fix indices $j_1, \ldots, j_n$ such that $p_{1,j_1}, \ldots, p_{n,j_n}$ are different from 0. The monomial $M = x_{1,j_1} \cdots x_{n,j_n}$ does not vanish at P, hence also the coordinates of Q $q_{1,j_1}, \ldots, q_{n,j_n}$ are all different from 0.

Call $\alpha = q_{1,j_1}/p_{1,j_1}$. We want to show that also $q_{1,i}/p_{1,i} = \alpha$ for $i = 1, \ldots, n_1$ i.e. that $P = Q$. Define $\beta = (q_{2,j_2} \cdots q_{n,j_n})/(p_{2,j_1} \cdots p_{n,j_n})$. Then $\beta \neq 0$ and: $\alpha \cdot \beta = (q_{1,j_1} \cdots q_{n,j_n})/(p_{1,j_1} \cdots p_{n,j_n})$. Since P, Q have the same image in the Segre map, then for all $i = 1, \ldots, a_1$, the monomials $M_i = x_{1,i} x_{2,j_2} \cdots x_{n,j_n}$ satisfy: $\alpha\beta M_i(P) = M_i(Q)$. As a consequence we have that $\alpha \cdot \beta \cdot (p_{1,i} \cdots p_{n,j_n}) = (q_{1,i} \cdots q_{n,j_n})$ so that $\alpha \cdot \beta \cdot p_{1,i} = q_{1,i}$ for all i . Thus $[p_{1,0} : \cdots : p_{1,a_1}] = [q_{1,0} : \cdots : q_{1,a_1}]$. To conclude the proof we can repeat the same argument used before for the remaining factors, obtaining $P = Q$. □

So, every Segre maps is the embedding of a multiprojective space $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ to a projective space $\mathbb{P}^N$. Moreover, as before, we call the image of a Segre map a *Segre variety.* The following result justifies this notation.

Proposition 1.1.23. *The image of the Segre map is a projective subvariety of $\mathbb{P}^N$. Moreover all the Segre maps are closed in the Zariski topology.*

Proof. We can find equations in $\mathbb{C}[x_{1,0} \ldots, x_{1,a_1}, \ldots, x_{n,0} \ldots, x_{n,a_n}]$ defining the Segre variety $s_{a_1,\ldots,a_n}(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n})$ as follows.

First we associate to every n-tuple $Y = (\alpha_1, \ldots, \alpha_n)$ with $0 \leq \alpha_i \leq a_i$ the multihomogeneous polynomial $M^Y \in \mathbb{C}[x_{1,0} \ldots x_{n,a_n}]$ of multidegree $(1, \ldots, 1)$ defined as:

$$M^Y = x_{1,\alpha_1} \cdots x_{n,\alpha_n}.$$

Then, given any subset $J \subset \{0, 1, \ldots, n\}$ and two n-tuple $A = (\alpha_1, \ldots, \alpha_n)$ and $B = (\beta_1, \ldots, \beta_n)$ with $0 \leq \alpha_i, \beta_i \leq a_i$ we define an n-tuple $C^J_{A,B} = (\gamma_1, \ldots, \gamma_n)$ with

$$\gamma_i = \begin{cases} \alpha_i & \text{if } i \in J \\ \beta_i & \text{otherwise} \end{cases} \tag{1.2}$$

In the same way we define $D = C^{J^c}_{AB}$ where $J^C = \{0, 1, \ldots, n\} \setminus J$.

Then $s_{a_1,\ldots,a_n}(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n})$ is defined by the polynomials of the form:

$$f^J_{A,B} = M^A M^B - M^C M^D$$

A complete proof of this fact can be found in Proposition 10.5.12 of [15].

In order to prove that Segre maps are closed we need to prove that the image in $s_{a_1,\dots,a_n}$ of a multiprojective subvariety $X \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ is a projective subvariety of $\mathbb{P}^N$.

As usual the multiprojective variety X is defined by multihomogeneous polynomials in $\mathbb{C}[x_{1,0},\dots,x_{n,a_n}]$ (with the usual partition of the variables).

As for the Veronese map, the proof is based on the following remark. If F is a monomial of multidegree $(d,\dots,d)$ in the variables $x_{i,j}$, then it can be written as a product of k multilinear forms in the unknowns $x_{i,j}$'s, which corresponds to a monomial of degree d in the coordinates $M_0,\dots,M_N$ of $\mathbb{P}^N$.

Thus, any form f of multidegree $(d,\dots,d)$ in the x_{ij} 's can be rewritten as a form of degree d in the coordinates M_j 's. For a complete proof see 10.5.16 of [15].

□

A direct consequence of Lemma 1.1.22 and Proposition 1.1.23 is that all multiprojective spaces are projective varieties via the Segre embedding, and that all multiprojective varieties can be identified with projective varieties.

There is a strong connection between the Veronese and Segre maps. In order to show this relation we need to define the *diagonal embedding*.

Definition 1.1.24. We say that a multiprojective space $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$ is *cubic* if there exists an $n \in \mathbb{N}$ such that $a_i = n$ for all i. We call a *diagonal embedding* the map δ from $\mathbb{P}^n$ to a multiprojective space of the form $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$ which send a point $P \in \mathbb{P}^n$ to the point $(P,\dots,P)$ of $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$.

Theorem 1.1.25. *Fix a cubic multiprojective space with $d > 1$ factors. Then, the Veronese embedding $\overline{v}_{n,d}$ of degree d corresponds to the composition of the diagonal embedding δ, the Segre embedding $s_{n,\dots,n}$ and one projection.*

Proof. Let $P = [p_0 : \cdots : p_n]$ be a point of $\mathbb{P}^n$ and consider $s_{n,\dots,n} \circ \delta(P) = s_{n,\dots,n}(P,\dots,P)$.

By definition of Segre embeddings it follows directly that all the coordinates of $s_{n,\dots,n}(P,\dots,P)$ corresponding to monomials $x_{1,\sigma(i_1)}\dots x_{d,\sigma(i_d)}$, where σ is any permutation of $[d] = \{1,\dots,d\}$, are equal.

A direct computation shows also that there are exactly $\binom{n+d}{d}$ non-repeated coordinates of $s_{n,\dots,n} \circ \delta(P)$ (with "non-repeated coordinates" we indicate the coordinates of $s_{n,\dots,n} \circ \delta(P)$ corresponding to different monomials). In particular, the non-repeated coordinates correspond to the multilinear forms $x_{1,i_1}\dots x_{d,i_d}$ such that $i_1 \leq i_2 \leq \cdots \leq i_d$.

We want to eliminate the repeated coordinates from $s_{n,\dots,n} \circ \delta(P)$. In other words, we want to compose $s_{n,\dots,n} \circ \delta(P)$ with the projection induced by the linear maps $\phi : \mathbb{C}^{N+1} \to \mathbb{C}^{\binom{n+d}{d}}$ which send a point $Q = s_{n,\dots,n}(P,\dots,P)$ seen as a point of $\mathbb{C}^{N+1}$ to the point $\overline{Q}$ of $\mathbb{C}^{\binom{n+d}{d}}$ obtained from Q by removing the repeated coordinates. In order to define correctly this projective map, we have to check that the projective kernel of ϕ is disjoint from the points of the form $(P,\dots,P)$.

The kernel of ϕ is the set of all the $(N+1)$-tuples in which the coordinates corresponding to linear forms $x_{1,i_1}\dots x_{d,i_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ are all zero. So, for all $P = [p_0 : p_1 : \cdots : p_n] \in \mathbb{P}^n$ we have that $s_{n,\dots,n} \circ \delta(P)$ cannot meet the kernel otherwise we would have that $p_0^{d_0} = \cdots = p_n^{d_n} = 0$ that is $p_0 = \cdots = p_n = 0$. Thus the map $\phi \circ s_{n,\dots,n} \circ \delta$ is well defined for all $P \in \mathbb{P}^n$.

Moreover, the coordinates of $s_{n,\ldots,n} \circ \delta(P)$ corresponding to $x_{1,i_1} \cdots x_{d,i_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ are equal to $p_0^{r_0} \cdots p_n^{r_d}$, where r_i is equal to the number of times in which the number i appears among $i_1, \ldots, i_d$. Then, $r_0 + \cdots + r_d = d$ and it is clear that the coordinates of $\phi \circ s_{n,\ldots,n} \circ \delta$ are the monomials of degree d in $x_0, \ldots, x_r$. So computing $\phi \circ s_{n,\ldots,n} \circ \delta(P)$ is equivalent to computing the Veronese map $v_{n,d}(P)$ defined as in Definition 1.1.15.

□

Remark 1.1.26. Let $A \subset \mathbb{P}^n$ be a finite set of points and let ϕ a injective projective map. As we have seen in Remark 1.0.2, A linearly independent means that the projective span $L(A)$ has dimension equal to the number of points minus 1. Moreover, if $\phi(A)$ is linearly independent then also A is linearly independent.

We conclude this section by mentioning an important Theorem about projective maps: Chow's Theorem.

Theorem 1.1.27. *Every multiprojective map $f : \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r} \to \mathbb{P}^m$ is Zariski closed.*

Proof. For the projective case the proof follows directly from the fact that every projective map can be written as the composition of a Veronese map, a change of coordinates and a projection. Indeed, from Remark 1.1.11 we know that there are $f_0, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$ of the same degree d which do not vanish simultaneously at any point of $\mathbb{P}^n$ and such that $f(P) = [f_0(P) : \cdots : f_m(P)]$ for every $P \in \mathbb{P}^n$. Since each f_i is a linear combination of monomials of degree d, we have that there exist a change of coordinates h of $\mathbb{P}^N$ such that f is the composition of the Veronese map $v_{n,d}$, the change of coordinates h and a projection π to the first $m+1$ factors. Notice that π is well defined since $f(P) = [f_0(P) : \cdots : f_m(P)] \neq 0$ for all $P \in \mathbb{P}^n$.

The result follows from the fact that changes of coordinates are closed maps and from Proposition 1.1.13 and Proposition 1.1.20.

The proof of the multiprojective case is similar and follows from the fact that every multiprojective map factors through a Veronese map, a Segre map, a change of coordinates and a projection. For a proof of this fact see Proposition 10.6.2 of [15]. Also in this case the conclusion follows from Proposition 1.1.20, Proposition 1.1.23 and Proposition 1.1.14 □

Remark 1.1.28. Theorem 1.1.27 holds also for multiprojective maps whose domain is a variety different from the whole space $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_r}$ (see Corollary 11.3.7 of [15]).

1.2 Dimension of a variety

In this section we give a definition of *dimension of a variety.* In order to recall this concept we will take as main reference [15].

In order to define what is the dimension of a variety we will use the notion of dimension of a linear projective subspace. First of all we recall the definition of a *fiber* of a projective map.

Definition 1.2.1. Let f be a projective map, $f : X \to Y$. We call *fiber* of f over a point P the inverse image $f^{-1}(P)$.

Notice in particular that since projective maps are continuous in the Zariski topology, the fiber over any point P is closed in the Zariski topology i.e. it is a projective variety.

Proposition 1.2.2. *Let $X \subsetneq \mathbb{P}^n$ be a projective variety. Then, there exists a linear subspace $L \subset \mathbb{P}^n$ not intersecting X such that the projection of X from L to a linear subspace L' is surjective with finite fibers.*

Proof. See proposition 11.2.3 of [15]. □

Now, we are able to define what is the dimension of a projective variety. First we will give the definition only for irreducible projective variables and then we will extend this concept using the fact that every projective variety can be seen as the union of a finite number of irreducible projective varieties.

Definition 1.2.3. Given $X \subset \mathbb{P}^N$ an irreducible projective variety, we say that X has dimension n if there exist a linear subspace $L \subset \mathbb{P}^N$ of dimension $N-n-1$ with does not meet X and such that the projection with center L which maps X to a linear subspace L' of dimension n is surjective with finite fibers.

We assign dimension -1 to the empty set and if $X = \mathbb{P}^n$ we consider valid to take $L = \emptyset$ and the projection equal to the identity. So, $\mathbb{P}^n$ has dimension n.

Example 1.2.4. Let $V \subset \mathbb{P}^2$ be a nonempty irreducible variety defined by an irreducible polynomial $g \neq 0$. V has dimension equal to 1. In fact, let P_0 be a point such that $g(P_0) \neq 0$ and consider the projection π from P_0 which maps C to $\mathbb{P}^1$. Thus for each $P \in V$ the fiber $\pi^{-1}(P)$ is a proper subvariety of $\mathbb{P}^1$ since $P_0 \notin \pi^{-1}(P)$. So $\pi^{-1}(P)$ is a finite union of points and all the fibers are finite. Moreover, since by Theorem 1.1.27 π is closed and since V is infinite we have that the image of π has to be infinite and so π is surjective.

We can prove that the dimension of an irreducible variety is unique. In order to state the next proposition, we recall the definition of *quotient field* of a ring.

Definition 1.2.5. Let R be a integral domain. We call the *quotient field* of R the smallest field in which R can be embedded. An effective construction of such a field can be found in Chapter 3 of [9].

In particular, given a prime ideal I corresponding to an irreducible variety X we have that $\mathbb{K}[x_0, \dots, x_n]/I$ is a domain. We will refer to the quotient field of $\mathbb{K}[x_0, \dots, x_n]/I$ as the quotient field of X.

Proposition 1.2.6. *Given an irreducible variety $X \subset \mathbb{P}^n$, if there is a projection $\Phi : X \to \mathbb{P}^m$ with finite fibers, then the transcendence degree of the quotient field of X is n. In particular, given $m' \neq m$ we cannot find a surjective projection $\Phi' : X \to \mathbb{P}^{m'}$ with finite fibers.*

Proof. See Corollary 11.2.12 of [15]. □

Now that we know that the dimension of an irreducible variety is well defined, we can give a definition of dimension for any variety.

Definition 1.2.7. Let $X \subset \mathbb{P}^n$ be a variety. We define the dimension of X as:

$$\dim(X) = \max\{\dim(X_i)\}$$

where each X_i is an irreducible component of X.

We want to give a result that estimates the dimension of the fiber of a projective map. First we give some preliminary definitions.

Definition 1.2.8. Given a projective function $f : X \to Y$, for all $P \in Y$ we define $\mu_f(P)$ as:

$$\mu_f(P) = \dim(f^{-1}(P)).$$

Definition 1.2.9. Let $f : X \to Y$ be a continuous map. We say that f is *dominant* if $f(Y)$ is dense in Y.

Definition 1.2.10. Given a projective variety $X \subseteq \mathbb{P}^n$ and a map $f : X \to \mathbb{Z}$ we say that g is *upper semicontinuous* if for every $n \in \mathbb{Z}$ the set of points $P \in X$ such that $f(P) \geq n$ is closed in the Zariski topology.

Now, we can state the following theorem that characterize the dimension of the fiber of a certain projective map.

Proposition 1.2.11. *Let X be a irreducible variety and let $f : X \to Y$ be a dominant projective map. Then we have that f is surjective, the function μ_f is upper semicontinuous and $\mu_f(P) \leq \dim(X) - \dim(Y)$ for all $P \in X$.*

Proof. See Theorem 11.3.5 of [15]. □

Corollary 1.2.12. *Let X be a irreducible variety and let $f : X \to Y$ be a dominant projective map with Y irreducible. If n is the dimension of the fiber $f^{-1}(P)$ with P general in Y, then $\dim X = \dim Y + n$.*

Proof. The proof follows directly from Proposition 1.2.11. □

1.3 Secant varieties

In this section we will introduce an important object useful for the study of problems concerning tensors: the *secant variety*. In particular, we will see that this concept is strictly related to the decomposition of a tensor.

Definition 1.3.1. Let $Y_1, \dots, Y_r$ be projective subvarieties of $\mathbb{P}^n$ and let $Y_1 \times \dots \times Y_r$ be a multiprojective variety of $(\mathbb{P}^n)^r = (\mathbb{P}^n) \times \dots \times (\mathbb{P}^n)$. The total join of $Y_1, \dots, Y_r$ is the subset $TJ(Y_1, \dots, Y_r) \subset Y_1 \times \dots \times Y_r \times \mathbb{P}^n$ of all $(r+1)$-tuples $(P_1, \dots, P_r, Q)$ such that the points $P_1, \dots, P_r, Q$ as points of $\mathbb{P}^n$ are linearly dependent.

Remark 1.3.2. Notice that if $r \geq n+1$ then the total join $TJ(Y_1, \dots, Y_r)$ is equal to $Y_1 \times \dots \times Y_r \times \mathbb{P}^n$. In order to avoid this trivial cases we will suppose $r \leq n$

In particular, a point $v = (P_1, \dots, P_r, Q)$ in the total join of $Y_1 \dots, Y_r \subset \mathbb{P}^n$ satisfies several conditions. In fact, every $P_i \in Y_i$ satisfies the set of equations defining Y_i. Moreover, using the condition of linear dependence we have the following result.

Proposition 1.3.3. *The total join of $Y_1, \dots, Y_r \subseteq \mathbb{P}^n$ is a multiprojective subvariety of $(\mathbb{P}^n)^{r+1}$.*

Proof. Let $P = (P_1, \dots, P_r, T)$ be a point in $TJ(Y_1, \dots, Y_r)$. We want to find multihomogeneous polynomial $f_i \in \mathbb{C}[x_{1,0}, \dots, x_{1,n}, \dots, x_{r+1,0}, \dots, x_{r+1,n}]$ which vanish at P for all P in $TJ(Y_1, \dots, Y_r)$.

Since each P_i belongs to Y_i for all $i = 0, 1, \dots, r$, we have that P satisfies multihomogeneous equations defining Y_i in the i-th set of unknowns $x_{i,0}, \dots, x_{i,n}$. Moreover since T is contained in the span of $p_1, \dots, p_r$, for a choice of homogeneous coordinates $(h_{i,0}, \dots, h_{i,n})$ for p_i and homogeneous coordinates $x_0, \dots, x_r$ for T, we have that all the $(r+1) \times (r+1)$ minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_r \\ h_{1,0} & h_{1,1} & \dots & h_{1,n} \\ \dots & \dots & \dots & \dots \\ h_{r,0} & h_{r,1} & \dots & h_{r,n} \end{pmatrix}$$

vanish.

These minors define multihomogeneous polynomials in the multihomogeneous coordinates in $\mathbb{C}[x_{1,0}, \dots, x_{1,n}, \dots, x_{r+1,0}, \dots, x_{r+1,n}]$. Since conversely every point which satisfies the previous equations lies in the total join, by definition, this concludes the proof.

□

Definition 1.3.4. Given a set of varieties $A = \{Y_1, \dots, Y_k\}$ of $\mathbb{P}^n$ we say that A is independent if we can find linearly independent points $P_1, \dots P_r$ such that $P_1 \in Y_1, \dots, P_r \in Y_r$. In particular if $Y_1 = \dots = Y_r = Y$ then $\{Y_1, \dots, Y_r\}$ is independent if and only if Y is not contained in a projective subspace of dimension $r - 1$.

Moreover, if Y is nondegenerate i.e. not contained in any hyperplane, then for every $r \leq n+1$ we can obtain a set of independent varieties simply by taking r copies of Y.

Remark 1.3.5. Let $Y_1, \dots, Y_r$ be varieties of $\mathbb{P}^n$. If there are points $P_1 \in Y_1, \dots, P_r \in Y_r$ which are linearly dependent, then for every $Q \in \mathbb{P}^n$ the point $(P_1, \dots, P_r, Q)$ belongs to the total join.

We want to exclude, whenever it is possible, the case shown in the previous remark. To do this, we have to recall some results.

Proposition 1.3.6. *The product of a finite number of irreducible projective varieties is irreducible.*

Proof. A proof can be found using the same argument of the proof of Theorem 5.8 of [32].

□

An important consequence of this Proposition is the following.

Theorem 1.3.7. *Let $Y_1, \dots, Y_k \subseteq \mathbb{P}^n$ be a set of irreducible varieties. Then, there exists a unique irreducible component Z of the total join $TJ(Y_1, \dots, Y_k)$ such that the restriction to Z of the projection π of $Y_1 \times \dots \times Y_k \times \mathbb{P}^n$ to the first k factors surjects onto $Y_1 \times \dots \times Y_k$.*

Proof. We know from Theorem 1.0.38 that $TJ(Y_1 \times \cdots \times Y_k) = C_1 \cup \cdots \cup C_r$ where each C_i is an irreducible closed set for all $i = 1, \dots, r$. Thus, since the projection $\pi : TJ(Y_1 \times \cdots \times Y_k) \to Y_1 \times \cdots \times Y_k$ is surjective and closed (by Proposition 1.1.14), we have that $Y_1 \times \cdots \times Y_k$ is contained in the union $\pi(C_1) \cup \cdots \cup \pi(C_r)$, where each $\pi(C_r)$ is a closed set. By Proposition 1.3.6 we know that $Y_1 \times \cdots \times Y_k$ is irreducible and so there exists an index i such that $Y_1 \times \cdots \times Y_k = \pi(C_i)$. Notice that C_i maps onto $Y_1 \times \cdots \times Y_k$ in π. We want to prove that such i is unique.

Since by hypothesis the set of varieties $Y_1, \dots, Y_k$ is independent we can find $p_1, \dots, p_k$ linearly independent points of $\mathbb{P}^n$ such that $p_i \in Y_i$ for $i = 1, \dots k$.

By definition of total join we have that the fiber $\pi^{-1}(p_1, \dots, p_k)$ is a subset of $\{p_1, \dots, p_k\} \times \mathbb{P}^n$ which is isomorphic to $p_1, \dots, p_k \times \Lambda$, where Λ is the linear space of dimension $k-1$ spanned by $p_1, \dots, p_k$. So, the space $\pi^{-1}(p_1, \dots, p_k)$ is irreducible since it is the product of some irreducible varieties and as a consequence it is contained in one irreducible component of the total join $TJ(Y_1, \dots, Y_k)$.

Let W_i be the set of k-tuples $(P_1, \dots, P_k) \subset Y_1 \times \cdots \times Y_k$ such that for all $T \in < P_1, \dots, P_k >$ the point $(P_1, \dots, P_k, T)$ belongs to C_i. Each W_i is a subvariety. Indeed we can find a set of equations defining W_i as follows. Since each C_i is irreducible, we can find an irreducible multihomogeneous polynomial f for C_i seen as a subvariety of the space $(\mathbb{P}^n)^k \times \mathbb{P}^n$. Now consider f as a polynomial in the variables of the last factor $\mathbb{P}^n$, with coefficients f_i's which are multihomogeneous polynomials in the coordinates of $(p_1, \dots, p_k) \in (\mathbb{P}^N)^k$. Since $(p_1, \dots, p_k, T) \in C_i$ for all $T \in < p_1, \dots, p_k >$ we have that $(p_1, \dots, p_k)$ annihilates all the polynomials f_i and vicecersa. This provides a set of multihomogeneous equations which defines W_i.

From the fact that each W_i is a subvariety, the result follows. Indeed suppose, by contradiction, that there are several irreducible components of the join, $C_1, \dots, C_m$, which map onto $Y_1 \times \cdots \times Y_k$ under π, and consider the sets $W_1, \dots, W_m$ as above. We know that the fibers $\pi^{-1}(p_1, \dots, p_k)$ belong to some C_i, whenever the points $p_1, \dots, p_k$ are independent. So, the union $\bigcup(C_i)$ contains the subset U of k-tuples $(p_1, \dots, p_k)$ in $Y_1 \times \cdots \times Y_k$ such that the p_i's are linearly independent. We can prove that U is open. In fact the complement of U i.e. the set of k-tuples $(p_1, \dots, p_k)$ such that the p_i's are linearly dependent can be defined by the $k \times k$ minors of the matrix obtained by taking as rows a set of coordinates of the P_i's. The set U is also non-empty since the Y_i's are independent, so U is dense in $Y_1 \times \cdots \times Y_k$, which is irreducible, by Proposition 1.3.6. It follows that $Y_1 \times \cdots \times Y_k$, i.e. the closure of U, is contained in the union $\pi(C_i)$, hence it is contained in some $\pi(C_i)$, say in $\pi(C_1)$. In particular, $\pi(C_1)$ contains an open, non-empty, hence dense, subset of $Y_1 \times \cdots \times Y_k$. Thus $\pi(C_1) = Y_1 \times \cdots \times Y_k$.

Assume that C_2 in another component which satisfies $\pi(C_2) = Y_1 \times \cdots \times Y_k$. We prove that $C_2 \subset C_1$, which contradicts the maximality of irreducible components. Namely $W = (Y_1 \times \cdots \times Y_k) \setminus U$ is closed in the product, so $C_2 \cap (\pi^{-1}(W))$ is closed in C_2 and it is a proper subset, since π restricted to C_2 surjects. Hence $C_2 \setminus (\pi^{-1}(W))$ is dense in C_2, which is irreducible. On the other hand if $(p_1, \dots, p_k, Q) \in C_2 \setminus \pi^{-1}(W)$, then $p_1, \dots, p_k$ are linearly independent, thus $(p_1, \dots, p_k, Q) \in C_1$ because C_1 contains the fiber of π over $(p_1, \dots, p_k)$. It follows that $C_2 \setminus \pi^{-1}(W) \subset C_1$ hence $C_2 \subset C_1$, a contradiction.

□

This Theorem guarantees that the following definition is correct.

Definition 1.3.8. Given an independent set of irreducible varieties $Y_1, \dots, Y_k$ we define the abstract join of the Y_i's as the unique irreducible component $AJ(Y_1, \dots Y_k)$ of the total join $TJ(Y_1, \dots Y_k)$ which maps onto $Y_1, \dots Y_k$ in the natural projection.

The (embedded) join $J(Y_1, \dots, Y_k)$ is the image of $AJ(Y_1, \dots Y, k)$ under the projection of $(\mathbb{P}^n)^k \times \mathbb{P}^n$ to the last copy of $\mathbb{P}^n$.

Definition 1.3.9. If we apply the previous definitions to the case $Y_1 = \cdots = Y_k = Y$ when Y is an irreducible variety of $\mathbb{P}^n$ we call the abstract join the abstract secant variety $AS_k(Y)$ and we call the (embedded) join the secant variety $S_k(Y)$.

Notice that both the secant variety and the abstract secant variety of an irreducible variety Y are irreducible.

Example 1.3.10. Consider the Veronese variety $V = v_{1,d}(\mathbb{P}^1)$. A general point of the abstract secant variety $AS_2(V)$ is of the form (P_0, P_1, Q) where Q belongs to the line spanned by P_0 and P_1. In particular, since P_0 and P_1 belong to the Veronese variety V, they are the image of two points $\overline{P}_0 = [a_0 : a_1]$ and $\overline{P}_1 = [b_0 : b_1]$ in $\mathbb{P}^1$, so they are of the form $P_0 = (a_0x_0 + a_1x_1)^d$ and $P_1 = (b_0x_0 + b_1x_1)^d$. Now, by using the condition of linear dependence, we have that $Q = \alpha_0(a_0x_0 + a_1x_1)^d + \alpha_1(b_0x_0 + b_1x_1)^d$.

So, the (embedded) secant variety $S_2(V)$ contains all the points of $\mathbb{P}^d$ (considered as the projective space of forms of degree d) that can be expressed as combinations of the d-th power of two linear forms.

We can verify that not all the points of $S_2(V)$ can be written as a combination of two linear forms. As an example consider the point of $AS_2(V)$ of the form $(P_0, P_1(t), Q) = (x_0^d, (x_0 + tx_1)^d, Q)$. For t that tends to 0 then $P_1(t)$ tends to P_0 and, since $AS_2(V)$ is closed, then the limit of a family of points still belongs to the abstract secant variety. Consider

$$P_0 - P_1 = (x_0 + tx_1)^d - x_0^d = (d-1)tx^{d-1}y + \cdots + t^d y^d.$$

Since we are working in a projective variety we can divide Q by t. Passing to the limit for t that tends to 0 we have that Q tends to $(d-1)x^{d-1}y$ which cannot be written as a sum of two d-th power of linear forms (see Example 12.1.10 of [15]).

We can estimate the dimension of the abstract join of some varieties as follows.

Proposition 1.3.11. *The abstract join $AJ(Y_1, \dots, Y_k)$ has dimension $(k-1) + \dim(Y_1) \cdots + \dim(Y_k)$. In particular, the abstract secant variety $AS_k(Y)$ of a variety Y of dimension n has dimension $k - 1 + nk$.*

Proof. The fiber of the projection $\pi : AJ(Y_1, \dots, Y_k) \to Y_1 \times \cdots \times Y_k$ over a general point $P = (p_1, \dots, p_n) \in Y_1 \times \cdots \times Y_k$ (such that the p_i are linear independent) has dimension $k - 1$, since $\pi^{-1}(P)$ corresponds to the projective span of $p_1, \dots, p_k$. The conclusion follows directly from Corollary 1.2.12. □

In particular, the previous Proposition provides an easy method to compute the dimension of an abstract secant variety.

We conclude this section by recalling the relation between secant varieties and tangent spaces.

We can give a definition of a tangent space as follows.

Definition 1.3.12. Let $X \subset \mathbb{P}^n$ be the hypersurface defined by the form:

$$f = x_0^{d-1}h_1 + x_0^{d-2}h_2 + \cdots + h_d$$

where each h_i is a homogeneous polynomial of degree i in the unknowns $x_1, \ldots, x_n$. Notice that $P_0 = [1 : 0 \cdots : 0]$ belongs to X. We define the (embedded) tangent space to be the linear subspace $T_X(P_0)$ of $\mathbb{P}^n$ defined by the equation $h_1 = 0$.

Now we extend this definition to a generic variety.

Definition 1.3.13. Let X be any variety of $\mathbb{P}^n$ containing $P_0 = [1 : 0 \cdots : 0]$ and consider the elements f_i of the homogeneous ideal $I(X)$. Each f_i defines an hypersurface $V(f_i)$ containing P_0. We define the (embedded) tangent space to X at P_0 the intersection of the tangent spaces $T_{V(f_i)}(P_0)$ with $f_i \in I(X)$.

Given a variety X of $\mathbb{P}^n$ and a point P in X, then we define the tangent space to X at P as:

$$T_X(P) = \phi^{-1}(T_{\phi(X)}(P_0))$$

where ϕ is the change of coordinates that sends P to P_0.

An important result about tangent spaces is the following.

Theorem 1.3.14. *For every $P \in X$ the following inequality holds.*

$$\dim(T_X(P)) \geq \dim(X).$$

Proof. We can find a general proof of this fact in Corollary 11.15 of [9]. $\square$

Example 1.3.15. Consider the Veronese embedding $v_{n,d}$ of $\mathbb{P}^n$ into $\mathbb{P}^N$ with $N = \binom{n+d}{d} - 1$ and let P be a point of $\mathbb{P}^n$ corresponding to a linear form L. We claim that the tangent space to $V = v_{n,d}(\mathbb{P}^n)$ at P is the ideal generated by

$$L^d, L^{d-1}x_1, \ldots, L^{d-1}x_n.$$

To show this, first we change coordinates sending P to $P_0 = [1 : 0 : \cdots : 0]$. Now P_0 correspond to the linear forms x_0 and it is mapped by $v_{n,d}$ onto the monomial $m^0 = x_0^d$.

As we have seen, the Veronese variety can be defined as the vanishing locus of quadratic equations of the form:

$$m^A m^B - m^C m^D$$

where each m^i is a monomial of multidegree i (see Theorem 1.1.20).

In particular, we can consider the quadratic equations of the form $m^0 m^B - m^C m^D$ where m^0 is the monomial x_0^d. We can notice that these equations are not trivial whenever m^B cannot be divided by x_0^{d-1}.

Thus, to every equation of the form $m^0 m^B - m^C m^D$ in the ideal of the Veronese variety it corresponds an equation of the form $m^i = 0$ for the tangent space to V at P_0, where m^i is a monomial in which the exponent of x_0 is smaller or equal than $d-2$.

So, it follows directly that the ideal $T_X(P_0)$ is contained in the space S generated by the monomial corresponding to $x_0^d, x_0^{d-1}x_1, \dots, x_0^{d-1}x_n$. Since S has dimension $n = \dim(V)$ (see Example 11.2.22 of [15]) and since we know from Theorem 1.3.14 that the dimension of $T_V(P_0)$ has to be at least n, we have that $T_V(P_0)$ is the subspace of the space of polynomials of degree d in $n+1$ variables generated by $x_0^d, x_0^{d-1}x_1, \dots, x_0^{d-1}x_n$. Our claim follows by changing the coordinates again.

Now we can state the following important results due to Terracini. The idea of this Lemma is that given a variety $X \subset \mathbb{P}^n$ and $x_1, \dots, x_r \in X$, if $P \in < x_1, \dots, x_r >$ is a general point, then the dimension of $S_k(X)$ is equal to its expected dimension minus the number of ways in which we can *move* the x_i such that $< x_1, \dots, x_k >$ still contains P.

Lemma 1.3.16 (Terracini)**.** *Let $X \subset \mathbb{P}^N$ be an irreducible, non degenerate projective variety of dimension n and let $k < N$ be a positive integer. Let $p_0, \dots, p_k$ be general points of X and let $P \in < p_0, p_1, \dots, p_k >$ be a general point in $S_k(X)$. Then we have:*

$$T_{S_k(X),P} =< \bigcup_{i=0}^{k} T_{X,p_i} > \tag{1.3}$$

where $T_{X,p}$ is the tangent space to X passing through p.

Proof. We give an idea of the proof.

Consider the map from $\Sigma : AS_k(X) \to \mathbb{P}^N$ defined as $\Sigma(p_1, \dots, p_k, p) = p$. It is clear by the definition of k-secant variety that

$$S_k(X) = V(\Sigma(AS_k(X)))$$

where $V(\Sigma(AS_k(X)))$ is the Zariski closure of $\Sigma(AS_k(X))$. Notice in particular that there exists an open set $U \subset S_k(X)$ such that all the points $P \in S_k(X)$ can be written as $P = a_1 + \cdots + a_k$ with $a_i \in Y_i$. Thus, for a generic point $(p_1, \dots, p_k, P) \in AS_k(X)$ we have that the space $T_{S_k(X),P}$ is given by the linear combination of polynomials obtained by differentiating curves $v_i(t) \subset Y_i$. The result follows.

The original proof of this Lemma can be found in the work of Terracini [47]. □

We will use several times this Lemma to study tensors. Moreover, thanks to this Lemma we can associate to every variety an *expected dimension*, defined as follows.

Definition 1.3.17. Given a variety $X \subset \mathbb{P}^n$ we define the expected dimension of $S_k(X)$ as

$$exdim_k(X) = \min\{n, k\dim(X) + k - 1\}.$$

Notice indeed that the expected dimension is the minimum between the dimension of the space $\mathbb{P}^n$ and the dimension of the abstract secant variety $AS_r(X)$, which by a simple count of parameters, is the maximum dimension that $S_k(X)$ can reach. In particular, we always have:

$$\dim(S_k(X)) \leq \min\{n, k\dim(X) + k - 1\} \tag{1.4}$$

Definition 1.3.18. When the inequality in 1.4 is strict, we say that X is *defective*. In this case we call $\delta_k(X) = \min\{n, k\dim(x) + k - 1\} - \dim(S_k(X))$ the $k-$defect of X.

The $k-$defect of a Veronese variety is described by the following theorem of Alexander and Hirschowitz (see [1]).

Theorem 1.3.19. *Let $v_{n,d}$ be a Veronese variety with $d \geq 2$. Then, we have:*

$$dim(S_k(v_{n,d})) = \min\left\{\binom{n+d}{d} - 1, kn + k - 1\right\}$$

except the following cases:

(1) $d = 2$, $n \geq 2$, $2 \leq k \leq n$, *where* $dim(S_k(v_n, d)) = \min\left\{\binom{n+2}{2} - 1, 2n + 1 - \binom{s}{2}\right\}$;

(2) $d = 3$, $n = 4$, $k = 7$ *where* $\delta_k = 1$*;*

(3) $d = 4$, $n = 2$, $k = 5$ *where* $\delta_k = 1$*;*

(4) $d = 4$, $n = 3$, $k = 9$ *where* $\delta_k = 1$*;*

(5) $d = 4$, $n = 4$, $k = 14$ *where* $\delta_k = 1$.

Proof. See the work of Alexander and Hirschowitz [1]. □

Chapter 2

Tensor Geometry

In this section, we will introduce the study of the problem of the *identifiability for symmetric tensors*. In this section we explain in details the problem, and we recall some classic results and definitions useful for our investigations.

We start this section by defining formally what is a Tensor.

Definition 2.0.1. A *tensor* T over $\mathbb{K}$ of dimension n and type $a_1 \times a_2 \times \cdots \times a_n$ is a multi-linear map (i.e. a map which is linear with respect to all the arguments) of the form:

$$T : \mathbb{K}^{a_1} \times \mathbb{K}^{a_2} \times \cdots \times \mathbb{K}^{a_n} \to \mathbb{K}.$$

Notice also that vectors and matrices are particular cases of tensors. In particular each vector represents a linear map and every matrix represents a bilinear map.

Fix the canonical base for each $\mathbb{K}^{a_j}$, i.e. a base made of vectors e_i^j whose coordinates are all equal to 0 except the i-th one that is equal to 1. We can represent a tensor T as a multidimensional array such that the entry of T corresponding to the multi-index $(i_1, \ldots, i_n)$ is $T(e_{i_1}^1, e_{i_2}^2, \ldots e_{i_n}^n)$.

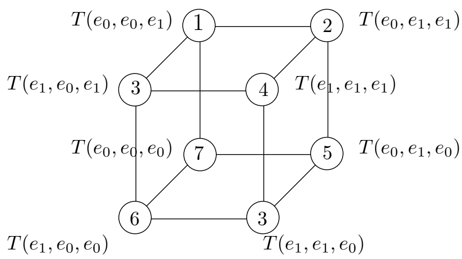

Figure 2.1: An example of a tensor seen as a multidimensional array.

The set of all tensors of fixed dimension n and type $a_1 \times \cdots \times a_n$ is a vector space, the operations being defined over elements with corresponding multi-indices. We indicate this space by $\mathbb{K}^{a_1,\ldots,a_n}$.

One basis for this vector space is obtained by considering all the multidimensional arrays with a 1 in only one place and a zero in every other place. If a unique 1 is in the position $(i_1, \dots, i_n)$, we refer to the basis tensor as $e(i_1, \dots, i_n)$. The space has dimension $a_1 \cdot \dots \cdot a_n$.

We can define also an operation between two tensors of different types, the *tensor product*.

Definition 2.0.2. Given $T \in \mathbb{K}^{a_1,\dots,a_n}$, $S \in \mathbb{K}^{a'_1,\dots,a'_m}$ we define the *tensor product* $T \otimes S$ as the tensor $U \in \mathbb{K}^{a_1,\dots,a_n,a'_1,\dots,a'_m}$ such that:

$$U(v_1, \dots, v_n, s_1, \dots, s_m) = T(v_1, \dots, v_n)S(s_1, \dots s_m).$$

We can extend this definition in order to consider more factors. Given any finite collection of tensors $T_j \in \mathbb{K}^{a^j_1,\dots,a^j_{n_j}}$, $j = 1, \dots, m$, we define their tensor product as the tensor $W = T_1 \otimes \dots \otimes T_m \in \mathbb{K}^{a^1_1,\dots,a^1_{n_1},\dots,a^m_1,\dots,a^m_{n_m}}$ such that

$$W(i^1_1, \dots, i^1_{n_1}, \dots, i^m_1, \dots, i^m_{n_m}) = T_1(i^1_1, \dots, i^1_{n_1}) \cdots T_m(i^m_1, \dots, i^m_{n_m}).$$

Example 2.0.3. Consider the following vectors v_1, v_2 corresponding to the bilinear map $v_1 : \mathbb{K}^3 \to \mathbb{K}$ and $v_2 : \mathbb{K}^4 \to \mathbb{K}$.

$$v_1 = (1, 3, 5) \qquad v_2 = (1, 2, 3, 5).$$

Then the tensor product $v_1 \otimes v_2$ is the matrix:

$$(1, 3, 5) \otimes (1, 2, 3, 5) = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 3 & 6 & 9 & 15 \\ 5 & 10 & 15 & 25 \end{pmatrix} \tag{2.1}$$

Remark 2.0.4. With a direct computation we can show that the following properties hold.

- $T \otimes (U \otimes V) = (T \otimes U) \otimes V$;
- $T \otimes (U + U') = T \otimes U + T \otimes U'$;
- $(T + T') \otimes U = T \otimes U + T' \otimes U$;
- $(\lambda T) \otimes U = T \otimes \lambda U = \lambda(T \otimes U)$.

So, we see that the tensor product is a multi-linear product in its factors.

Using the multilinearity of the tensor product we can prove that the vanishing law holds for the tensor products.

Lemma 2.0.5. *Let $T \in \mathbb{K}^{a_1,\dots,a_n}$, $U \in \mathbb{K}^{b_1,\dots,b_m}$ be tensors. Then $T \otimes U = 0$ if and only if either $T = 0$ or $U = 0$.*

Proof. Suppose (without loss of generality) that $T = 0$. Then, $T(i_1, \dots, i_n) = 0$ for all multi-indices $i_1, \dots, i_n$, so that

$$T \otimes U(i_1, \dots, i_n, j_1, \dots, j_m) = T(i_1, \dots, i_n)U(j_1, \dots, j_m) = 0.$$

Viceversa suppose that $T \neq 0$ and $U \neq 0$. So, there exist two multi-indices $(i_1, \dots, i_n)$ and $(j_1, \dots, j_m)$ such that $T(i_1, \dots, i_n) \neq 0$ and $U(j_1, \dots, j_m) \neq 0$. Thus, we have:

$$T \otimes U(i_1, \dots, i_n, j_1, \dots, j_m) = T(i_1, \dots, i_n)U(j_1, \dots, j_m) \neq 0.$$

$\square$

Now, we are interested in defining an equivalence relation given by the tensor product. It is easy to see that $T \otimes U = T' \otimes U'$ does not imply that $T = T'$ and $U = U'$. As an example, we can consider in the hypothesis of the previous Lemma that $T \otimes 0 = 0 \otimes U = 0$. However, we have the following proposition.

Proposition 2.0.6. *Let $T, T' \in \mathbb{K}^{a_1,\ldots,a_n}$ and $U, U' \in \mathbb{K}^{b_1,\ldots,b_m}$ be tensors. Suppose that $T \otimes U = T' \otimes U' \neq 0$. Then, there exists a coefficient $\alpha \in \mathbb{K}$ such that $T' = \alpha T$ and $U' = \frac{1}{\alpha} U$.*

More in general, given $T_i, U_i \in \mathbb{K}^{a_1^i,\ldots,a_n^i}$ for $i = 1, \ldots, s$ we have that, if

$$T_1 \otimes \cdots \otimes T_s = U_1 \otimes \cdots \otimes U_s \neq 0$$

then there exist coefficients $\alpha_1, \ldots, \alpha_s \in \mathbb{K}$ such that $U_i = \alpha T_i$ and $\alpha_1 \cdot \cdots \cdot \alpha_s = 1$.

Proof. We can prove the claim by induction on the number of factors. Since $T \otimes U = T' \otimes U' \neq 0$ we have that there exists a multi-index $(i_1, \ldots, i_n, j_1, \ldots, j_m)$ such that:

$$T \otimes U(i_1, \ldots, i_n, j_1, \ldots, j_m) = T(i_1, \ldots, i_n)U(j_1, \ldots, j_m) = \\ = T'(i_1, \ldots, i_n)U'(j_1, \ldots, j_m) \neq 0. \quad (2.2)$$

Thus, $U(j_1, \ldots, j_m) \neq 0$ and $U'(j_1, \ldots, j_m) \neq 0$. We call $\alpha = \frac{U(j_1,\ldots,j_m)}{U'(j_1,\ldots,j_m)} \neq 0$ and from 2.2 we have: $T' = \alpha T$. Similarly, if we call $\alpha' = \frac{T'(i_1,\ldots,i_n)}{T(i_1,\ldots,i_n)}$ we have that $U' = \alpha' U$. Using the multi-linearity of the tensor product we have:

$$T' \otimes U' = \alpha T \otimes \alpha' U = \alpha \cdot \alpha' T \otimes U.$$

So, since $T' \otimes U' = T \otimes U$ we have $\alpha \cdot \alpha' = 1$. Thus $\alpha' = \frac{1}{\alpha}$. If we have s factors, $T_1 \otimes \cdots \otimes T_s$ we repeat this proof using the fact that the tensor product is associative and the inductive hypothesis.

□

Remark 2.0.7. In our investigation, we consider only tensors defined over an algebraically closed field e.g. the complex field. This is due to the fact that studying tensors from a geometrical point of view is connected to the study of systems of homogeneous polynomials, so it is easier working on an algebraically closed field.

Now we want to define what is the *rank* of a tensor T, extending the notion of matrix rank. First we will define tensors of rank one. Then, we will use this notion to define the rank of general tensors.

Definition 2.0.8. A non zero tensor $T \in \mathbb{K}^{a_1,\ldots,a_n}$ has *rank 1* if there are vectors $v_i \in \mathbb{K}^{a_i}$ such that $T = v_1 \otimes \cdots \otimes v_n$.

We say that T has *rank r* if r is the minimum integer such that there exist r rank one tensors $T_1 \ldots T_r$ such that

$$T = T_1 + T_2 + \cdots + T_r.$$

We will denote the rank of T by $\text{rank}(T)$.

We call the expression $T_1 + \cdots + T_r$ a decomposition of T of length r.

By convention, we say that null tensors have rank equal to 0.

The notion of decomposition was first introduced by Hitchcock in [36] and it is often called *parafac* or *CP decomposition*, where C and P stands respectively for *Candecomp* and *Parafac*, that are the names given for this decomposition by Caroll and Chang in [16] and Harschman in [33].

In general, it is easy to verify when a given tensor has rank equal to 1.

Theorem 2.0.9. *Let $T \neq 0$ be a tensor of dimension n and type $a_1 \times \cdots \times a_n$. Then T has rank 1 if and only if for every subset $J \subset \{1, 2, \ldots, n\}$ and for every choice of multi-indices $I_1 = (l_1, \ldots, l_n), I_2 = (h_1, \ldots, h_n)$, T satisfies all the equalities of the form*

$$T_{I_1} T_{I_2} = T_{J(I_1,I_2)} T_{J'(I_1,I_2)}$$

where $J'(I_1, I_2) = \{1, 2, \ldots, n\} \setminus J(I_1, I_2)$ and for every $X \subset \{1, 2, \ldots, n\}$ we have that $X(I_1, I_2)$ is the multi-index $(a_1, \ldots a_n)$ where

$$a_i = \begin{cases} l_i \text{ if } i \in J \\ h_i \text{ otherwise.} \end{cases}$$

Proof. See Theorem 6.4.13 of [15]. □

Notice that this procedure is very expensive from a computational point of view, because the number of vanishing that one needs to control is huge.

Example 2.0.10. The tensor in figure 2.2 is a Tensor of rank equal to 1. In fact the determinant of the matrix

$$\begin{pmatrix} T_{I_1} & T_{J(I_1)} \\ T_{J'(I_2)} & T_{I_2} \end{pmatrix}$$

vanishes for every choice of multi-indices $I_1, I_2, J(I_1), J(I_2)$, so the hypotheses of Theorem 2.0.9 are satisfied. Indeed we can write T as the tensor product $(1, 2) \otimes (1, 2) \otimes (1, 1, 1)$.

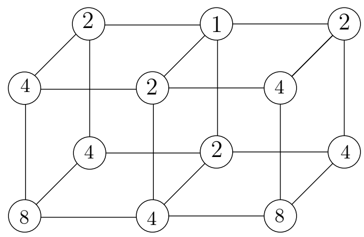

Figure 2.2: An example of a tensor of rank 1.

Despite the fact that it is easy to recognize rank 1 tensors, is not easy to determine when a certain tensor has rank bigger than one. To determine the rank of a tensor, it is convenient to study the relation between Segre maps and tensors.

Remark 2.0.11. There is a strong connection between the Segre embedding and the decomposition of a tensor T. In fact, given $([b_1], \dots, [b_n]) \in \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ and fixed a set of coordinates b_i for each projective point $[b_i]$, it follows directly by the definition of tensor product and Segre embedding that one can make the identification:

$$s_{a_1,\dots,a_n}(b_1, \dots, b_n) = b_1 \otimes \cdots \otimes b_n.$$

Moreover by using the properties of the tensor product described in Remark 2.0.4 it is easy to see that the previous equality holds independently from the choice of coordinates for the points $[b_i]$. Thus all the tensors with rank equal to one belong to the Segre variety.

Moreover, all the tensors of rank k belongs to the k-secant variety of the Segre variety. Indeed recall that $T = T_1 + T_2 + \cdots + T_r$ implies that $T \in < T_1, \dots, T_r >$.

In our work we take into account a specific class of tensors, the class of *(projective) symmetric tensors.*

The choice of the projective space is due to the fact that working in a compact algebraic ambient allows us to use much more powerful tools without losing the information encoded by the tensor T. In particular, we can rephrase the results in a probabilistic language, simply by imposing that the sum of some particular entries of the tensor is equal to one.

Now, we can give a precise definition of the spaces we are working with.

Definition 2.0.12. We say that a tensor $T \in \mathbb{K}^{a_1,\dots,a_d}$ is *cubic* if all the a_i are equal. In particular, if $a_i = n+1$ then T is of type $(n+1) \times (n+1) \times \cdots \times (n+1)$ d times.

We say that a cubic tensor is *symmetric* if for any multi-index $(i_1, \dots i_d)$ and for any permutation σ of the set $\{i_1, \dots, i_d\}$ T satisfies $T_{i_1,\dots,i_d} = T_{i_{\sigma(1)},\dots,i_{\sigma(d)}}$.

We call $Sym^d(\mathbb{K}^{n+1})$ the space of symmetric tensors $T \in \mathbb{K}^{(n+1)\times\cdots\times(n+1)}$ (d-times).

Remark 2.0.13. Notice that in particular the set of symmetric tensors is a linear subspace of $\mathbb{K}^{n+1 \dots n+1}$ defined by a set of linear equations:

$$T_{i_1,\dots,i_d} = T_{i_{\sigma(1)},\dots,i_{\sigma(d)}}.$$

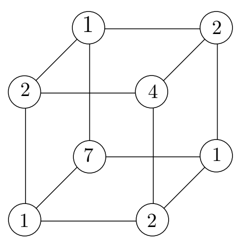

Figure 2.3: An example of symmetric tensor seen as an array.

Let T be the multidimensional array represented in figure 2.3. If we set the indices of T such that $T_{0,0,0} = 7$, $T_{1,1,1} = 4$, $T_{0,1,0} = T_{1,0,0} = T_{0,0,1} = 1$ and $T_{1,1,0} = T_{1,0,1} = T_{0,1,1} = 2$ then T represents also a symmetric tensor.

Remark 2.0.14. It is well known that the space $Sym^d(\mathbb{C}^{n+1})$ can be identified as the space of homogeneous polynomial in $n+1$ variables and degree d.

In fact, given a tensor $T \in \mathbb{K}^{(n+1),\dots,(n+1)}$ (d times) we can identify T with the polynomial F_T defined as:

$$F_T = \sum_{i_1,\dots i_n} T_{i_1,\dots,i_n} x_{i_1} \cdot \dots \cdot x_{i_n}.$$

Viceversa, given an homogeneous polynomial F we can associate to it a tensor $t(F)$ defined as:

$$t(F)(i_1,\dots i_d) = \frac{1}{m(i_1,\dots,i_d)} \cdot \alpha_{(i_1,\dots,i_d)}$$

where $\alpha_{(i_1,\dots,i_d)}$ is the coefficient of the monomial $x_{i_1} \cdot \dots \cdot x_{i_d}$ and $m(i_1,\dots,i_d)$ is the number of different permutations of the multi-index $(i_1,\dots,i_d)$. If some monomial $x_{i_1} \cdot \dots \cdot x_{i_d}$ does not appear in F then we put $t(F)(i_1,\dots,i_d) = 0$.

Example 2.0.15. The symmetric matrix $M = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 6 & 0 \\ 3 & 0 & 15 \end{pmatrix}$ corresponds to the form

$$F = \left(x_0, x_1, x_2\right) \cdot \begin{pmatrix} 1 & 0 & 3 \\ 0 & 6 & 0 \\ 3 & 0 & 15 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = x_0^2 + 6x_1^2 + 15x_2^2 + 6x_0x_2.$$

Example 2.0.16. The tensor in Figure 2.4 corresponds to the form $F = 3x_0^3 + 4x_1^3 + 3x_0^2x_1$.

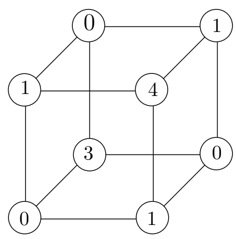

Figure 2.4

It follows directly that the space $Sym^d(\mathbb{C}^{n+1})$ has dimension equal to the number of monomials of degree d in $n+1$ variables, so it is equal to $\binom{n+d}{d}$. As a consequence we have that $\mathbb{P}\left(Sym^d(\mathbb{C}^{n+1})\right)$ has dimension equal to $\binom{n+d}{d} - 1$. This is the space we will work with.

By abuse of notation, we call $\mathbb{P}\left(Sym^d(\mathbb{C}^{n+1})\right)$ both the *space of symmetric tensors*, and the *space of homogeneous forms of degree d in $n+1$ variables*. We can consider $\mathbb{P}\left(Sym^d(\mathbb{C}^{n+1})\right)$ as a projective space $\mathbb{P}^N$ with $N = \binom{n+d}{d} - 1$.

Symmetric tensors of rank 1 can be well described not only using by Theorem 2.0.9 but also using the following Proposition.

Proposition 2.0.17. *Let T be a cubic tensor i.e. a tensor of type $n \times \cdots \times n$ d times. Then T is a symmetric tensor of rank 1 if and only if*

$$T = \lambda(v \otimes \cdots \otimes v)$$

where $\lambda \in \mathbb{K}$ is a non-zero scalar and $v \in \mathbb{K}^n$ is a non-zero vector.

Proof. Let $v = (v_1, \dots, v_n) \neq 0$ be a vector of $\mathbb{P}^n$ and let $T = \lambda(v \otimes \cdots \otimes v)$. By Lemma 2.0.5 we know that $T \neq 0$ thus it has rank equal to 1. Moreover, for any multi-indices $(i_1, \dots, i_n)$ and for any permutation σ of $\{1, 2, \dots, n\}$ we have

$$T_{i_1,\dots,i_n} = v_{i_1} \cdots v_{i_n} = T_{\sigma(i_1),\dots,\sigma(i_n)}$$

thus T is symmetric.

Viceversa let $T = a_1 \otimes \cdots \otimes a_n$ be a symmetric tensors of rank 1. By Lemma 2.0.5 we known that none of the $a_i = [a_{i,1} \dots, a_{i,n}]$ is zero. As a consequence there is a multi-index $(i_1, \dots, i_n)$ such that $T(i_1, \dots, i_n) = a_{1,i_1} \cdots a_{n,i_n} \neq 0$. Define $b_2 = a_{2,i_1}/a_{1,i_1}$. Then we claim that $a_2 = b_2 a_1$. In fact, for all j we have, by symmetry:

$$a_{1,i_1} a_{2,j} a_{3,i_3} \dots a_{n,i_n} = T_{i_1,j,i_3,\dots,i_n} = T_{j,i_1,i_3,\dots,i_n} = a_{1,j} a_{2,i_1} a_{3,i_3} \cdots a_{n,i_n}$$

which means that $a_{1,i_1} a_{2,j} = a_{1,j} a_{2,i_1}$, so that $a_{2,j} = b_2 a_{1,j}$. In the same way, we can define $b_3 = a_{3,i_1}/a_{1,i_1}, \dots, b_d = a_{d,i_1}/a_{1,i_1}$, and obtain that $a_3 = b_3 a_1, \dots, a_d = b_d a_1$. Thus, if we take $\lambda = b_2 \cdot b_3 \dots b_d$, we have

$$T = a_1 \otimes a_2 \otimes \cdots \otimes a_n = a_1 \otimes (b_2 a_1) \otimes \cdots \otimes (b_n a_1) = \lambda(a_1 \otimes a_1 \otimes \cdots \otimes a_1).$$

□

Notice in particular that if we work in an algebraic closed field, in the previous Proposition we can take λ equal to 1.

An important consequence of the previous proposition is the following result that give us a useful identification between symmetric tensors of rank 1 and powers of linear forms.

Corollary 2.0.18. *Let F be an homogeneous polynomial in n variables and let $t(F)$ be the associated cubic tensor of type $n \times \cdots \times n$ (d-times). Then $t(F)$ has rank 1 if and only if there exist a linear polynomial L in n variables such that $F = L^d$.*

Proof. The proof follows immediately by Proposition 2.0.17. For a complete proof see Proposition 7.4.2 of [15].

□

Now, we have two different concepts of decompositions of a symmetric tensor. In fact, given a symmetric tensor T we can be interested in finding a decomposition of T as sum of rank 1 tensors or as sum of *symmetric rank one tensors*. We will focus on the so-called *Waring decomposition* of a symmetric tensor.

Definition 2.0.19. Let $T \in \mathbb{P}\left(Sym^d(\mathbb{C}^{n+1})\right)$ be a form of degree d in $n+1$ variables. We say that T has a *Waring decomposition* of rank r and degree d if we can write T as the sum of r-th powers of linear forms i.e. if we can write T as:

$$T = \sum_{i=0}^{r} h_i^d$$

where each h_i are linear forms in $n+1$ variables.

If r is minimal, then we call it the *(symmetric) rank of* T.

The problem of determining if the symmetric rank of a certain tensor T is equal to the rank of T (defined in Definition 2.0.8) is still an open problem called the *Comon's problem.* Shitov in [45] gives an example of a tensor whose symmetric rank is different from the non symmetric one, but in general the Comon's problem is still not well understood.

We can give a different and more useful definition of the Waring's rank of a symmetric tensor by using the *Veronese map.* As a matter of fact, Corollary 2.0.18 tells us that all the symmetric tensors T of rank 1 and type $n \times \cdots \times n$ (d-times) belong to the Veronese variety $v_{n,d}(\mathbb{P}^n)$.

So, we can reinterpret the definition of a Waring decomposition of a tensor T as follows.

Definition 2.0.20. Let $A \subset \mathbb{P}^n$ be a finite set of cardinality $\ell(A)$. We say that A is a decomposition of the symmetric tensor $T \in \mathbb{P}(Sym^d(\mathbb{C}^{n+1}))$ if $T \in < v_{n,d}(A) >$ where $< v_{n,d}(A) >$ is the linear span of the image of the points of A through the Veronese map defined as in Definition 1.1.17. We call $\ell(A)$ the length of the decomposition.

We say that the decomposition A is non-redundant if for every $B \subset A$, T is not contained in the span of $v_{n,d}(B)$.

Notice in particular that if $v_{n,d}(A)$ is linearly dependent, then A cannot be non-redundant.

Definition 2.0.21. Let T be a symmetric tensor in $\mathbb{P}(Sym^d(\mathbb{C}^{n+1}))$ and let A be a decomposition of T. We say that A is minimal if there exists no $B \subset \mathbb{P}^n$ such that $\ell(B) < \ell(A)$ and $T \in < v_{n,d}(B) >$.

It follows directly from the definition of symmetric rank of a tensor and Remark 1.1.26 that if A is a minimal decomposition of a symmetric tensor T then, the rank of T is equal to $\ell(A)$.

Definition 2.0.22. Let T be a tensor $T \in \mathbb{P}(Sym^d(\mathbb{C}^{n+1}))$ and let A be a decomposition of T such that the rank of T is equal to $\ell(A)$. We say that T is *identifiable* if A is unique (up to scaling and rearranging the elements of A).

We call the *identifiability problem* the problem to determine if a given tensor T is identifiable or not.

Notice that as for the non symmetric case, the study of the identifiability of a symmetric tensor T of rank k is strictly connected to the study of the k-secant variety of the Veronese variety.

2.0.1 Generic identifiability for symmetric tensors

In this section we will recall some well known results about the identifiability problem for symmetric tensors. We start by recalling the state of the art for the generic case i.e. when the symmetric tensor T lies in the complement of a Zariski closed proper subset.

The identifiability of symmetric tensors, both in the generic and non-generic case, is strictly related to the study of defective varieties (see Definition 1.3.18).

As we have seen in the previous chapter, given $X \subset \mathbb{P}^N$ an irreducible non-degenerate projective variety of dimension n and $\{p_0, p_1, \ldots p_k\} \subset X$ a set of independent points we have that the $(k+1)$-*secant* of X is the span $\langle p_0, p_1 \ldots p_k \rangle$ and that $S_{k+1}(X)$ is the Zariski closure of the union of all the $k+1$-secant of X (see Definition 1.3.9). We recall also that $S_{k+1}(X)$ is an irreducible algebraic variety and the dimension of $S_{k+1}(X)$ is bounded by the inequality 1.4 that we re-write below.

$$\dim(S_{k+1}(X)) \leq \min\{(n+1)(k+1) - 1, N\}.$$

In order to study the identifiability of a generic symmetric tensor we need the notions of *border rank* and *generic rank*.

Definition 2.0.23. Let $Y \subset \mathbb{P}^n$ be an irreducible variety such that $< Y >= \mathbb{P}^n$. We say that $T \in \mathbb{P}^N$ has Y-*border rank* k if k is the minimum integer such that $T \in S_k(Y)$.

Notice that if k is the minimum such that $S_k(Y) = \mathbb{P}^n$, then for all $T \in \mathbb{P}^n$ the border rank of T is at most k.

Since in this section we deal with symmetric tensors, in Definition 2.0.23 we will take Y equal to the Veronese variety $v_{n,d}(\mathbb{P}^n)$ and, as a consequence, we will take as $\mathbb{P}^N$ the space of symmetric tensors $\mathbb{P}(Sym^d(\mathbb{C}^{n+1}))$.

Definition 2.0.24. Fix $n, d > 1$. We indicate with $r_{d,n}$ the unique value such that the set of tensors of rank $r_{d,n}$ is dense in $\mathbb{P}(Sym^d(\mathbb{C}^{n+1}))$. We call $r_{d,n}$ the *generic rank*.

Remark 2.0.25. Definition 2.0.23 and Definition 2.0.24 are particularly interesting from our point of view. Indeed, if a tensor $T \in \mathbb{P}(Sym^d(\mathbb{C}^{n+1}))$ has border rank equal to k then we have that $T \in S_k(Y)$ so the rank of T is bigger or equal than the border rank.

If the tensor T is *general* in $S_k(Y)$ we have that the rank of T is equal to its border rank. Indeed, the set of tensors of rank exactly equal to k is open in $AS_k(Y)$, thus a general tensor $T \in S_k(Y)$ has rank $\leq k$ hence it has rank equal to k.

If $r_{d,n}$ is the generic rank, no tensors $T \in \mathbb{P}^N$ can have border rank bigger than r because $T \in \mathbb{P}^N = S_{r_{d,n}}(Y)$. However, there are some known examples of tensors of rank bigger then the generic one. We can find some examples of tensors whose rank is bigger than the generic one in section 3 of [22] and Example 1.3.10 shows a tensor whose rank is bigger than its border rank. The problem of the maximal rank for tensors is still an open problem. Indeed also finding tensors of high rank is a difficult task since they belong to a set of measure zero.

The case of symmetric tensors whose rank is equal or smaller than the generic one has been almost completely studied.

From the definition of Waring rank (Definition 2.0.20), given a symmetric tensor $T \in \mathbb{P}(Sym^d(\mathbb{C}^{n+1}))$ and a decomposition $A \subset \mathbb{P}^n$ of T then T belongs to an r-secant of the Veronese $v_{n,d}(\mathbb{P}^n)$ for some $r \in \mathbb{N}$. In particular, all the symmetric tensors of rank r belongs to $S_r(v_d(\mathbb{P}^n))$. So in order to study the identifiability of a symmetric tensor it is useful to study the defectivity of the corresponding Veronese variety.

Definition 2.0.26. We say that a tensor T is *algebraically* r-identifiable with respect to X if T admits only a finite number of decompositions of length r.

Remark 2.0.27. A general tensor T of rank r is algebraically k-identifiable if and only if the dimension of the abstract secant variety $AS_k(X)$ is equal to the dimension of $S_k(X)$. In fact if X is non defective the generic fiber of the projection $\pi : AS_k(X) \to S_k(X)$ has dimension 0 i.e. is a finite union of points so, there exists only a finite number of decompositions for T.

Since $\dim AS_k(x) = k \dim(X) + (k-1)$, when the last number is bigger than n, then no tensor is algebraically r-identifiable when $k \dim(X) + (k-1) \leq n$. Thus a general tensor of border rank k is algebraically k-identifiable if and only if $\dim S_k(X) = expdim S_k(X)$ i.e. X is not k-defective.

From Theorem 1.3.19 we have that $v_{n,d}(\mathbb{P}^n)$ is non defective, except for few special cases. Thanks to this important result, the problem of the generic algebraic identifiability has been almost completely studied.

For identifiability itself, the case $r < r_{d,n}$ has been completely described by Chiantini, Ottaviani and Vannieuwenhoven in [21].

Theorem 2.0.28. *Fixed $d, r \geq 2$ and $n \in \mathbb{N}$ we have that the general symmetric tensor T in $\mathbb{P}(Sym^d(\mathbb{C}^{n+1}))$ of subgeneric rank $r < r_{r,d}$ is identifiable, unless it is one of the following cases:*

- *$d = 2$;*
- *$d = 6$, $n = 2$, and $r = 9$;*
- *$d = 4$, $n = 3$, and $r = 8$;*
- *$d = 3$, $n = 5$, and $r = 9$.*

In the first case there are infinitely many decompositions for T, in the other cases T has exactly two decompositions.

Proof. See Theorem 1.1. of [21]. □

The identifiability for the case in which the rank r is exactly equal to the generic rank has been completed studied by Galuppi and Mella in [30].

Theorem 2.0.29. *Let T be a generic symmetric tensor in $\mathbb{P}(Sym^d(\mathbb{C}^{n+1}))$ and let $r = r_{d,n}$. Then we have that T is identifiable if and only if:*

- *$n = 1$, $d = 2s - 1$ and $k = s$ with $s \in \mathbb{N}$;*
- *$n = 3$, $d = 3$ and $k = 5$;*
- *$n = 2$, $d = 5$ and $k = 7$.*

2.0.2 Specific identifiability

In this section we will deal with the problem of specific (non-generic) identifiability. We fix a tensor T and a minimal decomposition A and we want to know whether or not A is unique (up to scalar multiplication).

The main results that give us a criterion for the identifiability of tensors is the Kruskal Theorem (see [38]). In order to state correctly this result, we need some preliminary definition.

Definition 2.0.30. Let $A \subset \mathbb{P}^n$ be a finite set of points. We call the *Kruskal rank* of A the maximum integer k_A such that any subset $B \subset A$ of length $\ell(B) \leq k_A$ is linearly independent.

Remark 2.0.31. The Kruskal rank is maximal if $k_A = \min\{n+1, \ell(A)\}$. In this case we say that A is in *linear general position*. Thus $A \subset \mathbb{P}^n$ is in linear general position, if and only if any subset of A of cardinality at most $n+1$ is linearly independent. In this case, for any subset $B \subset A$ one has $k(B) = \min\{n+1, \ell(B)\}$.

More in general for any subset $B \subset A$, we have $k(B) \geq \min\{\ell(B), k(A)\}$

Example 2.0.32. The Kruskal rank of three points in $\mathbb{P}^n$, $n > 1$, is maximal if and only if they are not aligned.

Example 2.0.33. Consider the points $P_0 = [1:0:2:0:0]$, $P_1 = [0:1:3:0:0]$ $P_2 = [0:1:1:1:0]$ and $P_3 = [1:2:6:1:0]$. Then $A = \{P_0, P_1, P_2, P_3\}$ has Kruskal rank equal to 2. Notice in particular that to find the Kruskal rank it is necessary to find the rank of sub-matrices obtained from

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 1 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

by removing some columns. So, we can say that A has Kruskal rank equal to 3 if every sub-matrix of M obtained by removing a single column has rank equal to 3.

We can generalize the notion of Kruskal rank using Veronese maps.

Definition 2.0.34. Let $Z \subset \mathbb{P}^n$ be a finite set of points. We define the *d-th Kruskal rank* $k_d(A)$ of A as the Kruskal rank of $v_{n,d}(Z)$.

Remark 2.0.35. Let $Z \subset \mathbb{P}^n$ be a finite set of points. The d-th Kruskal rank $k_d(Z)$ is bounded above by $\min\{\ell(Z), \binom{n+d}{n}\}$. Moreover for any subset $B \subset Z$, we have $k_d(B) \geq \min\{\ell(B), k_d(Z)\}$.

Notice also that the Kruskal rank k_Z coincides with the first Kruskal rank $k_1(Z)$.

Remark 2.0.36. Let $A \subset \mathbb{P}^n$ be a finite set of points. In analogy to what we have seen in Example 2.0.33 we can notice that in order to find the Kruskal rank $k_d(A)$ it is necessary to find the dimension of all the subspaces spanned by $v_d(B)$ for all $B \subseteq A$.

Remark 2.0.37. We point out also that since projective spaces are irreducible, if A is sufficiently general then all the Kruskal ranks $k_d(Z)$ are maximal.

Example 2.0.38. Let $A = \{P_0, P_1, P_2, P_3\}$ with $P_0 = [1:0:0]$, $P_1 = [1:5:0]$, $P_2 = [1:2:2]$ and $P_3 = [3:4:4]$. Then we have that $k_1(A) = 2$ and $k_2(A) = 3$. Indeed let M be the matrix which columns are the coordinates of the points of A i.e.

$$M = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 5 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{pmatrix}.$$

$k_1(A)$ cannot be equal to 3 since the sub-matrix obtained from M removing the second column has rank equal to 2. Moreover all the sub-matrices obtained from M removing two columns have rank equal to 2 thus, $k_1(A) = 2$.

Now, let M' be the matrix which columns are the coordinates of the points of $v_{2,2}(A)$ i.e.

$$M' = \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 10 & 4 & 24 \\ 0 & 0 & 4 & 24 \\ 0 & 25 & 4 & 16 \\ 0 & 0 & 8 & 32 \\ 0 & 0 & 4 & 16 \end{pmatrix}.$$

M' has rank equal to 4 so $k_2(A) = 4$.

Now we can state the Kruskal criterion. This Theorem works for all kind of tensor, but we state the criterion only for the symmetric case.

Theorem 2.0.39. *[Kruskal, 1977]*
Let $T \in \mathbb{P}^m$ be a symmetric tensor of degree d in $n+1$ variables and let $A = \{P_1, P_2, \dots, P_r\}$ a decomposition of T. Let k_1 be the Kruskal rank of T. If $dk_1 - d + 1 \geq 2r$ then T is identifiable and it has rank equal to r.

Proof. The original proof of this Theorem can be found in [38]. □

In other words T is identifiable when its rank r is less or equal than $\frac{d \cdot k_1 - d + 1}{2}$. In particular, if a decomposition A of T of length $r > n+1$ is in linear general position, that is if the Kruskal rank of A is equal to $n+1$, we get the identifiability as soon as

$$r \leq \frac{dn+1}{2}.$$

In the same way if the Kruskal rank of A is equal to $k+1$ we get the identifiability as soon as

$$r \leq \frac{dk+1}{2}.$$

We point out that Kruskal Theorem gives us a method to determine the identifiability of a tensor which is *effective*. This means that, given a decomposition $A = \{P_1, P_2, \dots, P_r\}$ of a symmetric tensor T i.e. $T = \alpha_1 P_1 + \dots + \alpha_r P_r$, the Kruskal criterion certifies the uniqueness of the decomposition for every choice of the scalars α_i (see the work of Chiantini Ottaviani Vannieuwenhoven [19]).

Nevertheless, Kruskal theorem can be expensive from the point of view of computational costs, especially when the decomposition A is not in general position.

An important technique which is largely used to prove the identifiability of a given symmetric tensor is the *Reshaped Kruskal criterion.* This theorem, which is proved [20], is indeed the original Kruskal criterion applied to Veronese varieties instead of Segre varieties.

Theorem 2.0.40 (Reshaped Kruskal criterion)**.** *Let T be a form of degree d and let A be a non-redundant decomposition of T with $\ell(A) = r$. Fix a partition a, b, c of d and call k_a, k_b, k_c the Kruskal ranks of $v_a(A), v_b(A), v_c(A)$ respectively. If:*

$$r \leq \frac{k_a + k_b + k_c - 2}{2},$$

then T has rank r and it is identifiable.

Proof. See Section 4 of [20]. □

Reshaped Kruskal criterion is a useful way to determine if a given tensor is identifiable or not. Nevertheless, this criterion can be very expensive from a computational point of view.

Remark 2.0.41. Derksen proved in [27] that the Kruskal Theorem 2.0.39 is sharp i.e. if one only knows the first Kruskal rank k_1 and ignores the higher Kruskal ranks, then it is not possible to enlarge the bound given by the inequality in 2.0.39. With the use of the higher Kruskal ranks k_i the original Kruskal criterion is considerably refined. Nevertheless, as we will see in Chapter 5 it is not true that the Reshaped Kruskal criterion is sharp.

Chapter 3

Hilbert Functions

A powerful instrument used in Algebraic Geometry to study varieties and in particular finite sets is the *Hilbert function*. In order to define such a function we need some basic definitions.

From now on we indicate the graded ring $\mathbb{K}[x_0, x_1, \ldots, x_n]$ with R.

A concept that will be necessary in order to define the Hilbert function is the notion of *graded module*. First we recall what is a graded ring.

Definition 3.0.1. A *graded ring* is a ring A together with a family $(A_n)_{n\geq 0}$ of subgroups of the additive group of A such that $A = \bigoplus_{n=0}^{\infty} A_n$ and $A_m A_n \subseteq A_{m+n}$.

Example 3.0.2. The ring $\mathbb{K}[x_0, x_1, \ldots, x_n]$ is a graded ring if we define A_n as the vector space of all homogeneous polynomials of degree n.

Definition 3.0.3. Let A be a graded ring. A *graded* A*-module* is an A-module $M = (M^+, \mu)$ together with a family M_t with $t \in \mathbb{Z}$ of subgroups of M^+ such that:

- $M = \bigoplus_{t\in Z} M_t$.
- $A_m M_t \subseteq M_{m+t}$ for all $m, t \in \mathbb{Z}$.

Given an element x of graded A-module M we say that x is *homogeneous (of degree n)* if there exist an $n \in \mathbb{Z}^+$ such that $x \in M_n$. In particular all the elements of the module can be written uniquely as a sum of homogeneous elements.

Example 3.0.4. R is a graded R-module where each subgroup R_n is the set of homogeneous polynomials of degree n. In this case homogeneous elements coincide with homogeneous polynomials.

Example 3.0.5. A particular example of graded module is what we call a *free module*. We say that a module M is *free* if M is the direct sum of copies of $R = \mathbb{K}[x_0, \ldots, x_n]$ i.e. $M = R \oplus \cdots \oplus R$ r times with $r \geq 1$. For $r = 0$, we set $R^0 = (0)$.

Notice that, since R is a graded R-module, every free R-module R^n is indeed a graded module.

Example 3.0.6. Let I be an homogeneous ideal of R (See Definition 1.0.18). Then we have that I and R/I are graded R-modules.

Notice in particular that the R-module R^n is finitely generated i.e. there exists a finite set of elements of R^n which generate the whole module. For example R^3 is generated by $(1,0,0),(0,1,0),(0,0,1)$. Since we are working with a Noetherian ring, also I and R/I are finitely generated.

Remark 3.0.7. Given an homogeneous ideal I of R and fixed $d \geq 0$ we have that $(R/I)_d = R_d/I_d$. Moreover, R_d, I_d and $(R/I)_d$ can be seen as linear spaces over the field $\mathbb{K}$.

Remark 3.0.8. Let M be an R-module finitely generated. We can notice that, fixed $t \in \mathbb{Z}$, M_t can be seen as a $\mathbb{K}$-vector space. The proof is straightforward and follows directly by the definition of homogeneous elements.

We recall that given M, N two A-modules, a map $f : M \to N$ is an *A-module homomorphism* if:

- $f(x+y) = f(x) + f(y)$ for all $x + y \in M$;
- $af(x) = f(ax)$ for all $x \in M$ and $a \in A$.

Thus, f is a homomorphism of abelian groups which commutes with the multiplication by elements $a \in A$.

Definition 3.0.9. Let $M = \bigoplus_{t \in Z} M_t$ and $N = \bigoplus_{t \in Z} N_t$ be two graded A-modules and let f be an homomorphism of A-modules, $f : M \to N$. We say that f is a *graded morphism* of A-modules if and only if f preserves the degree i.e. $f(M_n) \subseteq N_n$ for all n.

Example 3.0.10. Let M be a graded A-module. Let $f : M \to N$ be the A-module homomorphism defined as: $f(x) = 2x$. It is easy to see that f is a graded morphism of A-modules

Example 3.0.11. Let M be a graded R-module. Fixed a homogeneous element $g \in R$ of degree $d \geq 1$ define $f : M \to M$ as $f(x) = g \cdot x$.

Then f is not a graded morphism of R-modules. In fact from the definition of graded R-module we have that if x is an homogeneous element of degree m i.e. $x \in M_m$ then $g \cdot x \in R_{m+n}$.

A good reason why graded morphisms are important follows from the property below.

Lemma 3.0.12. *Let M, N be two graded A-module and let $f : M \to M'$ be a graded A-module homomorphism. Then* $\ker(f)$ *and* $\operatorname{im}(f)$ *are graded A-modules.*

Proof. The proof is standard and follows directly from the definitions of graded homomorphism and graded module.

□

There is a trick that allows us to consider also the maps described in Example 3.0.11 as a graded morphisms. To do that we can change formally the grading of the domain of our maps in order to make our definition work. We formalize this fact as follows.

Definition 3.0.13. Given a graded A-module M with a decomposition $M = \bigoplus_{d\in\mathbb{Z}} M_d$ we define the *twisted module*, $M(l)$ by setting $M(l)_t = M_{t+l}$. Notice that $M(l)$ is still a well defined graded A-module for all $l \in \mathbb{Z}$.

Example 3.0.14. In the hypothesis of Example 3.0.11 if we consider f as a map $f : M(-d) \to M$, then it becomes a graded morphisms of A-modules. In fact now, if we take any element $x \in M(-d)_n$ we have that $x \in M_{n-d}$ and as a consequence $g \cdot x \in M_n$. So, $f(M(-d)_n) \subset M_n$.

Now it is possible to define the Hilbert function of a graded R-module.

Definition 3.0.15. Let M be a graded R-module finitely generated by elements of positive degree, and let M_t with $t \in \mathbb{Z}$ the family associated to M. The *Hilbert function* of M, denoted with $h_M(t)$ is the function:

$$h_M : \mathbb{Z} \to \mathbb{Z}^{\geq 0}$$

such that $h_M(t) = \dim_{\mathbb{K}} M_t$ where the $\dim_{\mathbb{K}} M_t$ is the dimension of M_t as a $\mathbb{K}$ vector space. We notice indeed that since M is finitely generated then M_t has finite dimension for all $t \in \mathbb{Z}$.

In particular we are interested in studying the Hilbert function of the so called *standard graded algebras*.

Definition 3.0.16. Suppose that A is a ring. R' is a *standard graded A algebra* if $R'_0 = A$ and R' is generated by the elements of R'_1.

Example 3.0.17. The ring of polynomials $R = \mathbb{K}[x_0, \dots, x_r]$ is a standard $\mathbb{K}$ algebra. Indeed R_0 are the constants and all the polynomials of R are generated by products and sums of linear polynomials.

The Hilbert function of a graded module is in general non-trivial to compute, so we would like to give a better interpretation of this map in order to use it more easily. A specific case in which it is easy to compute the Hilbert function is illustrated in the following example.

Example 3.0.18. We recall that an R-module M is *free* if M is the direct sum of copies of R (see Example 3.0.5). When $M = R^r$ we claim that

$$H_M(d) = r \cdot dim_{\mathbb{K}} R_d = r \cdot \binom{n+d}{d}$$

if $d \geq 0$ and $H_M(d) = 0$ otherwise.

To prove this is sufficient to recall that monomials of degree d in $n+1$ variables forms a base for each space R_t, and they are exactly $\binom{n+d}{d}$.

Example 3.0.19. Let $M(-\ell)$ be a free graded module whose degree are shifted by $-l$. Then using the same arguments used in the previous example we have

$$H_{M(-\ell)}(d) = r \cdot \binom{n+d-l}{d-l}$$

if $d - l \geq 0$ and $H_{M(-\ell)}(d) = 0$ otherwise.

Since it is easy compute the Hilbert function of free modules, the idea is to reduce the complexity of computing the Hilbert function of a certain module M by finding a *free resolution* of M.

Definition 3.0.20. We say that a sequence of A-modules and A-homomorphisms of the form

$$\cdots \to M_{i-1} \xrightarrow{\phi_i} M_i \xrightarrow{\phi_{i+1}} M_{i+1} \cdots \to \ldots$$

is *exact* if for every i we have that $Im(f_i) = \ker(f_{i+1})$.

Proposition 3.0.21. *Given an exact sequence of graded A-modules*

$$\cdots \to M_{i-1} \xrightarrow{\phi_i} M_i \xrightarrow{\phi_{i+1}} M_{i+1} \cdots \to \ldots$$

we have that the sequence of vector spaces

$$\cdots \to (M_{i-1})_{(t)} \xrightarrow{f_i} (M_i)_{(t)} \xrightarrow{f_{i+1}} (M_{i+1})_{(t)} \cdots \to \ldots$$

where each f_i is the restriction of ϕ_i to $(M_i)_t$ is exact for all $t \in \mathbb{Z}$.

Proof. The proof is straightforward. The fact that each f_i is a linear map between vector spaces follows directly from the definition of graded morphism. For the same reason we have that $\ker(f_i) = \ker(\phi)_i \cap (M_{i-1})_t$ and $\mathrm{im}(f_i) = \mathrm{im}(\phi)_i \cap (M_i)_t$. Moreover by definition of exact sequence we have $\mathrm{im}(\phi_i) = \ker \phi(i+1)$. Thus, since ϕ_i and ϕ_{i+1} are both graded morphisms we have that $(\ker \phi_{i+1})_t = (\mathrm{im}(\phi_i))_t \subseteq (M_i)_t$ and this concludes the proof.

□

Definition 3.0.22. A *free resolution* of a graded module M is an exact sequence of the form:

$$0 \to F_n \xrightarrow{\phi_n} \cdots \to F_i \xrightarrow{\phi_i} F_{i-1} \xrightarrow{\phi_{i-1}} \ldots \xrightarrow{\phi_1} F_0 \to M \to 0 \tag{3.1}$$

such that for all i, F_i is a free graded module and ϕ_i is a morphism of graded modules. We call the number n the *length of the resolution* i.e. the number of the graded modules F_i minus 1.

Definition 3.0.23. Let X be a class of A-modules and let ϕ a function from X to $\mathbb{Z}$. We say that ϕ is *additive* if for every short exact sequence

$$0 \to A \to B \to C \to 0$$

with $A, B, C \in X$ we have that $\phi(A) - \phi(B) + \phi(C) = 0$.

Example 3.0.24. Let X be the class of finitely generated vector spaces and let A, B, C three finite dimensional vector spaces. Then, we have that the dimension is additive i.e. if there exists the following exact sequence:

$$0 \to A \to B \to C \to 0$$

we have that $\dim(C) - \dim(B) + \dim(A) = 0$. In particular we can compute the dimension of C knowing only the dimensions of A and B.

An important result that we will use several times is the following.

Proposition 3.0.25. *Let X a family of finitely generated modules, ϕ an additive function on this families and let $0 \to M_n \to M_{n-1} \to \cdots \to M_0 \to 0$ be an exact sequence. Then we have:*

$$\sum_{i=0}^{n} \phi(M_i) = 0$$

Proof. See Proposition 2.11 of [9]. It is a consequence of the fact that the sequence

$$0 \to M_0 \to \cdots \to M_{i+1} \xrightarrow{\phi_{i+1}} M_i \xrightarrow{\phi_i} M_{i-1} \xrightarrow{\phi_{i-1}} \cdots \to M_0 \to 0$$

splits in a series of short exact sequences of the form

$$0 \to \ker \phi_i = \operatorname{im} \phi_{i+1} \to M_i \to \operatorname{im} \phi_i = \ker \phi_{i-1}.$$

□

Notice in particular that if the exact sequence 3.1 is finite, then since the dimension is additive we can compute the Hilbert function of M by computing the Hilbert function of all the free modules F_i.

Definition 3.0.26. Let M be a graded finitely generated R-module and let

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

be a free resolution of M. We call the modules F_i the *syzygy modules* of M.

The fact that a free resolution of a module ends after a finite number of steps is not trivial. We state below the theorem due to Hilbert that guarantee us the finiteness of this kind of sequences.

Theorem 3.0.27 (Hilbert Syzygy Theorem)**.** *Fix $R = \mathbb{K}[x_0, \ldots, x_n]$. We have that every finitely generated R-module M has a finite graded free resolution of length smaller or equal to $n+1$ by finitely generated free modules.*

Proof. See Theorem 1.13 of [29]. □

The Hilbert Syzygy Theorem guarantees that we can find a free resolution of the module R/I, so now the problem is how to determine it.

We give an idea of a possible algorithm that constructs a free resolution of a finitely generated graded modules M in the case in which M is generated by homogeneous elements. For example M can be an homogeneous ideal defining a variety.

Remark 3.0.28. Let M be a graded finitely generated R-modules and let $\{m_0, \ldots, m_k\}$ be a set of generators of M such that m_i has degree a_i.

Step 1 - We define a map ϕ_0 from the graded free module $F_0 = \bigoplus_i R(-a_i)$ to M such that $\phi(f) = \phi(f_0, \ldots, f_k) = f_0 m_0 + f_1 m_1 + \ldots f_k m_k$. Notice that this map is surjective.

Step 2 - Now we consider $M_1 = Ker(\phi_0)$. By the Hilbert basis theorem M_1 is a finitely generated R-module. We chose a finite sets of syzygies that generate M_1 and we define a map $\phi_1 : F_1 \to F_2$ as in point 1. Notice that the image of ϕ_1 is equal to the kernel of ϕ_0.

Step 3 - We repeat Step 2 until ϕ_i becomes injective. Notice that Theorem 3.0.27 guarantee us that this will happen after at least $n-1$ iteration of this step.

Indeed, step 3 is effective. In fact we can compute a set of generators by using the *Buchberger's algorithm* to finding a Gröbner basis for M_1. More informations about the Buchberger's algorithm can be found in Chapter 2 of [23].

Example 3.0.29. Let $I = I(x_1x_3 - x_2^2, x_1x_2 - x_0x_3, x_0x_2 - x_1^2)$ be the ideal defining the twisted cubic curve. Suppose we want to find a free resolution of I. In this case we have that $F_0 = R^3(-2)$ and ϕ_0 is the map which sends every point $(a_0, a_1, a_2) \in R^3(-2)$ to the row-column product $(a_0, a_1, a_2) \cdot (x_1x_3 - x_2^2, x_1x_2 - x_0x_3, x_0x_2 - x_1^2)^t$.

Moreover, we can verify that the kernel of ϕ_0 is generated by the elements $m'_0 = (x_0, x_1, x_2)$ and $m'_1 = (x_1, x_2, x_3)$. So F_1 is equal to $R^2(-3)$ and the map ϕ_1 is injective, so the free resolution is:

$$0 \to R^2(-3) \to R^3(-2) \to I \to 0.$$

Remark 3.0.30. The technique introduced in 3.0.28 helps us also to compute the Hilbert function of an R-module of the form R/I where I is an homogeneous ideal of I. In fact, by definition, we have that the Hilbert function of R/I is equal to:

$$h_{R/I}(d) = dim_{\mathbb{K}}(R/I)_d.$$

Consider the exact sequence of graded R-modules:

$$0 \to I_d \to R_d \to R/I_d \to 0.$$

Since the dimension is an additive function and since the sequence is exact we have that:

$$h_{R/I}(d) = \dim(R/I)_d = \dim R_d - \dim I_d. \tag{3.2}$$

In particular given a free resolution of the ideal I

$$0 \to M_1 \to \cdots \to M_k \to I \to 0$$

where each M_i is of the form $M_i = \oplus_j R^{n_j}(-a_j)$ with a_j, n_j positive integers and $k \leq n$ we have that:

$$h_{R/I}(d) = \dim(R/I)_d = \dim R_d - \sum_{j=0}^{k} (-1)^j \dim \left(M_j\right)_d. \tag{3.3}$$

As we have seen in Example 3.0.19 the dimension of each $(M_i)_d$ is easy to compute and it is equal to:

$$\dim(M_j)_d = \sum_j n_j q_j \tag{3.4}$$

where $q_j = \binom{n+d-a_j}{d-a_j}$ if $d - a_j \geq 0$ and $q_j = 0$ otherwise.

If we consider a degree d sufficiently large, all the summands in 3.3 will be different from 0 so we will obtain a sum of binomials $\binom{n+d-a_j}{d-a_j}$. These binomials are polynomials in d. So from d sufficiently large, we have that the Hilbert function of R/I, $h_{R/I}$ is indeed a polynomial. This fact is true also in general and it is described in the following theorem due to Hilbert and Serre.

Theorem 3.0.31 (Hilbert-Serre)**.** *Let M be a finitely generated R-module then, there is an unique polynomial $P_M(d) \in \mathbb{Q}[d]$ such that $P_m(d) = h_M(d)$ for $d >> 0$. We call such a polynomial the* Hilbert polynomial *of M.*

Proof. See Theorem 7.5 of chapter one of [34]. □

Example 3.0.32. The Hilbert function of the module R/I where I is the ideal generating the twisted cubic defined as in 3.0.29 is:

$$h_{R/I}(d) = \binom{d+3}{3} - 3\binom{d+1}{3} + 2\binom{d}{3}.$$

In this case the Hilbert polynomial is equal to $P_{R/I}(d) = 3d + 1$.

Remark 3.0.33. We will use the Hilbert function applied to a module of the form R/I in order to obtain informations about the variety $V(I)$. If I_V is the ideal associated to a variety V, we will write h_V instead of h_{R/I_V} and we will call the Hilbert function of R/I as the Hilbert function of V.

Definition 3.0.34. We define the *first difference of Hilbert function Dh_Z* of Z as:

$$Dh_M(j) = h_M(j) - h_M(j-1), \quad j \in \mathbb{Z}.$$

In the same way we can define $D^d h_M$ as

$$D^d h_M(j) = D^{d-1} h_M(j) - D^{d-1} h_M(j-1), \quad j \in \mathbb{Z}.$$

There is another way to define the Hilbert function of the ideal I of a certain finite variety V i.e. when V is a finite set of points. The idea is to compute directly the dimension of $(R/I)_d$ by computing the rank of the maps which evaluates a homogeneous polynomial on the points of V. We can formalize this idea in this way. We recall that the polynomials of degree d can be identified with the space $Sym^d(\mathbb{C}^{n+1})$ of symmetric tensors of degree d (see the previous chapter).

Definition 3.0.35. Let $Y \subseteq \mathbb{C}^{n+1}$ be a finite set of cardinality $\ell(Y) = k$ of vectors where we fix a certain ordering. Fix an integer $d \in \mathbb{N}$. The *evaluation map of degree d on Y* is the linear map

$$ev_Y(d) : R_d \to \mathbb{C}^k$$

which sends $F \in R_d$ to the evaluation of F at the vectors of Y.

Proposition 3.0.36. *Let $Z \subset \mathbb{P}^n$ be a finite projective variety and let $Y \subset \mathbb{C}^{n+1}$ be any ordered set obtained by choosing a set of homogeneous coordinates for the points of Z. The map $h'_Y : \mathbb{Z} \to \mathbb{N}$ such that :*

$$h'_Y(d) = \mathrm{rank}(ev_Y(d))$$

is equal to the Hilbert function *$h_Z(d)$ of $Z \subset \mathbb{P}^n$ for all $d \in \mathbb{N}$.*

Proof. To prove the equality is sufficient to notice that the rank of the evaluation map is equal to the dimension of $(\mathbb{C}[x_0, \dots, x_n])_d / \ker(ev)$ and that the kernel of the evaluation map is indeed equal to the homogeneous ideal I defining Z. $\square$

Remark 3.0.37. Let $V \subset \mathbb{P}^n$ be a finite projective variety and let Z be any ordered set obtained by choosing a set of homogeneous coordinates for the points of V. A direct consequence of Proposition 3.0.36 is that the rank of the evaluation map of a certain degree on Z does not depends neither on the ordering chosen for Z nor on the choice of coordinates for the points of V.

One can ask when a certain sequence of integers is the Hilbert function of an algebra. We can answer to this question by giving a relation between sequences of integers and the Hilbert function of standard graded algebras.

In order to show which are the sequences corresponding to Hilbert functions of graded algebras we have to recall a way to express a natural number in terms of binomial coefficients.

Lemma 3.0.38. *Let d be a positive integers and let $a \in \mathbb{N}^+$. Then a can be written uniquely in the form:*

$$a = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_m}{m}. \tag{3.5}$$

where $a_d > a_{d-1} > \cdots > a_m \geq m \geq 1$.

Proof. We can prove the existence of such an expression by induction on a.

For every d we have that $1 = \binom{d}{d}$.

Fix $a > 1$. We take a_d such that $\binom{a_d}{d}$ is the largest values smaller than a. Then, consider $a - \binom{a_d}{d}$. By inductive hypothesis there is a $(d-1)$-binomial expansion of $a - \binom{a_d}{d}$.

To prove the unicity of this expression we can use another easy inductive strategy.

In particular we claim that if we can write a in the form

$$a = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_m}{m}$$

with $a_d > a_{d-1} > \cdots > a_m$, then a_d is the largest integer with $\binom{a_d}{d} < a$.

For $a = 1$ the claim is trivial. Suppose $a > 1$ and suppose by contradiction that $\binom{a_d+1}{d} \leq a$. Then we have

$$\sum_{i=m}^{d-1} \binom{a_i}{i} \geq \binom{a_d+1}{d} - \binom{a_d}{d} = \binom{a_d}{d-1} \geq \binom{a_{d-1}+1}{d-1}$$

a contradiction with standard formulas on binomials. $\square$

Definition 3.0.39. Given the expression:

$$a = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_m}{m}$$

we define

$$a^{\langle d \rangle} = \binom{a_d+1}{d+1} + \binom{a_{d-1}+1}{d} + \cdots + \binom{a_m+1}{m+1}.$$

Macaulay proved in [39] the following theorem.

Theorem 3.0.40. *Let $\mathbb{K}$ be a field and $h : \mathbb{N} \to \mathbb{N}$ a numerical function. The following are equivalent:*

1) there exist an homogeneous ideal I such that the graded $k-$algebra R/I has the Hilbert function $h_{R/I}(t)$ equal to $h(t)$;

2) $h(0) = 1$, $h(n+1) \leq h(n)^{\langle d \rangle}$ for all $n \in \mathbb{N}$.

A sequence satisfying (2) is called an $O-$sequence.

Example 3.0.41. Consider the sequence $h = (1, 3, 6, 8, 10, 11, 11, \dots)$. We can notice that h is an O-sequence. Indeed we have that the first element of h is 1 and moreover we have that $3 = \binom{3}{1} \leq \binom{4}{2} = 6$, $6 = \binom{4}{2} \leq 8 \leq \binom{5}{3}$, $8 = \binom{4}{3} + \binom{3}{2} + \binom{1}{1} \leq 10 = \binom{5}{4} + \binom{4}{3} + \binom{2}{2}$ and finally $10 = \binom{5}{4} + \binom{4}{3} + \binom{2}{2} \leq 11 \leq \binom{6}{5} + \binom{5}{4} + \binom{3}{3} = 12$.

From Theorem 3.0.40 we know that h represents the Hilbert function of a finite set of points V. In particular, the first difference of the Hilbert function of V is equal to $Dh_V = (1, 2, 3, 2, 2, 1, 0, 0, \dots)$.

In Figure 3.1 is represented the graph of Dh_V. It is easy to verify that Figure 3.1 represents also a possible set of points whose Hilbert function is h.

Notice in particular that there are 10 points contained in a conic given by the product of the lines s and t.

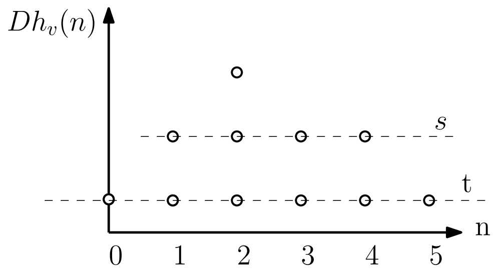

Figure 3.1

Since from now on we will focus on the Hilbert function of a finite set of points, we introduce some additional notation.

Definition 3.0.42. Given $Z \subset \mathbb{P}^n$ a finite set, for any $d \geq 0$, the value $h_Z(d)$ is also called the number of conditions that Z imposes to forms of degree d.

It is easy to understand why such a terminology is used. In fact $h_Z(d)$ is equal to $\dim(\mathbb{C}[x_0, \dots, x_n])_d$ minus the dimension of the degree d part of the ideal of polynomials which vanishes at 0. In particular $h_Z(d)$ can be seen as the number of independent conditions on polynomials of degree d vanishing at Z necessary to form a basis for the polynomial vanishing at Z.

Another useful definition is the following. Recall that for any finite set Z we denote with $\ell(Z)$ the cardinality.

Definition 3.0.43. We say that the points of Z are *separated* by forms of degree d or that Z imposes independent conditions to forms of degree d, if $h_Z(d) = \ell(Z)$.

Remark 3.0.44. Notice that Z is separated by curves of degree d when fixed a point $P \in Z$ we can find a curve of degree d which vanishes at all the point of $Z \setminus P$ and not in P. In fact since $h_Z(d) = \dim(R/I_Z)_d = \ell(Z)$ we have that the evaluation map $ev_Z(d)$ defined in remark 3.0.36 is surjective and the kernel is exactly equal to $(I_Z)_d$. Moreover to each element of the canonical base of $\mathbb{C}^{\ell(Z)}$ i.e. an element of the form $e_j = (0, \ldots, 1, 0, \ldots, 0)$ with 1 in the j-th position, correspond a point in Z. So since $ev_Z(d)$ is surjective we have that there exists a polynomial $F \in Sym^d(\mathbb{C}^n)$ such that $ev_Z(d)(F) = e_i$, so F vanishes at all the points of Z except P_i.

The following lemma collects some well known facts about Hilbert function and its first difference.

Lemma 3.0.45. *Let $Z \subset \mathbb{P}^n$ be a finite set of points and set $\ell = \ell(Z)$ the cardinality of Z. Then we have:*

1) $h_Z(d) \leq \ell$ for all d;

2) $Dh_Z(d) = 0$ for $d < 0$;

3) $h_Z(0) = Dh_Z(0) = 1$;

4) $Dh_Z(d) \geq 0$ for all d;

5) $h_Z(d) = \ell(Z)$ for all $d \geq \ell(Z) - 1$;

6) $h_Z(i) = \sum_{0 \leq d \leq i} Dh_Z(d)$;

7) $Dh_Z(d) = 0$ for $d >> 0$ and $\sum_d Dh_Z(d) = \ell(Z)$;

8) if $h_Z(d) = \ell(Z)$, then $Dh_Z(d+1) = 0$;

9) if $Z' \subset Z$, then, we have $h_{Z'}(d) \leq h_Z(d)$ and $Dh_{Z'}(d) \leq Dh_Z(d)$ for every $d \in \mathbb{Z}$.

Proof. We call $I(Z) = I_Z$ and, for any polynomial $g \in R$, we call $I_{Z,g}$ the ideal generated by the union of $I(Z)$ and g.

(1), **(2)**: These two points follow directly from the definition of Hilbert function.

(3): We have just to compute $h_Z(0) = \binom{0+n}{n} - I(Z)_0 = 1 - 0 = 1$ and by definition of Dh_Z and **2** we have that also $Dh_Z(0) = 1$.

(4): We have to show that $\dim(R/I)_d \geq \dim(R/I)_{d-1}$. First notice that I_{d-1} can be embedded in $\subseteq I_d$. In fact fixed a linear form g which does not vanish in Z we have that fg belongs to I_d. As a consequence notice that the multiplication by g induces an embedding $R/I_{d-1} \xrightarrow{g} R/I_d$. In fact given a form $F \in R_{d-1}$ which not vanish in Z we have that also Fg does not vanish in Z so the class Fg is non zero in $\dim(R/I_d)$. So $\dim(R/I_{d-1}) \leq \dim(R/I_d)$.

(5): We prove it just for the case $d = \ell(Z) - 1$. The other cases will follows from **(4)**. To prove that $\dim(R/I)_d = \ell(Z)$ we define a surjective map ϕ from R to $\mathbb{C}^{\ell(Z)}$ such that the kernel of this map is equal to I_d. In particular we can choose ϕ equal to the evaluation map defined in Remark 3.0.36. Fix for each point $P_i \in Z$ a linear form L_i which vanishes at P_i and does not vanish at any other points $P_j \in Z$. Then we construct $\ell(Z)$ forms F_j of degree $\ell(Z)-1$ such that F_j is the product of the linear forms L_i with $i \neq j$. F_j is a form of degree $\ell(Z)-1$, which vanishes at all the points of Z except P_j. Thus in this case the evaluation map is surjective.

(6): follows directly from the definition of the first difference of an Hilbert function.

(7) ,**(8)**: They are consequences of **(5)** and **(6)**.

(9): The first inequality follows directly from the fact that $I_{Z'} \supseteq I_Z$ and equation 3.2. As we have seen for point **(4)** there is a natural embedding between R/I_d and R/I_{d+1}. The following sequence

$$0 \to R/(I_Z)_d \xrightarrow{\phi_0} R/(I_Z)_{d+1} \to R/(I_{Z,g})_{d+1} \to 0 \tag{3.6}$$

where g is a form not vanishing in Z, $I_{Z,g}$ is the ideal spanned by I_z and g and ϕ_0 is the embedding induced by the multiplication of all the elements of I_d by g, is exact.

So we have that $Dh_Z(d) = \dim(R/(I_{Z,g})_{d+1})$.

Similarly we have that g induces an inclusion $(R/I_{Z'})_d$ in $(R/I_{Z'})_{d+1}$ and that the following sequence is exact.

$$0 \to R/(I_{Z'})_d \xrightarrow{\phi_0} R/(I_{Z'})_{d+1} \to R/(I_{Z',g})_{d+1} \to 0 \tag{3.7}$$

As before we have that $Dh_{Z'}(d) = \dim R/(I_{Z',g})_{d+1}$. Moreover we have that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \to & R/(I_Z)_d & \xrightarrow{\phi} & R/(I_Z)_{d+1} & \to & R/(I_{Z,g})_{d+1} & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & R/(I_{Z'})_d & \xrightarrow{\phi} & R/(I_{Z'})_{d+1} & \to & R/(I_{Z',g})_{d+1} & \to & 0 \end{array}$$

Since the map $R/(I_Z)_{d+1} \to R/(I_{Z'})_{d+1}$ surjects, by the snake Lemma (see Proposition 2.10 of [9]) we have that also the map $R/(I_{Z,g})_{d+1} \to R/(I_{Z',g})_{d+1}$ surjects and this conclude the proof. $\square$

We will use several times the well known fact that the first difference of Hilbert function, after a certain point, is not increasing.

Proposition 3.0.46. *Let $Z \subset \mathbb{P}^n$ be a finite set of points and assume that for some $j > 0$ we have $Dh_Z(j) \leq j$. Then:*

$$Dh_Z(j) \geq Dh_Z(j+1)$$

and as a consequence $Dh_Z(j) \geq Dh_Z(i)$ for all $i \geq j$. In particular, if for some $j > 0$,we have $Dh_Z(j) = 0$, then $Dh_Z(i) = 0$ for all $i \geq j$.

Proof. See section 3 of [13]. $\square$

Remark 3.0.47. In particular we are interested in studying the Hilbert function of the set of points A such that $v_{n,d}(A)$ is a Waring decomposition of a certain symmetric tensor T. In fact, since Veronese maps are injective as we saw in Remark 1.1.16 we can study directly the points of A in order to obtain information about the points of $v_{n,d}(A)$.

That is why we will refer both to A and to $v_{n,d}(A)$ as decomposition of T.

When we will try to compute the Hilbert function of a certain set of points Z which is the union of two different decompositions of a certain symmetric tensor T we will use several times the following proposition.

Lemma 3.0.48. *Let $Z \subset \mathbb{P}^n$ be a finite set of points with $\ell(Z)$. Then we have that $h_Z(1) < \ell(Z)$ if and only if Z is linearly dependent.*

Proof. To prove the lemma we have just to notice that, given a point $P \in Z$, P imposes a condition to the forms of degree 1 in $\mathbb{P}^n$. In particular, by definition, we have that Z is linearly independent if and only if the number of conditions to the forms of degree 1 imposed by Z is exactly equal to $\ell(Z)$. Since the number of conditions that Z imposes to the forms of degree 1 in $\mathbb{P}^n$ is equal to $h_Z(1)$ we have that $h_Z(1) < \ell(Z)$ if and only if Z is linearly dependent. $\square$

Proposition 3.0.49. *Let $v_{n,d}$ be the Veronese map from $\mathbb{P}^n$ to $\mathbb{P}^N$ with $N = \binom{n+d}{d} - 1$ and let $Z \subset \mathbb{P}^n$ be a finite set of points. Then we have that $h_Z(d) < \ell(Z)$ if and only if $v_d(Z)$ is linearly dependent.*

Proof. In order to prove this result we have just to notice that, by definition of the Veronese map, we have:

$$h_{v_{n,d}(Z)}(1) = h_Z(d).$$

The conclusion follows applying Lemma 3.0.48 to this particular setting.

$\square$

We recall that $R = \mathbb{C}[x_0, \ldots, x_n]$, I_Z is the homogeneous ideal generated by Z and that we indicate with M_d the homogeneous part of degree d of a graded module M.

Remark 3.0.50. Let $Z \subset \mathbb{P}^n$ be a finite set of points and let $(Z) \subset \mathbb{C}^{n+1}$ be a set of coordinates for the points of Z. Fix $s \in \mathbb{N}$.

Consider the sequence

$$0 \to I_{Z\,s} \xrightarrow{d_0} \mathbb{C}[x_0, \ldots, x_n]_s \xrightarrow{ev_{(Z)}(s)} \mathbb{C}^{\ell(Z)} \tag{3.8}$$

where d_0 is the natural embedding of $I_{Z\,s}$ and $ev_{(Z)}(s)$ is the evaluation map of degree d in (Z) defined before. We can notice that the sequence is *left-exact* i.e. the map d_0 is injective for every s. However, the map $ev_{(Z)}(s)$ in general is not surjective. We point out again that the rank of the evaluation maps does not depend from the choice of the coordinates of Z (see Remark 3.0.37).

Example 3.0.51. Consider a set $Z \subset \mathbb{P}^2$ of length $\ell(Z) = 7$ such that Z is contained in an irreducible conic g and take $s = 2$. Let (Z) be any set of coordinates for the points of Z. In this case we have that the sequence 3.8 is equal to:

$$0 \to I_{Z\,2} \xrightarrow{d_0} \mathbb{C}[x_0, x_1, x_2]_2 \xrightarrow{ev_{(Z)}(2)} \mathbb{C}^{\ell(Z)}. \tag{3.9}$$

Notice that from Bézout's theorem (see Chapter one Corollary 7.8 of [34]) $I_{Z\,2}$ is equal to $I_{(g)_2}$ that is the degree 2 part of the ideal generated by g. Thus $d_0(I_{Z\,2})$ can be seen as the linear subspace of the space of polynomial of degree two generated by g. Moreover we have $\dim(I_{Z\,2}) = 1$.

Thus, since the sequence 3.8 is exact we have that $d_0(I_{Z\,2})$ is equal to the kernel of the map $ev_{(Z)}$. $\mathbb{C}[x_0, x_1, x_2]_2$ is a linear space of dimension $\dim(\mathbb{C}[x_0, x_1, x_2]_2) = 6$. Thus d_0 cannot be surjective.

Another useful definition borrowed from the sheaf theory is the following.

Definition 3.0.52. Let $Z \subset \mathbb{P}^n$ be a finite set of points and let Z' be a set of coordinates for the points of Z. We define $h^1 I_Z(s)$ as

$$h^1 I_Z(s) = \ell(Z) - h_Z(s).$$

Remark 3.0.53. Let Z be a finite set of points. We have that for every $s \in \mathbb{Z}$ the Hilbert function of Z is equal to $h_Z(s) = \binom{n+s}{s} - I_{Z\,s}$. Moreover we have that $h^1(I_Z(i)) = \sum_{j>i} Dh_Z(j)$.

The first fact follows directly from the definitions of $I_{Z\,s}$, the other one from the definition of the first difference of the Hilbert function.

We can notice also that, given two disjoint decomposition $A, B \subset \mathbb{P}^n$ of a symmetric tensor T, $h^1_{A\cup B}(d)$ has an important interpretation. Indeed as we will see in the next proposition $h^1_{A\cup B}(d)$ represent the dimension of the intersection $< v_d(A) > \cap < v_d(B) >$.

Proposition 3.0.54. *Let $A, B \subset \mathbb{P}^n$ be two minimal non-redundant disjoint decompositions of a symmetric tensor T with $\ell(A) \geq \ell(B)$. Then for any d we have that*

$$\dim(< v_d(A) > \cap < v_d(B) >) + 1 = h^1_{A\cup B}(d).$$

Proof. Call $Z = A \cup B$. It is a straightforward fact that

$$< v_d(Z) >=< v_d(A) > + < v_d(B) > .$$

Thus, using the Grassmann's formula we have that

$$\dim < v_d(Z) >= \dim(< v_d(A) >) + \dim(< v_d(B) >) \\ - \dim(< v_d(A) > \cap < v_d(B) >) \quad (3.10)$$

Notice also that, from Remark 3.0.53, we can write the Hilbert function of Z at degree i as $h_Z(i) = \ell(Z) - h^1_Z(i)$. As a consequence, from the fact that $\dim < v_d(Z) >= h_Z(d) - 1$, follows that

$$\dim < v_d(Z) >= h_Z(d) - 1 = \ell(Z) - 1 - h^1_Z(d) = \ell(A) + \ell(B) - \ell(A\cap B) - 1 - h^1_Z(d). \quad (3.11)$$

Since A and B are non redundant, so that both $v_d(A)$ and $v_d(B)$ must be linearly independent, using the injectivity of the Veronese map, we have that $\ell(A) = \dim < v_d(A) > +1$ and $\ell(B) = \dim < v_d(B) > +1$. Finally, comparing 3.10 and 3.11 we obtain

$$\dim(< v_d(A) > \cap < v_d(B) >) = \ell(A \cap B) - 1 + h^1_Z(d).$$

The conclusion follows from the fact that A and B are disjoint.

□

The shape of first difference Hilbert function Dh_A gives us also some informations on how the points of $A \subseteq \mathbb{P}^n$ are located in the space. In particular, we cite the following Theorem of Bigatti Geramita and Migliore.

Theorem 3.0.55. *Let $Z \subset \mathbb{P}^n$ be a finite set. Assume that for some $s \leq j$, $Dh_Z(j) = Dh_Z(j+1) = s$. Then there exists a reduced curve C of degree s such that, setting $Z' = Z \cap C$ and $Z'' = Z \setminus Z'$:*

- *for $i \geq j-1$, $Dh_{Z'}(i) = Dh_Z(i)$*
- *for $i \leq j$, $h_{Z'}(i) = h_C(i)$*
- $Dh_{Z'} = \begin{cases} Dh_C(i) & for\ i \leq j+1 \\ Dh_Z(i) & for\ i \geq j \end{cases}$

In particular, $Dh_{Z'}(i) = s$ for $s \leq i \leq j+1$.

Proof. See Theorem 3.6 of [13]. □

Example 3.0.56. The following table represents the first difference of Hilbert function of a set of points $Z \subset \mathbb{P}^3$.

d	0	1	2	3	4	5	6	7	...
$Dh_Z(d)$	1	3	6	4	4	2	2	0	...

From Theorem 3.0.55 we have that there are at least 13 points of Z contained in a curve C of degree 2. Notice in fact that $Dh_Z(5) = Dh_Z(6) = 2$, thus we have a reduced curve C of degree 2 containing some points of Z. However, notice that $\ell(C \cap Z)$ can be bigger than 13. Indeed if C is the union of two skew lines we have that the first difference of the Hilbert function of C will be

d	0	1	2	3	4	5	6	...
$Dh_Z(d)$	1	3	2	2	2	2	2	...

Thus, from Theorem 3.0.55 we have $\ell(C \cap Z) = 14$.

On the other hand, if C is an irreducible plane conic, the first difference of the Hilbert function will be equal to

d	0	1	2	3	4	5	6	...
$Dh_Z(d)$	1	2	2	2	2	2	2	...

Thus, from Theorem 3.0.55 we have that $\ell(Z \cap C) = 13$.

3.1 The Cayley-Bacharach property.

Another property that we will use to describe the Hilbert function of a finite set of points is the *Cayley-Bacharach* property.

Definition 3.1.1. A finite set $Z \subset \mathbb{P}^n$ satisfies the *Cayley-Bacharach property in degree d*, abbreviated as $CB(d)$, if for any $P \in Z$ every form of degree d vanishing at $Z \setminus \{P\}$ also vanishes at P.

Example 3.1.2. Let $Z \subset \mathbb{P}^2$ be a set of 6 points contained in a irreducible conic curve. It is a direct consequence of the Bézout's theorem that Z satisfies $CB(2)$.

Remark 3.1.3. Notice that if a finite set of points $Z \subset \mathbb{P}^n$ satisfies $CB(d)$ then it satisfies also $CB(d-1)$. In fact suppose by contradiction that $CB(d-1)$ does not hold. Thus we can find an hypersurface $F \subset \mathbb{P}^n$ of degree $(d-1)$ such that F vanish at all the points of $Z \setminus \{P\}$ and does not vanish at P. Thus the product $F \cdot g$, where g is an hyperplane not-vanishing at P, is a curve of degree d which vanish at all the points of $Z \setminus \{P\}$ and not in P, a contradiction.

This property has a long history. Many classical problems can be reinterpreted using the Cayley-Bacharach property (see [28]).

As an example we can consider the following Theorem of Pappus.

Pappus Theorem: Given two points x, y in a plane we call $L[x, y]$ the line passing through x and y. Let $X_0 = \{A, B, C\}$ and $X_1 = \{a, b, c\}$ be two sets of aligned points. Then, we have that the points $X = L[A, b] \cap L[a, B], Y = L[A, c] \cap L[a, C], Z = L[B, c] \cap L[b, C]$ are collinear.

This theorem, that was originally proven by using Euclidean geometry, can be seen as a direct consequence of the so called *Cayley-Bacharach theorem.*

Theorem 3.1.4 (Cayley-Bacharach)**.** *Let $\gamma_1, \gamma_2 \subset \mathbb{C}^2$ be two cubic curves that intersect in precisely nine distinct points $\{A_1, \ldots, A_9\} \in \mathbb{C}^2$. Consider γ_3 another cubic polynomial that vanishes on eight of these points. Then γ_3 is a linear combination of γ_1, γ_2, and in particular vanishes on the ninth point A_9. In other word $\{A_1, \ldots, A_9\}$ satisfies $CB(3)$.*

Proof. A proof of this theorem can be found in Chapter 5 Corollary 4.5 of [34]. □

Notice that Pappus Theorem can be seen as a consequence of the Cayley-Bacharach Theorem. Indeed, in the hypothesis of Pappus theorem, the points A, B, C and a, b, c belongs to the intersection between the cubic curve $L[A, b] \cdot = L[A, c] \cdot L[B, c]$ and $L[a, B] \cdot = L[a, C] \cdot L[b, C]$. Thus from Cayley-Bacharach Theorem we have that the cubic $L[A, C] \cdot L[a, c] \cdot L[X, Z]$ has to vanish also in Y.

In particular, we can notice in this example that the fact that a certain set of points Z satisfies the Cayley-Bacharach properties give us some information about the ideal $I(Z)$ and, as a consequence, about the Hilbert function of Z.

We give a useful definition in order to simplify the notation.

Definition 3.1.5. Let $Z \subset \mathbb{P}^n$ be a finite set of points. We call *h-vector* of Z the vector

$$(h_0, h_1, h_2, \ldots, h_r)$$

where $h_i = Dh_Z(i)$ for all $i = 0, \ldots, r$ and where r is the largest positive integer such that the first difference of the Hilbert function of Z is different from 0.

As another example of how the Cayley-Bacharach property influences the shape of the h-vector of a given variety, consider the following Lemma.

Lemma 3.1.6. *Let $Z \subseteq \mathbb{P}^n$ be a finite set of points. Then, if $CB(d)$ holds, we have that for all $P \in Z$ there exists a $j > d$ such that the first difference of the Hilbert function of $Z \setminus \{P\}$ is equal to:*

$$1, Dh_Z(1), Dh_Z(2), \ldots, Dh_Z(d), \ldots, Dh_Z(j) - 1, \ldots, 0, 0, \ldots$$

Proof. In order to prove this Lemma we have just to show that, for every $P \in Z$, $Dh_Z(i) = Dh_{Z \setminus P}(i)$ for $i = 0, 1, \ldots, d$. Suppose that there is a point $\overline{P}$ for which there exist an $i < d+1$ such that the i-th value $Dh_{Z \setminus \{P\}}(i)$ is smaller than $Dh_Z(i)$. Then we would have that $\dim(I_Z(i)) < \dim(I_{Z \setminus \overline{P}}(i))$. In particular Z can be separated by the curves of degree i, so $CB(i)$ cannot hold for Z and, as a consequence $CB(d)$ cannot hold for Z, a contradiction. □

Conversely, we can describe the behaviour of a finite set of points Z which does not satisfies $CB(d)$ for some d.

Lemma 3.1.7. *Let $Z \subset \mathbb{P}^n$ be a finite set of points. If Z does not satisfy $CB(d)$ then, there exists a point $P \in Z$ such that $h^1_Z(d) = h^1_{Z \setminus P}(d)$.*

Proof. Let $i \leq d$ be the minimum integer such that $CB(i)$ does not hold for Z. Then, there exists a point $P \in Z$ and a curve F of degree i vanishing at $Z \setminus \{P\}$ and not in P. Thus $dim(I_Z(i)) < dim(I_{Z \setminus \overline{P}}(i))$ and so $Dh_Z(i) > Dh_{Z \setminus \overline{P}}(i)$. As a consequence using the trivial fact that $Z \setminus P \subset Z$ and Lemma 3.0.45 we have that

$$Dh_Z(j) = Dh_{Z \setminus \overline{P}}(j)$$

for all $j \neq i$. In particular since $i \leq d$ we have that $Dh_Z(j) = Dh_{Z \setminus \overline{P}}(j)$ for all $j > d$.

In conclusion using Remark 3.0.53 we have that we can always choose P such that:

$$h^1_Z(d) = h^1_{Z \setminus P}(d).$$

□

Another Theorem that shows us how the CB-properties characterize the shape of Dh_Z is the following.

Theorem 3.1.8. *If a finite set $Z \subset \mathbb{P}^n$ satisfies $CB(i)$, then for any j such that $0 \leq j \leq i+1$ we have*

$$Dh_Z(0) + Dh_Z(1) + \cdots + Dh_Z(j) \leq Dh_Z(i+1-j) + \cdots + Dh_Z(i+1).$$

Proof. See Theorem 4.9 of [7]. □

The first version of Theorem 3.1.8 was proven by Geramita Kreuzer and Robbiano in [31]. They proved this Theorem just taking i equal to the socle degree of Z i.e. taking i equal to the last integer such that $Dh_Z(i+1) > 0$. In 2018, Angelini, Chiantini and Vannieuwenhoven (see [7]) generalized the result of Geramita Kreuzer and Robbiano proving Theorem 3.1.8 just using the original theorem for the socle degree and using an inductive argument. Theorem 3.1.8 will be essential to prove our results about the identifiability of symmetric tensors.

We conclude this section by recalling an important Lemma that we will use several times in our investigations.

Lemma 3.1.9. *Let T be a symmetric tensor of degree d in $n+1$ variables and consider two non-redundant decompositions A, B of T. Set $Z = A \cup B$. If $A \cap B = \emptyset$, then Z has the Cayley-Bacharach property $CB(d)$.*

Proof. Suppose by contradiction that $CB(d)$ does not hold for Z. This means that there is a point $P \in Z$ and a hypersurface $F \subset \mathbb{P}^n$ of degree d such that F vanishes at all the points of $Z \setminus \{P\}$ and does not vanish at P. From Remark 3.1.7 we have that we can find P such that:

$$h^1_Z(d) = h^1_{Z\setminus\{P\}}(d).$$

Thus by Proposition 3.0.54 we have:

$$\begin{aligned}\dim(< v_d(A) > \cap < v_d(B) >) = h^1_Z(d) - 1 = h^1_{Z\setminus\{P\}}(d) - 1 = \\ = dim(< v_d(A \setminus \{P\}) > \cap < v_d(B \setminus \{P\}) >).\end{aligned} \quad (3.12)$$

Notice that either $A \setminus \{P\} \neq A$ or $B \setminus \{P\} \neq B$ and that T still belongs to $< v_d(A \setminus \{P\}) > \cap < v_d(B \setminus \{P\}) >$. Thus we have that $T \in v_d(A \setminus \{P\})$ and $T \in < v_d(B \setminus \{P\}) >$ but this is a contradiction because we have assumed A, B non redundant. $\square$

Chapter 4

A new proof of Kruskal Theorem.

4.0.1 Preparatory results

Our goal is to prove the Kruskal criterion using the properties of the Hilbert function of a decomposition $A \subset \mathbb{P}^n$ of a given symmetric tensor T of degree d in $n+1$ variables.

In particular we have the following Lemmas which describe the behaviour of the Hilbert function of a set of points A in linear general position.

Lemma 4.0.1. *Given $A = \{P_1, P_2, \dots P_r\} \subset \mathbb{P}^n$, if the first Kruskal rank k_1 of A is equal to $k+1$ then we have:*

- $h_A(i) \geq 1 + ik$ *if* $1 + ik \leq r$
- $h_A(i+1) = r$ *otherwise.*

Proof. We prove that when $r \geq 1 + ik$ we can separate $1 + ik$ points of A using curves of degree i then, as a consequence, we will have that $h_A(i) \geq 1 + ik$.

So take $A_0 = \{P_1, P_2, \dots, P_{1+ik}\} \subset A$ so that $\ell(A_0) = 1 + ik$. From the fact that the first Kruskal rank of A is equal to $k+1$, we also have that the first Kruskal rank of A_0 is at least $k+1$ by Remark 2.0.35. Suppose we want to separate a point $P_j \in A_0$. We can take i disjoint subset formed by k points of $A_0 \setminus \{P_j\}$ and, for each of this subsets, we take an hyperplane passing through all the points of the subset. We call these hyperplanes $L_1, L_2, \dots, L_i$. Thus, we know that P_j is contained in none of these hyperplanes because we know that the first Kruskal rank of A_0 is $k_1 = k+1$. So, P_j is not contained in the hypersurface of degree i defined by $L_1 \cdot L_2 \cdot \ldots \cdot L_i$ which contains $A_0 \setminus \{P_j\}$. So we have that $h_{A_0}(i) \geq 1 + ik$ for all the subsets A_0 of A of cardinality $1 + ik$. The statement follows directly from point 9 of Lemma 3.0.45.

The proof of the second claim is the same of the first one, except for the fact that one of the subsets of A_0 can be of cardinality smaller then i. □

The previous Lemma gives us also some information about the first difference of the Hilbert function of a set of points A.

Lemma 4.0.2. *Let T be a symmetric tensor of degree d in $n+1$ variables and let $A = \{P_1, P_2, \dots, P_r\} \subset \mathbb{P}^n$ be a decomposition of T so that the Kruskal rank of A is equal to $k+1$ and such that $r \leq \frac{dk+1}{2}$.*

Then we have:

- $Dh_A(\frac{d}{2}+1) = 0$ *if d is even.*
- $Dh_A(\frac{(d+1)}{2}+1) = 0$ *if d is odd.*

Moreover we have that $Dh_A(\frac{d}{2}) \leq k-1$ when d is even, $Dh_A(\frac{d+1}{2}) \leq \frac{k-1}{2}$ when d and k are both odd and $Dh_A(\frac{d+1}{2}) \leq \frac{k}{2}-1$ when d is odd and k is even.

Proof. Suppose d even. Let $i \in \mathbb{N}$ be the maximum such that $1+ik \leq r$. We can prove that $i < \frac{d}{2}$. Indeed if $i = \frac{d}{2}$ then we have $1 + \frac{dk}{2} \leq r$ but this is a contradiction since we know by hypothesis that $r \leq \frac{dk+1}{2}$ and $1 + \frac{dk}{2} > \frac{dk+1}{2}$.

Thus, from Lemma 4.0.1 we have:

$$h_A\left(\frac{d}{2}\right) \geq \min\left(\frac{dk}{2}+1, r\right) = r.$$

So, $h_A\left(\frac{d}{2}\right) = r$ and as a consequence we have $Dh_A\left(\frac{d}{2}+1\right) = 0$.

Moreover, we can prove that $Dh_A\left(\frac{d}{2}\right) \leq k-1$. In fact, suppose by contradiction that $Dh_A\left(\frac{d}{2}\right) \geq k$. By definition of the first difference of Hilbert function we have that:

$$h_A\left(\frac{d}{2}\right) = h_A\left(\frac{d}{2}-1\right) + Dh_A\left(\frac{d}{2}\right).$$

If $h_A\left(\frac{d}{2}-1\right) = r$ then we have also that $Dh_A\left(\frac{d}{2}\right) = 0$, a contradiction because we have supposed $Dh_A\left(\frac{d}{2}\right) \geq k$.

If $h_A\left(\frac{d}{2}-1\right) < r$ then from Lemma 4.0.1 and from our assumption we have:

$$h_A\left(\frac{d}{2}-1\right) + Dh_A(\frac{d}{2}) \geq \left(\frac{d}{2}-1\right)\cdot k + 1 + k = \frac{dk}{2} - k + 1 + k = \frac{dk}{2} + 1 > r.$$

a contradiction.

Suppose now d odd. Let $i \in \mathbb{N}$ be the maximum such that $1+ik \leq r$. As before we can prove that $i < \frac{d+1}{2}$. Indeed if $i = \frac{d+1}{2}$ then we have $1+\frac{(d+1)k}{2} \leq r$ but this is a contradiction since we know by hypothesis that $r \leq \frac{dk+1}{2}$ and $1 + \frac{(d+1)k}{2} > \frac{dk+1}{2}$.

Thus, from Lemma 4.0.1 we have:

$$h_A\left(\frac{d+1}{2}\right) \geq \min\left(\frac{(d+1)k}{2}+1, r\right) = r.$$

So, $h_A(\frac{d+1}{2}) = r$ and as a consequence $Dh_A(\frac{d+1}{2}+1) = 0$.

If d is odd and k is even we have that $Dh_A(\frac{d+1}{2}) \leq \frac{k}{2}-1$. In fact suppose by contradiction that $Dh_A(\frac{d+1}{2}) \geq \frac{k}{2}$.

By definition of the first difference of Hilbert function we have:

$$h_A\left(\frac{d+1}{2}\right) = h_A\left(\frac{d+1}{2}-1\right) + Dh_A\left(\frac{d+1}{2}\right).$$

As before, if $h_A\left(\frac{d+1}{2}-1\right)=r$ then we have $Dh_A\left(\frac{d+1}{2}\right)=0$ a contradiction because we have supposed $Dh_A\left(\frac{d+1}{2}\right)\geq\frac{k}{2}$.

If $h_A\left(\frac{d+1}{2}-1\right)<r$ then from Lemma 4.0.1 we have:

$$h_A\left(\frac{d+1}{2}\right)=h_A\left(\frac{d+1}{2}-1\right)+Dh_A\left(\frac{d+1}{2}\right)\geq \geq\frac{(d-1)k}{2}+1+\frac{k}{2}=\frac{dk}{2}+1>r \quad (4.1)$$

a contradiction.

Suppose that d and k are both odd. We prove that $Dh_A\left(\frac{d+1}{2}\right)\leq\frac{k-1}{2}$.

Suppose by contradiction that $Dh_A\left(\frac{d+1}{2}\right)\geq\frac{k-1}{2}+1$.

We know that:

$$h_A\left(\frac{d+1}{2}\right)=h_A\left(\frac{d-1}{2}\right)+Dh_A\left(\frac{d+1}{2}\right).$$

As before, if $h_A\left(\frac{d+1}{2}-1\right)=r$ then we have $Dh_A\left(\frac{d+1}{2}\right)=0$ a contradiction because we have supposed $Dh_A\left(\frac{d+1}{2}\right)\geq\frac{k-1}{2}+1$.

If $h_A\left(\frac{d+1}{2}-1\right)<r$ from Lemma 4.0.1 we have:

$$h_A\left(\frac{d+1}{2}\right)=h_A\left(\frac{d-1}{2}\right)+Dh_A\left(\frac{d+1}{2}\right)\geq \geq\frac{(d-1)k}{2}+1+\frac{k-1}{2}+1=\frac{dk}{2}+\frac{1}{2}+1>r \quad (4.2)$$

a contradiction.

□

Given $A,B\subset\mathbb{P}^n$ two sets of points, if we know the h-vector of A we can also have some information on the shape of the h-vector of $A\cup B$. In particular we have the following Lemma.

Lemma 4.0.3. *Let $A,B\subset\mathbb{P}^n$ be two nonempty subsets with $A\cap B=\emptyset$. Call $Z=A\cup B$ and let $(h_0,h_1,\dots,h_\alpha)$ be the h-vector of Z. Fix $s\in\mathbb{N}$ such that $s\leq\alpha$ and $h^1I_A(s)=0$. Then, there exist a subset B_0 of B such that the h-vector of $Z\setminus B_0$ is equal to $(h_0,h_1,\dots h_s,0)$.*

Proof. We construct the set B_0 removing from Z one point of B at a time and verifying that the new h-vector satisfies the condition of the statement. Suppose by contradiction that there is no choice of a point $P\in B\setminus B_i$ for which the h-vector decreases in h_j for some $j\geq s+1$ where B_i is the set points that we have removed up to the i-th step ($B_0=\emptyset$). Thus, there is an i for which $dim(I_{Z\setminus B_i})(s)<dim(I_{((Z\setminus B_i))\setminus P})(s)$. So $CB(s)$ does not hold for $Z\setminus B_i$ because P is separated by curves of degree s. In particular, for a point P in $B\setminus B_i$ there exist a curve C of degree s such that C vanishes in all the points of $(Z\setminus B_i)\setminus P$ and C does not vanish in P. We can do this for all the points in $B\setminus B_i$.

Our claim is that $h^1I_{Z\setminus B_i}(s)=0$. In order to prove this fact we have just to show that the evaluation map $ev_{Z\setminus B_i}(s):R_s\to\mathbb{C}^{\ell(Z\setminus B_i)}$ is surjective. In other word we have just to show that there are polynomials $C_1,\dots,C_{\ell(Z\setminus B_i)}\in\mathcal{O}_{\mathbb{P}^n}(s)$ such that the image of $C_1\dots C_{\ell(Z\setminus B_i)}$ forms a base for $\mathbb{C}^{\ell(Z\setminus B_i}$.

Fix a set of coordinates and an ordering for $Z \setminus B_i$, say that $\{P_1, \dots, P_\gamma\}$ are coordinates of the points of A and $\{P_{\gamma+1}, \dots, P_\beta\}$ are coordinates for the points of $B \setminus B_i$.

We proved that any $P_j \in B \setminus B_i$ is separated by polynomials of degree s thus, we have that for any $P_j \in B_i$ $(j = (\gamma+1), \dots, \beta)$ there exists a polynomial $C_j \in R_s$ such that $ev_{Z\setminus B_i}(s)(C_j) = (0, \dots, 0, \gamma_j, 0, \dots, 0)$ where $0 \neq \gamma_j \in \mathbb{C}$ is the j-th coordinate of $ev_{Z\setminus B_i}(s)(C_j)$.

Thus, from the fact that $h^1 I_A(s) = 0$ and so that the evaluation map $ev_A(s)$ is surjective, we have that the matrix associated to the evaluation map $ev_{Z\setminus B_i}$ is a block-matrix of the form:

$$\left(\begin{array}{c|c} T' & 0 \\ \hline T & I \end{array}\right)$$

where I is the identity matrix and T' is a triangular matrix of full rank. Thus the evaluation map $ev_{Z\setminus B_i}$ is surjective. As a consequence we have that $h^1 I_{A\cup(B\setminus B_i)}(s) = 0$ and so $Dh_{A\cup(B\setminus B_i)}(s+1) = 0$. Thus, we have that the h-vector of $Z \setminus B_i$ is equal to $(h_0, h_1, \dots, h_s)$ and this conclude the proof. $\square$

Remark 4.0.4. In the hypothesis of the previous Lemma, if $\ell(B) \geq 2$, we can always find a set B_0 such that $Z \setminus B_0$ has h-vector equal to $(h_1, h_2, \dots, h_s, 1)$. Indeed, as in the proof of the lemma, we have just to remove one point of B at a time until we obtain a set B' such that $h^1 I_{(Z\setminus B')} = 1$. This means that if we remove another points from B we obtain an h-vector for Z minus the removed points of the form $(h_1, h_2, \dots, h_s, 0)$.

The conclusion follows since from Proposition 3.0.46 we know that if $Dh_Z(n) = 0$ for some $n \in N$ then, $Dh_Z(j) = 0$ for all $j \geq n$.

4.1 Kruskal Theorem (for symmetric tensors) revisited

Now, we are able to prove that a tensor T of degree d in $n+1$ variables with a decomposition $A \subset \mathbb{P}^n$ of length $\ell(A) \leq \frac{dk+1}{2}$ and Kruskal rank equal to $k+1$ is identifiable by using the properties of the Hilbert function and the Cayley-Bacharach property.

Theorem 4.1.1. *Let T be a symmetric tensor of degree d and let $A \subset \mathbb{P}^n$ be a decomposition of T such that $\ell(A) = r \leq \frac{dk+1}{2}$. If the Kruskal rank of A is $k+1$ then T is identifiable.*

In order to prove this proposition, first we prove that, under our hypothesis, if there exist two different decompositions of a tensor T then these two decompositions cannot be disjoint. We will prove this fact separately for tensors of degree d even and odd. Then, we will use an inductive strategy to deduce the identifiability of T.

Proposition 4.1.2. *Assume we have a symmetric tensor T of degree d and $A \subset \mathbb{P}^n$ a decomposition of T such that $\ell(A) = r \leq \frac{dk+1}{2}$ and first Kruskal rank $k_1 = k+1$ with $k \in \mathbb{N}$. Then, T cannot have two disjoint decompositions.*

Proof.

Case 1. *d even.*

Suppose that T is a symmetric tensor of degree $d = 2m$. We start by proving that a symmetric tensor of degree even and rank bounded by $\frac{dk+1}{2}$ cannot have two disjoint decompositions.

Suppose by contradiction that T is not identifiable and call B another decomposition of B such that $\ell(B) \leq \ell(A)$ and $A \cap B = \emptyset$. We call $Z = A \cup B$. Let $(a_0, \dots, a_m)$ be the h-vector of A and let $(h_0, \dots, h_n)$ be the h-vector of Z. From Theorem 3.1.9 we know that $Z = A \cup B$ satisfies $CB(d)$.

Since Z contains A, then from Lemma 3.0.45 we have that

$$Dh_Z(0) + Dh_Z(1) + \cdots + Dh_Z(m) \geq Dh_A(0) + Dh_A(1) + \cdots + Dh_A(m) = r.$$

Thus, from the fact that $\ell(A) \geq \ell(B)$ and since from Theorem 3.1.8 we know that

$$Dh_Z(m+1) + \cdots + Dh_Z(d+1) \geq Dh_Z(0) + \cdots + Dh_Z(m) = r$$

we have that:

$$\begin{aligned} 2r \geq \ell(Z) = \ell(A) + \ell(B) &\geq \sum_{i=0}^{m} Dh_Z(i) + \sum_{i=m+1}^{d+1} Dh_Z(i) \geq \\ &\geq \sum_{i=0}^{m} Dh_A(i) + \sum_{i=m+1}^{d+1} Dh_Z(i) = r + \sum_{i=m+1}^{d+1} Dh_Z(i) \geq 2r. \end{aligned} \tag{4.3}$$

Thus, we have necessarily that:

$$2r = \ell(A) + \ell(B) = \sum_{i=0}^{m} Dh_A(i) + \sum_{i=m+1}^{d+1} Dh_Z(i) = r + \sum_{i=m+1}^{d+1} Dh_Z(i) \tag{4.4}$$

and so we have $\ell(B) = \sum_{i=m+1}^{d+1} Dh_Z(i) = r$, $Dh_Z(d+2) = 0$, $a_m > 0$ and $Dh_Z(i) = Dh_A(i)$ for all $i \leq m$.

Moreover, from Lemma 4.0.2 we know that $h_m = a_m \leq k - 1$.

Now, we remove from Z all the points of B but one. The set Z_0 that we obtain has cardinality $\ell(Z_0) = r + 1$ and contains A. Thus, its h-vector is $h_{Z_0} = (1, k, h_2, \dots h_m, 1) = (1, k, a_2, \dots, a_m, 1)$. We can notice that $CB(m)$ does not hold for Z_0. In fact, if $CB(m)$ hold we would have by Theorem 3.1.8 $Dh_{Z_0}(m) + Dh_{Z_0}(m+1) \geq Dh_{Z_0}(0) + Dh_{Z_0}(1)$. But $h_m + 1 \leq k - 1 + 1 < k + 1$ a contradiction.

Now we remove points from Z_0 to preserve the value 1 in degree $m + 1$ in order to obtain a subset Z_0' having the Cayley-Bacharach property $CB(m)$ (see Remark 4.0.4). In doing so, since $a_m \leq k - 1$, we are forced to decrease the value in degree 1 (otherwise we have a contradiction with Theorem 3.1.8). So, the set Z_0' has h-vector $h_{Z_0'} = (1, a_1', a_2', \dots, a_m', 1)$ with $a_m' < k$, hence $a_1' < k$ because of the $CB(m)$ property. Since $m \geq 2$ and $a_m' > 0$ (otherwise we would have a contradiction by Proposition 3.0.46), then Z_0' is contained in a subspace of dimension $a_1' < k$ of $\mathbb{P}^n$ and contains at least $a_1' + 2$ points of A. This contradicts the fact that the Kruskal rank of A is equal to $k + 1$.

So $CB(d)$ cannot hold and as a consequence T cannot have two disjoint decompositions.

Case 2. *d odd.*

Now we analyse the cases where the symmetric tensor T is of odd degree d. It turns out that this case is more difficult than the previous one.

Suppose by contradiction that $CB(d)$ holds for Z.

From the fact that A is in linear general position by the definition of the Hilbert function, we know that the $h-$vector of A is of the form:

$$(1, k, h_2, \dots, h_{(\frac{d+1}{2})})$$

where $h_{(\frac{d+1}{2})} \le \frac{k}{2} - 1$ if k is even and $h_{(\frac{d+1}{2})} \le \frac{k-1}{2}$ otherwise (see lemma 4.0.2).

Moreover, from the fact that $\ell(A) \ge \ell(B)$ we have that the h-vector of Z is of the form $(1, k, h_2, h_3, h_4, \dots, h_{d+1})$ with $h_{(\frac{d+1}{2})} \le k - 1$. In fact, from Lemma 4.0.2, Theorem 3.1.8 and Proposition 3.0.46 we know that:

$$\sum_{i=\frac{d+3}{2}}^{d+1} Dh_Z(i) \ge \sum_{i=0}^{\frac{d-1}{2}} Dh_Z(i) \ge \sum_{i=0}^{\frac{d-1}{2}} Dh_A(i) = h_A(\frac{d-1}{2}) \ge k \cdot \frac{d-1}{2} + 1,$$

and so we have:

$$Dh_Z(\frac{d+1}{2}) \le 2r - \sum_{i=0}^{\frac{d-1}{2}} Dh_Z(i) - \sum_{i=\frac{d+3}{2}}^{d+1} Dh_Z(i) \le dk + 1 - dk + k - 2 = k - 1.$$

If $r \le k \cdot (\frac{d-1}{2}) + 1$ from Lemma 4.0.2 we have that

$$k \cdot (\frac{d-1}{2}) + 1 \le \sum_{i=0}^{\frac{d-1}{2}} Dh_A(i) \le \sum_{i=0}^{\frac{d-1}{2}} Dh_Z(i) \le \sum_{i=\frac{d+3}{2}}^{d+1} Dh_Z(i).$$

So from the fact that $\ell(A) \ge \ell(B)$ we have $Dh_Z(\frac{d+1}{2}) = 0$, a contradiction with Proposition 3.0.46.

Thus, suppose $r > k \cdot (\frac{d-1}{2}) + 1$.

From Lemma 3.1.6 and the fact that $CB(d)$ holds we have that the h-vector of $Z \setminus P_0$, where P_0 is any point in B, is:

$$1, k, h_2, \dots, h_{d+1} - 1.$$

The set $Z \setminus P_0$ must satisfy $CB(d-1)$, otherwise Z could not satisfy $CB(d)$ (see Remark 3.1.3). Thus, if we remove any other point $P' \in Z \setminus P_0$, we get an h-vector $(1, k, h_2, \dots, h'_d, h'_{d+1})$, with $h'_d + h'_{d+1} = h_d + h_{d+1} - 2$.

For any pair of points $P_0, P' \in B$, we can apply Lemma 4.0.3 and find a subset $\overline{B} = \overline{B}(P_0, P') \subset B$, with $P_0, P' \in \overline{B}$, such that the h-vector of $Z \setminus \overline{B}$ is equal to

$$(1, k, h_2, \dots, h_{(\frac{d+1}{2})}, k - h_{(\frac{d+1}{2})}).$$

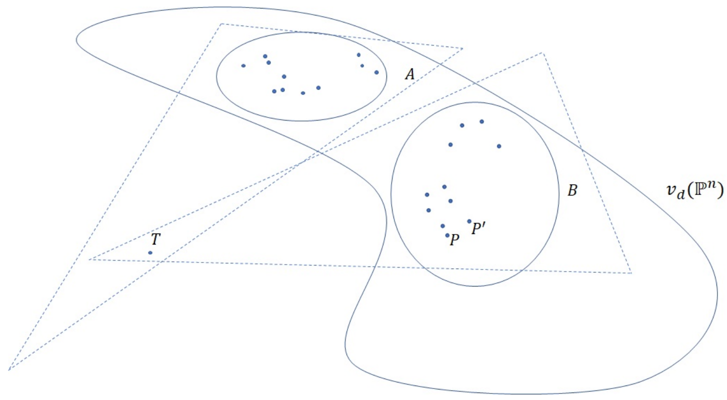

Figure 4.1

We can notice that $Z \setminus \overline{B}$ does not satisfy $CB(\frac{d+1}{2})$ otherwise we will get a contradiction with Theorem 3.1.8.

Since $r = \ell(A) = \sum_{i=0}^{\frac{d+1}{2}} Dh_A(i)$, from Proposition 3.0.46, we can find the following two inequalities. If k is odd we have:

$$\ell(A) + \ell(B) - \ell(\overline{B}) = \sum_{i=0}^{\frac{d+1}{2}} h_i + k - h_{(\frac{d+1}{2})} \geq$$

$$\geq \sum_{i=0}^{(\frac{d+1}{2})} Dh_A(i) + h_{(\frac{d+1}{2})} - Dh_A(\frac{d+1}{2}) + k - h_{(\frac{d+1}{2})} \geq r + k - \frac{k}{2} + 1 \geq r + \frac{k}{2} + 1.$$

Otherwise, if k is even we have:

$$\ell(A) + \ell(B) - \ell(\overline{B}) = \sum_{i=0}^{\frac{d+1}{2}} h_i + k - h_{\frac{d+1}{2}} \geq$$

$$\geq \sum_{i=0}^{\frac{d+1}{2}} Dh_A(i) + h_{\frac{d+1}{2}} - Dh_A(\frac{d+1}{2}) + k - h_{\frac{d+1}{2}} \geq$$

$$\geq r + k - \frac{k-1}{2} = r + \frac{k+1}{2}.$$

In both cases we have that $\ell(\overline{B}) \leq \frac{(d-1)k}{2}$.

We write $\overline{B}$ as the union of $\frac{(d-1)}{2}$ disjoint subsets $\overline{B}_1, \dots, \overline{B}_{\frac{d-1}{2}} \subset \overline{B}$ such that $\ell(\overline{B}_i) \leq k$ and $\bigcup_{i=1}^{\frac{d-1}{2}} \overline{B}_i = \overline{B}$.

Since $CB(\frac{d+1}{2})$ does not hold for $Z \setminus \overline{B}$ there exists a point $Q_1 \in Z \setminus \overline{B}$ and a curve S_1 of degree $\frac{d+1}{2}$ which contains $(Z \setminus \overline{B}) \setminus \{Q_1\}$ and misses Q_1.

We show that $v_{\frac{d+1}{2}}(Q_1) \in \bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) >$. In fact, if $v_{\frac{d+1}{2}}(Q_1) \notin \bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) >$ we could take the product of $\frac{d-1}{2}$ hyperplanes $L_1 \cdot \ \dots \ \cdot$

$L_{\frac{d-1}{2}}$ such that each L_i contains $< \overline{B}_i >$ and misses Q_1. Thus, the product $S_1 \cdot L_0 \cdot \ \ldots \ \cdot L_{\frac{d-1}{2}}$ contains $Z \setminus Q_1$ and misses Q_1, contradicting the $CB(d)$ property for Z.

Note also that, from the fact that $CB(\frac{d+1}{2})$ does not hold for $Z \setminus \overline{B}$, the point Q_1 preserves the value $k - h_{\frac{d+1}{2}}$ of the h-vector i.e the last value of the h-vector of the set of points $(Z \setminus \overline{B}) \setminus \overline{Q}_1$ is still equal to $k - h_{\frac{d+1}{2}}$.

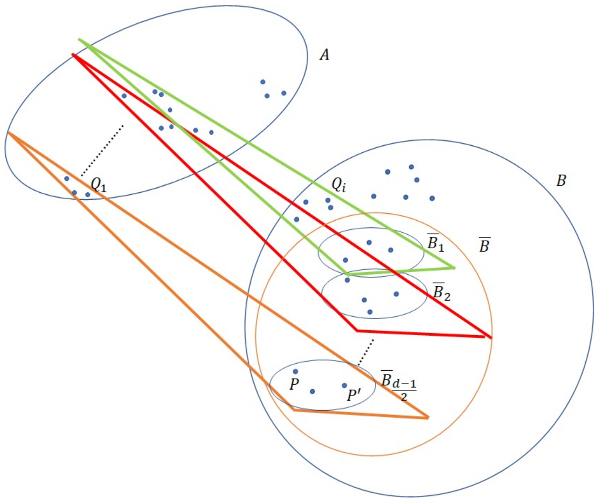

Figure 4.2

Now, we eliminate Q_1 from $Z \setminus \overline{B}$. The set $Z_1 = (Z \setminus \overline{B}) \setminus \{Q_1\}$ contains at least $r - 1 \geq k + 1$ points of A, thus $v_{\frac{d+1}{2}}(Z_1)$ cannot be contained in $n - k$ independent hyperplanes, because the first Kruskal rank k_1 of A is equal to $k+1$. The h-vector of Z_1 is equal to $(1, k, h'_2, \ldots, h'_{\frac{d+1}{2}}, k - h_{\frac{d+1}{2}})$, where there is an i for which $h'_i = h_i - 1$ and $h'_j = h_j$ for all $j \neq i$. We can notice that $CB(\frac{d+1}{2})$ still does not hold for Z_1, otherwise we will have a contradiction with Theorem 3.1.8.

We continue dropping points of Z_1 and preserving the last value of the h-vector of $Z \setminus \overline{B}$, until we get a set of points with Kruskal rank less than $k + 1$.

These sets Z_i that we find at any step cannot satisfy $CB(\frac{d+1}{2})$. Thus at any step one can find a point $Q_i \in Z_i$ and a curve S_i of degree $\frac{d+1}{2}$ passing through $Z_i \setminus Q_i$ and missing Q_i. Moreover all the points Q_i belong to the union of the spans $< v_{\frac{d+1}{2}}(\overline{B}_i) >$. At the next step, we can remove Q_i and still preserve the last value $k - h_{\frac{d+1}{2}}$ in the h-vector.

The process stops when we get a subset with Kruskal rank less than $k+1$, but this can happen only after removing at least $r-k-1$ points of A. Thus for any pair of points $P_0, P' \in B$ there exists a subset $\overline{B}$ of $h_{\frac{d+3}{2}} + \cdots + h_{d+1} - (k - h_{\frac{d+1}{2}})$ points of B, with $P_0, P' \in \overline{B}$, and a partition of $\overline{B} = \overline{B_1} \cup \cdots \cup \overline{B}_{\frac{d-1}{2}}$ such that $\ell(\overline{B_i}) \leq k$ and $< v_{\frac{d+1}{2}}(\overline{B}_1) > \cup \cdots \cup < v_{\frac{d+1}{2}}(\overline{B}_{\frac{d-1}{2}}) >$ contains at least $r - k - 1$ points of A. Call $C_0 = B \setminus (\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) >) \subset B$. The set C_0 has cardinality at most $\ell(B) - \ell(\overline{B})$. By construction and using again Lemma 4.0.1

and Theorem 3.1.8 we have that:

$$\begin{aligned}\ell(\overline{B}) &= h_{\frac{d+3}{2}} + h_{\frac{d+5}{2}} + \cdots + h_{d+1} - k + h_{\frac{d+1}{2}} \\ &\geq Dh_A(0) + Dh_A(1) + \cdots + Dh_A\left(\frac{d-1}{2}\right) - k + Dh_A\left(\frac{d+1}{2}\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \geq r-k\end{aligned}$$

So we have that:

$$\ell(C_0) \leq \ell(B) - \ell(\overline{B}) \leq r - r + k \leq k.$$

Now, we prove that $A \setminus (\bigcup_{i=1}^{\frac{d-1}{2}} < \overline{B}_i > \cup < C_0 >) = \emptyset$.

Suppose by contradiction that $A \setminus (\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i >) \cup < C_0 >)$ is non-empty. Then from the fact that $\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) >$ contains at least $r-k-1$ points of A we can find a hyperplane H'' which contains all the points of $A \setminus (\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) > \cup < C_0 >)$ but one point T'.

Now, consider the curve $S' = L_1 \cdot L_2 \cdot \ \ldots \ \cdot L_{\frac{d-1}{2}}$ such that $< \overline{B}_i > \subset L_i$ defined as before, an hyperplane $L_{\frac{d+1}{2}}$ containing $< C_0 >$ and avoiding T, and H''. We can notice that since $\ell(C_0) \leq k \leq n$ it is always possible to find an hyperplane passing through $\langle C_0 \rangle$ and avoiding T. Moreover from the definition of the L_i it is clear that T does not belong to $(\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i >) \cup < C_0 >)$.

Thus, we have a curve $L_1 \cdots L_{\frac{d+1}{2}} \cdot H''$ of degree $\frac{d+3}{2}$ that contains $Z \setminus \{T\}$ and misses T, a contradiction with $CB(d)$.

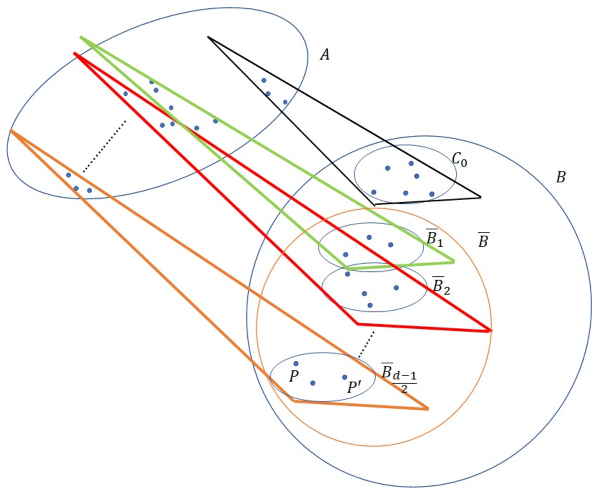

Figure 4.3

So, A is contained in the union $\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) > \cup < C_0 >$. Since $< C_0 >$ and $\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) >$ can contain at most $\ell(C_0)$ and $\ell(\overline{B})$ points of

A, because the first Kruskal rank of A is $k+1$ and $\ell(C_0) \leq k$ and $\ell(\overline{B_i}) \leq k$, then we have:

$$\ell(A) = r \leq \ell(\overline{B}) + \ell(C_0) \leq \ell(\overline{B}) + \ell(B) - \ell(\overline{B}) \leq r. \tag{4.5}$$

So, we have necessarily $\ell(B) = r$ and that $\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) >$ contains $\ell(\overline{B})$ points of A while $< C_0 >$ contains the remaining $\ell(C_0)$ points of A. Moreover we have that $\ell(B) = \ell(\overline{B}) + \ell(C_0)$ and from the fact that $C_0, \overline{B} \subset B$, we have that $\overline{B}$ and C_0 are disjoint and as a consequence C_0 is equal to $B \setminus \overline{B}$. From the definition of C_0 we have also $(\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) >) \cap B = \overline{B}$. Moreover, $A \cap (\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) >)$ and $A \cap < C_0 >$ are disjoint.

Now, repeat the procedure varying the choice of the point P_0. So, we take another point P_1, different from P_0, in $B \setminus \overline{B}$, and we define the sets $\underline{B}$ and C_1 corresponding respectively to $\overline{B}$ and C_0.

We can notice that $\ell(\underline{B}) = \ell(\overline{B})$. As before we can find a partition of $\underline{B}$ of the form $\underline{B}_i$ with $0 \leq i \leq \frac{d-1}{2}$ such that $\ell(\underline{B}_i) \leq k$ and $\bigcup_{i=1}^{\frac{d-1}{2}} \underline{B}_i = \underline{B}$.

Moreover we have that all the properties of the sets $\overline{B}$ and $\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) >$ are also properties of $\underline{B}$ and $\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\underline{B}_i) >$. In particular, $\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\underline{B}_i) >$ contains $\ell(\overline{B}) = \ell(\overline{B})$ points of A and we can define a set $C_1 = B \setminus (\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\underline{B}_i) >)$ such that $< C_1 >$ contains the remaining points of A. As before we have also that $A \cap (\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\underline{B}_i) >)$ and $A \cap < C_1 >$ are disjoint

We want to prove that the new set C_1 defines a span $< C_1 >$ which is different from the span of C_0.

In fact, suppose by contradiction that $< C_0 > = < C_1 >$. Then, we have $< C_0 > \cap A = < C_1 > \cap A$ and as a consequence we have

$$\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) > \cap A = \bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\underline{B}_i) > \cap A.$$

Moreover, we know that:

$$\ell(\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) > \cap A) = \ell(\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\underline{B}_i) > \cap A) = \ell(\overline{B}) = \ell(\underline{B}).$$

From the fact that the first Kruskal rank of A is $k+1$ and that $\ell(\underline{B}_i) \leq k$, we can take among the generators for $< \underline{B}_i >$ the points in $< \underline{B}_i > \cap A$ and among the generators for $< \overline{B}_i >$ the points in $< \overline{B}_i > \cap A$. Thus, from the fact that we made no assumptions on how the points of $\underline{B}$ are divided on the sets $\underline{B}_i$, we can suppose that each $\langle \underline{B}_i \rangle$ is generated by the same generators of $\overline{B_i}$. So we have that also $< v_{\frac{d+1}{2}}(\overline{B}_i) > = < v_{\frac{d+1}{2}}(\underline{B}_i) >$ and for what we have seen before we have

$$\overline{B} = (\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) >) \cap B = (\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\underline{B}_i) >) \cap B = \underline{B}$$

a contradiction.

In particular we have that $\langle C_0\rangle \cap A$ must be different from $\langle C_1\rangle \cap A$, otherwise we obtain a contradiction in the same way as before.

Thus $< C_1 >$ must miss at least one point T of $A\setminus < C_0 >$ and as a consequence $A_0 = (A \setminus \bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B_i}) >)\setminus < C_1 >$ is non empty. We recall that $\ell(A_0)$ is at most $k+1$ since $\bigcup_{i=1}^{\frac{d-1}{2}} < v_{\frac{d+1}{2}}(\overline{B}_i) >$ contains at least $r-k-1$ points of A.

Now, we can take the product of hyperplanes containing $\bigcup_{i=1}^{\frac{d-1}{2}} < \overline{B}_i >$ and $< C_1 >$ that avoid T and a general hyperplane containing all the points of A_0 but the point T. We obtain a curve of degree $\frac{d+3}{2}$ that contains $Z \setminus \{T\}$ and misses T. A contradiction.

□

4.2 Inductive step: non empty intersection

Now we are able to prove again the Kruskal criterion using only the previous results and an inductive strategy.

Proposition 4.2.1. *Given a symmetric tensor T of degree d and $A \subset \mathbb{P}^n$ a decomposition of T such that $\ell(A) = r \leq \frac{dk+1}{2}$. Then, if the Kruskal rank of A is $k+1$ then T is identifiable.*

Proof. Suppose by contradiction that there exist another decomposition B of T of length $\ell(B) = r' \leq \ell(A)$. Of course we may assume that B is non-redundant.

If $A \cap B = \emptyset$ we have a contradiction from Proposition 4.1.2. So, assume $A \cap B \neq \emptyset$.

Thus we can write, without loss of generality, $B = \{P_1, \dots P_i, P'_{i+1}, \dots, P'_{r'}\}$ i.e. we may assume that $A \cap B = \{P_1, \dots, P_i\}$, $i > 0$. Then there are coefficients $a_1, \dots, a_r, b_1, \dots, b_{r'}$ such that:

$$\begin{aligned} T = a_1 v_d((P_1)) + \cdots + a_i v_d((P_i)) + a_{i+1} v_d((P_{i+1})) \cdots + a_r v_d((P_r)) = \\ = b_1 v_d((P_1)) + \cdots + b_i v_d((P_i)) + b_{i+1} v_d((P'_{i+1})) + \cdots + b_k v_d((P'_{r'})) \end{aligned}$$

where we indicate with (P_i) a possible choice of affine coordinates for the projective point P_i. Consider the form

$$T_0 = (a_1 - b_1) v_d((P_1)) + \cdots + (a_i - b_i) v_d((P_i)) + a_{i+1} v_d((P_{i+1})) + \cdots + a_r v_d((P_r)),$$

which is also equal to $b_{i+1} v_d((P'_{i+1})) + \cdots + b_k v_d((P'_{r'}))$. Thus T_0 has two decompositions A and $B' = \{P'_{i+1}, \dots, P'_{r'}\}$, which are disjoint. If A and B' are both non-redundant, then by Lemma 3.1.9 applied to A, B' and T_0, we get that $A \cup B'$ satisfies $CB(d)$. Since $A \cup B' = A \cup B = Z$, and we know by Proposition 3.1.9 that Z does not satisfies $CB(d)$, we find that either A or B' are redundant.

Assume that B' is a redundant decomposition of T_0. Then we can find a point of B', say $P'_{r'-1}$, such that T_0 belongs to the span of $v_d(B' \setminus \{P'_{r'}\})$. Since $T = T_0 + b_1 v_d((P_1)) + \cdots + b_i v_d((P_i))$, this would mean that T belongs to the span of $v_d(B' \setminus \{P'_{r'}\})$, which contradicts the fact that B is a non-redundant decomposition of T.

Assume that A is a redundant decomposition of T_0 and in particular that T_0 belongs to the span of $v_d(A \setminus \{P_j\})$, for some $j > i$. As above, since $T = T_0 + b_1 v_d((P_1)) + \cdots + b_r v_d((P_r))$, this would mean that T belongs to the span of $v_d(A \setminus \{P_j\})$, which contradicts the fact that A is a non-redundant decomposition of T.

Assume that A is redundant, and T_0 belongs to the span of $v_d(A \setminus \{P_j\})$, for some $j \leq i$, say $j = 1$. Then $T_0 = \gamma_2 v_d((P_2)) + \cdots + \gamma_r v_d((P_r))$, for some choice of the coefficients γ_j. Since $v_d(A)$ is linearly independent, because A is non-redundant, this is only possible if $a_1 - b_1 = 0$. So there exists a proper subset $A' \subset A$ which provides a non-redundant decomposition of T_0, together with B'. Moreover $\ell(A') \leq \frac{dn+1}{2}$ and $A' \cap B' = \emptyset$.

From the fact that $A' \subset A$, we also know that the Kruskal rank of A' is $\min\{k+1, \ell(A')\}$.

From Proposition 4.1.2 the existence of B' yields a contradiction.

We conclude that T is identifiable.

□

Chapter 5

Extending Kruskal criterion

In this chapter we want to extend the bound given by the Reshaped Kruskal criterion 2.0.40 using a strategy similar to the one used in our proof of Kruskal criterion. We present here the results obtained in [6].

The first case we want to analyse is an extension of the Kruskal criterion for the case of symmetric tensors of type $3 \times \cdots \times 3$ (7 times), or equivalently we want to study forms T of degree 7 in 3 variables. We suppose to know a non-redundant decomposition $A = \{P_1, \dots, P_{11}\} \subset \mathbb{P}^2$ of T with $\ell(A) = 11$ and we will give a criterion for the identifiability of A.

We need some assumptions on the points of A. In particular we will make the hypothesis that no 3 points of A are aligned and that no 10 points of A are contained in a cubic curve. We can describe these two conditions in terms of Kruskal ranks as follows.

- The Kruskal rank k_1 of A is 3;
- the third Kruskal rank k_3 of A is 10 or equivalently the Kruskal rank of $v_{2,3}(A)$ is 10.

Notice that these assumptions hold in a Zariski open subset of $(\mathbb{P}^2)^{11}$.

Remark 5.0.1. The Reshaped Kruskal criterion 2.0.40 does not cover the case of tensors of degree 7 in 3 variables and $\ell(A) = 11$. Consider the partition of 7 of the form $a + b + c$ and compute the corresponding Kruskal ranks in order to find the bound given by the Theorem.

Consider the partition $7 = 3 + 3 + 1$. Under our hypothesis the given decomposition A of T satisfies $k_3 = 10$, $k_1 = 3$, but:

$$11 \not\leq \frac{10 + 10 + 3 - 2}{2}.$$

Another exemplifying case is when we take the partition $7 = 2 + 2 + 3$. In this case we know that $k_3 = 10$ and that $k_2 \leq 6$. On the other hand we have that

$$11 \not\leq \frac{6 + 6 + 10 - 2}{2}.$$

A similar computation holds for any other partition of 7.

The goal of this section is to present the proof of the following result originally presented in Theorem 5.1 of [6].

Theorem 5.0.2. *Let T be a symmetric tensors and let A be a decomposition of T of length $\ell(A) = 11$ such that the first Kruskal rank k_1 of A is equal to 3 and the third Kruskal rank of A is 10. Then T has rank 11 and it is identifiable.*

The proof of this theorem is based on the fact that under this hypothesis we have a complete knowledge of the first difference of the Hilbert function of A. Indeed we can represent Dh_A using the following table.

j	0	1	2	3	4	5	6
$Dh_A(j)$	1	2	3	4	1	0 ...	

Our strategy is similar to the one used in the previous chapter. The first fact that we prove is that if there is another decomposition B of T with $\ell(B) \leq \ell(A)$ then A and B cannot be disjoint.

Proposition 5.0.3. *Let $Z = A \cup B$ the union of two non-redundant decompositions for a symmetric tensors T in 3 variables. Suppose also that $\ell(A) = 11$ and that $\ell(B) \leq \ell(A)$. Then Z cannot satisfy $CB(7)$ so, from Lemma 3.1.9, A and B cannot be disjoint.*

Proof. Assume by contradiction that Z satisfies $CB(7)$. From Lemma 3.0.45, Proposition 3.0.49 and the fact that $A \subset Z$ we know that Hilbert function of Z can be represented as follows.

j	0	1	2	3	4	5	6	7	8	...
$Dh_Z(j)$	1	2	3	4	h_4	h_5	h_6	h_7	h_8	...

where h_8 is different from 0. Moreover from Proposition 3.0.46 we know that also $h_4, h_5, h_6, h_7 > 0$.

By Proposition 3.1.8 we know that:

$$h_5 + h_6 + h_7 + h_8 \geq 1 + 2 + 3 + 4 = 10.$$

So, from the fact that $\ell(Z) \leq 22$, we have that $h_4 \leq 2$.

But then, by Proposition 3.0.46, $h_5, \ldots, h_8 \leq 2$. Thus $h_5 + h_6 + h_7 + h_8 \leq 2 + 2 + 2 + 2 = 8$, a contradiction. Thus $CB(7)$ cannot hold and the conclusion follows directly from Lemma 3.1.9. $\square$

Now, using an inductive strategy, we can prove the main Theorem 5.0.2 of this section.

Proof of Theorem 5.0.2. Suppose we have another non-redundant decomposition B of T of length smaller or equal to A. By 5.0.3 we know that A and B cannot be disjoint. and in particular by Lemma 3.1.9 we know that $Z = A \cup B$ cannot satisfy $CB(7)$.

After rearranging the points of A, B, we may assume

$$A = \{P_1, \ldots, P_i, P_{i+1}, \ldots, P_{11}\} \qquad B = \{P_1, \ldots, P_i, P'_{i+1}, \ldots, P'_q\},$$

where $q = \ell(B) \leq 11$, $\ell(A \cap B) = i < q$, and the set $B_0 = \{P'_{i+1}, \ldots, P'_q\}$ is disjoint from A, i.e. $B_0 = B \setminus A$.

Now, as we did in Proposition 4.2.1, we construct another tensor T_0 starting from A and B as follows. By definition of decomposition we can write T as:

$$\begin{aligned} a_1 v_7((P_1)) + \cdots + a_{11} v_7((P_{11})) &= T \\ &= b_1 v_7((P_1)) + \cdots + b_i v_7((P_i)) + b_{i+1} v_7((P'_{i+1})) + \cdots + b_q v_7((P'_q)) \end{aligned}$$

where none of the coefficients a_i, b_i is 0 (otherwise A and B cannot be non-redundant) and where we indicate with (P_i) a choice of affine coordinates for the projective point P_i.

We define T_0 as:

$$\begin{aligned} T_0 = (a_1 - b_1) v_7((P_1)) + \cdots + (a_i - b_i) v_7((P_i)) + a_{i+1} v_7((P_{i+1})) + \cdots + a_{11} v_7((P_{11})) \\ = b_{i+1} v_7((P'_{i+1})) + \cdots + b_q v_7((P'_q)). \quad (5.1) \end{aligned}$$

Notice in particular that $T = T_0 + b_1 v_7((P_1)) + \cdots + b_i v_7((P_i))$.

We want to prove that both decompositions of T_0 have less than 11 summands. We already know that $\ell(B_0) = \ell(B \setminus A) \leq 10$ since $A \cap B \neq \emptyset$. Repeating the argument used to prove Lemma 3.1.9 we can prove that there exists a point $P \in Z$ such that

$$< v_7(A \setminus \{P\}) > \cap < v_7(B_0 \setminus \{P\}) > = < v_7(A) > \cap < v_7(B_0) > .$$

As a consequence T_0 belongs to $< v_7(A \setminus \{P\}) > \cap < v_7(B_0 \setminus \{P\}) >$. P cannot belong to $< B_0 >$ otherwise we would have that T is spanned by $v_7(B \setminus \{P\})$, which contradicts the fact that B is non-redundant. Similarly, if $P = P_j$ with $j > i$, i.e. if $P \in A \setminus B$ then T is spanned by $v_7(A \setminus \{P\})$, which contradicts the fact that A is non-redundant. The only possibility left is that P is a point in $A \cap B$. We assume without loss of generality that $P = P_1$.

Thus we can write T_0 as $T_0 = c_2 v_7((P_2)) + \cdots + c_{11} v_7((P_{11}))$. Notice that the coefficient c_i can be chosen equal to the one of equation 5.1. Indeed if $a_1 - b_1 \neq 0$, this implies that $v_7(A)$ is linearly dependent, which contradicts the fact that A is non-redundant. Thus $a_1 = b_1$, so that T_0 is spanned by $v_7((P_2)), \ldots, v_7((P_{11}))$ and can be written as:

$$T_0 = (a_2 - b_2) v_7((P_2)) + \cdots + (a_i - b_i) v_7((P_i)) + a_{i+1} v_7((P_{i+1})) + \cdots + a_{11} v_7((P_{11}))$$

where not all the coefficients $(a_j - b_j)$ are necessarily different from 0.

Thus T_0 has a non-redundant decomposition $A' \subset A$ such that $11 - i \leq \ell(A') \leq 10$. Moreover T_0 has a second decomposition, B_0, of length $\ell(B_0) = k - i \leq 11 - i \leq \ell(A')$.

Since $k_1 = 3$ and $k_3 = 10$ and A' consists of at most 10 points of A, we have that both A' and $v_3(A')$ are linearly independent. In other words we can say that the third Kruskal rank k'_3 of $v_3(A')$ is $\ell(A')$, while the Kruskal rank k'_1 of A' is $\min\{\ell(A'), 3\}$. Moreover $\ell(A') > 1$, otherwise we would have $i = 10$, $k = 11$ and $T_0 = a_{11} v_7((P_{11})) = b_{11} v_7((P'_{11}))$, which means that, as projective points, $P_{11} = P'_{11}$, a contradiction.

Thus one computes:

$$\ell(A') \leq \frac{k'_3 + k'_3 + k'_1 - 2}{2}.$$

It follows that by the Reshaped Kruskal criterion 2.0.40 T_0 is identifiable. Thus the existence of B_0 yields a contradiction.

□

5.1 Decompositions on cubics

The arguments used to prove the identifiability of symmetric tensors of degree 7 in 3 variables of rank 11 can be slightly generalized to prove the identifiability of other classes of symmetric tensors. In this paragraph, we present another case proven originally in Proposition 6.4 of [6].

Remark 5.1.1. We will consider symmetric tensors T of degree $7+2k$ in three variables, which satisfy the following assumptions:

- Fix the integer $k \geq 0$ and let T be a symmetric tensor of degree $d = 7+2k$ in 3 variables, with a decomposition $A = \{P_1, \ldots, P_r\} \subset \mathbb{P}^2$, of length $r = \ell(A) \leq 10 + 3k$ such that A is contained in a plane cubic curve C (thus the third Kruskal rank is not maximal).
- Assume that the Kruskal rank of A is $k_1 = \min\{3, r\}$ and the $(k+3)$-th Kruskal rank k_{k+3} of A is equal to $\min\{r, 3k+9\}$.

Under this hypothesis we have a complete knowledge of the first difference of the Hilbert function of A.

Remark 5.1.2. Assume $r = 3k + 10$.

Since $A \subset \mathbb{P}^2$ is contained in a cubic curve, the difference of the Hilbert function of A satisfies $Dh_A(0) = 1$, $Dh_A(1) \leq 2$, $Dh_A(2) \leq 3$ and also $Dh_A(3) \leq 3$. Moreover, by Proposition 3.0.46, we also know $Dh_A(i) \leq 3$ for $i > 3$.

From the fact that $k_1 = 3$ follows directly that the Hilbert function $h_A(1)$ is equal to 3 hence $Dh_A(1) = 2$. Moreover we can show that $Dh_A(i) = 3$ for $2 \leq i \leq k+3$. Assume by contradiction that there exist an $i \in \mathbb{N}$ such that $2 \leq i \leq k+3$ and $Dh_A(i) < 3$. Then by definition of first difference of Hilbert function we have:

$$h_A(k+3) = \sum_{j=0}^{k+3} Dh_A(j) \leq 1 + 2 + 3(k+1) + 2 < 3k+9,$$

which contradicts the assumption $k_{k+3} = 3k + 9$.

It follows that $\sum_{j=0}^{k+3} Dh_A(j) = 3k+9$. As a consequence $Dh_A(k+4)$ cannot be 0, otherwise by Proposition 3.0.46 we would have $Dh_A(j) = 0$ for $j \geq k+4$ hence $\sum_{j=0}^{\infty} Dh_A(j) < 3k+10$, a contradiction. It follows that $Dh_A(k+4) = 1$ and $Dh_A(j) = 0$ for $j > k+4$. In particular, we get $h_A(2) = 6$ and $h_A(3) = 9$, which means that A is contained in no conics and in exactly one cubic curve.

Now we use the same strategy used before to exclude the existence of a second decomposition B of T of length $\ell(B) \leq 3k + 10$. So assume, by contradiction, that B exists, and assume, as above, that B is non-redundant. Define as above $Z = A \cup B$, so that $\ell(Z) \leq 6k + 20$.

Using Lemma 3.1.9, we can prove that the intersection $A \cap B$ cannot be empty.

Proposition 5.1.3. *The Cayley-Bacharach property $CB(d)$ cannot hold for $Z = A \cup B$. Thus $A \cap B \neq \emptyset$.*

Proof. The proof is similar to the one of Proposition 5.0.3 and consist on a computation based on Theorem 3.1.8 and Lemma 3.1.9. For a complete proof see Proposition 6.3 of [6]

□

Now we can prove the main results of this section (see Proposition 6.4 of [6]).

Theorem 5.1.4. *Let T be a symmetric tensors of degree $7+2k$ in three variables, $k \in \mathbb{N}$ and let $A \subset \mathbb{P}^2$ be a decomposition of T of length $\ell(A) \leq 10+3k$.*

Assume that A is contained in a plane cubic curve C. and that the Kruskal rank of A is $k_1 = \min\{3, r\}$ and the $(k+3)$-th Kruskal rank k_{k+3} of A is equal to $\min\{r, 3k+9\}$. Then T is identifiable and has rank $r = \ell(A)$.

Proof. Suppose by contradiction that there exists another decomposition B of T. We know from Proposition 5.1.3 that $A \cap B$ cannot be empty. Then, the same procedure of Theorem 5.0.2 yields a contradiction.

□

Remark 5.1.5. We can apply the previous procedure even for the case $k = -1$. In this case we get ternary forms of degree $7+2k = 5$ and rank $r \leq 10+3k = 7$. In particular, notice that 7 is the rank of a generic quintic form, by [1]. Notice also that, for $k = -1$, the assumption that A is contained in a cubic is unnecessary, for any set of cardinality ≤ 7 in $\mathbb{P}^2$ lies in a cubic. Thus, we get back, from our procedure, the classically well known fact that a general ternary form T of degree 5 has a unique decomposition with 7 powers of linear forms.

Indeed, we can give a more precise notion of *generality* for T: the uniqueness holds if a decomposition A of T has Kruskal ranks $k_1 = 3$ and $k_2 = 6$.

5.2 Reducing the computational cost, via the Terracini Lemma

One of the problems of the reshaped Kruskal criterion is that it can be rather demanding from a computational point of view. So we would like to use another strategy to verify if a tensor is identifiable that works at least in the same range of Kruskal. In order to do that we will user a nice consequence of the Terracini Lemma 1.3. The new aspect of this result is that it takes care of tensors whose Kruskal ranks are not maximal. This enables to construct algorithms which guarantee the identifiability, with a computational cost much smaller than algorithms exclusively based on the Kruskal ranks.

In this paragraph we will present results originally proven in [40].

In particular, the goal of this section is to describe an algorithm that guaranties the identifiability of certain symmetric tensors T such that, fixed a natural number $q \in \mathbb{N}$, we have:

1) T is a tensor of degree $d = 8+2q$ in three variables;
2) we know a priori a decomposition $A = \{P_1, \dots, P_r\} \subset \mathbb{P}^2$ of T of length $r = \ell(A) \leq 11+3q$.

Notice that it is possible to prove the identifiability of a symmetric tensor T satisfying condition 1) and 2) using the Reshaped Kruskal criterion 2.0.40, if we just add some hypothesis on the Kruskal ranks of the decomposition A of T.

Proposition 5.2.1. *Let T be a symmetric tensor of degree $d = 8 + 2q$ in three variables and let A be a non-redundant decomposition of T such that fixed $q \in \mathbb{N}$*

- *$A = \{P_1, \dots, P_r\} \subset \mathbb{P}^2$ is a non-redundant decomposition of T of length $r = \ell(A) \leq 11 + 3q$;*
- *the second Kruskal rank k_2 of A is $k_2 = \min\{6, r\}$,*
- *the $(q+3)$-th Kruskal rank k_{q+3} of A is $k_{q+3} \geq \{r, 3q + 9\}$.*

Then, T is identifiable.

Proof. The proof of this proposition is based on the Reshaped Kruskal criterion and it consists on an easy computation (see Proposition 1 of [40]).

□

Remark 5.2.2. Through this section, we will always assume that A is non-redundant, a condition that it is easy to check: it suffices to prove that $v_d(A)$ is linearly independent and all the coefficients a_i's of the decomposition are non-zero.

In the next remark we estimate the computational cost of applying the Reshaped Kruskal criterion.

Remark 5.2.3. Let T be a symmetric tensor of degree $d = 8 + 2q$ and let A be a decomposition of T of length $\ell(A) \leq 11 + 3q$.

In order to apply Proposition 5.2.1 we have to check that the hypothesis for A are satisfied, so we need to compute that:

- $k_2(A) = \min\{r, 6\}$;
- $k_{q+3}(A) \geq \min\{r, 9 + 3q\}$;

and verify that the inequality $r \leq \frac{k_1+k_2+k_3-2}{2}$ holds.

We determine the rank of matrices derived by the coordinates of the points of some Veronese images of A.

The standard method to find the rank of a matrix is the Gauss elimination method. The computational cost of computing the rank of a matrix $\mathbb{C}^{m \times n}$ using this method is of $\frac{2}{3}m^2 \cdot n$ flops (see Chapter 3.3 of [42]). In particular, if the matrix is a $n \times n$ square matrix then the Gauss elimination method has a cost in the order of $\frac{2}{3}n^3$.

In order to verify that $k_2(A) = \min\{6, r\}$ we have to compute the rank of all the 6×6 sub-matrices of the matrix $[v_2(P_1), \dots, v_2(P_r)]$. So, we need to compute the rank of $\binom{r}{6} \approx \frac{r^6}{6!}$ matrices. By using the Gauss elimination algorithm, we see that the computational cost is of about

$$\frac{2 \cdot 6^3}{3} \cdot \frac{r^6}{6!} = \frac{r^6}{5}.$$

In the same way, to verify that $k_{q+3}(A) \geq \min\{9 + 3q, r\}$ we may have to compute the rank of all the submatrices $\binom{q+5}{2} \times 3q + 9$ of the matrix:

$$[v_{q+3}(P_1), \dots, v_{q+3}(P_r)].$$

So, the worst case is when $r = 11 + 3q$, where we have to find the rank of $\binom{r}{2} \approx \frac{r^2}{2!}$ matrices and the computational cost is of about

$$\frac{2}{3} \cdot (r-2)^2 \cdot \binom{r/3+4}{2} \cdot \frac{r^2}{2} \approx \frac{r^6}{54}.$$

Hence, for a general set A which verifies the conditions of Proposition 5.2.1, the total cost expected for the computation is about

$$\frac{59r^6}{270}.$$

Remark 5.2.4. In Proposition 5.2.1 instead of $k_{q+3} \geq \min\{r, 3q+9\}$ we can suppose that the $(q+3)$-th Kruskal rank is exactly equal to $\min\{r, 3q+9\}$. Indeed in Remark 4 of [40] it is proved that the bound obtained by the Reshaped Kruskal criterion in the case $k_{q+3} = \min\{r, 3q+9\}$ is maximal. Notice also that, if we choose k_{q+3} exactly equal to $\min\{r, 3q+9\}$, nothing changes from a computational point of view; in particular the computational cost remains the same. So, from now on we will consider k_{q+3} exactly equal to $\min\{r, 3q+9\}$.

Fix $q \geq 0$ and let T be a symmetric tensor of degree $d = 8 + 2q$ in three variables, with a decomposition $A = \{P_1, \ldots, P_r\} \subset \mathbb{P}^2$ of length $r = \ell(A) \leq 11 + 3q$. We proceed by showing that in this situation T is either identifiable or has an infinite family of decompositions. Then we will define an algorithm to detect if such an infinite family of decompositions exists.

We will prove that T is either identifiable or has an infinite family of decompositions in three steps.

Step 1- We start our analysis with the case in which we cannot find $5 + q$ points of A aligned or $9 + 2q$ points of A in a conic curve.

Step 2- Then, we will analyse separately the cases in which there are $5 + q$ points of A aligned and the case in which there are $9 + 2q$ points of A contained in a conic curve

Step 3- Finally, we will use an inductive strategy to prove the main result of the section.

Proposition 5.2.5. *Suppose that $r = \ell(A)$ is at most $11 + 3q$ and A is non-redundant. If A does not contain $5+q$ points on a line or $9+2q$ points in a conic curve, then $CB(d)$ cannot hold for $A \cup B$ where B is another non-redundant decomposition of T of length $\ell(B) \leq r$. Thus, $A \cap B \neq \emptyset$.*

In order to prove Proposition 5.2.5 we will use Theorem 1.5.1 of [14].

Theorem 5.2.6. *Let T be a symmetric tensor of degree d in $n + 1$ variables, with a non-redundant decomposition A. If $\ell(A) \leq \frac{d+1}{2}$ then A is minimal and T is identifiable.*

Proof. We give a short proof of this result which was originally proven in Theorem 1.5.1 of [14]. Suppose by contradiction to have another decomposition B of T such that $\ell(B) \leq \ell(A)$. Call $Z = A \cup B \subset \mathbb{P}^n$. We know from Proposition 3.0.49 that $Dh_Z(d+1) > 0$. Thus, from the fact that $Dh_Z(0) = 1$, Proposition 3.0.46 and the fact that $\ell(B) \leq \ell(A)$, follows directly that there exist an

$i \leq d+1$ such that $Dh_Z(i) = 0$. This contradicts either Proposition 3.0.46 or Proposition 3.0.49.

□

Now, we have ready to prove Proposition 5.2.5.

Proof of Proposition 5.2.5 : Set $A = \{P_1, P_2, \dots P_r\}$ a non-redundant decomposition of A contained in a cubic curve such that $r \leq 11 + 3q$.

We analyse the following cases:

1) $r < 5 + q$.

2) $5 + q \leq r < 9 + 2q$.

3) $9 + 2q \leq r \leq 11 + 3q$.

If $r < 5 + q$ the results follows directly from Theorem 5.2.6.

Suppose $5 + q \leq r < 9 + 2q$. From the fact that there are not $5 + q$ aligned points of A we know that A itself is not aligned. Assume that $CB(d)$ holds for $Z = A \cup B$. Thus, by Theorem 3.1.8 we have

$$\sum_{j=0}^{1} Dh_Z(j) \leq \sum_{j=2q+8}^{2q+9} Dh_Z(j).$$

So, $\sum_{j=2q+8}^{2q+9} Dh_Z(j) \geq \sum_{j=0}^{1} Dh_Z(j) \geq \sum_{j=0}^{1} Dh_A(j) = 3$ (Lemma 3.0.45) then from Proposition 3.0.46 we have that $Dh_Z(2q+8)$ is at least equal to 2 otherwise $Dh_A(2q+8) + Dh_A(2q+9)$ would be less or equal to 2. From the fact that A is not aligned we know that $Dh_Z(1) \geq Dh_A(1) = 2$. We get:

$$\begin{aligned} 18 + 4q > \ell(Z) \geq \sum_{j=0}^{2q+9} Dh_Z(j) = \\ = \sum_{j=0}^{1} Dh_Z(j) + \sum_{j=2}^{2q+7} Dh_Z(j) + \sum_{j=q+8}^{2q+9} Dh_Z(j) \geq \\ \geq 2\sum_{j=0}^{1} Dh_Z(j) + \sum_{j=2}^{2q+7} Dh_Z(j) \geq 6 + \sum_{j=2}^{2q+7} Dh_Z(j). \end{aligned}$$

Thus $\sum_{j=2}^{2q+7} Dh_Z(j) < 12 + 4q$. But then, by Proposition 3.0.46 and from the fact that $Dh_Z(2) \geq 2$ and $Dh_Z(2q+7) \geq 2$, we have:

$$12 + 4q > \sum_{j=3}^{2q+7} Dh_Z(j) \geq 2 \cdot (2q + 6)$$

a contradiction.

The second claim follows by Lemma 3.1.9 applied to A, B and T.

Suppose now $9 + 2q \leq r \leq 11 + 3q$. From the fact that A does not contain $9 + 2q$ points in a conic, we know that also A lies in no conic curves. Thus, we know from the definition of the Hilbert function and Lemma 3.0.45 that $Dh_Z(2) \geq Dh_A(2) = 3$.

Moreover, we know that $Dh_A(4 + q) < 3$. In fact, if $Dh_A(4 + q) \geq 3$ then we have that:

$$\ell(A) = \sum_{j=0}^{\infty} Dh_A(j) \geq \sum_{j=0}^{4+q} Dh_A(j) \geq 3(q + 4) > \ell(A),$$

a contradiction.

If $Dh_A(4 + q) = 0$ then $CB(d)$ cannot hold for Z. In fact, suppose by contradiction that $CB(d)$ holds for Z. Then, by Theorem 3.1.8 we have:

$$\ell(A) = \sum_{j=0}^{3+q} Dh_A(j) \leq \sum_{j=0}^{3+q} Dh_Z(j) \leq \sum_{j=6+2q}^{9+2q} Dh_Z(j).$$

From $\ell(A) \geq \ell(B)$ we have $2\ell(A) \geq \ell(Z)$. Thus, we can find the following inequality:

$$2\ell(A) \leq \sum_{j=0}^{3+q} Dh_Z(j) + \sum_{j=6+2q}^{9+2q} Dh_Z(j) \leq \ell(Z). \tag{5.2}$$

So, we have $2\ell(A) = \ell(Z)$ and as a consequence we have that:

$$\sum_{j=0}^{3+q} Dh_Z(j) + \sum_{j=6+2q}^{9+2q} Dh_Z(j) = \ell(Z).$$

We can notice that $Dh_Z(4+q)$ must be equal to 0 and this is a contradiction. In fact, from Proposition 3.0.46 we have that if $Dh_Z(4 + q) = 0$ then $Dh_Z(j) = 0$ for all $j \geq 4 + q$. Thus, we can conclude that $CB(d)$ does not hold for Z. As before, the second claim of the statement follows by Lemma 3.1.9 applied to A, B and T.

Suppose now $Dh_A(4 + q) = 1$. As before, from Proposition 3.0.46 we know that also $Dh_A(3 + q) \neq 0$.

If $Dh_A(3 + q) = Dh_A(4 + q) = 1$ then, from Theorem 3.0.55 we have that $5 + q$ points of A are aligned. So, A does not satisfies the hypothesis of the proposition.

If $Dh_A(3 + q) = 2$ and $Dh_A(4 + q) = 1$ we have two possibilities: by Proposition 3.0.46 $Dh_A(5 + q)$ can be equal either to 1 or 0 .

If $Dh_A(5 + q) = 1$ then, from Theorem 3.0.55 we have that $6 + q$ points of A are contained in a line and so, A does not satisfies the hypothesis of the proposition.

If $Dh_A(5 + q) = 0$ then $CB(d)$ cannot hold for Z.

In fact, suppose by contradiction that $CB(d)$ holds for Z. From the fact that A is contained in no conic curve, from $Dh_A(4 + q) = 1$ and Proposition 3.0.46 we have:

$$\ell(A) = \sum_{j=0}^{4+q} Dh_A(j) = Dh_A(0) + Dh_A(1) + Dh_A(2) + \sum_{j=3}^{4+q} Dh_A(j) \geq$$
$$\geq 1 + 2 + 3 + 2 + q = 8 + q.$$

Thus by Theorem 3.1.8 we have that

$$\ell(A)=\sum_{j=0}^{4+q} Dh_A(j)\leq \sum_{j=0}^{4+q} Dh_Z(j)\leq \sum_{j=5+q}^{9+2q} Dh_Z(j)\leq \ell(Z).$$

So, we have that $\sum_{j=5+q}^{9+2q} Dh_Z(j)\geq 8+q$ and as a consequence we have that $Dh_Z(5+q)>1$ otherwise we would have by Proposition 3.0.46 that $\sum_{j=5+q}^{9+2q} Dh_Z(j)=5+q$.

Moreover, from the fact that $\ell(A)\geq \ell(B)$ we have that $\ell(Z)\leq 2\ell(A)$ and as a consequence:

In particular we have

$$2\ell(A)\leq \sum_{j=0}^{4+q} Dh_A(j)+\sum_{j=5+q}^{9+2q} Dh_Z(j)\leq \sum_{j=0}^{4+q} Dh_Z(j)+\sum_{j=5+q}^{9+2q} Dh_Z(j)= \\ =\sum_{j=0}^{9+2q} Dh_Z(j)\leq \ell(Z)\leq 2\ell(A) \quad (5.3)$$

and as a consequence we have $\ell(B)=\sum_{j=5+q}^{9+2q} Dh_Z(j)=r$, $\sum_{j=0}^{4+q} Dh_A(j)=\sum_{j=0}^{4+q} Dh_Z(j)$. In particular we have $Dh_A(4+q)=Dh_Z(4+q)=1$. This is a contradiction. In fact, from Proposition 3.0.46 we cannot have $Dh_Z(4+q)=1$ and $Dh_Z(5+q)>1$. So $CB(d)$ cannot hold for Z.

As before, the second claim of the statement follows by Lemma 3.1.9 applied to A, B and T.

If $Dh_A(4+q)=2$ we can have either $Dh_A(3+q)=2$ or $Dh_A(3+q)=3$. If $Dh_A(3+q)=2$ then from Theorem 3.0.55 we have that $9+2q$ points of A are contained in a conic curve, so the hypothesis of the proposition are not satisfied by A.

If $Dh_A(3+q)=3$ then $CB(d)$ cannot hold for Z. In fact, suppose by contradiction that $CB(d)$ hold for Z. As before, from the fact that $Dh_A(4+q)=2$, from $Dh_A(3+q)=3$ and Proposition 3.0.46 we have that $\ell(A)\geq 11+3q$ so, by hypothesis, we have that $\ell(A)=11+3q$. Moreover, we know from Proposition 3.0.46 and Lemma 3.0.45 that $Dh_Z(3)\geq Dh_A(3)\geq 3$.

Thus, by Theorem 3.1.8 we have:

$$9\leq \sum_{j=0}^{3} Dh_Z(j)\leq \sum_{j=2q+6}^{d+1} Dh_Z(j). \quad (5.4)$$

We know that $Dh_Z(3)\geq Dh_A(3)=3$ and since $\sum_{j=0}^{3} Dh_Z(j)\geq \sum_{j=0}^{3} Dh_A(j)=9$, we get $\sum_{j=2q+6}^{d+1} Dh_Z\geq 9$. Furthermore, from the fact that Dh_Z is not increasing we have $Dh_Z(2q+6)\geq 3$ (otherwise we would have $\sum_{j=2q+6}^{d+1} Dh_Z\leq 8$, by Proposition 3.0.46).

Using the inequality 5.4, we have

$$6q+22 \geq \ell(Z) \geq \sum_{j=0}^{2q+9} Dh_Z(j) =$$

$$\sum_{j=0}^{3} Dh_Z(j) + \sum_{j=4}^{2q+5} Dh_Z(j) + \sum_{j=2q+6}^{2q+9} Dh_Z(j)$$

$$\geq 9 + \sum_{j=4}^{2q+5} Dh_Z(j) + 9,$$

thus $\sum_{j=4}^{2q+5} Dh_Z(j) \leq 6q+4$. Then, by Proposition 3.0.46, since $Dh_Z(3) \geq 3$ and $Dh_Z(2q+6) \geq 3$, we have $\sum_{j=4}^{2q+5} Dh_Z(j) \geq 3(2q+2)$ for some i. Thus:

$$6q+4 \geq \sum_{j=4}^{2q+5} Dh_Z(j) \geq 6q+6,$$

a contradiction.

As before, the second claim of the statement follows by Lemma 3.1.9 applied to A, B and T.

□

Next, we analyse the behavior of a symmetric tensor T for which Proposition 5.2.5 does not hold. In particular, given a decomposition $A = \{P_1, P_2, \dots, P_r\}$ of T, we analyse the following cases:

1) There is a subset A' of A such that A' is aligned and $\ell(A') \geq 5+q$.
2) There is a subset A' of A such that $\ell(A') \geq 9+2q$ and A' is contained in a conic curve.

In case 1 we are going to prove that T has always an infinite family of decompositions. To do that, we recall Proposition 4.9 of [17].

Proposition 5.2.7. *Assume that a decomposition A of length $\ell(A) = r$ of T is contained in a projective curve $C \in \mathbb{P}^n$ which is mapped by ν_d to a space $\mathbb{P}^m$, with $m < 2r-1$. Then there exists positive dimensional family of different decompositions $\{A_t\}$ of T, such that $A_0 = A$.*

Proof. By definition of decomposition we have that $T \in < v_d(A) >$ and moreover by hypothesis we know that $< v_d(A) > \subset < v_d(C) > \subset \mathbb{P}^m$. The abstract secant variety of C, $AS_r(C)$, has dimension equal to $2r-1$ (see Chapter 1) thus all the fibers of the projection map $AS_r(C) \to \mathbb{P}^m$ are positive dimensional (see Chapter 1) and this concludes the proof.

□

As a consequence, we can prove the following lemma.

Lemma 5.2.8. *Given a symmetric tensor T and a decomposition $A = \{P_1, P_2, \dots, P_r\}$ of T, such that there exists a subset A' of A with $\ell(A') = r' \geq 5+q$ and A' is aligned, then there exists a positive dimensional family $\{A_t\}$ of decompositions of T such that $A_0 = A$.*

Proof. We may assume that $A' = \{P_1, P_2, \dots, P_{5+q}\} \subset L$ such that L is a line. If $T = a_1 v_d(P_1) + a_2 v_d(P_2) + \cdots + a_r v_d(P_r)$ we define T_0 as follows:

$$T_0 = a_1 v_d(P_1) + a_2 v_d(P_2) + \cdots + a_{5+q} v_d(P_{5+q}).$$

The image of L through v_d is the composition of v_1 and v_d applied to $\mathbb{P}^1$. Thus, $v_d(C)$ is embedded in a $\mathbb{P}^d$.

Moreover, we have that the inequality

$$d = 8 + 2q \leq 2r' - 1$$

holds for all r' such that $5 + q \leq r' \leq 11 + 3q$. In fact:

$$2r' - 1 \geq 2(5 + q) - 1 > 8 + 2q.$$

Thus, from Proposition 5.2.7, T_0 has an infinite family of decompositions. If we add $a_{6+q} v_d(P_{6+q}) + \cdots + a_r v_d(P_r)$ to all the decompositions A'_t of T_0 we find an infinite family of decompositions A_t for T.

□

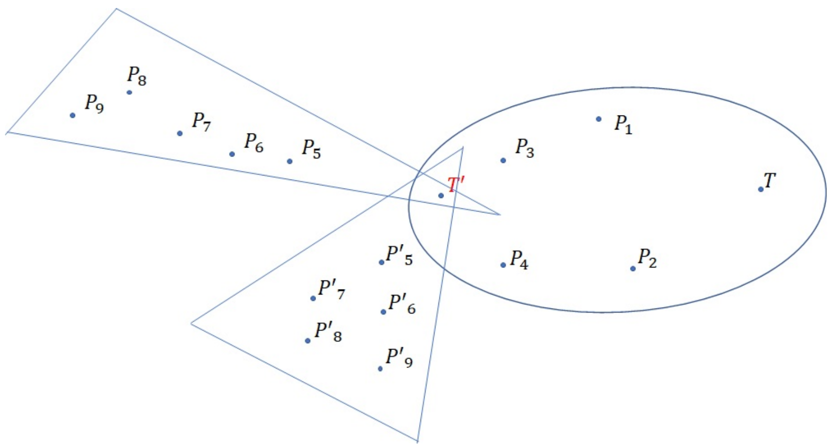

Figure 5.1

Case 2 is similar to case 1. When there are at least $9 + 2n$ points of A contained in a conic we can find the existence of an infinite family of decompositions for T.

Lemma 5.2.9. *Given a symmetric tensor T and a decomposition $A = \{P_1, P_2, \dots, P_r\}$ of T, such that there exists a subset A' of A with $\ell(A') = r' \geq 9 + 2q$ and A' is contained in a conic curve, then there exists a positive dimensional family $\{A_t\}$ of decompositions of T such that $A_0 = A$.*

Proof. As before, we may assume that $A' = \{P_1, P_2, \dots P_{9+2q}\} \subset C$ such that C is a conic curve.

If $T = a_1 v_d(P_1) + a_2 v_d(P_2) + \cdots + a_r v_d(P_r)$ we define T_0 as follows:

$$T_0 = a_1 v_d(P_1) + a_2 v_d(P_2) + \cdots + a_{9+2q} v_d(P_{9+2q}).$$

As before, a conic curve C is a rational normal curve, so it is an image through v_2 of $\mathbb{P}^1$. Moreover, the image of C through v_d is the composition of v_2 and v_d applied to $\mathbb{P}^1$. So $v_d(C)$ is embedded in a $\mathbb{P}^{2d}$ and, as before, the inequality $2d = 16+4q \leq 2r'-1$ holds for all r' such that $9+2q \leq r' \leq 10+3q$. In fact:

$$2r' - 1 \geq 2(9 + 2q) - 1 > 16 + 4q.$$

Thus, from Proposition 5.2.7 T_0 has an infinite family of decompositions. If we add $a_{10+2q}v_d(P_{10+2q})+\cdots+a_r v_d(P_r)$ to all the decompositions A'_t of T_0 we find an infinite family of decompositions A_t for T.

□

Now, we are able to describe the behaviour of all symmetric tensors of degree $r \leq 11 + 3q$ with a decomposition contained in at least one cubic curve. This result extends a similar result proven in [12].

Theorem 5.2.10. *Given a symmetric tensor T of rank $r = \ell(A) \leq 11+3q$ such that $A = \{P_1, P_2, \ldots, P_r\}$ is a non-redundant decomposition of T, then either A is unique or there is an infinite family A_t of decompositions of length $11 + 3q$, such that $A_0 = A$.*

Proof. Suppose by contradiction that there exists another decomposition B of T. By Proposition 5.2.5 we know that A and B are disjoint. Thus, the same inductive strategy of Theorem 5.0.2 yields a contradiction.

□

The previous result cannot be extended directly to higher value of r. In fact, we can find an example of a tensor T of rank 12 in degree 8 that is not identifiable and for which there are exactly two decompositions. This follows from a result proved in [18].

Example 5.2.11. Take T a symmetric tensor and $A = \{P_1, P_2, \ldots, P_{12}\}$ a decomposition of T in degree 8 contained in an unique irreducible, smooth plane cubic curve C. This case is outside our numerical bound for the length of the decomposition. Then T has two different decompositions (so that our range is sharp).

The proof is the same of Theorem 5.1 of [18] and it is a direct consequence of Theorem 2.4, Theorem 2.10 and Proposition 5.2 of [18].

From our point of view, we can prove the claim as follows. From the fact that A is contained in a unique smooth cubic curve C, we know from Theorem 3.1.8 and from Theorem 3.0.55 that all the other decompositions B of T lie in C and, moreover, the function Dh_Z of $Z = A \cup B$ is symmetric, i.e. it is:

i	0	1	2	3	4	5	6	7	8	9	10 ...
$Dh_Z(i)$	1	2	3	3	3	3	3	3	2	1	0 ...

This implies that Z is a complete intersection of the cubic C with a curve C' of degree 8 (this follows directly from the fact that C irreducible. For a complete proof see the main theorem of [24]). Furthermore, we obtain that $\ell(B)$ is 12 and the intersection of the span of $v_8(A)$ with the span of $v_8(B)$ is only T, because $\sum_{i=9}^{\infty}(Dh_Z(i)) = 1$ (see section 6 of [7]). One computes that the sets of 12 points B in the plane which, together with A, are a complete intersection of type $3, 8$, are parametrized by a projective space of dimension 11 i.e. $\mathbb{P}\left(\frac{I(A)_8}{I(C)_8}\right)$. This space maps birationally to the span of $P_1, ..., P_{12}$ (this can be obtained by a direct computation on one specific point T, see e.g. [5] Claim 4.4).

Thus a general T in the span of $P_1, \ldots, P_{12}$ has two decompositions. Notice that, in this situation, we know that the rank of T is equal to 12.

Numerically, if we know that A is not contained in a cubic curve, then we can conclude the identifiability. Thus, In order to repeat the proof of 3.7 for $\ell(A) = 3q + 12$ we must have that no subset of 10 points of A sits in a cubic curve. To control that, we need to compute k_3, which we would like to avoid, to maintain a cheap computational cost.

5.3 Excluding the existence of a family of decompositions

The main difference between Theorem 5.2.10 and Proposition 5.2.1 is that in order to check the hypothesis of Theorem 5.2.10 we do not need to compute the Kruskal ranks k_2 and k_{q+3}, but only to determine the non existence of a family of decompositions A_t for T. This can be done by means of the *Terracini test* on A. We briefly exemplify how this is possible.

We recall some definitions related to the secant varieties of a Veronese embedding already given in Chapter 1 for general varieties.

Definition 5.3.1. We denote with S_r the closure of the subset of tensors of rank r in $\mathbb{P}(Sym^d(\mathbb{C}^{n+1}))$. Moreover we indicate with $(\mathbb{P}^n)^r$ the product $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$ (r-times).

Definition 5.3.2. We define the abstract secant variety AS_r as the subvariety of $\mathbb{P}(Sym^d(\mathbb{C}^{n+1})) \times (\mathbb{P}^n)^r$ which is the Zariski closure of the set of pairs $(T, [P_1 : \cdots : P_r])$ such that the set $\{v_d(P_1), \ldots, v_d(P_r)\}$ spans a subspace of dimension $r-1$ in $\mathbb{P}(Sym^d(\mathbb{C}^{n+1}))$ and T belongs to the span of $v_d(A)$.

We define the $r-$th secant map s_r as the first projection

$$s_r : AS_r \to \mathbb{P}(Sym^d(\mathbb{C}^{n+1})).$$

We note that the image of the secant map is S_r and that the inverse image of a tensor T in the secant map is the set of decompositions of cardinality r of T. Furthermore, since $\mathbb{P}^n$ is a smooth variety, then $(\mathbb{P}^n)^r$ is smooth outside the diagonals. Thus, if U is the open set of $(\mathbb{P}^n)^r$ of sets $[P_1, \ldots, P_r]$ such that $\{v_d(P_1), \ldots, v_d(P_r)\}$ is linearly independent, then AS_r is a $\mathbb{P}^{r-1}$ bundle over U, thus $s_r^{-1}(U)$ is smooth, of dimension $(r - 1 + rn)$ $(= \dim AS_r)$.

We can now define the Terracini space τ.

Definition 5.3.3. Let T be a symmetric tensor and let $A = \{P_1, \ldots, P_r\}$ be a decomposition of T. We call the Terracini space of a decomposition A the image of the tangent space to AS_r at the point $(T, [P_1, \ldots, P_r])$ in the differential of s_r. Thus τ is a linear subspace of $\mathbb{P}^N = \mathbb{P}(Sym^d(\mathbb{C}^{n+1}))$. It is the linear space spanned by the tangent spaces to $v_d(\mathbb{P}^n)$ at the points $v_d(P_1), \ldots, v_d(P_n)$.

The Terracini Lemma (see Chapter 1 1.3) says that for a general choice of $T \in S_r$ and for $r \leq N$ the Terracini space is the tangent space to S_r at T.

Remark 5.3.4. We recall that the dimension of the Terracini space τ is naturally bounded:

$$dim(\tau) \leq (n+1)r - 1$$

and the equality means that the tangent spaces to $v_d(\mathbb{P}^n)$ at the points $v_d(P_i)$'s are linearly independent. By [1], as soon as $d > 2$ and $r > 5$ we know that for a *general* choice of the set A, the dimension of the Terracini space equals the expected dimension.

The main link with our problem is given by the following observation.

Proposition 5.3.5. *Let A be a non-redundant decomposition of T of length r and assume that there exists a non trivial family A_t of decompositions of T, such that $A_0 = A$. Then the Terracini space τ of A has dimension strictly smaller than $(n+1)r - 1$.*

Proof. A_t determines a positive dimensional subvariety W in the fiber of s_r over T (see Proposition 2.3 of [19]). Thus, there exists a tangent vector to AS_r at $(T, [A])$, where $[A]$ is the point of the symmetric product corresponding to A, which is killed by the differential of s_r at $(T, [A])$. So τ can not have maximal dimension and this concludes the proof.

□

The converse of the previous proposition does not hold in general. However, the proposition implies that, in order to exclude the existence of the family, it is sufficient to control that the dimension of the Terracini space of A attains the expected value. We can collect our results in the following.

Theorem 5.3.6. *Let A be a non-redundant decomposition of a ternary form T of degree $d = 2q + 8$. Assume that $\ell(A) \leq 3q + 11$. If the dimension of the Terracini space of A equals the expected dimension $3r - 1$, then A is minimal. Thus T has rank $r = \ell(A)$ and it is identifiable.*

In the following remark, we explain how the dimension of the Terracini space can be computed, in practice. The claims below on the structure of tangent spaces to Veronese embedding are standard, and it is based on Example 1.3.15. A complete proof of this fact can be found e.g. in [37] (see also [19]).

Remark 5.3.7. As we noticed in the introduction, a decomposition $A = \{P_1, \ldots, P_r\}$ of $T \in \mathbb{P}(Sym^d(\mathbb{C}^{n+1}))$ corresponds to the datum of r linear forms $L_1, L_2, \ldots, L_r$ in three variables.

The tangent spaces to $v_d(\mathbb{P}^n)$ at $v_d(P_i)$ can be identified with the degree d homogeneous piece of the ideal spanned by $L_i^{d-1}m$, where m is the ideal generated by the variables.

It follows that the Terracini space can be identified with the degree d homogeneous piece of the ideal spanned by

$$L_1^{d-1}m, \ldots, L_1^{d-1}m.$$

Thus, in our case, we have that the Terracini space is the ideal spanned by $L_1^{7+2q}m, \ldots L_r^{7+2q}m$.

The computation of the dimension of the d-th piece of this ideal corresponds to the computation of the rank of the matrix of coefficients of the forms $L_i^{7+2q}x_j$, where the x_j's are the coordinates of $\mathbb{P}^2$.

We are now able to write an algorithm for detecting the identifiability of T.

5.3.1 The algorithm

Let T be a ternary form of degree $d = 8 + 2q$.

Assume that we are given a decomposition A of T, $A = \{P_1, P_2, \dots, P_r\} \subset \mathbb{P}^2$. Assume that $\ell(A) \leq 3q+11$ and that, if $T = \{a_1 v_d P_1 + a_2 v_d P_2 + \dots + a_r v_d P_r\}$, none of the a_i are equal to zero. Then, in order to prove that A is minimal and so T is identifiable, the following steps could be taken.

S0. Compute the rank of the matrix M_d of coordinates of the points $v_d(P_j)$'s.

S0.1 If the rank of M_d is smaller than r, then A is redundant, and the algorithm terminates and states that T has rank $< r$.

S0.2 If the rank of M_d is r, then A is non-redundant, and the algorithm continues.

S1. If $r \leq 4 + q$, the algorithm terminates and states that T is identifiable.

S2. Perform the Terracini test as follows:

S2.1 Compute the linear form $L_1 \dots L_r$ associated with $P_1, \dots P_r$.

S2.2 For $i = 1, 2, \dots r$ and $j = 0, 1, 2$ compute the rank of the matrix of coefficients of the forms $x_j L_i^{7+2q}$, and call it q

S2.3 If $q < 3r$ then the algorithm terminates claiming that it cannot prove the identifiability of T.

S2.4 If $q = 3r$ the algorithm terminates and states that T is identifiable.

Now we can show that using the method exposed below we can reduce considerably the computational cost.

Remark 5.3.8. In order to find the dimension of the Terracini space, the crucial step is to compute the rank of the matrix made by $[x_i(L_j)^{7+2q}]$ with $i = 1, 2, 3$ and $j = 1, \dots, r$. So, we have to compute the rank of a $\binom{9+2q}{2} \times 3r$ matrix. Using the Gauss elimination method, we have that the computational costs of this process is in the order of

$$\frac{2}{3} \cdot \frac{(2r/3 + 27/22)^2}{2} \cdot 9r^2 \approx \frac{4}{3} r^4.$$

Notice indeed that to verify that A is contained in a cubic curve, so to compute $h_A(3)$, we have to find the rank of the matrix $[v_3(P_1), \dots, v_3(P_r)]$. With the Gauss elimination method, we have a computational cost in the order of

$$\frac{2}{3} 10^2 \cdot r.$$

So, the total computational cost is in the order of

$$\frac{4}{3} r^4.$$

Then, comparing the two methods, we have that Terracini test can be much quicker than computing the Kruskal ranks for high value of r.

Chapter 6

Tensors in 4 variables

Using the same techniques presented in the previous chapter one can find some examples in which we can prove the identifiability of tensors in a range for the rank which is larger than the range in which the reshaped Kruskal criterion 2.0.40 applies.

In this final section we will give an example concerning tensors in 4 variables, in particular we consider a symmetric tensor of type $4 \times \cdots \times 4$ (5 times).

We suppose to know a non-redundant decomposition $A \subset \mathbb{P}^3$ of T with $\ell(A) \leq 12$ and we want to prove that T is identifiable.

As usual, we will make some assumptions on the points of A. In particular we suppose that:

- the Kruskal rank k_1 of A is $\min\{4, \ell(A)\}$;
- the second Kruskal rank k_2 of A is $\min\{10, \ell(A)\}$.

This case is not covered completely by the Reshaped Kruskal criterion 2.0.40.

Indeed, in order to apply the bound given by the theorem one must consider partitions of 5 of the form $a + b + c$ and compute the corresponding Kruskal ranks.

Consider the partition $5 = 2 + 2 + 1$. Under our hypothesis the given decomposition A of T satisfies $k_2 = 20$, $k_1 = 4$, but:

$$11 = \frac{10 + 10 + 4 - 2}{2}.$$

The other case is when we take the partition $5 = 3 + 1 + 1$. In this case we know that $k_3 \leq \min\{20, \ell(A)\} = \ell(A) \leq 12$ and that $k_2 = \min\{10, \ell(A)\} \leq 10$. On the other hand we have that

$$\frac{12 + 4 + 4 - 2}{2} = 9.$$

In particular, the Reshaped Kruskal criterion 2.0.40 guarantee the identifiability only if $\ell(A) \leq 11$. We want to present a method that certifies the identifiability of T also when $\ell(A) = 12$.

As usual, the first fact that we prove is that if there is another decomposition B of T with $\ell(B) \leq \ell(A)$ then A and B cannot be disjoint.

To handle this case we need an useful result that can be found in the work of Bigatti Geramita and Migliore [13].

Theorem 6.0.1. *Let $Z \subset \mathbb{P}^n$ be a finite set of points such that the first difference of the Hilbert function $Dh_Z(i) = \binom{i+m-1}{i}$ and $Dh_Z(i+1) = \binom{i+m}{i+1}$ Then we have:*

a) Z is the disjoint union of two sets Z_1 and Z_2 where Z_1 lies in a $\mathbb{P}^m = \Lambda$

b) the first difference of the Hilbert function of Z_1 is equal to:

$$Dh_{Z_1}(t) = \begin{cases} Dh_\Lambda(t) & for\ t \leq i+1 \\ Dh_Z(t) & for\ t \geq i. \end{cases}$$

Proof. See Theorem 3.3 of [13]. □

The following Proposition, which is a direct consequence of Theorem 3.0.55, will allow us to exclude a lot of cases simply using the fact that we can suppose A non-redundant.

Proposition 6.0.2. *Let T be a symmetric tensor of degree d and let $A \subset \mathbb{P}^n$ be a decomposition of T. Let $B \subset \mathbb{P}^n$ be another decomposition of T disjoint from A such that $\ell(B) \leq \ell(A)$ and suppose that there exist $i, j \in \mathbb{N}$ with $j < i$, such that $Dh_Z(i) = Dh_Z(i+1) \leq i$ and $Dh_A(j) > Dh_Z(i)$. Then, A is redundant.*

Proof. As usual call $Z = A \cup B$. From Theorem 3.0.55 we know that there exists a curve C of degree i such that the first difference of the Hilbert function of the set $Z' = Z \cap C$ is equal to

- $Dh_{Z'}(t) = \begin{cases} Dh_C(t) & for\ t \leq i+1 \\ Dh_Z(t) & for\ t \geq i. \end{cases}$

In particular since $Dh_A(j) > Dh_Z(i)$ for some $j < i$ we know that $Z \cap C$ does not contains all the points of A. But from Proposition 3.0.54 and Theorem 3.0.55 we know that $h^1_Z(d) = h^1_{Z\cap C}(d)$ thus in particular $\dim(< v_d(A) > \cap < v_d(B) >) = \dim(< v_d(A \cap C) > \cap < v_d(B \cap C) >)$ and since $(A \cap C) \subset A$ and $(B \cap C) \subset B$ follows that $< v_d(A) > \cap < v_d(B) >=< v_d(A \cap C) > \cap < v_d(B \cap C) >$. Thus by definition of decomposition we have that $A \cap C$ is also a decomposition of T of length $\ell(A \cap C) < \ell(A)$ and as a consequence A is redundant. □

Now we are ready to prove that under the hypothesis specified on the first part of this chapter, if $\ell(A) \leq 12$ then T is identifiable and so, in particular, the rank of T is equal to $\ell(A)$. As usual, first we prove that T cannot have two disjoint decompositions.

Proposition 6.0.3. *Let $Z = A \cup B$ the union of two non-redundant decompositions for a symmetric tensor T in 4 variables such that $\ell(A) \leq 12$ and $\ell(B) \leq \ell(A)$. Suppose also that, the first Kruskal rank of A is equal to $k_1 = 4$ and that the second Kruskal rank of A is equal to $k_2 = 10$. Then Z cannot satisfy $CB(5)$ so A and B cannot be disjoint.*

Proof. Assume by contradiction that Z satisfies $CB(5)$.

From Lemma 3.0.45, Proposition 3.0.49 and the fact that $A \subset Z$ we know that Hilbert function of Z can be represented as follows.

j	0	1	2	3	4	5	6	...
$Dh_Z(j)$	1	h_1	h_2	h_3	h_4	h_5	h_6	...

where h_6 is different from 0 and $h_2 = \min\{3, \ell(A)\}$ and $h_1 = \min\{6, \ell(A)\}$. Moreover from Proposition 3.0.46 we know that also $h_i > 0$ for all $i = 1, \dots, 6$.

By Proposition 3.1.8 we know that:

$$h_4 + h_5 + h_6 \geq 1 + h_1 + h_2. \tag{6.1}$$

If $\ell(A) \leq 10$, from the inequality 6.1 and the fact that $\ell(Z) \leq 20$, it follows that $h_3 = 0$ but this is a contradiction. If $\ell(A) = 11, 12$, since $k_1 = 4$ and $k_2 = 10$ we have that the Hilbert function of Z is of the form:

j	0	1	2	3	4	5	6	...
$Dh_Z(j)$	1	3	6	h_3	h_4	h_5	h_6	...

In particular, from the inequality 6.1, we have that $h_4 + h_5 + h_6 \geq 10$ and from the fact that $\ell(Z) \leq 23$, we have that $h_3 \leq 4$.

If $h_3 \leq 3$ we have a contradiction with Proposition 3.0.46. In fact we would have $h_4, h_5, h_6 \leq 3$, so $h_4 + h_5 + h_6 \leq 9$ and this contradicts inequality 6.1. So we can suppose $h_3 = 4$.

Notice that $4 = \binom{4}{3}$ thus for Theorem 3.0.40 h_4 can be at most $\binom{5}{4} = 5$. If $h_4 \leq 3$, from Proposition 3.0.46 we have a contradiction as above. Indeed we would have $h_4 + h_5 + h_6 \leq 9$ but this contradicts equation 6.1 since in this case $1 + h_1 + h_2 = 10$.

If $h_4 = 4$ then, from equation 6.1 we have $h_5 = 4$ and $h_6 = 3$, a contradiction with Proposition 6.0.2.

If $h_4 = 5$ we have maximal growth in degree 3 thus, from Theorem 6.0.1, Z contains a subset Z_1 which lies in a $\mathbb{P}^2$ and whose Hilbert function is equal to

j	0	1	2	3	4	5	6	...
$Dh_Z(j)$	1	2	3	4	5	h_5	h_6	...

In particular Z_1 contains at least 19 points of Z and so, there would be at least 7 points of A contained in a plane. A contradiction with the fact that $k_1 = 4$. □

Now using the usual inductive strategy we can prove the following Theorem.

Theorem 6.0.4. *Let T be a symmetric tensor of degree 5 in 4 variables and let $A \subset \mathbb{P}^3$ be a decomposition of T of length $\ell(A) = 12$ such that the first Kruskal rank k_1 of A is equal to 4 and the second Kruskal rank of A is 10. Then T is identifiable and it has rank 12.*

Proof. Suppose to able another decomposition B with $\ell(B) \leq \ell(A)$. Then, from Proposition 6.0.3, $A \cap B \neq \emptyset$ and $\ell(Z) \leq 23$. Thus, using the same inductive strategy of Theorem 5.0.2, we obtain that the existence of B yields a contradiction. □

6.1 Work in progress: the rank of tensors with a decomposition of length 13

In this last paragraph we present briefly another result we are still working on.

We study a symmetric tensor T of type $4 \times \cdots \times 4$ (5 times) which has a non-redundant decomposition $A = \{P_1, \dots, P_r\} \subset \mathbb{P}^3$ of length $\ell(A) \leq 13$ and we want to certify that the rank of A is equal to $\ell(A)$. In the previous section we proved that if the first and the second Kruskal ranks of A are maximal and if $\ell(A) \leq 12$ then T is identifiable. Thus, under the previous hypothesis, we are sure that the rank of T is equal to $\ell(A)$.

When the decomposition has length $\ell(A) = 13$ the problem becomes much more complicated. The knowledge of the Kruskal ranks of A is no more longer sufficient neither to prove the identifiability of T nor to certify if the rank of A is 13. Indeed if the first and the second Kruskal ranks of A are maximal we have that the Hilbert function of A is of the form:

j	0	1	2	3	4	5
$Dh_A(j)$	1	3	6	3	0	...

Thus we want to exclude, e.g., that we can construct a tensor with two decompositions, A and B, one of length $\ell(A) = 13$ and the other one of length $\ell(B) = 12$ such that the Hilbert function of $A \cup B$ is equal to

j	0	1	2	3	4	5	6
$Dh_A(j)$	1	3	6	5	5	4	1

Indeed, a case in which our method cannot certify that the rank of T is 13 is when A and B can be found in the intersection of an elliptic curve of degree 5 with a surface of degree 5.

In order to avoid this problem we have to strengthen our hypothesis. In particular we suppose that:

1) the Kruskal rank k_1 of A is 4;

2) the second Kruskal rank k_2 of A is 10;

3) A is not contained in an elliptic curve of degree 5.

Notice that the first two conditions are easy to verify from the coordinates of the decomposition, at least theoretically, The third condition is much harder to treat, and we will not enter here in the details.

As usual, the first fact that we prove is that if there is another decomposition B of T with $\ell(B) < \ell(A)$ then A and B cannot be disjoint.

The case in which $\ell(B) \leq 11$ is easy to handle. Indeed, since $\ell(Z) = \ell(A \cap B) = 24$, we are in a setting similar to the one of the previous section. As a consequence we can use the same proof used in Proposition 6.0.3.

Suppose that $\ell(B)$ equal to 12. We can simplify our problem using Proposition 6.0.2 and Proposition 3.0.46. Indeed, with the methods illustrated in the previous paragraph, we obtain that the only possible h-vectors for Z are

- $H_1 = (1, 3, 6, 5, 6, 3, 1)$
- $H_2 = (1, 3, 6, 5, 5, 4, 1)$
- $H_3 = (1, 3, 6, 5, 5, 3, 2)$
- $H_4 = (1, 3, 6, 4, 5, 4, 2)$
- $H_5 = (1, 3, 6, 4, 5, 5, 1)$
- $H_6 = (1, 3, 6, 4, 5, 3, 2, 1)$.

The case in which the h-vector of $Z = A \cup B$ is equal to H_1, H_4, H_5 and H_6 are easy to handle. Indeed we can notice that there is *maximal-growth* from degree 3 to degree 4 i.e. the first difference of the Hilbert function of Z achieve the maximal value allowed by Theorem 3.0.40. Thus, we can use once again Theorem 6.0.1 to eliminate these cases.

The case in which the h-vector of Z is equal to H_3 requires an extra effort. We cannot eliminate this case directly using just properties of the Hilbert function. However, using strongly the Cayley-Bacharach property and a strategy similar to the one of Proposition 4.1.2, we can handle also this case. The idea is to reconstruct how the points of $Z = A \cup B$ are disposed in the space removing one by one some points either from A or B. Studying the new h-vectors we obtain at several step we can show that in this case we would have A and B redundant.

The case in which the h-vector of Z is equal to H_2 is the more complicated one.

In order to eliminate this case we need a completely different approach. We studied the degree 3 part of the ideal I_Z of Z. Since the h-vector of Z is equal to H_2 we know that $\dim(I_Z)_3 = 5$, thus $(I_Z)_3$ is generated by 5 cubic surfaces. Call Q the intersection of 5 cubic surfaces generating $(I_Z)_3$. Notice that $\dim(Q) \leq 2$ and that Q is not necessarily reduced.

The idea of our proof is to show that Q and $A \setminus Q$ impose too many conditions to the ideal $(I_Z)_3$ and so that $(I_Z)_3$ cannot have dimension equal to 5.

We can prove that Q cannot be a set of points. Indeed if Q is 0-dimensional we can prove that Z is contained in a complete intersection of type $(3, 3, 3)$ and the Hilbert function of such an intersection has to be symmetric, a contradiction since we are supposing the h-vector of Z equal to H_2.

It is easy to see that Q cannot contain a surface S. Indeed if the dimension of Q is two, then Z has to be contained in the union of a hyperplane with a surface of degree two. But this yields a contradiction with the conditions on the Kruskal ranks of A.

If Q contains a curve C of degree d there are many different cases to handle. Notice that the degree of C can be at most 9 since it has to be contained in the intersection of two cubic surfaces.

A direct consequence of the Riemann-Roch theorem is that C cannot have degree equal to $2 \leq d \leq 9$ unless it is an elliptic curve of degree 5. However, if C is an elliptic curve of degree 5, we can prove that C contains the whole A and this is excluded by our hypothesis.

Finally the case in which Q contains a line can be handled by studying divisors of a cubic surface containing Z. Up to now, we can only exclude this case by assuming that Z is contained in a smooth cubic surface S. Indeed if S is smooth its well known that S is isomorphic to a $\mathbb{P}^2$ blown up at 6 points and the divisors of S are known. We are not yet able to get rid of the assumption on the smoothness of S. Since quintics containing Z, when H_2 holds, form a 5-dimensional subspace of $(I_A)_3$, we cannot exclude a priori that Z lies in no smooth cubic, nor we have an easy way to exclude the case by computations on A. Since all irreducible singular cubic surfaces are classified, then one can use the classification and the geometry of Weil divisors to get a contradiction even when S is singular. Due to the mass of examples and the complexity of the relative Picard groups, the task is demanding and we did not solve all the details. Probably there is a more direct strategy for the case in which Q is a line, that can provide the conclusion of the computations. This is the part that remains work in progress.

Once the case in which A,B are disjoint has been settled, we can exclude the case in which $A \cap B \neq \emptyset$, with the same inductive strategy introduced in the previous chapter.

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