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THE CONDITIONAL *C*-CONVOLUTION MODEL AND THE THREE STAGE QUASI MAXIMUM LIKELIHOOD ESTIMATOR

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Abstract

In this paper, we present the asymptotic results of the quasi maximum likelihood estimator of the parameters of a C-convolution model based on the conditional copula (Patton [11]). The C-convolution operator determines the distribution of the sum of two dependent random variables with the dependence structure given by a copula function. We focus in particular on the case where the vector of parameters may be partitioned into elements relating only to the marginals and elements relating to the copula. We propose a three-stage quasi maximum likelihood estimator (3SQMLE) and we provide conditions under which the estimator is asymptotically normal. We examine the small sample properties via Monte Carlo simulation. Finally, we propose an empirical application to explain how our model works.

1. Introduction

In this paper, we consider the estimation of parametric multivariate density models where the data generating process is given by a conditional C-convolution. The C-convolution is an operator introduced

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by Cherubini et al. [5], which recovers the distribution function of the sum of two dependent random variables. The dependence structure is provided by a copula function (see among others Nelsen [10] or Joe [9] for a detailed discussion on copulas and their properties). A bivariate copula function allows to decompose a joint distribution into its two univariate marginal distributions and a copula which describes the (non-linear) dependence between the variables. In statistical and econometric modelling, the main advantage of the use of copulas is in the construction of exible bivariate distributions, where we can combine different marginal distributions with any copula to create a specific bivariate distribution. In our case, the choice of the dependence structure reects on the distribution of the sum through the C-convolution.

The importance of the sum of two random variables in financial econometrics is clear if we consider a concrete example given by a managed fund. Consider a managed fund Z promising to yield an excess return, say Y, with respect to a benchmark index X. Assuming we know the dynamics and the distribution of the excess return of X, the distribution of the excess return on the managed fund Z will depend on the investment policy implemented by the fund manager that will add up to the return on the benchmark $X + Y$. Moreover, the market return X and the excess return promised by the manager Y will be dependent, and such a dependence may be modelled by a copula function. So, the distribution of the managed fund return is given by a C-convolution.

However, in financial econometrics, the data cannot be assumed to be independent and identically distributed (i.i.d.) since we develop the applications using time series. Therefore, we employ the concept of conditional copula, introduced by Patton [11], which allows us to handle conditioning variables. In this framework, we extend the definition of C-convolution in order to recover the conditional distribution of the sum of two random variables whose dependence structure is given by a conditional copula.

The main contribution of this paper is to provide the asymptotic behaviour of a three-stage quasi maximum likelihood estimator of the parametric conditional C-convolution model. The result is based on Theorem 6.10 in White [13]. Moreover, an application of the model to the sum of daily log-returns of two assets from Italian financial market (Eni and Fiat) is provided. In particular, we focus on the case where the conditional dependence structure is described by two of the most used archimedean copula functions: The Frank copula and the Clayton copula. We find significant evidence that the conditional C-convolution is the distribution of the sum of daily log-returns in the case of Frank copula.

The remainder of the paper is organized as follows. Section 2 introduces the C-convolution and its conditional version. In Section 3, we present the three-stage quasi maximum likelihood estimator of the parameters of the model. In Section 4, we prove the asymptotic normality of the estimator. Section 5 discusses the small sample properties in two particular cases via Monte carlo simulation. Section 6 presents an empirical application to a data set of daily log-returns of two stocks from Italian financial market. Section 7 concludes. Assumptions required for the asymptotic normality of the estimator and the proof of the main theorem of the paper are presented in the Appendix.

2. The Conditional *C*-convolution

Our data generating process is based on the C-convolution operator introduced by Cherubini et al. [5]. The C-convolution determines the distribution function of a sum of two dependent and continuous random variables X and Y. The dependence structure between X and Y is modelled by a copula function. The copula technique allows to write every joint distribution as a function of the marginal distributions. In our case, for example, we can represent the joint distribution of X and Y, say $\Pr(X \leq a, Y \leq b)$, with $a, b \in \Re$ as a function of $F_X(a) \equiv \Pr(X \leq a)$ and $F_Y(a) \equiv \Pr(Y \leq b)$. More formally, there exists a function $C_{X,Y}(u, v)$ such that

$$\Pr(X \leq a, Y \leq b) = C_{X,Y}(F_X(a), F_Y(b)).$$

Conversely, given two distribution functions F_X and F_Y and a suitable bivariate function $C_{X,Y}$, we may build joint distribution for the returns. The requirements to be met by this function are that: (i) it must be grounded $(C(u, 0) = C(0, v) = 0)$; (ii) it must have uniform marginals $(C(1, v) = v$ and $C(u, 1) = u)$; (iii) it must be 2-increasing (meaning that the volume $C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2)$ for $u_1 > u_2$ and $v_1 > v_2$ cannot be negative). The one to one relationship that results between copula functions and joint distributions is known as Sklar theorem.

While the concept of copula functions and their use both in finance and econometrics is well assessed, the application at hand raises a problem that must be handled with special caution. In fact, given the copula function $C_{X,Y}(u, v)$ linking X and Y, we are required to recover the copula function linking X and $X + Y$. Moreover, given the marginal distributions of X and Y and their dependence, we must come up with the probability distribution of the sum $X + Y$. It is clear that such a distribution results from the convolution of X and Y. This problem was solved in Cherubini et al. [5], with the introduction of the concept of convolution-based copulas. If $X \, e \, Y$ be two real-valued random variables with corresponding copula $C_{X,Y}$ and continuous marginals F_X and F_Y, then the distribution function of the sum $X + Y$ is given by

$$F_{X+Y}(z) = \int_0^1 D_1 C_{X,Y}(w, F_Y(z - F_X^{-1}(w)))dw, \tag{1}$$

where $D_1 C_{X,Y}(u, v)$ denotes $\dfrac{\partial C_{X,Y}(u, v)}{\partial u}$.

The main disadvantage of this approach is that the C-convolution may not be used when we model economic or financial variables. In fact, an economic time series is not generated by an independent and identically distributed sequence of random variables. In the other words, F_{X+Y} is a static distribution. For that reason, we need to extend the notion of C-convolution in order to handle dynamic time series. One possible way to proceed is to employ the notion of conditional copula introduced by Patton [11].

The theoretical framework is the following: X and Y are the variables of interest and Z is the conditioning variable. Let F_{XYZ} be the distribution of the random vector (X, Y, Z), $F_{XY|Z}$ be the conditional distribution of (X, Y) given Z and $F_{X|Z}$ and $F_{Y|Z}$ be the conditional marginal distributions of X given Z and of Y given Z, respectively. The conditional copula of (X, Y) given $Z = z$, where X given $Z = z$ has distribution $F_{X|Z=z}(\cdot|z)$ and Y given $Z = z$ has distribution $F_{Y|Z=z}(\cdot|z)$, is the conditional distribution function of $U \equiv F_{X|Z}(X|z)$ and $V \equiv F_{Y|Z}$ $(Y|z)$ given $Z = z$. Moreover, an extension of the Sklar's theorem is proved by Patton [11]: there exists a unique conditional copula $C(\cdot, \cdot|z)$ such that

$$F_{XY|Z}(x, y|z) = C(F_{X|Z}(x|z), F_{Y|Z}(y|z)|z), \forall (x, y) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}, \forall z \in \mathcal{Z},$$

where $\mathcal{Z}$ is the support of the conditioning variable Z. The key point of the 'conditional' version of Sklar's theorem is that the conditioning variable Z must be the same for both marginal distributions and the copula and this is a fundamental condition.

In the spirit of econometric time series analysis, we condition the variables involved to the information available at the time $t-1$, say $\mathcal{F}_{t-1}$. Let C_t be the conditional copula of $(X_t, Y_t)|\mathcal{F}_{t-1}$, G_t^x be the conditional marginal distribution of $X_t|\mathcal{F}_{t-1}$ and G_t^y be the conditional marginal distribution of $Y_t|\mathcal{F}_{t-1}$, then the conditional C-convolution will be the distribution of the sum $X_t|\mathcal{F}_{t-1} + Y_t|\mathcal{F}_{t-1}$, i.e., by applying Equation (1)

$$F_t(z_t) = \int_0^1 D_1 C_t\left(w, G_t^y\left(z_t - G_t^{x,-1}(w)\right)\right) dw,$$

where $D_1 C_t$ denotes the first partial derivative of the copula function. From a statical point of view, the data generating process is provided by (G_t^x, G_t^y, C_t) whereas the conditional distribution F_t is the convolution of the three elements.

3. The Estimator

Let us suppose that the conditional marginal distributions of $X_t|\mathcal{F}_{t-1}$ and $Y_t|\mathcal{F}_{t-1}$ be parameterized as $G_t^x(\cdot; \psi^0)$ and $G_t^y(\cdot; \gamma^0)$, where ψ^0 and γ^0 are vectors of parameters belonging to compact spaces Ψ and Γ, respectively. We assume that G_t^x and G_t^y are known but that the true parameters ψ^0 and γ^0 must be estimated. By the conditional C-convolution, we determine the distribution of the sum $Z_t|\mathcal{F}_{t-1}$ in the case where the dependence structure of $(X_t, Y_t)|\mathcal{F}_{t-1}$ is described by a conditional copula $C_t(u, v; \theta^0)$ characterized by the true parameter θ^0 belonging to a compact space Θ.

The conditional distribution of Z_t depends on ψ^0, γ^0, and θ^0. For estimation purposes, it is useful to emphasize the dependence of the conditional C-convolution on the parameters. So we will write

$$F_t(z_t; \psi^0, \gamma^0, \theta^0) = \int_0^1 D_1 C_t\left(w, G_t^y\left(z_t - G_t^{x,-1}(w, \psi^0); \gamma^0\right); \theta^0\right) dw.$$

We can see that the copula parameter θ may be a constant or time-dependent in a manner similar to Patton [11], which uses a Gaussian copula with a correlation parameter at time t as a function of a constant, the correlation parameter at time $t-1$ and the average of the products $\Phi^{-1}(u_{t-1})\Phi^{-1}(v_{t-1})$ over 10 lags

$$\rho_t = \Lambda\left(\alpha + \beta\rho_{t-1} + \delta \frac{1}{10}\sum_{j=1}^{10}\Phi^{-1}(u_{t-j})\Phi^{-1}(v_{t-j})\right).$$

In this case, we have that $\theta = (\alpha, \beta, \delta)$. Similarly, for Archimedean copulas, the dependence parameter may be modelled taking into account the correspondence value of the Kendall's τ coefficient. In what follows, we consider the case where the parameter is simply an autoregressive process.

It will not always follow that the parameters may be decomposed into three components associated with the two margins and the copula. However, examples where such a decomposition is possible are frequent in financial applications (first of all the Garch models).

Our main purpose is to provide the estimate of the parameters by three steps. In the first two steps, we estimate the vectors of parameters of the marginal distributions, ψ and γ, by a quasi-maximum likelihood method. Let g_t^x and g_t^y be the conditional density function of G_t^x and G_t^y, respectively. Then the logarithm of the quasi-likelihood functions are

$$\ell_{1n}(\psi) = \frac{1}{n}\sum_{t=1}^{n}\ln g_t^x(x_t;\, \psi),$$

$$\ell_{2n}(\gamma) = \frac{1}{n}\sum_{t=1}^{n}\ln g_t^y(y_t;\, \gamma).$$

Therefore, the quasi-maximum likelihood estimators of ψ and γ are

$$\hat{\psi}_n = \arg\max_{\Psi} \ell_{1n}(x^n; \psi),$$

and

$$\hat{\gamma}_n = \arg\max_{\Gamma} \ell_{2n}(y^n; \gamma).$$

The third step is given by the estimator of the copula parameter θ, which may be obtained by the maximization of the quasi-likelihood function relative to the conditional C-convolution. The density function of the conditional C-convolution is the derivative of (1) with respect to z_t, i.e.,

$$f_t(z_t; \psi, \gamma, \theta) = \int_0^1 c_t\Big(w, G_t^y\Big(z_t - G_t^{x,-1}(w, \psi); \gamma\Big); \theta\Big)$$

$$\times\, g_t^y\Big(z_t - G_t^{x,-1}(w, \psi); \gamma\Big)dw,$$

where c_t is the copula density. Denote by $\ell_{3n}(\hat{\psi}_n, \hat{\gamma}, \theta)$ the logarithm of the quasi-likelihood function of the C-convolution computed at $\hat{\psi}_n$ and $\hat{\gamma}_n$. We have

$$\ell_{3n}(\hat{\psi}_n, \hat{\gamma}_n, \theta) = \frac{1}{n}\sum_{t=1}^{n} \ln f_t(z_t; \hat{\psi}_n, \hat{\gamma}_n, \theta).$$

We can see that by construction ℓ_{3n} is a function of θ only. Therefore, the quasi-maximum likelihood estimator of θ will be

$$\hat{\theta}_n = \arg\max_{\Theta} \ell_{3n}(z^n; \hat{\psi}_n, \hat{\gamma}_n, \theta).$$

We call the vector of estimators $\hat{\eta}_n = (\hat{\psi}_n, \hat{\gamma}_n, \theta)$ the three-stage quasi maximum likelihood estimator (3SQMLE).

4. Asymptotic Normality of the 3SQMLE

In this section, we study the behaviour of the 3SQMLE as $n \to +\infty$. The assumptions listed in the appendix are sufficient to ensure that $\hat{\eta}_n$ has a Gaussian limit distribution. This is very important in econometric and statistical applications to generate robust statistical tests on the parameters of the model.

Our main result is the following:

Theorem 4.1. *Under Assumptions* 1-11 (*see the Appendix* 1) *the estimator* $\hat{\eta}_n = \begin{pmatrix} \hat{\psi}_n \\ \hat{\gamma}_n \\ \hat{\theta}_n \end{pmatrix}$ *satisfies*

$$\sqrt{n}(\hat{\eta}_n - \eta^0) = -(A_n^0)^{-1}\sqrt{n}\begin{pmatrix} \nabla_{\psi}\ell_{1n}(X^n;\, \psi^0) \\ \nabla_{\gamma}\ell_{2n}(Y^n;\, \gamma^0) \\ \nabla_{\theta}\ell_{3n}(Z^n;\, \psi^0,\, \gamma^0,\, \theta^0) \end{pmatrix} + o_{\mathbb{P}_0}(1),$$

and

$$\sqrt{n}(\overline{B}_n^0)^{-1/2} A_n^0(\hat{\eta}_n - \eta^0) \xrightarrow{d} N(0,\, I),$$

where I *is an identity matrix of appropriate dimension,* $A_n^0 = \mathbb{E}[H_n^0]$, *where* H_n^0 *is the block matrix of the second order partial derivatives*

$$H_n^0 =$$

$$\begin{pmatrix} \nabla_{\psi\psi}\ell_{1n}(X^n;\psi^0) & 0 & 0 \\ 0 & \nabla_{\gamma\gamma}\ell_{2n}(Y^n;\gamma^0) & 0 \\ \nabla_{\psi\theta}\ell_{3n}(Z^n;\psi^0,\gamma^0,\theta^0) & \nabla_{\gamma\theta}\ell_{3n}(Z^n;\psi^0,\gamma^0,\theta^0) & \nabla_{\theta\theta}\ell_{3n}(Z^n;\psi^0,\gamma^0,\theta^0) \end{pmatrix},$$

and $\overline{B}_n^0 = \mathbb{E}[B_n^0]$, *where*

$$B_n^0 =$$

$$\begin{pmatrix} \frac{1}{n}\sum_{t=1}^{n}[s_{1t}^0 \cdot (s_{1t}^0)^T] & \frac{1}{n}\sum_{t=1}^{n}[s_{1t}^0 \cdot (s_{2t}^0)^T] & \frac{1}{n}\sum_{t=1}^{n}[s_{1t}^0 \cdot (s_{3t}^0)^T] \\ \frac{1}{n}\sum_{t=1}^{n}[s_{2t}^0 \cdot (s_{1t}^0)^T] & \frac{1}{n}\sum_{t=1}^{n}[s_{2t}^0 \cdot (s_{2t}^0)^T] & \frac{1}{n}\sum_{t=1}^{n}[s_{2t}^0 \cdot (s_{3t}^0)^T] \\ \frac{1}{n}\sum_{t=1}^{n}[s_{3t}^0 \cdot (s_{1t}^0)^T] & \frac{1}{n}\sum_{t=1}^{n}[s_{3t}^0 \cdot (s_{2t}^0)^T] & \frac{1}{n}\sum_{t=1}^{n}[s_{3t}^0 \cdot (s_{3t}^0)^T] \end{pmatrix},$$

where $s_{1t}^0 = s_{1t}(x_t; \psi^0) = \nabla_\psi \ln g_t^x(x_t; \psi^0)$, $s_{2t}^0 = s_{2t}(y_t; \gamma^0) = \nabla_\gamma \ln g_t^y(y_t; \gamma^0)$, *and* $s_{3t}^0 = s_{3t}(z_t; \psi^0, \gamma^0, \theta^0) = \nabla_\theta \ln f_t(z_t; \psi^0, \gamma^0, \theta^0)$ *are the score functions.*

Following White [13], the asymptotic covariance matrix of the estimator is given by

$$AV\ ar(\hat{\eta}_n) = (A_n^0)^{-1}\overline{B}_n^0(A_n^0)^{-1},$$

and it may be consistently estimated using the Hessian and the outer product matrix of the scores evaluated at the 3SQMLE

$$\widehat{AV\ ar}(\hat{\eta}_n) = \hat{H}_n^{-1}\hat{O}P_n\hat{H}_n^{-1},$$

where $\hat{H}_n$ is the Hessian matrix evaluated at $[\hat{\psi}_n, \hat{\gamma}_n, \hat{\theta}_n]$ and $\hat{O}P_n$ is given by

$$\hat{O}P_n = \begin{pmatrix} \frac{1}{n}\sum_{t=1}^{n}[\hat{s}_{1t} \cdot \hat{s}_{1t}^T] & \frac{1}{n}\sum_{t=1}^{n}[\hat{s}_{1t} \cdot \hat{s}_{2t}^T] & \frac{1}{n}\sum_{t=1}^{n}[\hat{s}_{1t} \cdot \hat{s}_{3t}^T] \\ \frac{1}{n}\sum_{t=1}^{n}[\hat{s}_{2t} \cdot \hat{s}_{1t}^T] & \frac{1}{n}\sum_{t=1}^{n}[\hat{s}_{2t} \cdot \hat{s}_{2t}^T] & \frac{1}{n}\sum_{t=1}^{n}[\hat{s}_{2t} \cdot \hat{s}_{3t}^T] \\ \frac{1}{n}\sum_{t=1}^{n}[\hat{s}_{3t} \cdot \hat{s}_{1t}^T] & \frac{1}{n}\sum_{t=1}^{n}[\hat{s}_{3t} \cdot \hat{s}_{2t}^T] & \frac{1}{n}\sum_{t=1}^{n}[\hat{s}_{3t} \cdot \hat{s}_{3t}^T] \end{pmatrix},$$

where $\hat{s}_{1t} = s_{1t}(x_t; \hat{\psi}_n)$, $\hat{s}_{2t} = s_{2t}(y_t; \hat{\gamma}_n)$, and $\hat{s}_{3t} = s_{3t}(z_t; \hat{\psi}_n, \hat{\gamma}_n, \hat{\theta}_n)$.

Assumption 4 in the Appendix requires that the density of the conditional C-convolution is continuously differentiable of order 2 with respect to the parameters. Since $f_t(z_t; \cdot, \cdot, \cdot)$ depends on the copula function a reasonable requirement for the validity of Assumption 4 is that the integrand function and its partial derivatives are bounded from above by an integrable function, i.e.,

$$\left|c_t\left(w, G_t^y\left(z_t - G_t^{x,-1}(w, \psi); \gamma\right); \theta\right) g_t^y\left(z_t - G_t^{x,-1}(w, \psi); \gamma\right)\right| \leq h_0(z_t, w),$$

$$\left|\frac{\partial}{\partial\theta} c_t\left(w, G_t^y\left(z_t - G_t^{x,-1}(w, \psi); \gamma\right); \theta\right) g_t^y\left(z_t - G_t^{x,-1}(w, \psi); \gamma\right)\right| \leq h_1(z_t, w),$$

$$\left|\frac{\partial^2}{\partial\theta^2} c_t\left(w, G_t^y\left(z_t - G_t^{x,-1}(w, \psi); \gamma\right); \theta\right) g_t^y\left(z_t - G_t^{x,-1}(w, \psi); \gamma\right)\right| \leq h_2(z_t, w),$$

where $h_0(z_t, w), h_1(z_t, w)$, and $h_2(z_t, w)$ are integrable with respect to w for all $t = 1, 2, \ldots.$ Now, suppose that the conditional density g_t^y and its partial derivatives are bounded for all $t = 1, 2\ldots,$ then if the copula density and its partial derivatives are bounded from above by a constant, i.e.,

$$|c_t(u, v; \theta)| \leq k_0, \quad \forall (u, v),$$

$$\left|\frac{\partial}{\partial\theta} c_t(u, v; \theta)\right| \leq k_1, \quad \forall (u, v),$$

$$\left|\frac{\partial^2}{\partial\theta^2} c_t(u, v; \theta)\right| \leq k_2, \quad \forall (u, v),$$

Assumption 4 is ensured. Gaussian copula, Student's t copula and Frank copula satisfy the conditions above. Differently, for other Archimedean copulas like Clayton copula and Gumbel copula, the reader may find a detailed discussion on scores assumptions in Genest et al. [8] and Chen and Fan [2].

5. Small Sample Properties

In this section, we present the results of a Monte Carlo simulation of the small sample properties of the estimators discussed above in the case where both marginals are conditionally Gaussian with the same Garch (1,1) specifications, whose parameters are designed to reect the high persistence conditional volatility. The simulated marginal models are

$$X_t = \mu_x + h_{x,t}e_t,$$

$$h_{x,t}^2 = \omega_{x,0} + \omega_{x,1}(X_{t-1} - \mu_x)^2 + \omega_{x,2}h_{x,t-1}^2,$$

$$e_t|\mathcal{F}_{t-1} \overset{\text{iid}}{\sim} N(0, 1),$$

$$Y_t = \mu_y + h_{y,t}q_t,$$

$$k_{y,t}^2 = \omega_{y,0} + \omega_{y,1}(Y_{t-1} - \mu_y)^2 + \omega_{y,2}k_{y,t-1}^2,$$

$$q_t|\mathcal{F}_{t-1} \overset{\text{iid}}{\sim} N(0, 1),$$

with $\mu_x = \mu_y = 0.01$, $\omega_{x,0} = \omega_{y,0} = 0.05$, $\omega_{x,1} = \omega_{y,1} = 0.1$, and $\omega_{x,2} = \omega_{y,2} = 0.85$. The values of parameters are taken from Patton [11]. As for the dependence structure between the two variables, we examine the case of the Frank copula with two different dynamics for the parameter. Firstly, we consider the case where X_t and Y_t are linked by a Frank copula with a time invariant parameter chosen so as to imply a Kendall's τ coefficient of 0.25 and 0.50. The correspondent values of the parameter are $\theta = 2.37$ and $\theta = 5.74$. The relationship between the copula parameter and the Kendall's τ coefficient is obtained by inverting the "Debye" function (see for more details Cherubini et al. [4] p. 126). In a second case, we develop simulations for studying a time-varying conditional dependence. More precisely, suppose that θ_t has an autoregressive dynamics

$$\theta_t = \Lambda(\alpha + \beta\theta_{t-1} + \varepsilon_t),$$

where ε_t is a random disturbance normally distributed with zero mean and standard deviation σ_ε and $\Lambda(x) = \frac{3}{1 + e^{-x}}$ in order to ensure that the corresponding Kendall's τ remains in the interval $(0, 0.6)$. We develop simulations with $\alpha = 0.2$, $\beta = 0.8$ and $\sigma_\varepsilon = 0.1$. We consider two possible sample sizes: $n = 500, 1000$. We present only the results relative to the estimation of the copula parameter, which is the third step of the 3SQMLE. Tables 1 and 2 contain the averages of estimates and the mean squared error (MSE). The simulation results suggest that the level of dependence between the marginal variables affects the accuracy of the estimate in the case of constant parameter. A greater dependence leads to higher MSE for the same sample size. However, as the sample size raises the MSE rapidly decreases. Differently, in the case of time-varying dependence structure, the estimates show less accuracy even if the estimate of the autoregressive parameter is acceptable. The volatility σ_ε is overestimate in both sample sizes.

Table 1. Average value and relative MSE of the Monte Carlo estimates for three different values of the sample size and two different values of the Frank copula parameter

	$n = 500$	$n = 1000$
$\theta = 2.37$	2.5363(0.6738)	2.4383(0.3473)
$\theta = 5.74$	6.0671(2.4969)	5.9889(1.6427)

Table 2. Average value and relative MSE of the Monte Carlo estimates for two different values of the sample size and two different copula functions in the case of time-varying conditional dependence

	$n = 500$	$n = 1000$
$\alpha = 0.2$	0.2883(0.4580)	0.2663(0.3865)
$\beta = 0.8$	0.5665(0.3801)	0.6213(0.3125)
$\sigma_\varepsilon = 0.1$	0.3575(0.4174)	0.2512(0.3454)

6. Application

In this section, we apply the model discussed above to a data set of daily log-returns of prices of two stocks from Italian financial market: Eni and Fiat. The data employed runs from 1st January 2008 to 31 December 2010. The total number of observations is $n = 750$. Our technique requires the estimation of conditional marginal distributions. Since the standard Normal Garch (1,1) does not overcome the Kolmogorov-Smirnov (KS) test of goodness of fit, we decide to model the margins with a Student's t Garch (1,1) process; the functional forms are

$$X_t = \mu_x + e_{x,t},$$

$$h_{x,t}^2 = \omega_{x,0} + \omega_{x,1} e_{x,t}^2 + \omega_{x,2} h_{x,t-1}^2,$$

$$\sqrt{\frac{\nu_x}{h_{x,t}^2(\nu_x - 2)}} e_{x,t} | \mathcal{F}_{t-1} \overset{\text{iid}}{\sim} t_{(\nu_x)},$$

$$Y_t = \mu_y + e_{y,t},$$

$$h_{y,t}^2 = \omega_{y,0} + \omega_{y,1} e_{y,t}^2 + \omega_{y,2} h_{y,t-1}^2,$$

$$\sqrt{\frac{\nu_y}{h_{y,t}^2(\nu_y - 2)}} e_{y,t} | \mathcal{F}_{t-1} \overset{\text{iid}}{\sim} t_{(\nu_y)}.$$

Observe that in our notation we have $\psi = (\mu_x, \omega_{x,0}, \omega_{x,1}, \omega_{x,2}, \nu_x)$ and $\gamma = (\mu_y, \omega_{y,0}, \omega_{y,1}, \omega_{y,2}, \nu_y)$. The quasi-maximum likelihood estimates are reported in Table 3. The KS statistics show that both models overcome the goodness of fit test. Given that margins we can estimate the conditional *C*-convolution. For sake of comparison, we employ two copula functions to describe the dependence structure between the two assets: Frank copula and Clayton copula. First, we consider the case where the parameter is constant over time. The parameters estimates are $\hat{\theta} = 2.1328$ for the Frank copula and $\hat{\theta} = 0.4709$ for the Clayton copula which correspond to a Kendall's τ coefficient equal to 0.2269 and 0.1906, respectively. Since the most interesting case is the time-varying conditional dependence parameter, we select two possible autoregressive dynamics similar to the case studied in the Monte Carlo simulation. In particular, for the Frank copula,

$$\theta_t = \Lambda_1(\alpha + \beta\theta_{t-1} + \varepsilon_t),$$

where ε_t is a sequence of i.i.d. normally distributed r.vs. with zero mean and standard deviation σ_ε and $\Lambda_1(x) = \dfrac{5}{1 + e^{-x}}$ in such a way that the parameter remains in the interval $[0, 5]$. Similarly for the Clayton copula,

$$\theta_t = \Lambda_2(\alpha + \beta\theta_{t-1} + \varepsilon_t),$$

where $\Lambda_2(x) = \dfrac{3}{1 + e^{-x}}$ in such a way that the parameter remains in the interval $[0, 3]$. In our notation $\theta = (\alpha, \beta, \sigma_\varepsilon)$. Table 4 reports the quasi-maximum likelihood estimates which are all significantly different from zero. In the other words, we find significant evidence of time variation in the conditional dependence. The parameter σ_ε provides a measure of the variability of the dependence structure in the spirit of autoregressive processes. Figure 1 displays the evolution of the time-varying parameter.

Now, thanks to parameter estimates, we construct the estimated version of the conditional C-convolution, that is, the estimated distribution of the sum of daily log-returns of the two assets

$$\hat{F}_t(z_t) = \int_0^1 D_1 C_t\Big(w,\ \hat{G}_t^y\Big(z_t - \hat{G}_t^{x,-1}(w;\ \hat{\vartheta});\ \hat{\gamma}\Big);\ \hat{\theta}\Big)dw,$$

where $\hat{G}_t^x$ and $\hat{G}_t^y$ denote the estimated Student's t Garch (1,1) margins and C_t may be the Frank copula or the Clayton copula. Figure 2 shows the distribution with both copulas: We see that the Clayton copula highlights a heavy left-tail. Nevertheless, the Kolmogorov-Smirnov test for the shape of the distribution indicates that only the C-convolution constructed with the Frank copula is accepted, whereas the case of Clayton copula is rejected. In fact, the p-value from the KS test for the first case is 0.1809 (we accept the null hypothesis) and for the second case is 0.0064 (we reject the null hypothesis).

Table 3. Quasi-maximum likelihood estimates of the marginal returns distribution model. The asterisk denotes the parameters which are significantly different from zero at the 5% level

Eni	Fiat
$\mu_x = 6.0154 \times 10^{-4}$	$\mu_y = 8.2173 \times 10^{-4}$
$\omega_{x,0} = 7.7796 \times 10^{-6*}$	$\omega_{2,0} = 1.4582 \times 10^{-5}$
$\omega_{x,1} = 0.8744^*$	$\omega_{y,1} = 0.9184^*$
$\omega_{x,2} = 0.1088^*$	$\omega_{2,2} = 0.0689^*$
$\nu_x = 5.6735^*$	$\nu_y = 6.2341^*$
KS $= 0.0398 (p = 0.1808)$	KS $= 0.0330 (p = 0.3799)$

Table 4. Quasi-maximum likelihood estimates of conditional dependence parameters for both copulas. The asterisk denotes the parameters which are significantly different from zero at the 5% level

Frank	Clayton
$\hat{\alpha} = -0.2617^{*}$	$\hat{\alpha} = -0.6030^{*}$
$\hat{\beta} = -0.0552^{*}$	$\hat{\beta} = -0.8987^{*}$
$\hat{\sigma}_{\varepsilon} = 0.0764^{*}$	$\hat{\sigma}_{\varepsilon} = 0.0498^{*}$

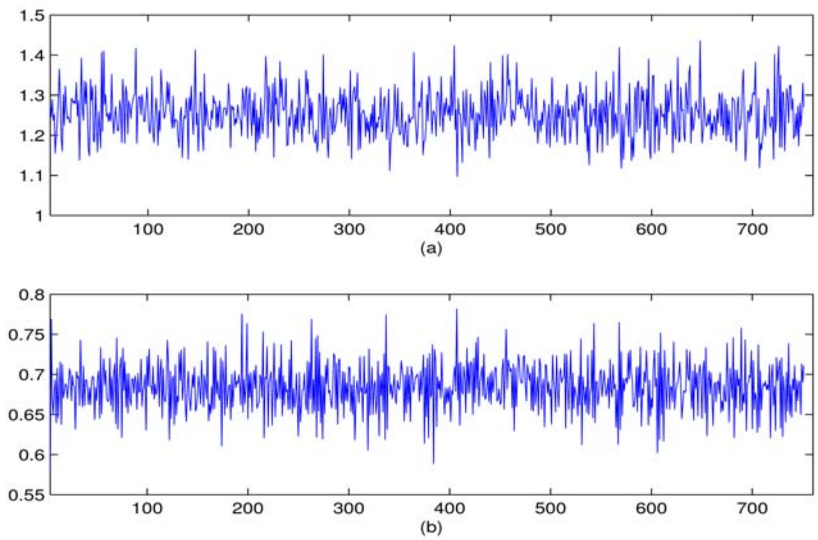

Figure 1. Conditional copula parameters dynamics: (a) Frank copula; (b) Clayton copula.

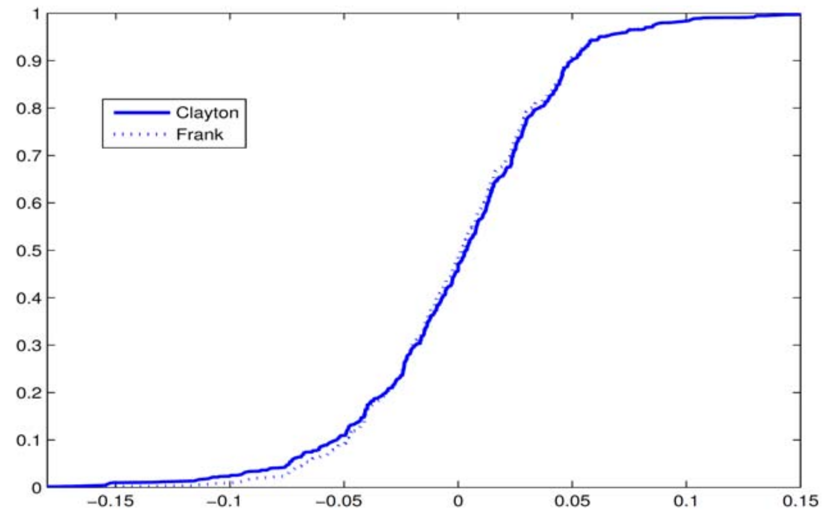

Figure 2. Comparison between estimated conditional C-convolutions.

7. Conclusion

This paper presented a three-stage maximum likelihood estimator of the parameters of a conditional C-convolution model for time series and determined its asymptotic normality. The assumptions required are not restrictive and they are standard in econometrics. The use of the conditional C-convolution is required because economic time series are not i.i.d. and conditioning variables are necessary. Moreover, numerous situations exist where the sum of two variables is required and the conditional C-convolution provides a model to recover the distribution of the sum. We performed the small sample properties of the 3SQMLE in a simulated model with marginal variables given by Garch processes and dependence structure given by a Frank copula with time-varying parameter. We found that the efficiency of the estimates rapidly increased with the sample size. Finally, our application focused on the Italian financial market with two time series of daily log-returns relative to Eni and Fiat. After estimating the appropriate conditional margins, we estimated the conditional C-convolution and we found evidence that this is the appropriate distribution of the sum of log-returns when their dependence structure is described by a Frank copula with time-varying parameter.

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Appendix 1

We present in this appendix the assumptions required for Theorem 4.1. Most of these assumptions are based on those presented in White [13].

In order to make more easier the reading of the assumptions below, we recall the notation. Let

$$s_{1t}(x_t;\psi) = \nabla_\psi \ln g_t^x(x_t;\psi),$$

$$\bar{\ell}_{1n}(\psi) = \mathbb{E}[\ell_{1n}(X^n;\psi)],$$

$$\nabla_\psi \ell_{1n}(x^n;\psi) = \frac{1}{n}\sum_{t=1}^{n} \nabla_\psi \ln g_t^x(x_t;\psi) = \frac{1}{n}\sum_{t=1}^{n} s_{1t}(x_t;\psi),$$

$$s_{2t}(y_t;\gamma) = \nabla_\gamma \ln g_t^y(y_t;\gamma),$$

$$\bar{\ell}_{2n}(\gamma) = \mathbb{E}[\ell_{2n}(Y^n;\gamma)],$$

$$\nabla_\gamma \ell_{2n}(y^n;\gamma) = \frac{1}{n}\sum_{t=1}^{n} \nabla_\gamma \ln g_t^y(y_t;\gamma) = \frac{1}{n}\sum_{t=1}^{n} s_{2t}(y_t;\gamma),$$

$$s_{3t}(z_t;\psi,\gamma,\theta) = \nabla_\theta \ln f_t(z_t;\psi,\gamma,\theta),$$

$$\bar{\ell}_{3n}(\psi,\gamma,\theta) = \mathbb{E}[\ell_{3n}(Z^n;(\psi,\gamma,\theta)],$$

$$\nabla_\theta \ell_{3n}(z^n;\psi,\gamma,\theta) = \frac{1}{n}\sum_{t=1}^{n} \nabla_\theta \ln f_t(z_t;\psi,\gamma,\theta) = \frac{1}{n}\sum_{t=1}^{n} s_{3t}(z_t;\psi,\gamma,\theta).$$

Assumption 1.

- (a) The sets of parameters Ψ, Γ and Θ are compacts.

- (b) $g_t^x(x_t;\cdot)$, $g_t^y(y_t;\cdot) f_t(z_t;\cdot,\cdot,\cdot)$ are continuous on Ψ, Γ and $\Psi \otimes \Gamma \otimes \Theta$ respectively, a.s., for all $t = 1, 2, \ldots$.

Assumption 2.

• (a) $\mathbb{E}[\ln g_t^x(X_t;\psi)]$, $\mathbb{E}[\ln g_t^y(Y_t;\gamma)]$, and $\mathbb{E}[\ln f_t(Z_t;\psi,\gamma,\theta)]$ exist and are finite for each $\psi \in \Psi$, $\gamma \in \Gamma$ and for each $(\psi,\gamma,\theta) \in \Psi \otimes \Gamma \otimes \Theta$.

• (b) $\mathbb{E}[\ln g_t^x(X_t;\psi)]$, $\mathbb{E}[\ln g_t^y(Y_t;\gamma)]$, and $\mathbb{E}[\ln f_t(Z_t;\psi,\gamma,\theta)]$ are continuous on Ψ, Γ and $\Psi \otimes \Gamma \otimes \Theta$, respectively, for all x_t, y_t, z_t.

• (c) $\{\ln g_t^x(X_t;\psi)\}_t$, $\{\ln g_t^y(Y_t;\gamma)\}_t$ and $\{\ln f_t(Z_t;\psi,\gamma,\theta)\}_t$ obey the weak uniform law of large numbers.

Assumption 3.

• (a) $\overline{\ell}_{1n}(\psi)$ and $\overline{\ell}_{2n}(\gamma)$ are $O(1)$ uniformly on Ψ and Γ. Moreover, $\overline{\ell}_{1n}(\cdot)$ and $\overline{\ell}_{2n}(\cdot)$ have unique maximizers $\psi^0 \in \overline{\Psi}$ and $\gamma^0 \in \overline{\Gamma}$, where $\overline{\Psi}$ and $\overline{\Gamma}$ are the interior of Ψ and Γ.

• (b) $\overline{\ell}_{3n}(\psi,\gamma,\theta)$ is $O(1)$ uniformly on $\Psi \otimes \Gamma \otimes \Theta$.

Assumption 4. $g_t^x(x_t;\cdot)$, $g_t^y(y_t;\cdot)$, and $f_t(z_t;\cdot,\cdot,\cdot)$ are continuously differentiable of order 2 on Ψ, Γ and $\Psi \otimes \Gamma \otimes \Theta$, respectively, a.s., for all $t = 1, 2, \ldots$.

Assumption 5.

• (a) $\mathbb{E}[\nabla_\psi \ell_{1n}(X^n;\psi)]$, $\mathbb{E}[\nabla_\gamma \ell_{2n}(Y^n;\gamma)]$, and $\mathbb{E}[\nabla_\theta \ell_{3n}(Z^n;\psi,\gamma,\theta)]$ are finite for each $\psi \in \Psi$, $\gamma \in \Gamma$ and for each $(\psi,\gamma,\theta) \in \Psi \otimes \Gamma \otimes \Theta$, respectively.

• (b) $\mathbb{E}[\nabla_\psi \ell_{1n}(X^n;\psi)]$, $\mathbb{E}[\nabla_\gamma \ell_{2n}(Y^n;\gamma)]$, and $\mathbb{E}[\nabla_\theta \ell_{3n}(Z^n;\psi,\gamma,\theta)]$ are continuous for each $\psi \in \Psi$, $\gamma \in \Gamma$ and for each $(\psi,\gamma,\theta) \in \Psi \otimes \Gamma \otimes \Theta$, respectively, uniformly in n.

Assumption 6.

- (a) $\mathbb{E}[\nabla_{\psi\psi}\ell_{1n}(X^n;\psi)]$ is finite for each $\psi \in \Psi$. $\mathbb{E}[\nabla_{\gamma\gamma}\ell_{2n}(Y^n;\gamma)]$ is finite for each $\gamma \in \Gamma$. $\mathbb{E}[\nabla_{\theta\theta}\ell_{3n}(Z^n;\psi,\gamma,\theta)]$, $\mathbb{E}[\nabla_{\psi\theta}\ell_{3n}(Z^n;\psi,\gamma,\theta)]$ and $\mathbb{E}[\nabla_{\gamma\theta}\ell_{3n}(Z^n;\psi,\gamma,\theta)]$ are finite for each $(\psi,\gamma,\theta) \in \Psi \otimes \Gamma \otimes \Theta$.

- (b) $\mathbb{E}[\nabla_{\psi\psi}\ell_{1n}(X^n;\psi)]$ is continuous for each $\psi \in \Psi$, uniformly in n. $\mathbb{E}[\nabla_{\gamma\gamma}\ell_{2n}(Y^n;\gamma)]$ is continuous for each $\gamma \in \Gamma$, uniformly in n. $\mathbb{E}[\nabla_{\theta\theta}\ell_{3n}(Z^n;\psi,\gamma,\theta)]$, $\mathbb{E}[\nabla_{\psi\theta}\ell_{3n}(Z^n;\psi,\gamma,\theta)]$, and $\mathbb{E}[\nabla_{\gamma\theta}\ell_{3n}(Z^n;\psi,\gamma,\theta)]$ are continuous for each $(\psi,\gamma,\theta) \in \Psi \otimes \Gamma \otimes \Theta$ uniformly in n.

Assumption 7.

- (a) $\{s_{1t}(X_t;\psi)\}_t$, $\{s_{2t}(Y_t;\gamma)\}_t$, and $\{s_{3t}(Z_t;\psi,\gamma,\theta)\}_t$ obey the weak uniform law of large numbers.

- (b) $\{\nabla_{\psi}s_{1t}(X_t;\psi)\}_t$, $\{\nabla_{\gamma}s_{2t}(Y_t;\gamma)\}_t$, $\{\nabla_{\theta}s_{3t}(Z_t;\psi,\gamma,\theta)\}_t$, $\{\nabla_{\psi}s_{3t}(Z_t;\psi,\gamma,\theta)\}_t$, and $\{\nabla_{\gamma}s_t(Z_t;\psi,\gamma,\theta)\}_t$ obey the weak uniform law of large numbers.

Assumption 8. $A_n^0 = \mathbb{E}[H_n^0]$ is $O(1)$ and non-singular uniformly in n.

Assumption 9. $\bar{\ell}_{3n}(\hat{\psi}_n, \hat{\gamma}_n, \theta)$ has a unique maximizer $\theta^0 \in \bar{\Theta}$.

Assumption 10. $\left\{\frac{1}{\sqrt{n}}s_{1t}(X_t;\psi^0); \frac{1}{\sqrt{n}}s_{2t}(Y_t;\gamma^0); \frac{1}{\sqrt{n}}s_{3t}(Z_t;\psi^0,\gamma^0,\theta^0)\right\}$ obeys the central limit theorem with covariance matrix $\bar{B}_n^0$, where $\bar{B}_n^0$ is $O(1)$ and positive definite.

Assumption 11.

- (a) The elements of $\bar{B}_n^0$ are finite and continuous on $\Psi \otimes \Gamma \otimes \Theta$, uniformly in n.

- (b) The sequence

$$\Big\{[s_{1t}(X_t;\psi^0);\ s_{2t}(Y_t;\gamma^0);\ s_{3t}(Z_t;\psi^0,\gamma^0,\theta^0)]$$

$$\cdot\,[s_{1t}(X_t;\psi^0)^T,\ s_{2t}(Y_t;\gamma^0)^T,\ s_{3t}(Z_t;\psi^0,\gamma^0,\theta^0)^T\,]\Big\}_t,$$

obeys the weak uniform law of large numbers.

Andrews [1], Gallant and White [7] and White [13] provide some results on uniform laws of large numbers and on central limit theorems for dependent, heterogeneously distributed random variables that may be used to justify Assumptions 2(c), 7, 10, and 11(b). White [13] also provides a review on these topics in a wide variety of situations.

Appendix 2

The proof of the Theorem 4.1 is based on some results due to Domowitz and White [6] and White [13] which we report here in form of lemmas. In particular, we refer to Theorems 6.10, 3.5, and 3.10 of White [13].

Lemma 9.1. *Given a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a compact subset* $\mathcal{H}$ *of* $\mathbb{R}^p$, $p \in \mathbb{N}$, *let* $\xi_n : \Omega \times \mathcal{H} \to \mathbb{R}^p$ *be a random function such that* $\xi_n(\cdot, \eta)$ *is* $\mathcal{F}$*-measurable for each* $\eta \in \mathcal{H}$ *and* $\xi_n(\omega, \cdot)$ *is continuously differentiable on* $\mathcal{H}$, $\mathbb{P}$*-a.s.*, $n = 1, 2, \ldots$ *Let* $\hat{\eta}_n : \Omega \to \mathcal{H}$ *be* $\mathcal{F}$*-measurable*, $n = 1, 2, \ldots$, *such that* $\hat{\eta}_n \xrightarrow{\mathbb{P}} \eta^0$, *where* η^0 *is interior of* $\mathcal{H}$. *Moreover* $\sqrt{n}\xi_n(\cdot, \hat{\eta}_n) \xrightarrow{\mathbb{P}} 0$. *Suppose there exists a sequence of* $p \times p$ *matrices* B_n^0 *that is* $O(1)$ *and uniformly positive definite such that* $(B_n^0)^{-1/2}\sqrt{n}\xi_n(\cdot, \hat{\eta}_n) \to N(0, I_p)$. *If there exists a sequence* $A_n : \mathcal{H} \to \mathbb{R}^{p\times p}$ *such that* A_n *is continuous on* $\mathcal{H}$ *uniformly in n and* $\nabla_\eta \xi_n(\cdot, \hat{\eta}_n) - A_n(\eta) \xrightarrow{\mathbb{P}} 0$ *uniformly in* $\mathcal{H}$ *and* $A_n^0 = A_n(\eta^0)$ *is uniformly non-singular, then*

$$\sqrt{n}(\hat{\eta}_n - \eta^0) = -(A_n^0)^{-1}\sqrt{n}\xi_n(\cdot, \hat{\eta}^n) + o_{\mathbb{P}}(1),$$

and

$$(B_n^0)^{-1/2}(A_n^0)^{-1}\sqrt{n}(\hat{\eta}_n - \eta^0) \xrightarrow{d} N(0, I_p).$$

Lemma 9.2. *Given a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a compact subset* $\mathcal{H}$ *of* $\mathbb{R}^p$, $p \in \mathbb{N}$, *let* $\{Q_n\}_n$ *be a sequence of random functions continuous on* $\mathcal{H}$ $\mathbb{P}$-*a.s. and* $\hat{\eta}_n = \arg\max_{\mathcal{H}} Q_n(\cdot; \eta)$. *If* $Q_n(\cdot; \eta) - \overline{Q}_n(\eta) \to 0$, $\mathbb{P}$-*a.s. uniformly on* $\mathcal{H}$ *and if* $\overline{Q}_n : \mathcal{H} \to \mathbb{R}$ *has identifiable unique maximizer* η^0, *then* $\hat{\eta}_n - \eta^0 \to 0$ $\mathbb{P}$-*a.s.*.

Lemma 9.3. *Given a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *let* $\{X_n\}_n$ *be a sequence of random vectors and* X *be a random vector. Then* $X_n - X \xrightarrow{\mathbb{P}} 0$ *if and only if for any subsequence* n', *there exits a further subsequence* n'' *such that* $X_{n''} - X \to 0$, $\mathbb{P}$-*a.s.*.

Proof. Proof of the Theorem 4.1. The proof is based on Lemma 9.1 on random functions when we set the positions $\eta = (\psi, \gamma, \theta)$, $\mathcal{H} = \Psi \otimes \Gamma \otimes \Theta$ and

$$(\omega, \eta) \mapsto \xi_n(\omega, \eta) = \begin{pmatrix} \nabla_{\psi}\ell_{1n}(X^n(\omega), \psi) \\ \nabla_{\gamma}\ell_{2n}(Y^n(\omega), \gamma) \\ \nabla_{\theta}\ell_{3n}(Z^n(\omega), \psi, \gamma, \theta) \end{pmatrix}.$$

Here we verify the assumptions required to prove the statements. For simplicity, we omit the checking of measurability conditions (see White [13] for more details). Lemma 9.1 requires four crucial assumptions to be applied. The first one is that the estimator $\hat{\eta}_n$ is consistent for η^0 and this is ensured by Assumptions 1, 2, 3, and 9 and Lemma 9.2. In fact, the first two components $\hat{\psi}_n$ and $\hat{\gamma}_n$ are strongly consistent for ψ^0 and γ^0, respectively, if we apply Lemma 9.2 setting $\mathcal{H} = \Psi$ and $Q_n(\cdot;\psi) = \ell_{1n}(\cdot;\psi)$ and $\mathcal{H} = \Gamma$ and $Q_n(\cdot;\gamma) = \ell_{2n}(\cdot;\gamma)$, respectively. Given that, the strong consistency of the third component, $\hat{\theta}_n$, is ensured by applying Lemma 9.2 with the positions $\mathcal{H} = \Theta$ and $Q_n(\cdot;\theta) = \ell_{3n}(\cdot;\hat{\psi}_n, \hat{\gamma}_n, \theta)$. Under Assumption 10, we have

$$(\overline{B}_n^0)^{-1/2}\sqrt{n}\left(\begin{pmatrix} \nabla_\psi \ell_{1n}(X^n, \psi^0) \\ \nabla_\gamma \ell_{2n}(Y^n, \gamma^0) \\ \nabla_\theta \ell_{3n}(Z^n, \psi^0, \gamma^0, \theta^0) \end{pmatrix} - \mathbb{E}\begin{pmatrix} \nabla_\psi \ell_{1n}(X^n, \psi^0) \\ \nabla_\gamma \ell_{2n}(Y^n, \gamma^0) \\ \nabla_\theta \ell_{3n}(Z^n, \psi^0, \gamma_0, \theta^0) \end{pmatrix}\right) \xrightarrow{d} N(0, I),$$

but under Assumption 5,

$$\mathbb{E}\begin{pmatrix} \nabla_\psi \ell_{1n}(X^n, \psi^0) \\ \nabla_\gamma \ell_{2n}(Y^n, \gamma^0) \\ \nabla_\theta \ell_{3n}(Z^n, \psi^0, \gamma^0, \theta^0) \end{pmatrix} = \begin{pmatrix} \nabla_\psi \overline{\ell}_{1n}(\psi^0) \\ \nabla_\gamma \overline{\ell}_{2n}(\gamma^0) \\ \nabla_\theta \overline{\ell}_{3n}(\psi^0, \gamma^0, \theta^0) \end{pmatrix} = 0.$$

Thus

$$(\overline{B}_n^0)^{-1/2}\sqrt{n}\begin{pmatrix} \nabla_\psi \ell_{1n}(X^n, \psi^0) \\ \nabla_\gamma \ell_{2n}(Y^n, \psi^0) \\ \nabla_\theta \ell_{3n}(Z^n, \psi^0, \gamma^0, \theta^0) \end{pmatrix} \xrightarrow{d} N(0, I),$$

which is the second hypothesis of Lemma 9.1.

Under Assumptions 6 and 7, we have $H_n(\psi, \gamma, \theta) - A_n(\psi, \gamma, \theta) \xrightarrow{\mathbb{P}} 0$ uniformly on $\Psi \otimes \Gamma \otimes \Theta$, where $H_n(\psi, \gamma, \theta)$ denotes the block matrix of the second order partial derivatives evaluated at (ψ, γ, θ) and $A_n(\psi, \gamma, \theta) = \mathbb{E}[H_n(\psi, \gamma, \theta)]$. This is the third hypothesis of Lemma 9.1.

Finally, the last assumption to be verified in order to apply Lemma 9.1 is that $\sqrt{n}\begin{pmatrix} \nabla_{\psi}\bar{\ell}_{1n}(\hat{\psi}_n) \\ \nabla_{\gamma}\bar{\ell}_{2n}(\hat{\gamma}_n) \\ \nabla_{\theta}\bar{\ell}_{3n}(\hat{\psi}_n, \hat{\gamma}_n, \hat{\theta}_n) \end{pmatrix} \xrightarrow{\mathbb{P}} 0$ and this is ensured by Lemma 9.3 and Assumptions 3 and 9.

■

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