Note

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A Height-Based Multidimensional Extension of the Lorenz Preorder for Integer-Valued Distributions

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Abstract:
We study multidimensional inequality on integers. In such a setting, a Lorenz-like preorder and a suitably adapted version of the Muirhead-Pigou-Dalton transfers are defined, and a counterpart of some classic results on inequality measurement is established in this multivariate setting.

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1 Introduction

1.1 Motivation

It is well established that the Lorenz criterion (and its related induced-ordering) occupies a prominent role in discussions of the measurement of inequality of *univariate* (income) distributions because it admits several mathematically interesting characterizations, and related substantive justifications. Nonetheless, inequality of access to advantages and resources is arguably an inherently multidimensional phenomenon whose measurement has recently attracted a considerable amount of work. Unfortunately, it turns out that producing suitable inequality criteria in a multivariate setting is not at all an easy task. Indeed, the current economic approach substantially consists in generalizing classes of (unidimensional) inequality indices to a multidimensional evaluative space following a two-steps procedure. Individuals are first represented by an aggregate evaluation (e.g. utility) function of all attributes they are endowed with. Thus, one obtains an (univariate utility) vector. Then, an inequality index is applied to such a vector distribution in order to evaluate a (utility) distribution in terms of inequality (see e.g. Kolm 1977; Koshevoy & Mosler 1996; Tsui 1995).[1] This exercise is quite problematic as long as the choice of aggregating evaluation functions demands access to private information that is either not verifiable, or simply not available. Therefore, the outcome may well result in a net information loss.

1.2 Content

In the present note, we propose a redefinition of the Lorenz preorder for the *integer-valued multivariate case* which avoids the aforementioned drawbacks of the multidimensional inequality literature and is easy to implement. We focus on multivariate distributions whose marginals have identical means.

In order to address the problem of assessing inequality among multidimensional distributions, we consider as a first suitable restriction that people are endowed with *purely private indivisible goods*. In fact, an individual endowment usually consists of many commodities which may be owned in many copies. A hypothetical vector $x_i = (3, 5, 1)$ represents a situation in which individual i has 3 units of a type-α-good (e.g. she owns three entitlements to the benefit of public health care for her and her two children), 5 units of a type-β-good (five

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annual scholarships to access to a Ph.D program) and 1 unit of a type-γ-good (one car).[2] In other words, a set of n vectors like $x_i$is the discrete counterpart of a multivariate distribution and can be represented as a rectangular *integer-valued* matrix whose generic row i denotes the assignment of the annual vector of *indivisible* goods to the i-th agent. Such an analytical structure enables us to study *multidimensional inequality* for the *special case* when people, endowed with several copies of different private indivisible goods which are *rivalrous* in use and *complementary*, are compared in terms of economic disparity.

In such a *specific setting*, the problem of extending the concept of majorization (or the dual Lorenz) preorder, as defined by Muirhead (1903) for vectors of integers, to a multiattribute framework is well-posed, but that is by no means an easy matter. Indeed, one of the main reasons why the assessment of inequality in resource allocation by means of Lorenz preorders is highly problematic for multivariate distributions is that, while the univariate case allows a natural *total* order of personal attributes (e.g. income, wealth, consumption expenditure etc.), any multidimensional distribution typically admits only *partial* rankings of individual endowments as natural and non-controversial. However, Lorenz-based comparisons of vector distributions require the complete comparability of individual resources. On the contrary, as observed above, individual endowments, namely *distribution vectors* of personal private goods, do *not* allow a natural *total preorder*: when people differ for at least two attributes, one of them could be better-off than another one in terms of the relative affluence of the first characteristic and worse-off with respect to the second. So, if a disputable dispersion index is not to be used, one is forced either to start from a class of *partial orderings* on the distributions of individual endowments or to define anew *a suitable total preorder*. Hence, we first define a *total* preorder that extends the dominance partial order, and takes into account complementarity between goods. Then, we represent that preorder by using one of *its own parameters*: the *height* (to be defined below), that enables us to provide a *natural* re-definition of the Lorenz preorder in our multidimensional setting. Indeed, the height of an ordering is the length of the longest chain of comparable vectors. Therefore, it provides *an objective numerical scale for assessing the value of an individual endowment*. The height is well defined for every graph or binary relation on a set, though it is mostly used when the relevant relation is an ordering, namely a transitive and non-symmetric binary relation.[3] In such a way, we obtain a counterpart of the classic unidimensional Lorenz ranking for our class of multidimensional distributions, without using any controversial aggregator (utility function).

A concept closely related to the Lorenz preorder is that of equality-enhancing (income) transfers (see Marshall, Olkin & Arnold 2011 , Ch. 1). A number of different proposals have been advanced in order to extend the unidimensional transfer principle to the case of multiple individual attributes (see e.g. Kolm 1977; Marshall, Olkin & Arnold 2011 , Ch. 15 and Savaglio 2006). What we propose here[4] is a straightforward *multidimensional extension* of the classic Muirhead-Pigou-Dalton transfer principle, under the assumption of *resource indivisibility*, hence for the case of integer vectors. Namely, we consider the set of all bilateral transfers that take place from the richer (in terms of the height of her endowment vector) to the poorer, leaving the total amount of each good in the multivariate distribution unchanged. Moreover, we focus on transfers which involve one unit *for some* component of the personal distribution of goods. In other words, we focus on those transfers such that *for some* good the endowment of the richer person is diminished by one unit and the corresponding endowment of the poorer one is increased by the same amount (without reversing the respective ranks as induced by heights). As a result, we show that a finite sequence of *minimal* transfers entails Lorenz dominance.[5]

1.3 Related Literature

A number of distinguished contributions that are closely related to the present work address either the problem of the multidimensional extension of the Lorenz preorder or the task of defining the latter for integer-valued distributions.

Among the former, Koshevoy (1995) and Koshevoy (1998) and Koshevoy and Mosler (1996) develop a geometric approach to order of multivariate distributions. A new definition of Lorenz curve on more dimensions is introduced by considering the convex symmetric polyhedron (namely the zonotope) generated by the Minkowski's sum of all the marginals of a multivariate distribution. Nesting of Lorenz zonotopes describes the high-dimensional Lorenz ordering precisely.

Among the latter, Fishburn and Lavalle (1995) work with a finite set of evenly-spaced points and define stochastic dominance relations for probability distributions that are based on successive partial sums of probabilities and on classes of utility functions defined on the aforementioned set of points. Chakravarty and Zoli (2012) derive the counterpart of the integer-Lorenz preorder for variable-sum comparisons, establishing the equivalence of some suitable conditions in this setting.

Savaglio and Vannucci (2007) introduce height-based extensions of certain threshold-induced partial preorders of opportunity sets to define a Lorenz preorder on opportunity profiles. In that special setting, heights essentially counts the size of an opportunity set above the relevant opportunity threshold. Vannucci (2013) provides a characterization of such height-based extensions of opportunity sets that includes Pattanaik and Xu's characterization of the cardinality preorder (Pattanaik & Xu 1990) as a special case.

The rest of the paper is organized as follows: we introduce the analytical setting in Section 2, state and discuss the main results in Section 3 and conclude with some brief remarks concernings both our results and directions for future research.

2 Notation and Definitions

Let $N = \{1, ..., n\}$ denote a finite fixed population of agents with $n \geq 2$ and $M = \{1, ..., m\}$ a set of types of commodities. A multidimensional discrete distribution can be represented as a matrix $X = \left(x^1, ..., x^m\right)$, namely a collection of column vectors x^j of length n. The ith row of X is denoted x_i and the jth column x^j or alternatively as $x_{i,\cdot}$ and $x_{\cdot,j}$ respectively. We assume that the marginal distributions across matrices have identical means, namely the column vectors represent distributions of fixed total amounts of resources. All the vectors considered are in general row-vectors, thus, if there is no possibility of confusion, we write x for x_i and $x_{i,\cdot}$. A row-vector $x_i = \left(x_{i,1}, ..., x_{i,m}\right)$ represents the bundle of private goods of individual i and the element $x_{i,j}$ is an *integer* representing the number of units of a good of type j that i is endowed with. We suggest interpreting a row-vector such as $x_i = (3, 5, 1)$ as representing an individual i that has 3 commodities of type α, 5 of type β and 1 of type γ. This essentially means two things. First, goods are indivisible: half a car (commodity of type γ, for instance), is not a car. Second, numbers simply represent how many copies of each type of good individuals are endowed with: if we change the measurement unit (from years to semester), of say, a type-β good, for example a 5-year scholarship, we are only rescaling our distribution (individual i has now has an allocation of 10 six-month instead of 5 annual scholarships), without affecting inequality within and between populations. Thus, X is an element of $\mathbb{Z}_+^{n,m}$, namely the (non-negative orthant of the) $n \times m$ -dimensional integer space.

The following question immediately arises:" *Given two distribution matrices $X, Y \in Z_+^{n,m}$, which exhibits the lower level of inequality?*" . To answer this question, the current economic literature has extended classes of (unidimensional) inequality indices to a multidimensional evalutative space. This approach is problematic (see e.g. Dardanoni 1995; Savaglio 2006) as long as the choice of an aggregating evaluation function is arbitrary and needs information on preferences (that may be unreliable and difficult to elicit). On the contrary, we explore the possibility of extending the *Lorenz* preorder for vectors to a multivariate framework. In order to do that, we define the Lorenz dominance criterion for vectors as follows:

Definition 1: Let $\hat{x}, \hat{y}$ be the vectors obtained from $x, y \in \mathbb{R}^n$ by re-arranging the components of the latter in non-decreasing order. Then, *$\hat{x}$ Lorenz-dominates $\hat{y}$*, written $\hat{x} \succcurlyeq \hat{y}$, whenever

$$\sum_{i=1}^{k} \hat{x}_i \geq \sum_{i=1}^{k} \hat{y}_i \text{for} k = 1, ..., n-1 \text{and} \sum_{i=1}^{n} \hat{x}_i = \sum_{i=1}^{n} \hat{y}_i.$$

In a multidimensional setting, the elements of a distribution (which are full-comparable in the unidimensional (income) case), are now vectors of individual endowments. These are usually compared according to the dominance vector order $\gg$ namely:

Definition 2: (*Dominance Vector Order - DVO*). For any $x_i, x_j \in \mathbb{Z}_+^m$, $x_i \gg x_j$ if and only if $x_{i,l} \geq x_{j,l}$ for every $l = \{1, ..., m\}$.

In order to reproduce the Lorenz preorder in the present multidimensional setting, we need however a *total* ordering of $\mathbb{Z}_+^m$, since the Lorenz preorder is induced by the natural *total order* of the elements in an univariate (income) distribution. Moreover, we shall focus on the special case of *complementary goods* as explained in the Introduction. So, in order to compare multivariate distributions of goods by using the Lorenz criterion, we introduce a total preorder $\sqsupseteq$ on $\mathbb{Z}_+^m$ that extends[6] the dominance order and takes properly into account complementarity, namely:

Definition 3: For any $x_i, x_j \in \mathbb{Z}_+^m$, $x_i \sqsupseteq x_j$ if and only if $\min_{l=\{1,...,m\}} x_{i,l} \geq \min_{l=\{1,...,m\}} x_{j,l}$.[7]

Finally, we use the *height*, an *intrinsic parameter* of $\left(\mathbb{Z}_+^m, \sqsupseteq\right)$, in order to represent the latter preorder by a numerical index.[8] The *height* is an integer-valued function that assigns to each $x \in \mathbb{Z}_+^m$ a non-negative number, namely the length of the longest strictly ascending chain $\mathbb{C}$ of $\left(\mathbb{Z}_+^m, \sqsupseteq\right)$ having x as its maximum.[9] Formally:

Definition 4: The *height function* $h_{\sqsupseteq} : \mathbb{Z}_+^m \to \mathbb{Z}_+$ of the totally preordered set $\left(\mathbb{Z}_+^m, \sqsupseteq\right)$ is defined as follows:

$$h_{\sqsupseteq}(z) = \max \left\{ \begin{array}{l} |\mathbb{C}| + 1 : \mathbb{C} \text{is a} \sqsupseteq \text{-chain, such that} \\ z \in \mathbb{C} \text{and} z \sqsupset z' \text{for any} z' \in \mathbb{C} \setminus \{z\} \end{array} \right\} \text{for every} z \in \mathbb{Z}_+^m.$$

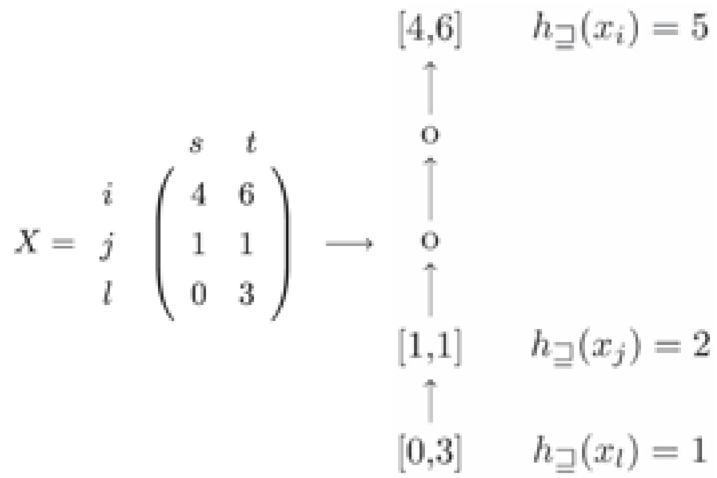
Thus, the height of a vector x_i (i.e. the endowment of goods of individual i in $\mathbb{Z}_+^m$), counts the number of vectors (bundles of goods) which stand below x_i according to $\sqsupseteq$, when considering the longest path ending in x_i. Therefore, in our setting, *heights amount to a straightforward expression of the 'distance' between population units in the given distribution*. We propose the following handy formula to compute heights, which will prove to be very useful in the ensuing analysis:

Height computation rule: For $\left(\mathbb{Z}_+^m, \sqsupseteq\right)$,

$$h_{\sqsupseteq}(z) = \min_{j=\{1,\dots,m\}} z_j + 1 \text{for} z \in \mathbb{Z}_+^m. \tag{1}$$

Expression 1 simply says that the height function identifies the resource in which the relevant agent is least affluent and assigns a value to it. We suggest the following interpretation of our rule for computing heights: the set of types of commodities that individuals are endowed with are *complementary* (primary) goods, all of which are necessary for enhancing equality. Precisely, because each good is taken to be a complement of every other good, the value of the individual endowment is assumed to be the *minimum value* of $x_i = \left(x_{i,1}, \dots, x_{i,m}\right)$ for $i \in \{1, \dots, n\}$. To put it in other terms, the application of the height function to $\mathbb{Z}_+^m$ consists in counting *how many steps* we have to go down before getting to the value zero for the minimal component of the endowment vector under examination. Thus, the height of $\left(\mathbb{Z}_+^m, \sqsupseteq\right)$ allows us to replicate, in the present more general context, the celebrated Lorenz preorder for the (income) inequality measurement.

The definition and computation of the height may be illustrated with the help of a Hasse diagram[10] as follows:



3 A Multidimensional Extension of the Lorenz Criterion

Following our approach to inequality rankings of profiles of individual endowments, we now apply the Lorenz preorder to the height vectors of endowment profiles. Such a preorder induces a further preorder, which is in fact an inequality ranking of the underlying profiles of individuals' endowments, and is a counterpart of the Lorenz ranking of (income) distributions.

Thus, a (counterpart of the) Lorenz preorder in our framework is defined as follows:

Definition 5: Let $X, Y \in \mathbb{Z}_+^{n,m}$ be two profiles of individual endowments of goods, $h_{\sqsupseteq}$ the height function on $\mathbb{Z}_+^m$ and $h_{\sqsupseteq}(X), h_{\sqsupseteq}(Y)$ the height-profiles of X and Y arranged in non-decreasing order. Then, we say that X *h-Lorenz dominates* Y, denoted $X \succcurlyeq_h Y$, if:

$$h_{\sqsupseteq}(X) = \left(h_{\sqsupseteq}(x_1), \dots, h_{\sqsupseteq}(x_n)\right) \succcurlyeq \left(h_{\sqsupseteq}(y_1), \dots, h_{\sqsupseteq}(y_n)\right) = h_{\sqsupseteq}(Y), \text{namely:}$$

$$\sum_{i=1}^{k} h_{\sqsupseteq}(x_i) \geq \sum_{i=1}^{k} h_{\sqsupseteq}(y_i) \text{ for} k = 1, \dots, n-1, \text{and} \sum_{i=1}^{n} h_{\sqsupseteq}(x_i) = \sum_{i=1}^{n} h_{\sqsupseteq}(y_i).$$

In words, starting from a domain of multivariate distributions, we map the multidimensional space of individual endowments into the *totally* ordered set of heights to which we can apply the (standard) Lorenz preorder

$\succcurlyeq$. As it is to be expected in a multidimensional setting, a significant set of multivariate distributions are not comparable by height (see also a particular case below).

We now provide a suitable counterpart of the Muirhead-Pigou-Dalton (MPD) transfer principle by considering that the distance between agents of a given distribution provided by the heights can be expressed in terms of *elementary transfers*:

Definition 6: (*Minimal Elementary Transfer (MET)*). Let $X \in \mathbb{Z}_+^{n,m}$ be a profile of individual endowments of goods, $i, j \in N$ be such that $h_{\sqsupseteq}(x_i) > h_{\sqsupseteq}(x_j) + 1$ and $J \subseteq I_m = \{1, ..., m\}$, $J_i(x) = \left\{s^* : x_{i,s^*} = \min_{s^*=\{1,...,m\}} x_{i,s^*}\right\}$ and $J_j(x) = \left\{s^* : x_{j,s^*} = \min_{s^*=\{1,...,m\}} x_{j,s^*}\right\}$ for $x \in \mathbb{Z}_+^m$. Take $J \subseteq I_m = \{1, ..., m\}$ such that $J \cap J_i(x) \neq \emptyset$ and $J \cap J_i(x) \neq \emptyset$ and a $Y \in \mathbb{Z}_+^{n,m}$, defined as follows:

$$\begin{cases} y_{i,l} = x_{i,l} - 1 \\ y_{j,l} = x_{j,l} + 1 \end{cases} \text{for any} l \in J$$

$$\begin{aligned} \text{and} y_{t,l} &= x_{t,l} \left[\text{for any} t \neq i, j \text{and any} l \in \{1, ..., m\}\right] \\ &\quad \text{and} \left[\text{for any} t \in \{i, j\} \text{ and any} l \in I_m \backslash J\right]. \end{aligned}$$

Then, Y is said to arise from X through a *minimal elementary transfer* (MET).[11]

Moreover, we say that Y is MET-related to X, denoted to as $\mathcal{T}(Y, X)$, when Y is reachable through X by a finite sequence of MET. A MET requires a transfer of units on some dimensions that include the minimum components of both the donors and the recipients' endowments.

Finally, we denote by $\mathbb{H}$ the set of (X, Y) pairs of endowments profiles that are MET-related and have the same-mean height vectors, namely

$$\mathbb{H} = \left\{ \begin{array}{c} \left(z', z''\right) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n : \text{there exists} X, Y \in \mathbb{Z}_+^{n,m}, \text{with} z' = h(X) \text{and} z'' = h(Y) \\ \text{such that} \mathcal{T}(X, Y) \text{ and } \sum_{i=1}^n h_{\sqsupseteq}(x_i) = \sum_{i=1}^n h_{\sqsupseteq}(y_i). \end{array} \right\}$$

It is important to remark that the MET principle avoids some drawbacks of other criteria by allowing transfers only in the meaningful and desiderable case in which one individual is richer than another one for at least some of her attributes. In other words, MET only allows transfers whose inequality-decreasing effect, assessed in terms of heights, is undisputable (as opposed to other transfer criteria as e.g. Uniform Majorization (see e.g. Savaglio (2006)).

The use of the definitions we have introduced is clarified in the following:

Example 1: Let us suppose that the set of available goods is composed of six units of good s and nine units of good t, distributed over a population of three agents $\{i,j,l\}$ so as to generate the following distribution matrix X:

$$X = \begin{array}{c} \\ i \\ j \\ l \end{array} \begin{array}{c} \begin{array}{cc} s & t \end{array} \\ \begin{pmatrix} 5 & 6 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \end{array}$$

Then, the corresponding $\sqsupseteq$-induced height function will be $h_{\sqsupseteq}(X) = (6, 2, 1)$. Now, suppose that, according to Definition 6, a transfer takes place on $J = \{s, t\}$ from richer i to poorer l in order to get the new multidimensional distribution, namely

$$Y = \begin{array}{c} \\ i \\ j \\ l \end{array} \begin{array}{c} \begin{array}{cc} s & t \end{array} \\ \begin{pmatrix} 4 & 5 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} \end{array}.$$

The corresponding height-vector $h_{\sqsupseteq}(Y)$ is $(5, 2, 2)$. It is obvious that $Y \succcurlyeq_h X$.

In general, it is possible to show that:

Proposition 1: Let $X, Y \in \mathbb{Z}_+^{n,m}$ be two profiles of individual endowments of goods such that $\sum_{i=1}^n h_{\sqsupseteq}(x_i) = \sum_{i=1}^n h_{\sqsupseteq}(y_i)$. If $\mathcal{T}(X, Y)$, then $X \succcurlyeq_h Y$.

Proof: We proceed by induction. Take a $Y \in \mathbb{Z}_+^{n,m}$ and two individual endowments $y_i, y_j \in Y$ such that $h_{\sqsupseteq}(y_i) > h_{\sqsupseteq}(y_j) + 1$. Then, define $X^* \in \mathbb{Z}_+^{n,m}$ to be a new profile of individual endowments such that:

$$\left\{ \begin{array}{l} x^*_{i,l} = y_{i,l} - 1 \\ x^*_{j,l} = y_{j,l} + 1 \end{array} \right. \text{ for every} l \in J$$

$$\begin{array}{rcl} \text{and} y_{t,l} & = & x_{t,l} \left[\text{for any} t \neq i, j \text{and any} l \in \{1, ..., m\}\right] \\ & & \text{and} \left[\text{for any} t \in \{i, j\} \text{ and any } l \in I_m \backslash J\right], \end{array}$$

where $J \subseteq I_m = \{1, ..., m\}$, and $y_{i,j^\circ} = \min_{j=\{1,...,m\}} y_{i,j}$ is such that $j^\circ \in J$.

Hence, by definition of the height function, $h_{\sqsupseteq}\left(x_j^*\right) = h_{\sqsupseteq}\left(y_j\right) + 1$ and $h_{\sqsupseteq}\left(x_i^*\right) = h_{\sqsupseteq}(y_i) - 1$. It follows that $\left(h_{\sqsupseteq}\left(x_1^*\right), ..., h_{\sqsupseteq}\left(x_n^*\right)\right) \succcurlyeq \left(h_{\sqsupseteq}(y_1), ..., h_{\sqsupseteq}(y_n)\right)$, namely $X^* \succcurlyeq_h Y$. A similar argument applies to the induction step of the proof and provides the desired result. □

Remark 1: In general, the converse of Proposition 1 does not hold. Indeed, consider the following two matrices:

$$X = \begin{array}{c} \\ i \\ j \\ l \end{array} \begin{array}{c} \begin{array}{ccc} \text{s} & \text{t} & \text{v} \end{array} \\ \left(\begin{array}{ccc} 5 & 5 & 6 \\ 6 & 3 & 0 \\ 0 & 4 & 5 \end{array} \right) \end{array} \quad Y = \begin{array}{c} \\ i \\ j \\ l \end{array} \begin{array}{c} \begin{array}{ccc} \text{s} & \text{t} & \text{v} \end{array} \\ \left(\begin{array}{ccc} 8 & 4 & 2 \\ 2 & 4 & 2 \\ 1 & 4 & 7 \end{array} \right) \end{array}$$

such that $h_{\sqsupseteq}(X) = (6, 1, 1)$, $h_{\sqsupseteq}(Y) = (3, 3, 2)$, and therefore $h_{\sqsupseteq}(Y) \succcurlyeq h_{\sqsupseteq}(X)$. Hence, an admissible MET can only involve i as a possible donor and both j and l as possible recipients. However, the final distribution in Y assigns endowment $(8, 4, 2)$ that requires i to be a recipient of a transfer of good s from some $u \in \{j, l\}$. But then, i should be entitled to be a recipient of a MET with u as a donor and this is only possible if i's height becomes suitably smaller than u's height. But, this is forbidden by the definition of a MET. □

It is also worth noticing that:

Proposition 2: For any $h, h' \in \mathbb{H}$, such that $h \succcurlyeq h'$, there exist two profiles $X, Y \in \mathbb{Z}_+^{n,m}$ such that $h = h_{\sqsupseteq}(X) \succcurlyeq h_{\sqsupseteq}(Z) = h'$ which are MET-related.

Proof: Take h and construct a multidimensional distribution $X \in \mathbb{Z}_+^{n,m}$ such that $\min_{j=1,...,k} x_{i,j} = h_i - 1$ for any $i = 1, ..., n$. Since h Lorenz dominates h' implies that $h_i - h_i' = \delta_i$ for any i, such that $\sum_{i=1}^n \delta_i = 0$, it is possible to define the number of sign-changes $S(g)$ of the function $g(\cdot)$, defined on $\left(h - h'\right)$ and corresponding to the condition that h' is more variable than h. Karlin (1968) ,Vol. 1, Ch. 5 has shown that a vector distribution Lorenz dominates another one if and only if $S = 1$ with the sign sequence $+, -$. Without loss of generality, this argument allows us to restrict our analysis to only one single transfer. So, let $h \succcurlyeq h'$ such that $h' = \left(h_1, ..., h_{i-1}, h_i - 1, h_{i+1}, ..., h_{j-1}, h_j + 1, h_{j+1}, ..., h_n\right)$. Then take the corresponding MET on X in order to obtain a Y such that $h(Y) = h'$ and we are done. □

Thus, Propositions 1 and 2 imply the statement $\succcurlyeq = \mathbb{H}$, i.e. we have produced a representation of the Lorenz preorder on integers through the MET principle. In other words, we have a new representation of the Lorenz preorder on $\mathbb{Z}_+^n$ in terms of reachability by a finite sequences of elementary transfers for multidimensional endowments profiles which embody some features of the classical Muirhead-Pigou-Dalton transfers.

We also observe that our Lorenz-like multidimensional extension is clearly not immune from criticisms. As an example, consider now the following two matrices (with the same total sum of the marginals):

$$X_1 = \left(\begin{array}{cc} 8 & 3 \\ 0 & 7 \end{array} \right) \text{with} h_{\sqsupseteq}(X_1) = (4, 1), Y_1 = \left(\begin{array}{cc} 4 & 5 \\ 4 & 5 \end{array} \right) \text{with} h_{\sqsupseteq}(Y_1) = (5, 5).$$

The latter represents the perfect egalitarian distribution, but it cannot be compared with X_1because the total sum of the heights in the former is greater than the one in the latter and therefore the (standard) Lorenz preorder cannot be applied.

Conclusions

The synthetic measurement of individual well-being on the real-valued scale although convenient, must be considered a possibly misleading simplification. Our alternative approach is conducive to a version of the Lorenz preorder for profiles of individual endowments that extends the classic unidimensional analysis of income inequality via the Lorenz criterion to a certain class of multivariate contexts. It relies on an *objective evaluation function*, namely the height function, which is an intrinsic parameter of our model. This approach does not require external value judgements and allows us to reproduce the Lorenz preorder for a certain class of multivariate distributions.

Of course, this result has its own obvious limitations. First of all, *it only applies to integer distributions; very few multidimensional distributions can be compared* and some *paradoxical situation* can arise when a height-based Lorenz dominance approach is adopted. Moreover, our model is focused on the special case of complementary resources. On the other hand, it should be remarked that $\succcurlyeq_h$ can also be applied to pairs of (multidimensional) distributions that do not have marginals with the same distribution mean, as usually required in the literature on multidimensional inequality (see e.g. Koshevoy 1995; Koshevoy 1998; Maasoumi 1986; Tsui 1995 among others).

To conclude, much work remains to be done in this relatively unexplored field, but this demanding task is best left as a possible topic for further research.

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Notes

[1]Gajdos and Weymark (2005) reverse the procedure aggregating first the marginal distributions using generalized Gini social evaluation functions and then applying a second-stage aggregator whose functional form is induced by a suitable comonotonic additivity axiom.

[2]Notice that, following the foregoing example, our general model can be related to the evaluation of well-being as measured by the human development index (HDI), which computes a country's average achievements in three basic aspects of human welfare: longevity, knowledge, and a decent standard of living. It should be remarked that the foregoing aspects might be regarded as *complementary*.

[3]See Haskins and Gudder (1972) for a through treatment of the heights of partial ordered sets and graphs.

[4]A similar but less general proposal can be found in the works of Fleurbaey and Trannoy (2003) and Diez et al. (2007) .

[5]Of course, the height-function (and its induced Lorenz preorder) is not immune from criticism. Below, a paradoxical situation, that further illustrates this point, is discussed.

[6]Namely, $x_i \gg x_j$ implies $x_i \sqsupseteq x_j$. Notice that $\sqsupseteq$ is a weak -not a strict- extension of $\gg$, since it does not preserve the asymmetric component of $\gg$. However, $\sqsupseteq$ is not symmetric; hence it provides a *non-trivial* extension of the dominance order.

[7]As usually, we denote by $\sqsupset$ the asymmetric component of $\sqsupseteq$.

[8]Notice the difference between our index and indices such as utility functions that require the elicitation of private information concerning individual preferences.

[9]A chain of a preorder $(X, \succcurlyeq)$ is a subset $Z \subseteq X$ such that $(Z, \succcurlyeq)$ is a totally preordered set. A strictly ascending chain is said to be maximal when it cannot be extended without changing one of its extrema.

[10]The Hasse diagram of a binary relation $\gtrsim$ on a set A is a graph whose vertices are the elements of $(A, \gtrsim)$ and whose edge are the pairs (x, y) for which x covers y, namely such that no elements of $(A, \gtrsim)$ lie between x and y . It is usually draws so that each element is placed higher than an element it covers.

[11]By analogy with the Muirhead-Pigou-Dalton principle, our definitions require that transfers of goods be not large enough to reverse the relative height-induced positions of the donor and recipient, namely if, for any $X, Y \in \mathbb{Z}_+^{n,m}$, Y is obtained from X by a MET that involves $i, j \in N$such that$y_i \neq x_i$ and$y_j \neq x_j$ if $h_{\sqsupseteq}(x_i) \geq h_{\sqsupseteq}(x_j)$, then $h_{\sqsupseteq}(y_i) \geq h_{\sqsupseteq}(y_j)$.

References

Chakravarty, S., and C. Zoli. 2012. "Stochastic Dominance Relations for Integer Variables." *Journal of Economic Theory* 147: 1331–1341.

Dardanoni, V. 1995. "On Multidimensional Inequality Measurement." In *Income Distribution, Social Welfare, Inequality, and Poverty Vol. 6 of Research on Economic Inequality*., edited by C. Dagum, and A. Lemmi, 201–207. Stanford, CT: JAI Press.

Diez, H., M. C. Lasso de la Vega, A. de Sarachu, and A. M. Urrutia. Topics. 2007. "A Consistent Multidimensional Generalization of the Pigou-Dalton Transfer Principle: An Analysis." *The B.E. Journal of Theoretical Economics* 7 (1).

Fishburn, P., and I. Lavalle. 1995. "Stochastic Dominance on Unidimensional Grids." *Mathematics of Operations Research* 20: 513–525.

Fleurbaey, M., and A. Trannoy. 2003. "The Impossibility of A Paretian Egalitarian." *Social Choice and Welfare* 21: 243–263.

Gajdos, T., and J. A. Weymark. 2005. "Multidimensional Generalized Gini Indices." *Economic Theory* 26: 471–496.

Haskins, S., and S. Gudder. 1972. "Height on Posets and Graphs." *Discrete Mathematics* 2: 357–382.
Karlin, S. 1968. *Total Positivity*, Vol. 1. Stanford: University Press.
Kolm, S.C. 1977. "Multidimensional Egalitarianisms." *Quarterly Journal of Economics* 91: 1–13.
Koshevoy, G. 1995. "Multivariate Lorenz Majorization." *Social Choice and Welfare* 12: 93–102.
Koshevoy, G. 1998. "The Lorenz Zonotope and Multivariate Majorizations." *Social Choice and Welfare* 15: 1–14.
Koshevoy, G., and K. Mosler. 1996. "The Lorenz Zonoid of a Multivariate Distribution." *Journal of the American Statistical Association* 91: 873–882.
Maasoumi, E. 1986. "The Measurement and Decomposition of Multidimensional Inequality." *Econometrica* 54: 991–997.
Marshall, A., I. Olkin, and B. Arnold. 2011. *Inequalities: Theory of Majorization and its Applications*., 2nd. New York: Springer.
Muirhead, R.F. 1903. "Some Methods Applicable to Identities and Inequalities of Symmetric Algebraic Functions of *n* Letters." *Proceeding Edinburgh Mathematical Society* 21: 144–157.
Pattanaik, P., and Y. Xu. 1990. "On Ranking Opportunity Sets in Terms of Freedom of Choice." *Recherches Économiques de Louvain* 56: 383–390.
Savaglio, E. 2006. "Three Approaches to the Analysis of Multidimensional Inequality." In *Inequality and Economic Integration*., edited by F. Farina, and E. Savaglio. London: Routledge.
Savaglio, E., and S. Vannucci. 2007. "Filtral Preorders and Opportunity Inequality." *Journal of Economic Theory* 132: 474–492.
Tsui, K. Y. 1995. "Multidimensional Generalizations of the Relative and Absolute Inequality Indices: The Atkinson-Sen-Kolm Approach." *Journal of Economic Theory* 67: 251–265.
Vannucci, S. 2013. "A Characterization of Height-Based Extensions of Principal Filtral Opportunity Rankings." *Cuadernos de Economía* 61: 803–815.