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Degrees of bi-embeddable categoricity of equivalence structures

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Abstract

We study the algorithmic complexity of embeddings between bi-embeddable equivalence structures. We define the notions of computable bi-embeddable categoricity, (relative) Δ^0_α bi-embeddable categoricity, and degrees of bi-embeddable categoricity. These notions mirror the classical notions used to study the complexity of isomorphisms between structures. We show that the notions of Δ^0_α bi-embeddable categoricity and relative Δ^0_α bi-embeddable categoricity coincide for equivalence structures for $\alpha = 1, 2, 3$. We also prove that computable equivalence structures have degree of bi-embeddable categoricity $\mathbf{0}$, $\mathbf{0}'$, or $\mathbf{0}''$. We furthermore obtain results on the index set complexity of computable equivalence structure with respect to bi-embeddability.

Keywords Computable categoricity · Bi-embeddability · Degrees of categoricity · Degrees of bi-embeddable categoricity

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1 Introduction

The systematic study of the complexity of isomorphisms between computable copies of structures was initiated in the 1950s by Fröhlich and Shepherdson [1] and independently by Maltsev [2]. The notions of computable categoricity (in the Russian tradition also called autostability) and relative computable categoricity are probably the most prominent in this line of research. A computable structure $\mathcal{A}$ is *computably categorical* if for every computable copy $\mathcal{B}$ there is a computable isomorphism from $\mathcal{B}$ to $\mathcal{A}$. A structure $\mathcal{A}$ is *relatively computably categorical* if for every copy $\mathcal{B}$ of $\mathcal{A}$ there is a $\deg(\mathcal{A} \oplus \mathcal{B})$-computable isomorphism from $\mathcal{B}$ to $\mathcal{A}$. For a survey of this topic see Fokina et al. [3].

In this article we study the algorithmic complexity of embeddings between bi-embeddable equivalence structures. The relation of bi-embeddability has attracted a lot of attention of specialists in computable structure theory and descriptive set theory in the recent years. Two structures $\mathcal{A}$ and $\mathcal{B}$ are *bi-embeddable* if there is an embedding of $\mathcal{A}$ in $\mathcal{B}$ and *vice versa*. In the literature $\mathcal{A}$ and $\mathcal{B}$ are sometimes called equimorphic [4–6]. Montalbán [4] showed that any hyperarithmetic linear ordering is bi-embeddable with a computable one and together with Greenberg [6] they showed the same for hyperarithmetic Boolean algebras, compact metric spaces, and Abelian p-groups. Fokina et al. [7] studied degree spectra with respect to the bi-embeddability relation and noticed that any countable equivalence structure is bi-embeddable with a computable one. For this reason, the study of the algorithmic complexity of embeddings is particularly interesting for this class of structures.

In this paper we focus on computability-theoretic properties of equivalence structures. An equivalence structure $\mathcal{A} = (A, E)$ has a computable subset A of the natural numbers as its universe and is equipped with an equivalence relation E on A. We identify such a structure $\mathcal{A}$ with its atomic diagram, which is the set of atomic formulas and negations of atomic formulas with parameters from A that are true of $\mathcal{A}$ under a fixed Gödel numbering. Our computability theoretic notions are standard and as in Soare [8].

Calvert et al. [9] initiated the study of computable categoricity for equivalence structures. Given a structure $\mathcal{A}$ and a structure $\mathcal{B}$ bi-embeddable with $\mathcal{A}$, we say that $\mathcal{B}$ is a *bi-embeddable copy* of $\mathcal{A}$. We study the complexity of embeddings through the following notions analogous to computable categoricity and relative computable categoricity.

Definition 1.1

- A computable structure $\mathcal{A}$ is **d**-*computably bi-embeddably categorical* if any computable bi-embeddable copy of $\mathcal{A}$ is bi-embeddable with $\mathcal{A}$ by **d**-computable embeddings. Here **d** is a Turing degree. For the case $\mathbf{d} = \mathbf{0}^{(n)}$, we usually use the term "Δ_{n+1}^0 *bi-embeddably categorical*" instead.
- A countable (not necessarily computable) structure $\mathcal{A}$ is *relatively* Δ_n^0 *bi-embeddably categorical* if for any bi-embeddable copy $\mathcal{B}$, $\mathcal{A}$ and $\mathcal{B}$ are bi-embeddable by $\Delta_n^{\mathcal{A} \oplus \mathcal{B}}$ embeddings. A computable structure is relatively computably bi-embeddably categorical if $n = 1$.

The definitions can be extended to arbitrary hyperarithmetic α in the usual way. In this paper we only focus on the cases of $n = 0, 1, 2$.

Fokina et al. [10] introduced the notion of a (strong) degree of categoricity of a structure. Given a computable structure $\mathcal{A}$ that is **d**-computably categorical in the classical sense, the *degree of categoricity* of $\mathcal{A}$ is the least degree that computes an isomorphism between any two computable isomorphic copies of $\mathcal{A}$. It is the *strong degree of categoricity* of $\mathcal{A}$ if there are two computable copies $\mathcal{A}_0$ and $\mathcal{A}_1$ such that for any $f\colon \mathcal{A}_0 \cong \mathcal{A}_1$, $f \geq_T \mathbf{d}$. Note that $\mathcal{A}$ might not have a (strong) degree of categoricity [11,12]. A Turing degree **d** is a (strong) degree of categoricity if there exists a structure having **d** as its (strong) degree of categoricity. This notion has seen a lot of interest over the last years. Fokina et al. [10] showed that all strong degrees of categoricity are hyperarithmetical. Csima et al. [13] extended this result by showing that all degrees of categoricity are hyperarithmetical. Miller [11] exhibited a field that does not have a degree of categoricity, and Fokina et al. [12] gave an example of a rigid structure without degree of categoricity. Recently, Csima and Stephenson [14], and independently, Bazhenov et al. [15,16] found examples of structures that have degree of categoricity but no strong degree of categoricity. The question whether there exists a degree of categoricity that is not strong is still open. We give an analogous definition for bi-embeddability.

Definition 1.2 The degree of bi-embeddable categoricity of a computable structure $\mathcal{A}$ is the least Turing degree **d** that, if it exists, computes embeddings between any computable bi-embeddable copies of $\mathcal{A}$. If, in addition, $\mathcal{A}$ has two computable bi-embeddable copies $\mathcal{A}_0$, $\mathcal{A}_1$ such that for all embeddings $\mu\colon \mathcal{A}_0 \hookrightarrow \mathcal{A}_1$, $\nu\colon \mathcal{A}_1 \hookrightarrow \mathcal{A}_0$, $\mu \oplus \nu \geq_T \mathbf{d}$, then **d** is the *strong degree of bi-embeddable categoricity* of $\mathcal{A}$.

Csima and Ng [unpublished] showed that a computable equivalence structure has strong degree of categoricity $\mathbf{0}$, $\mathbf{0}'$, or $\mathbf{0}''$. Our main result reflects theirs in our setting.

Theorem 1.1 *Let $\mathcal{A}$ be a computable equivalence structure.*

(1) *If $\mathcal{A}$ has bounded character and finitely many infinite equivalence classes, then its degree of bi-embeddable categoricity is $\mathbf{0}$.*
(2) *If $\mathcal{A}$ has unbounded character and finitely many infinite equivalence classes, then its degree of bi-embeddable categoricity is $\mathbf{0}'$.*
(3) *If $\mathcal{A}$ has infinitely many infinite equivalence classes, then its degree of bi-embeddable categoricity is $\mathbf{0}''$.*

Thus, the degree of bi-embeddable categoricity of equivalence structures is either $\mathbf{0}$, $\mathbf{0}'$, or $\mathbf{0}''$. Furthermore, the degrees of bi-embeddable categoricity of equivalence structures are strong.

The proof of Theorem 1.1 combines various theorems proved in Sects. 2 and 3. In these sections we also obtain results on the relation between classical notions of categoricity and bi-embeddable categoricity, summarized in Fig. 1.

In Sect. 2 we characterize the computably bi-embeddably categorical equivalence structures. In Sect. 3 we study Δ_2^0 and Δ_3^0 bi-embeddably categorical and relatively
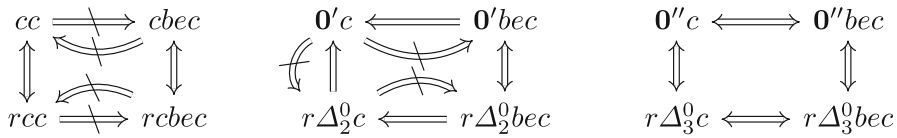
Fig. 1 Relations between categoricity for computable equivalence structures

Δ_2^0 and Δ_3^0 bi-embeddably categorical equivalence structures. We show that all equivalence structures are relatively Δ_3^0 categorical. We prove (2) and (3) of Theorem 1.1 and study the relations between those notions summarized in Fig. 1.

In Sect. 4 we obtain results on the complexity of the index sets of equivalence structures with degrees of bi-embeddable categoricity $\mathbf{0}, \mathbf{0}'$, and $\mathbf{0}''$.

2 Computable bi-embeddable categoricity

Given an equivalence structure $\mathcal{A}$ and $a \in A$ we write $[a]^{\mathcal{A}}$ for the equivalence class of a; if it is clear from the context which structure is meant, we omit the superscript. The following notions are central to our analysis.

Definition 2.1 Let $\mathcal{A}$ be an equivalence structure. A set $T \subseteq A$ is a *transversal* of $\mathcal{A}$ if

(1) for $x, y \in T$, if $x \neq y$, then $x \notin [y]^{\mathcal{A}}$,
(2) and $A = \bigcup_{x \in T} [x]^{\mathcal{A}}$.

Proposition 2.1 *Let $\mathcal{A}$ be an equivalence structure, then there is a transversal T of $\mathcal{A}$ such that $T \leq_T \mathcal{A}$.*

Proof For each equivalence class, we choose the least element in the class. We can do this computably in (the atomic diagram of) $\mathcal{A}$. □

Definition 2.2 (Calvert et al. [9]) Let $\mathcal{A}$ be an equivalence structure.

(1) We say that $\mathcal{A}$ *has bounded character*, or simply *is bounded*, if there is some finite k such that all finite equivalence classes of $\mathcal{A}$ have size at most k. If $\mathcal{A}$ has bound k on the sizes of its finite equivalence classes, we say that $\mathcal{A}$ is *k-bounded*.
(2) $Inf^{\mathcal{A}} = \{a \in A: [a]^{\mathcal{A}} \text{ is infinite}\}$ $Fin^{\mathcal{A}} = \{a \in A: [a]^{\mathcal{A}} \text{ is finite}\}$.

We will use the following relativization of [9, Lemma 2.2].

Lemma 2.2 *Let $\mathcal{A}$ be an equivalence structure, then*

(1) *For $k \in \omega$, $|[a]^{\mathcal{A}}| \leq k$ is $\Pi_1^{\mathcal{A}}$, $|[a]^{\mathcal{A}}| \geq k$ is $\Sigma_1^{\mathcal{A}}$, $|[a]^{\mathcal{A}}| = k$ is $\Delta_2^{\mathcal{A}}$,*
(2) *$Inf^{\mathcal{A}}$ is $\Pi_2^{\mathcal{A}}$, and $Fin^{\mathcal{A}}$ is $\Sigma_2^{\mathcal{A}}$,*

Proof Ad (1). A $\Pi_1^{\mathcal{A}}$ definition for $|[a]^{\mathcal{A}}| \leq k$ is

$$|[a]^{\mathcal{A}}| \leq k \Leftrightarrow \forall x_1, \ldots, x_{k+1} \bigwedge_{1 \leq i \leq k+1} x_i E a \to \bigvee_{1 \leq i < j \leq k+1} x_i = x_j,$$

a $\Sigma_1^{\mathcal{A}}$ definition for $|[a]^{\mathcal{A}}| \geq k$ is

$$|[a]^{\mathcal{A}}| \geq k \Leftrightarrow \exists x_1, \ldots, x_k \bigwedge_{1 \leq i \leq k} x_i E a \wedge \bigwedge_{1 \leq i < j \leq k} x_i \neq x_j,$$

and a $\Delta_2^{\mathcal{A}}$ definition for $|[a]^{\mathcal{A}}| = k$ is then just the conjunction of $|[a]^{\mathcal{A}}| \leq k$ and $|[a]^{\mathcal{A}}| \geq k$.

Ad (2). The property $a \in Inf^{\mathcal{A}}$ has a $\Pi_2^{\mathcal{A}}$ definition by $\forall k \, |[a]^{\mathcal{A}}| \geq k$. It follows immediately that $a \in Fin^{\mathcal{A}}$ has a $\Sigma_2^{\mathcal{A}}$ definition. $\qquad\square$

Our first goal is to characterize computably bi-embeddably categorical equivalence structures. In [9] the following characterization of computably categorical equivalence structures was given.

Theorem 2.3 (Calvert et al. [9]) *Let $\mathcal{A}$ be a computable equivalence structure, then $\mathcal{A}$ is computably categorical if and only if*

(1) *$\mathcal{A}$ has finitely many finite equivalence classes,*
(2) *or $\mathcal{A}$ has finitely many infinite classes, bounded character, and at most one finite k such that there are infinitely many classes of size k.*

Theorem 2.4 *An equivalence structure $\mathcal{A}$ is computably bi-embeddably categorical if and only if it has finitely many infinite equivalence classes and bounded character.*

Proof ($\Leftarrow$). Let $\mathcal{A}$ be k-bounded and let l be the size of the largest equivalence class such that $\mathcal{A}$ has infinitely many equivalence classes of size l (notice that l might be 0). Then the restriction $\mathcal{A}_{>l}$ of $\mathcal{A}$ to equivalence classes of size larger than l is computably categorical, as the number of equivalence classes in $\mathcal{A}_{>l}$ is finite, i.e., the bi-embeddability type and the isomorphism type of $\mathcal{A}_{>l}$ coincide. Hence, if $\mathcal{B}$ is a bi-embeddable copy of $\mathcal{A}$, then $\mathcal{B}_{>l}$ is isomorphic to $\mathcal{A}_{>l}$. This furthermore implies that l is invariant in the bi-embeddability type of $\mathcal{B}$. Non-uniformly fix a computable isomorphism $f \colon \mathcal{A}_{>l} \to \mathcal{B}_{>l}$.

Let $T_{\mathcal{A}_{>l}}$ be a transversal of $\mathcal{A}_{>l}$, and $T_{\mathcal{B}_{>l}}$ one of $\mathcal{B}_{>l}$. Clearly both $T_{\mathcal{A}_{>l}}$ and $T_{\mathcal{B}_{>l}}$ are finite and hence computable. Furthermore the equivalence classes of size l have a c.e. transversal as

$$|[a]^{\mathcal{A}}| \leq l \Leftrightarrow \forall x \in T_{\mathcal{A}_{>l}} \, a \notin [x]^{\mathcal{A}}$$

and $|[a]^{\mathcal{A}}| \geq l$ is Σ_1. Let $(b_i)_{i \in \omega}$ be an enumeration of the transversal of the equivalence classes of size l in $\mathcal{B}$ and let $(a_i)_{i \in \omega}$ be a computable enumeration of A. We can define a computable embedding $\nu \colon \mathcal{A} \hookrightarrow \mathcal{B}$ by recursion as follows.

$$\nu(a_i) = \begin{cases} f(a_i) & \exists y \in T_{\mathcal{A}_{>l}} \, a_i \in [y]^{\mathcal{A}} \\ b_k, \; k = \mu l[\forall j < i \; \nu(a_j) \notin [b_l]^{\mathcal{B}}] & \forall j < i \; a_i \notin [a_j]^{\mathcal{A}} \\ \mu x \in B[x \in [\nu(a_j)]^{\mathcal{B}} \wedge \forall l < i \; x \neq \nu(a_l)] & \exists j < i \; a_i \in [a_j]^{\mathcal{A}} \end{cases}$$

The embedding of $\mathcal{B}$ in $\mathcal{A}$ is defined similarly.

($\Rightarrow$). We show that computable equivalence structures with unbounded character and without infinite equivalence classes are not computably bi-embeddably categorical. The proof for equivalence structures with finitely many infinite equivalence classes is analogous. By Corollary 3.4 below, equivalence structures with infinitely many infinite equivalence classes are not even Δ_2^0 bi-embeddably categorical.

Note that any two equivalence structures with unbounded character and the same number of infinite equivalence classes are bi-embeddable and that any embedding needs to map elements to elements in equivalence classes of at least the same size. Consider the equivalence structure $\mathcal{A}$ with universe $\bigcup_{i \in \omega}\{\langle i, n \rangle \colon n \leq i\}$ where all elements with the same left column are in the same equivalence class. This structure is clearly a computable equivalence structure with computable size function $| \cdot |$. We build a computable equivalence structure $\mathcal{B} = (\omega, E^{\mathcal{B}})$ in stages such that no partial computable function is an embedding of $\mathcal{B}$ in $\mathcal{A}$. We want to satisfy the following requirements.

$$P_e \colon \quad \varphi_e \text{ is not an embedding of } \mathcal{B} \text{ in } \mathcal{A}$$

We say that a requirement P_e *needs attention* at stage s if the restriction of the approximation $\varphi_{e,s}$ of φ_e to elements in the structure $\mathcal{B}_s$ is a partial embedding of $\mathcal{B}$ in $\mathcal{A}$ not equal to $\emptyset$. The structure $\mathcal{B}$ is the limit of the structures $\mathcal{B}_s$ constructed as follows.
Construction
Stage $s = 0$: $\mathcal{B}_0$ is the singleton $\langle 0, 0 \rangle$.
Stage $s + 1$: Check if there is a requirement P_e, $e \leq s$ that needs attention. If such P_e exists, do the following. Choose the least requirement P_e that needs attention. Then $\varphi_{e,s}(\langle s, 0 \rangle) \downarrow = \langle i, n \rangle$ for some $\langle i, n \rangle$. Let $\mathcal{B}_{s+1} = \mathcal{B}_s \cup \{\langle s, s + j \rangle \colon 0 \leq j \leq i\} \cup \{\langle s + 1, 0 \rangle\}$; put all elements with s in the left column in the equivalence class of $\langle s, 0 \rangle$, and let $\langle s + 1, 0 \rangle$ be a singleton.

If no P_e needs attention set $\mathcal{B}_{s+1} = \mathcal{B}_s \cup \{\langle s + 1, 0 \rangle\}$ and let $\langle s + 1, 0 \rangle$ be a singleton.
Verification Assume towards a contradiction that φ_e is an embedding of $\mathcal{B}$ in $\mathcal{A}$ and no φ_j, $j < e$ is an embedding. Then there is a stage s such that $\varphi_{e,s}$ is a partial embedding of $\mathcal{B}_s$ in $\mathcal{A}$ and no P_j, $j < e$, needs attention. Thus, P_e receives attention at stage $s + 1$ and $\varphi_{e,s}(\langle s, 0 \rangle) \downarrow$; say $\varphi_{e,s}(\langle s, 0 \rangle) = \langle i, n \rangle$. Then, by construction, the equivalence class of $\langle s, 0 \rangle$ in $\mathcal{B}_{s+1}$ is bigger than the one of $\langle i, n \rangle$. Thus, φ_e can not be an embedding. However, by construction of $\mathcal{B}$, every equivalence class is grown only once and has the size of an equivalence in $\mathcal{A}$ plus one. Thus, $\mathcal{B}$ is unbounded without infinite equivalence classes and hence, is bi-embeddable with $\mathcal{A}$. $\square$

The following corollaries follow directly from Theorems 2.3 and 2.4.

Corollary 2.5 *There is a computably bi-embeddably categorical equivalence structure that is not computably categorical.*

Corollary 2.6 *There is a computably categorical equivalence structure that is not computably bi-embeddably categorical.*

Calvert et al. [9] showed that a computable equivalence structure is computably categorical if and only if it is relatively computably categorical. The analogous result holds in the context of bi-embeddability.

Proposition 2.7 *Let $\mathcal{A}$ be a computable equivalence structure. Then $\mathcal{A}$ is computably bi-embeddably categorical if and only if it is relatively computably bi-embeddably categorical.*

Proof Relativization of the proof of Theorem 2.4 ensures the result. □

3 Δ_2^0 and Δ_3^0 bi-embeddable categoricity

In this section we characterize Δ_2^0 and Δ_3^0 bi-embeddably categorical equivalence structures. We will show that a computable equivalence structure is Δ_2^0 (Δ_3^0) bi-embeddably categorical if and only if it relatively so. We will also see that all equivalence structures are relatively Δ_3^0 bi-embeddably categorical. This, together with the fact that any countable equivalence structure is bi-embeddable with a computable one [7] gives a complete structural characterization of bi-embeddable categoricity for equivalence structures. We also establish the remaining parts of Theorem 1.1. At first we characterize Δ_2^0 bi-embeddably categorical equivalence structures. We start by exhibiting a class of equivalence structures that is relatively Δ_2^0 bi-embeddably categorical.

Theorem 3.1 *If $\mathcal{A}$ has finitely many infinite equivalence classes, then $\mathcal{A}$ is relatively Δ_2^0 bi-embeddably categorical.*

Proof By Theorem 2.4 and Proposition 2.7 equivalence structures with bounded character are relatively computably bi-embeddably categorical and thus relatively Δ_2^0 bi-embeddably categorical. It remains to show that equivalence structures with unbounded character and finitely many infinite equivalence classes are relatively Δ_2^0 bi-embeddably categorical.

Let $\mathcal{A}$ have finitely many infinite equivalence classes and unbounded character, and let $\mathcal{B}$ be a bi-embeddable copy of $\mathcal{A}$. Note that $\mathcal{B}$ must have the same number of infinite equivalence classes as $\mathcal{A}$. Fix transversals $T^{\mathcal{A}}$ and $T^{\mathcal{B}}$ of $Inf^{\mathcal{A}}$ and $Inf^{\mathcal{B}}$, respectively. Let $f \colon T^{\mathcal{A}} \to T^{\mathcal{B}}$ be a bijection. As $T^{\mathcal{A}}$ and $T^{\mathcal{B}}$ are finite sets, they are computable. We define a $\Delta_2^{\mathcal{A} \oplus \mathcal{B}}$ embedding $\nu \colon \mathcal{A} \hookrightarrow \mathcal{B}$ by recursion. Let $(a_i)_{i \in \omega}$ be a computable enumeration of A.

$$\nu(a_0) = \begin{cases} f(t) & \text{if } \exists t \in T^{\mathcal{A}} \ a_0 \in [t]^{\mathcal{A}} \\ \mu x \in B[|[x]^{\mathcal{B}}| \geq |[a_0]^{\mathcal{A}}| \wedge \forall t \in T^{\mathcal{B}} \ x \notin [t]^{\mathcal{B}}] & \text{otherwise} \end{cases}$$

Assume ν has been defined for $a_0, \dots a_s$. We define $\nu(a_{s+1})$; there are three cases.

Case 1 a_{s+1} is equivalent to an element for which ν has already been defined, i.e., $\exists j \leq s \ a_{s+1} \in [a_j]^{\mathcal{A}}$. Then

$$\nu(a_{s+1}) = \mu x \in B[x \in [\nu(a_j)]^{\mathcal{B}} \wedge \forall i \leq s \ x \neq \nu(a_i)].$$

Case 2 a_{s+1} is not equivalent to any element for which ν has been defined and its equivalence class is infinite, i.e., $\exists t \in T^{\mathcal{A}} \; a_{s+1} \in [t]^{\mathcal{A}}$, then

$$\nu(a_{s+1}) = \mu x \in B[x \in [f(t)]^{\mathcal{B}} \wedge \forall i \leq s \; x \neq \nu(a_i)].$$

Case 3 a_{s+1} is not equivalent to any element for which ν has been defined and its equivalence class is finite. Then

$$\nu(a_{s+1}) = \mu x \in B[|[x]^{\mathcal{B}}| \geq |[a_{s+1}]^{\mathcal{A}}| \wedge \forall t \in T^{\mathcal{B}} \; x \notin [t]^{\mathcal{B}} \wedge \forall i \leq s \; x \notin [\nu(a_i)]^{\mathcal{B}}].$$

As $\mathcal{A}$ and $\mathcal{B}$ are both unbounded, at any stage s of the construction we can find an element in B with an equivalence class greater than or equal to the one of a_s in $\mathcal{A}$. Therefore, ν is an embedding. As $T^{\mathcal{A}}, T^{\mathcal{B}}$, and f are computable and comparing the size of two equivalence classes is $\Delta_2^{\mathcal{A} \oplus \mathcal{B}}$, ν is $\Delta_2^{\mathcal{A} \oplus \mathcal{B}}$. □

The following is the relativization of the classical computability theoretic concepts of immune and simple sets to $\mathbf{0}'$.

Definition 3.1 An infinite set A is $\mathbf{0}'$-*immune* if it contains no infinite set which is computably enumerable in $\mathbf{0}'$. A Σ_2^0 set A is $\mathbf{0}'$-*simple* if it is the complement of a $\mathbf{0}'$-immune set.

Theorem 3.2 *There is a computable equivalence structure $\mathcal{A}$ with infinitely many infinite equivalence classes such that $Fin^{\mathcal{A}}$ is $\mathbf{0}'$-simple. Hence, $Inf^{\mathcal{A}}$ is $\mathbf{0}'$-immune.*

Proof We build $\mathcal{A}$ with universe ω such that $Fin^{\mathcal{A}}$ is $\mathbf{0}'$-simple, i.e., for every infinite Σ_2^0 set S the intersection of $Fin^{\mathcal{A}}$ and S is nonempty. It has to satisfy the following requirements.

$$P_e: \quad |W_e^{\emptyset'}| = \infty \quad \Rightarrow \quad W_e^{\emptyset'} \cap Fin^{\mathcal{A}} \neq \emptyset$$

and the overall requirement that any transversal $T_{Inf^{\mathcal{A}}}$ of $Inf^{\mathcal{A}}$ is infinite.

$$G: \quad T_{Inf^{\mathcal{A}}} \text{ is infinite.}$$

Our strategy to satisfy a requirement P_e is to pick a witness x_e for $W_e^{\emptyset'}$ and prevent the equivalence class of x_e from growing any further.

We will construct $\mathcal{A}$ in stages. Elements and equivalence classes can be in one of three states. An element is *blocked by P_e* if it is equivalent to a witness picked by P_e. During the construction we also *designate* unblocked elements for expansion, i.e., we allow the equivalence class of such elements to grow in a later stage. Elements which are neither designated nor blocked are *fresh*, these elements have equivalence classes of size 1.

The set $W_{e,s}^{\emptyset'}$ is the Σ_2 approximation of $W_e^{\emptyset'}$ at stage s and the set $Fin_s^{\mathcal{A}}$ is the set of blocked elements at stage s; we will have that $Fin^{\mathcal{A}} = \lim_s Fin_s^{\mathcal{A}}$. A strategy P_e *needs attention* at stage $s + 1$ if

$$W_{e,s}^{\emptyset'} \cap Fin_s^{\mathcal{A}} = \emptyset \quad \& \quad \exists x > e^3 \; (x \in W_{e,s}^{\emptyset'}). \quad (\star)$$

Construction
Stage $s = 0$: Let $A = \omega$ and $E^{\mathcal{A}} = \{(x, x) \colon x \in \omega\}$. Define $Fin_s^{\mathcal{A}} = \emptyset$.
Stage $s + 1$: Assume we have built $\mathcal{A}_s$.

(1) Choose the least $e < s$ such that P_e needs attention. Take the least $x > e^3$ satisfying the second part of the matrix in Eq. ($\star$). Check if $x \leq s$. If so, then take the element y, $e^3 < y \leq s$ which has been in the approximation the longest without interruption and declare its equivalence class as blocked by P_e. If not, then declare the equivalence class of x as blocked by P_e. If P_e receives attention for the first time, designate the least fresh element.
(2) Add to all designated equivalence classes a fresh element bigger than s.
(3) Check if for any $e < s$ there is an element x blocked by P_e that is not blocked by any P_j, $j < s$, and $x \in W_{e,s-1}^{\emptyset'}$ but $x \notin W_{e,s}^{\emptyset'}$. If so, declare the equivalence class of x as designated.

Verification It is clear from the construction that $\mathcal{A} = \lim_s \mathcal{A}_s$ is a computable equivalence structure. The following two claims establish that it has the desired properties.

Claim 3.2.1 *Every requirement P_e is eventually satisfied.*

Proof By construction, no requirement P_e injures a requirement P_j, $j \neq e$, so it is sufficient to consider them in isolation. As $W_{e,s}^{\emptyset'}$ is a Σ_2 approximation we have that $x \in W_e^{\emptyset'}$ iff there is a stage s_0 such that for all $t > s_0$, $x \in W_{e,t}^{\emptyset'}$. In particular, there is a stage after which an element $x_0 \in W_e^{\emptyset'}$, $x_0 > e^3$ will be in the approximation longer than any element $y \notin W_e^{\emptyset'}$. This element will be chosen by our strategy the next time that P_e receives attention (which it will if its current witness is not in $W_e^{\emptyset'}$). Hence, P_e is satisfied in the limit. □

Claim 3.2.2 *The requirement G is satisfied.*

Proof Assume that P_e is the maximum requirement that acted at some stage in the construction. At most e equivalence classes are blocked at this stage and because at every stage every designated equivalence class grows by one element, at most e^2 out of e^3 elements are blocked. By the same reasoning at most e^2 elements are designated for expansion. Hence, at least $e^3 - 2e^2$ fresh elements are left to expand the equivalence classes of designated elements and so, for $e > 2$ there are enough fresh elements smaller than e^3 left to expand the designated elements. As every requirement that receives attention for the first time designates one fresh element, in the limit there are infinitely many infinite equivalence classes. □

It is immediate that two equivalence structures $\mathcal{A}$ and $\mathcal{B}$ with infinitely many infinite equivalence classes are bi-embeddable. To obtain an embedding of $\mathcal{A}$ in $\mathcal{B}$ just map all equivalence classes of $\mathcal{A}$ to infinite equivalence classes of $\mathcal{B}$.

Proposition 3.3 *If an equivalence structure $\mathcal{A}$ has infinitely many infinite equivalence classes, then it is not Δ_2^0 bi-embeddably categorical.*

Proof Let $\mathcal{A}$ be a computable equivalence structure with infinitely many infinite equivalence classes and no finite equivalence classes, and take $\mathcal{B}$ as in Theorem 3.2. Then, by the above argument, they are bi-embeddable and every embedding of $\mathcal{A}$ in $\mathcal{B}$ has as range an infinite subset of $Inf^{\mathcal{B}}$.

Now, assume that $\nu \colon \mathcal{A} \hookrightarrow \mathcal{B}$ is $\mathbf{0}'$-computable. Then, its range is a Σ^0_2 set and an infinite subset of $Inf^{\mathcal{B}}$. But $Inf^{\mathcal{B}}$ is $\mathbf{0}'$-immune, a contradiction. Hence, no embedding of $\mathcal{A}$ in $\mathcal{B}$ is $\mathbf{0}'$-computable and therefore $\mathcal{A}$ is not Δ^0_2 bi-embeddably categorical. □

Corollary 3.4 *An equivalence structure $\mathcal{A}$ is Δ^0_2 bi-embeddably categorical if and only if it has finitely many infinite equivalence classes.*

Calvert et al. [9] characterized relatively Δ^0_2 categorical computable equivalence relations. Their result relativizes.

Proposition 3.5 (Relativization of [9, Corollary 4.8]) *A countable equivalence structure $\mathcal{A}$ is relatively Δ^0_2 categorical if and only if $\mathcal{A}$ has finitely many infinite equivalence classes or $\mathcal{A}$ has bounded character.*

Corollary 3.6 *An equivalence structure with bounded character and infinitely many infinite equivalence classes is relatively Δ^0_2 categorical but not relatively Δ^0_2 bi-embeddably categorical.*

Corollary 3.7 *Let $\mathcal{A}$ be a computable equivalence structure. Then $\mathcal{A}$ is Δ^0_2 bi-embeddably categorical if and only if it is relatively Δ^0_2 bi-embeddably categorical.*

Proof By Theorem 3.1 computable equivalence structures with finitely many infinite equivalence classes are relatively Δ^0_2 bi-embeddably categorical and therefore, also Δ^0_2 bi-embeddably categorical. It follows from Proposition 3.3 that these are all equivalence structures which are Δ^0_2 bi-embeddably categorical. As relatively Δ^0_2 bi-embeddably categorical equivalence structures have the same characterization the result follows. □

Kach and Turetsky [17] gave an example of a Δ^0_2 categorical but not relatively Δ^0_2 categorical equivalence structure. Downey et al. [18] showed that an equivalence structure $\mathcal{A}$ is Δ^0_2 categorical iff the structure containing only one equivalence class of $\mathcal{A}$ for each finite size and all its infinite equivalence classes is Δ^0_2-computably categorical.

We now proceed with the study of possible degrees of categoricity for equivalence structures.

Definition 3.2 A function f is *limitwise monotonic* if there is a computable approximation function $h_f(\cdot, \cdot)$ such that

(1) $f(x) = \lim_s h_f(x, s)$
(2) for all x, s $h_f(x, s) \leq h_f(x, s + 1)$

It is not hard to see that for any limitwise monotonic function f, $f \leq_T \mathbf{0}'$. For more on limitwise monotonic functions and their applications see Downey et al. [19].

Theorem 3.8 *The degree of bi-embeddable categoricity of computable equivalence structures with unbounded character and finitely many infinite equivalence classes is $\mathbf{0}'$.*

Proof First notice that there are countably many bi-embeddability types of equivalence structures with unbounded character and finitely many infinite equivalence classes. Namely exactly one for each number of equivalence classes of infinite size. We prove the theorem for equivalence structures with no infinite equivalence classes. However, the argument can be easily modified to accomodate equivalence structures with finitely many infinite equivalence classes.

We define the following function.

$$f(x) := 1 + \sum_{0 \leq i \leq x, \varphi_i(x)\downarrow} \varphi_i(x)$$

Clearly f is limitwise monotonic and dominates every partial computable function. By the domination theorem (see [8, Theorem 4.5.4]) it holds that for any set D such that $f \leq_T D$, $D \geq_T \mathbf{0}'$; hence, in particular, $f \equiv_T \mathbf{0}'$.

We build a computable equivalence structure $\mathcal{A}_f$ with universe ω and no infinite equivalence classes in stages such that

$$|[\langle x, 0 \rangle]^{\mathcal{A}_f}| = f(x).$$

Let h_f be the computable approximation for f. At stage 0 of the construction, let the universe of the approximation $\mathcal{A}_f$ be ω and put $\langle 0, n \rangle$ in the equivalence class of $\langle 0, 0 \rangle$ for $n < h_f(0, 0)$. At stage $s+1$ check if for any $\langle x, 0 \rangle$ $x \leq s$, $|[\langle x, 0 \rangle]| < h_f(x, s+1)$. If so, add $\langle x, s+1 \rangle$ to the equivalence class.

Now consider the equivalence structure $\mathcal{A}$ with universe $\bigcup_{i \in \omega} \{\langle i, n \rangle : n \leq i\}$ and where all elements with the same left column are in the same equivalence class. This structure is clearly a computable equivalence structure bi-embeddable with $\mathcal{A}_f$ and computable size function $|\cdot|$. Any embedding $\nu \colon \mathcal{A}_f \to \mathcal{A}$ must map $[\langle x, 0 \rangle]^{\mathcal{A}_f} \mapsto [\langle y, 0 \rangle]^{\mathcal{A}}$ with $|[\langle y, 0 \rangle]^{\mathcal{A}}| \geq |[\langle x, 0 \rangle]^{\mathcal{A}_f}| = f(x)$. Consider the function $g(x) = |[\nu(\langle x, 0 \rangle)]^{\mathcal{A}}|$; as $|\cdot|$ is computable, $g \equiv_T \nu$ and as $\forall x \, g(x) \geq f(x)$, $g \equiv_T \nu \geq_T \mathbf{0}'$ by the domination theorem. As by Corollary 3.4 every computable equivalence structure with finitely many infinite equivalence classes is Δ_2^0 bi-embeddably categorical the theorem follows. □

Theorem 3.9 *Equivalence structures are relatively Δ_3^0 bi-embeddably categorical.*

Proof By Theorem 3.1, equivalence structures with finitely many infinite equivalence classes are relatively Δ_2^0 bi-embeddably categorical. Thus, it suffices to show that equivalence structures with infinitely many infinite classes are relatively Δ_3^0 bi-embeddably categorical. Let $\mathcal{A}$ and $\mathcal{B}$ be equivalence structures with infinitely many infinite classes. Recall that any two such equivalence structures are bi-embeddable. There is an embedding of $\mathcal{A}$ in $\mathcal{B}$ that maps every equivalence class in $\mathcal{A}$ to an infinite equivalence class in $\mathcal{B}$. As $\mathrm{Inf}^{\mathcal{B}}$ is $\Pi_2^{\mathcal{B}}$ there is at least one such embedding which is $\Delta_3^{\mathcal{A} \oplus \mathcal{B}}$. By the same argument there is a $\Delta_3^{\mathcal{A} \oplus \mathcal{B}}$ embedding of $\mathcal{B}$ in $\mathcal{A}$. □

The analogous result about classical relative Δ_3^0 categoricity of equivalence struc-
tures is also true [9], as every equivalence structure has a Σ_3^c Scott family.

We close by proving the remaining parts of Theorem 1.1.

Theorem 3.10 *The degree of bi-embeddable categoricity of computable equivalence
structures with infinitely many infinite equivalence classes is $\mathbf{0}''$.*

Proof We first build a computable equivalence structure $\mathcal{A}$ with the property that
any infinite partial transversal of $Inf^{\mathcal{A}}$ computes $\mathbf{0}''$. Let $(\sigma_i)_{i \in \omega}$ be a computable
$1 - 1$ enumeration of $2^{<\omega}$ and associate to every σ_i an infinite set of witnesses
$\{\langle i, x, y \rangle : x, y \in \omega\}$. Elements of the form $\langle i, x, 0 \rangle$ will serve as witnesses while
all other elements will be used to grow the equivalence classes. We will build $\mathcal{A}$
using a Π_2 approximation to $\overline{\emptyset''}$. Let $\overline{\emptyset''}_s$ be the Π_2 approximation at stage s of our
construction.

Construction

Stage $s = 0$: Let $A = \omega$ and $E^{\mathcal{A}} = \{(x, x) : x \in \omega\}$. Furthermore, for all strings σ_i
in our computable enumeration of $2^{<\omega}$ designate witnesses $\langle i, 0, 0 \rangle$.

Stage $s + 1$: Assume we have built $\mathcal{A}_s$.

(1) For all witnesses with left column $i < s + 1$ check if for some x with $\sigma_i(x) =$
    0, $x \in \overline{\emptyset''}_{s+1}$. If $\langle i, j, 0 \rangle$ is a witness for such σ_i, discard it (never touch its
    equivalence class again during the construction) and designate the witness $\langle i, s +$
    $1, 0 \rangle$.
(2) For any $\sigma_i, i < s+1$ grow the equivalence class of its designated witness $\langle i, j, 0 \rangle$
    to match $\min\{|\{t : t \le s, x \in \overline{\emptyset''}_t\}| : \sigma_i(x) = 1\}$ using fresh elements $\langle i, j, r \rangle$
    with $r > s$.

Verification We have to show that any infinite partial transversal of $Inf^{\mathcal{A}}$ computes $\mathbf{0}''$.
The following claim establishes the crucial part.

Claim 3.10.1 $\sigma_i \prec \overline{\emptyset''} \Leftrightarrow \exists y \langle i, y, 0 \rangle \in Inf^{\mathcal{A}}$.

Proof ($\Rightarrow$). Assume $\sigma_i \prec \overline{\emptyset''}$. As we have a Π_2 approximation to $\overline{\emptyset''}$ there is a stage
s such that no $x < |\sigma_i|, x \notin \overline{\emptyset''}$ enters $\overline{\emptyset''}_t$ at any stage $t > s$. Hence, by construction
there is a $j < s$ such that $\langle i, j, 0 \rangle$ has infinite equivalence class.

($\Leftarrow$). Assume $|[\langle i, j, 0 \rangle]| = \omega$ and let $\tau \prec \overline{\emptyset''}$ with $|\tau| = |\sigma_i|$. Assume, for some x,
$\tau(x) = 0$ and $\sigma_i(x) = 1$. Then there are only finitely many stages t such that $x \in \overline{\emptyset''}_t$.
Thus, by construction, no equivalence class of elements with i in the left column can
become infinite. Therefore, if $\tau(x) = 0$, then $\sigma_i(x) = 0$ as well. Now assume that
$\tau(x) = 1$ and $\sigma_i(x) = 0$. Then there are infinitely many stages s such that $x \in \overline{\emptyset''}_s$.
Hence, by construction, there can not be finite j such that $|[\langle i, j, 0 \rangle]| = \omega$ because for
any j if $\langle i, j, 0 \rangle$ is designated at stage $s = j$ there is a stage $t > s$ such that $x \in \overline{\emptyset''}_t$.
Let t_0 be the least such stage for given j, then $\langle i, t_0, 0 \rangle$ will be designated at stage t_0
and no new elements will be added to the equivalence class of $[\langle i, j, 0 \rangle]$ at any stage
$t \ge t_0$. Thus $[\langle i, j, 0 \rangle]$ is finite, $\tau(x) = 1 \Rightarrow \sigma_i(x) = 1$ and hence $\tau = \sigma_i$. $\square$

Consider an infinite partial transversal T of $Inf^{\mathcal{A}}$. To check whether some fixed
$x \in \mathbf{0}''$, consider an enumeration of T. As all elements in T code an initial segment
of $\overline{\emptyset''}$ in their left column there is a finite stage t and y, z, such that $\langle i, y, z \rangle \in T_t$

and $|\sigma_i| \geq x$. Hence, $x \in \mathbf{0}'' \Leftrightarrow \sigma_i(x) = 0$ and we can find σ_i uniformly. Therefore, $T \geq_T \mathbf{0}''$.

Now consider the structure $\mathcal{B}$ with universe ω and where $\forall x \forall n \ \langle x, n \rangle \in [\langle x, 0 \rangle]^{\mathcal{B}}$. It is a computable equivalence structure consisting only of infinite equivalence classes and it clearly embeds into $\mathcal{A}$. To compute $\mathbf{0}''$ from any embedding $\nu \colon \mathcal{B} \hookrightarrow \mathcal{A}$ look at the strings coded by the left column of the images of elements of the form $\langle x, 0 \rangle$. By the above argument, after enumerating a finite number of images of such elements we can decide whether $x \in \mathbf{0}''$, hence $\nu \geq_T \mathbf{0}''$. Since, by Theorem 3.9, any equivalence structure is relatively Δ_3^0 bi-embeddably categorical the theorem follows. □

At last we put together the pieces that prove Theorem 1.1.

Proof of Theorem 1.1 (1) follows directly from Theorem 2.4. Theorem 3.8 proves that $\mathbf{0}'$ is the degree of bi-embeddable categoricity of equivalence structures with unbounded character and finitely many infinite equivalence classes. The two equivalence structures $\mathcal{A}_f$ and $\mathcal{A}$ constructed in the proof witness that $\mathbf{0}'$ is a strong degree of bi-embeddable categoricity for equivalence structures with unbounded character and no infinite equivalence classes. Similar structures can be easily constructed for equivalence structures with any finite number of infinite classes. This proves (2). Theorem 3.10 shows that $\mathbf{0}''$ is the degree of bi-embeddable categoricity of equivalence structures with infinitely many infinite classes. To see that it is a strong degree of bi-embeddable categoricity, consider the structures $\mathcal{A}$ and $\mathcal{B}$ constructed in the proof. Any embedding $\nu \colon \mathcal{B} \hookrightarrow \mathcal{A}$ computes $\mathbf{0}''$, hence, for any $\mu \colon \mathcal{A} \hookrightarrow \mathcal{B}$, $\mu \oplus \nu \geq_T \mathbf{0}''$. □

4 Index sets

Let $(\mathcal{C}_e)_{e \in \omega}$ be an enumeration of the partial computable equivalence structures, i.e., given a computable function $\varphi_e \colon \omega \times \omega \to \{0, 1\}$, $\mathcal{C}_e$ has universe ω and

$$x E^{\mathcal{C}_e} y \colon \Leftrightarrow \varphi_e(x, y) = 1.$$

Note that it is Π_1^0 to check whether $\mathcal{C}_e$ is indeed an equivalence structure.

We say that a set is D_n^0 if it is the difference of two Σ_n^0 sets, or equivalently, the intersection of a Σ_n^0 and a Π_n^0 set. We start by recalling a simple observation.

Lemma 4.1 *For computably bi-embeddably categorical equivalence structures $\mathcal{A}$ we have that $\mathcal{B} \approx \mathcal{A}$ if and only if*

(1) *$\mathcal{B}$ has the same number of infinite equivalence classes as $\mathcal{A}$,*
(2) *$\mathcal{B}$ has the same bound as $\mathcal{A}$,*
(3) *and if $\mathcal{A}$ has infinitely many equivalence classes of size n and for all $k > n$ there are only finitely many equivalence classes of size k, then for every $m \geq n$, $\mathcal{B}$ has the same number of equivalence classes of size m as $\mathcal{A}$.*

Theorem 4.2 *Let $\mathcal{A}$ be a computable, computably bi-embeddably categorical equivalence structure.*

(1) *If $\mathcal{A}$ is finite then the index set $\{\mathcal{C}_e \colon \mathcal{C}_e \approx \mathcal{A}\}$ is D_1^0-complete.*

(2) *If $\mathcal{A}$ has infinitely many equivalence classes of size n for some $n < \omega$, and no infinite equivalence classes, then the index set $\{\mathcal{C}_e \colon \mathcal{C}_e \approx \mathcal{A}\}$ is Π_2^0-complete.*

(3) *If $\mathcal{A}$ has $r > 0$ infinite equivalence classes, then the index set $\{\mathcal{C}_e \colon \mathcal{C}_e \approx \mathcal{A}\}$ is Π_2^0-complete.*

Proof Ad (1). Assume $\mathcal{A}$ is finite, say $|A| = m$. Let $\theta_{\mathcal{A}}$ be the formula obtained from the atomic diagram by replacing the constants from A by variables. Then the index set is definable by the following D_1^0 formula.

$$\mathcal{C}_e \approx \mathcal{A} \Leftrightarrow \mathcal{C}_e \cong \mathcal{A} \Leftrightarrow \exists x_1, \dots x_m (\theta_{\mathcal{A}}(x_1, \dots, x_m))$$

$$\wedge \forall x_1, \dots x_{m+1} \left(\bigvee_{1 \leq i < j \leq m+1} x_i = x_j \right).$$

To see that the index set is D_1^0-hard, we define a computable function g such that $\mathcal{C}_{g(e,i)} \approx \mathcal{A}$ if and only if $e \in \emptyset'$ and $i \in \overline{\emptyset'}$. Fix an element a of $\mathcal{A}$. Let $\mathcal{B}$ be a computable equivalence structure isomorphic to $\mathcal{A} \setminus [a]^{\mathcal{A}}$; say $|[a]^{\mathcal{A}}| = n$. We build a computable function f and a structure $\mathcal{E}_{f(e,i)}$ disjoint from $\mathcal{B}$ in stages such that $\mathcal{C}_{g(e,i)} = \mathcal{B} \oplus \mathcal{E}_{f(e,i)}$.

Let $\emptyset'_s, \mathcal{E}_{f(e,i),s}$ be the approximations to $\emptyset'$ and $\mathcal{E}_{f(e,i)}$, respectively, at stage s. Let $\mathcal{E}_{f(e,i),0} = \emptyset$ and assume we have defined $\mathcal{E}_{f(e,i),s}$. To define $\mathcal{E}_{f(e,i),s+1}$ check if (i) $e \searrow \emptyset'_s$ and (ii) $i \searrow \emptyset'_s$. The structure $\mathcal{E}_{f(e,i),s+1}$ extends $\mathcal{E}_{f(e,i),s}$ as follows. If (i), add a new equivalence class of size n to $\mathcal{E}_{f(e,i),s+1}$ by using the elements $2(s + j)$ for $j \in 1, \dots, n$. If (ii), add a new equivalence class of size $n + 1$ by using the elements $2(s + j) + 1$ for $j \in 1, \dots n + 1$.

Let $\mathcal{E}_{f(e,i)} = \lim_s \mathcal{E}_{f(e,i),s}$. It is now easy to see that $\mathcal{C}_{g(e,i)} = \mathcal{B} \oplus \mathcal{E}_{f(e,i)}$ is bi-embeddable with $\mathcal{A}$ (in this case even isomorphic) if $e \in \emptyset'$ and $i \in \overline{\emptyset'}$.

Ad (2). Assume n is the maximal size such that $\mathcal{A}$ has infinitely many equivalence classes of size n. Let $\mathcal{A}_{>n}$ be the substructure of $\mathcal{A}$ restricted to classes bigger than n. Then $\mathcal{A}_{>n}$ is finite. The index set is definable by the following Π_2^0 formula.

$$\mathcal{C}_e \approx \mathcal{A} \Leftrightarrow \forall x \exists y > x ([y]^{\mathcal{C}_e} \geq n \wedge \forall z < y \ \neg zEy) \wedge \mathcal{C}_{e,>n} \cong \mathcal{A}_{>n}$$

To see that it is hard, consider the Π_2^0 complete set $\mathrm{Inf} = \{e \colon W_e \text{ is infinite}\}$. We will build a computable function g such that $\mathcal{C}_{g(e)} \approx \mathcal{A} \Leftrightarrow e \in \mathrm{Inf}$. Fix a computable equivalence structure $\mathcal{B}$ isomorphic to $\mathcal{A}_{>n}$. Our desired structure $\mathcal{C}_{g(e)}$ will be the disjoint union of $\mathcal{B}$ with the structure $\mathcal{E}_{f(e)}$, where f is a computable function constructed as follows. Let $W_{e,s}$ be the computable approximation to W_e after s stages of our construction; we make the standard assumption that $x \in W_{e,s} \Rightarrow x < s$. Assume we have defined $\mathcal{E}_{f(e),s}$ and are at stage $s + 1$ of the construction. The structure $\mathcal{E}_{f(e),s+1}$ extends $\mathcal{E}_{f(e),s}$ by a new equivalence class of size n for every $x \searrow W_{e,s}$, i.e., if $x \searrow W_{e,s}$ let $\langle x, s \rangle, \dots, \langle x, s + n \rangle \in \mathcal{E}_{f(e),s+1}$ and set them to be equivalent. This finishes the construction; let $\mathcal{E}_{f(e)} = \lim_s \mathcal{E}_{f(e),s}$.

By construction $\mathcal{E}_{f(e)}$ has only equivalence classes of size n and it has infinitely many of those if and only if W_e is infinite. As $\mathcal{B} \cong \mathcal{A}_{>n}$, we have that

$$\mathcal{C}_{g(e)} = \mathcal{B} \oplus \mathcal{E}_{f(e)} \approx \mathcal{A} \Leftrightarrow e \in \mathrm{Inf} .$$

Thus the index set is Π_2^0 complete.

Ad (3). To see that it is in Π_2^0 one has to consider two cases. Either the finite part of $\mathcal{A}$ is as in (1) or as in (2). In any case, let k be the bound. If we are in case (1), let m be the number of elements in the finite part of $\mathcal{A}$. If we are in case (2), let m be the number of elements in the finite part of $\mathcal{A}$ restricted to equivalence classes bigger than n, where n is as above.

If we are in case (1) we can define the index set by

$$\mathcal{C}_e \approx \mathcal{A} \Leftrightarrow \forall y_1, \ldots, y_{m+1} \left(\bigwedge_{1 \leq i \leq m+1} [y_i] \leq k \rightarrow \bigvee_{1 \leq i < j \leq m+1} y_i = y_j \right)$$

$$(*)$$

$$\wedge \forall y_1, \ldots, y_{r+1} \left(\bigwedge_{1 \leq i \leq r+1} [y_i] > k \rightarrow \bigvee_{1 \leq i < j \leq r+1} y_i E y_j \right)$$

$$\wedge \exists x_1, \ldots, x_m (\theta_{\mathcal{A}_{fin}}(x_1, \ldots, x_m))$$

$$\wedge \forall y_1, \ldots, y_m \left(\left(\bigwedge_{1 \leq i \leq m} [y_i] \leq k \wedge \bigwedge_{1 \leq i < j \leq m} y_i \neq y_j \right) \rightarrow \Theta_{\mathcal{A}_{fin}}(y_1, \ldots, y_m) \right)$$

$$\wedge \exists x_1, \ldots, x_r \left(\bigwedge_{1 \leq i \leq r} [x_i] > k \wedge \bigwedge_{1 \leq i < j \leq r} \neg x_i E x_j \right)$$

$$\wedge \forall x ([x] > k \rightarrow \exists y > x \, y E x)$$

where $\theta_{\mathcal{A}_{fin}}$ is the formula obtained from the atomic diagram of the finite part of $\mathcal{A}$ by replacing the constants by variables and $\Theta_{\mathcal{A}_{fin}}$ is the disjunction over $\theta_{\mathcal{A}_{fin}}(x_1, \ldots, x_m)$ permuting over all variables. Let the formula in Eq. ($*$) be $\varphi_{\mathcal{A}}$. If we are in case (2) the defining formula is

$$\mathcal{C}_e \approx \mathcal{A} \Leftrightarrow \forall x \exists y > x ([y]^{\mathcal{C}_e} \geq n) \wedge \varphi'_{\mathcal{A}_{>n}}$$

where $\varphi'_{\mathcal{A}_{>n}}$ is as above with the slight difference that the second and third universal quantifiers now range over all x where $n < [x] \leq k$ instead of only $[x] \leq k$. The two formulas are easily seen to be Π_2^0.

For the hardness consider the Π_2^0 complete set $\mathrm{Inf} = \{e \colon W_e \text{ is infinite}\}$ and fix a computable structure $\mathcal{B}$ without infinite equivalence classes isomorphic to the finite part of $\mathcal{A}$. We build a computable function g such that

$$\mathcal{C}_{g(e)} = \mathcal{B} \oplus \mathcal{E}_{f(e)} \approx \mathcal{A} \Leftrightarrow e \in \mathrm{Inf}$$

where f is again a computable function. We prove the hardness for the case that $\mathcal{A}$ has one infinite equivalence class, the case for $r > 1$ is similar. The construction of $\mathcal{E}_{f(e)}$ is in stages, at stage 0 the universe of $\mathcal{E}_{f(e),0}$ is empty. Assume we have defined $\mathcal{E}_{f(e),s}$ and are at stage $s + 1$ of the construction.

For any $x < s$ such that $x \searrow W_{e,s}$, add $\langle x, s+1 \rangle$ to $\mathcal{E}_{f(e),s+1}$ and make it equivalent to all elements already in $\mathcal{E}_{f(e),s}$. It is easy to see that the structure $\mathcal{E}_{f(e)} = \lim_s \mathcal{E}_{f(e),s}$ has an infinite equivalence class if and only if $e \in \text{Inf}$ and thus $\mathcal{C}_{g(e)} \approx \mathcal{A}$ if and only if $e \in \text{Inf}$. □

Theorem 4.3 *Let $\mathcal{A}$ be an equivalence structure with degree of categoricity $\mathbf{0}'$. Then the following holds.*

(1) *If $\mathcal{A}$ has no infinite equivalence classes, then the index set $\{\mathcal{C}_e : \mathcal{C}_e \approx \mathcal{A}\}$ is Π_3^0-complete.*
(2) *If $\mathcal{A}$ has $0 < k < \omega$ infinite equivalence classes, then the index set $\{\mathcal{C}_e : \mathcal{C}_e \approx \mathcal{A}\}$ is D_3^0-complete.*

Proof Recall that an equivalence structure has degree of cateogricity $\mathbf{0}'$ if and only if it has finitely many infinite classes and is unbounded. Notice that two unbounded equivalence structures are bi-embeddable if and only if they have the same number of infinite classes and assume that $\mathcal{A}$ has $0 < k < \omega$ infinite equivalence classes. Then the index set is definable by

$$\mathcal{C}_e \approx \mathcal{A} \Leftrightarrow \forall x \left(x \in \text{Fin}^{\mathcal{C}_e} \rightarrow \exists y \left(y \in \text{Fin}^{\mathcal{C}_e} \wedge [y] \geq [x] \right) \right)$$

$$\wedge \exists x_1, \ldots, x_k \in \text{Inf}^{\mathcal{C}_e} \left(\bigwedge_{1 \leq i < j \leq k} \neg x_i E x_j \right)$$

$$\wedge \forall x_0, x_1, \ldots, x_k \in \text{Inf}^{\mathcal{C}_e} \left(\bigvee_{1 \leq i < j \leq k} x_i E x_j \right).$$

The part of the formula defining the finite equivalence classes is Π_3^0 and the part defining the infinite equivalence classes is Σ_3^0. Thus, the formula is D_3^0. If $\mathcal{A}$ has no infinite equivalence classes then the part of the defining formula defining the infinite classes becomes $\forall x \ x \in Fin^{\mathcal{C}_e}$, a Π_3^0 formula. Hence, in this case the above formula is Π_3^0.

To show the completeness of (2) we will define a computable function f such that for every Π_3^0 set P and every Σ_3^0 set S

$$\mathcal{C}_{f(p,e)} \approx \mathcal{A} \Leftrightarrow p \in P \wedge e \in S.$$

By Π_2^0 completeness of $\text{Inf} = \{e : W_e \text{ is infinite}\}$ we have that there is a computable function g such that

$$e \in S \Leftrightarrow \exists x \ W_{g(x,e)} \text{ is infinite.}$$

In the same vein, as $\mathrm{Fin} = \overline{\mathrm{Inf}}$ is Σ_2^0 complete we have that there is a computable function h such that

$$p \in P \Leftrightarrow \forall x \ W_{h(x,p)} \text{ is finite.}$$

We may assume without loss of generality that

$$e \in S \Leftrightarrow \exists ! x \ W_{g(x,e)} \text{ is infinite.}$$

In other words, we may assume that there is a unique x witnessing that $e \in S$. See for instance [8, Theorem 4.3.11] for a proof of this fact. We build the structure $\mathcal{C}_{f(p,e)}$ in stages.

Construction

Stage 0: The universe of $\mathcal{C}_{f(p,e)}$ is ω and $\mathcal{C}_{f(p,e)}$ has exactly one equivalence class for each finite size, i.e., for each $x \in \omega$, we set $\langle 2x, 0, 0 \rangle E \langle 2x, 0, i \rangle$ for $i \leq x$. All other elements are singletons.

Stage $s+1 = 2j$: For every $x < j$, if for some $t < j, t \searrow W_{h(x,p),j}$, then put $\langle 2x, i, j \rangle$ into the equivalence class of $\langle 2x, i, 0 \rangle$ for all i with $0 \leq i \leq k$.

Stage $s + 1 = 2j + 1$: For every $x < j$, if for some $t < j$, $t \searrow W_{g(x,e),j}$, then put $\langle 2x + 1, i, j \rangle$ into the equivalence class of $\langle 2x + 1, i, 0 \rangle$ for all i with $0 \leq i < k$.

Verification

$p \in P, e \in S$: The construction at even stages, taking care of the Π_3^0 part, will not contribute any infinite equivalence classes and $\mathcal{C}_{f(p,e)}$ is unbounded from the beginning. To see that it has k infinite equivalence classes just notice that by our assumption above there is exactly one x such that $W_{g(x,e)}$ is infinite. Thus, the construction at odd stages guarantees that the equivalence classes of elements $\langle 2x+1, i, 0 \rangle$ with $0 \leq i < k$ become infinite in the limit.

$p \notin P, e \in S$: Then there is an x such that $W_{h(x,p)}$ is infinite. By construction the equivalence class of $\langle 2x, 0, 0 \rangle$ is infinite and our strategy for S builds k infinite equivalence classes. Thus, $\mathcal{C}_{f(p,e)}$ has more than k infinite equivalence classes and hence, $\mathcal{C}_{f(p,e)}$ is not bi-embeddable with $\mathcal{A}$.

$p \in P, e \notin S$: Then for no x the set $W_{g(x,e)}$ is infinite. So, by construction, no equivalence class of elements with an odd number in the left column will grow to be infinite and all equivalence classes with an even number in the left column have finite equivalence classes as $W_{h(x,p)}$ is finite for all x.

$p \notin P, s \notin S$: There will be some x such that $W_{h(p,x)}$ is infinite. By construction the equivalence classes of $\langle x, i, 0 \rangle$ with $0 \leq i \leq k$ will be infinite and thus $C_{f(p,e)}$ will have $k + 1$ infinite equivalence classes.

To prove completeness for (1), at every stage, we apply the strategy described above for even stages. □

Theorem 4.4 *The index set* $\{e \colon \mathcal{C}_e$ *is computably bi-embeddably categorical*$\}$ *is* Σ_2^0 *complete.*

Proof Recall that an equivalence structure is computably bi-embeddably categorical if and only if it has bounded character and finitely many infinite equivalence classes.

Thus the index set is definable by the following computable Σ_2 formula.

$$\mathcal{C}_e \text{ is computably bi-embeddably categorical}$$

$$\Leftrightarrow \exists k \bigvee_{r \in \omega} \left(\forall x_1, \ldots, x_{r+1} \left(\bigwedge_{1 \leq i \leq r+1} |[x_i]| \geq k \to \bigvee_{1 \leq i < j \leq r+1} x_i \, E \, x_j \right) \right).$$

To see that it is hard consider the classical Σ_2^0 complete set $\text{Fin} = \{e: W_e \text{ is finite}\}$. We build a computable function f such that

$$\mathcal{C}_{f(e)} \text{ is computably bi-embeddably categorical} \Leftrightarrow W_e \text{ is finite}.$$

Let $W_{e,s}$ be the computable approximation of W_e at stage s. We construct $\mathcal{C}_{f(e)}$ in stages. At stage 0, $\mathcal{C}_{f(e),0}$ has universe ω and the equivalence relation is the identity relation. Assume we have defined $\mathcal{C}_{f(e),s}$. To define $\mathcal{C}_{f(e),s+1}$ check for $x < s$ if $x \searrow W_{e,s}$. If so declare $\langle x, s+i \rangle$ for $i \in 0, \ldots, x$ to be equivalent. This finishes the construction. Let $\mathcal{C}_{f(e)} = \lim_s \mathcal{C}_{f(e),s}$. It follows directly from the construction that $\mathcal{C}_{f(e)}$ has bounded character if and only if W_e is finite. This completes the proof. $\square$

Theorem 4.5 *The index set $\{e: \mathcal{C}_e \text{ has degree of b.e. categoricity } \mathbf{0}'\}$ is Σ_4^0 complete.*

To prove that Theorem 4.5 is Σ_4^0 hard we use the function constructed in the proof of Theorem 4.7. We will state and prove this theorem after proving the following representation lemma for Π_4^0 sets.

Lemma 4.6 *Let P be a Π_4^0 set, then there is a computable function g such that the following two conditions hold.*

$$x \in P \Leftrightarrow \forall y \exists! z \, W_{g(x,y,z)} \text{ is infinite} \tag{1a}$$

$$\forall x, y \left(\exists z \, W_{g(x,y,z)} \text{ is infinite} \to \exists! z \, W_{g(x,y,z)} \text{ is infinite} \right) \tag{1b}$$

$$\forall x, y \left(\forall z \, W_{g(x,y,z)} \text{ is finite} \to \forall z \, W_{g(x,y+1,z)} \text{ is finite} \right) \tag{2}$$

Proof Using a proof similar to the one of [8, Theorem 4.3.11] we get a computable function h satisfying (1a) and (1b) if we replace g by h. For $y > 0$ let

$$f(x, y, s) = \begin{cases} \mu(z < s)[|W_{h(x,y-1,z),s}| > |W_{h(x,y-1,z),s-1}|] & \text{such } z < s \text{ exists} \\ s & \text{otherwise} \end{cases}$$

Now at stage $s + 1$, for $y = 0$ let $W_{g(x,y,z),s} = W_{h(x,y,z),s}$ and otherwise let $u = f(x, y, s)$ and enumerate into $W_{g(x,y,u),s}$ all elements in $W_{h(x,y,z),s}$.
Verification

$x \in P$: Then for all y there is one z_y such that $W_{h(x,y,z_y)}$ is infinite. Hence, there is a stage s such that for given y, $u < z_{y-1}$ and $t > s$, $W_{h(x,y-1,u),t} = W_{h(x,y-1,u),s}$.

Thus, for infinitely many stages t, z_{y-1} is the least such that $|W_{h(x,y-1,z_{y-1}),t}| > |W_{h(x,y-1,z_{y-1}),t-1}|$ and therefore by construction there is z such that $W_{g(x,y,z)}$ is infinite. By definition of h there can be at most one such z.

$x \notin P$: Then there is a least y_0 such that for all z, $W_{h(x,y_0,z)}$ is finite. By construction $\lim_{s\to\infty} f(x, y_0 + 1, s)$ does not exist and thus no $W_{g(x,y_0+1,z)}$ can be infinite. By induction the same holds for all $y > y_0$. $\square$

Theorem 4.7 *The index set $\{e : \mathcal{C}_e$ has degree of b.e. categoricity $\mathbf{0}''\}$ is Π_4^0-complete.*

Proof By Theorem 3.10 the index set of the equivalence structures having degree of bi-embeddability categoricity $\mathbf{0}''$ is the same as the index set

$$\{e: \mathcal{C}_e \text{ has infinitely many infinite equivalence classes}\}.$$

This index set is clearly Π_4^0. To see that it is complete consider a Π_4^0 set P, then there is a computable function g such that

$$e \in P \Leftrightarrow \forall x \exists y \ W_{g(e,x,y)} \text{ is infinite.}$$

We may assume that g satisfies the matrices in Lemma 4.6. We build a computable function f such that

$$\mathcal{C}_{f(e)} \text{ has infinitely many infinite equivalence classes.} \Leftrightarrow e \in P$$

The construction is in stages; the universe of $\mathcal{C}_{f(e),0}$ is ω and $E^{\mathcal{C}_{f(e),0}} = \{(x, x) : x \in \omega\}$. Assume we have defined $\mathcal{C}_{f(e),s}$ and are at stage $s + 1$ of the construction. The structure $\mathcal{C}_{f(e),s+1}$ extends $\mathcal{C}_{f(e),s}$ as follows. For each $x, y \leq s + 1$, if there is $u < s + 1$ such that $u \searrow W_{g(e,x,y),s+1}$, then add $\langle x, y, s + 1 \rangle$ to the equivalence class of $\langle x, y, 0 \rangle$. Then proceed to the next stage.

The desired structure $\mathcal{C}_{f(e)}$ is the structure in the limit of the construction. If $e \in P$, then $\mathcal{C}_{f(e)}$ has infinitely many infinite equivalence classes as for every x there is a y such that $W_{g(e,x,y)}$ is infinite and by construction the elements having the same first and second column are in the same equivalence class. Assume $e \notin P$, then there exists an x_0 such that for no y_0 the above set is infinite; so no equivalence class of elements with x_0 in the left column will be infinite. Then by Lemma 4.6 the same holds for all $x > x_0$ and for all $x_1 < x$ there is exactly one y with $W_{g(e,x_1,y)}$ infinite. Hence, $\mathcal{C}_{f(e)}$ has only finitely many infinite equivalence classes. $\square$

Proof of Theorem 4.5 The index set is definable by the Σ_4^0 formula

$$\forall k (\exists x (x \in Fin^{\mathcal{C}_e} \wedge [x] \geq k)) \wedge \exists r (T_{\mathrm{Inf}^{\mathcal{C}_e}} \leq r).$$

where $T_{\mathrm{Inf}^{\mathcal{C}_e}}$ is a Π_2^0 transversal of $\mathrm{Inf}^{\mathcal{C}_e}$. For the hardness we use the strategy used in the proof of Theorem 4.7. There, given a Π_4^0 set P, we define a computable function f such that

$$\mathcal{C}_{f(e)} \text{ has degree of b.e. categoricity } \mathbf{0}'' \Leftrightarrow e \in P.$$

Clearly, $\overline{P}$ is Σ_4^0 and using the same function we have that

$$\mathcal{C}_{f(e)} \text{ has degree of b.e. categoricity } \mathbf{0} \text{ or } \mathbf{0}' \Leftrightarrow e \notin P$$

since if $e \notin P$, f produces an equivalence structure with finitely many infinite equivalence classes. Notice that $\mathcal{C}_{f(e)}$ need not have unbounded character and thus might have degree of categoricity $\mathbf{0}$. Therefore, define a computable unbounded equivalence structure $\mathcal{B}$ and a function g such that

$$\mathcal{C}_{g(e)} = \mathcal{C}_{f(e)} \oplus \mathcal{B}.$$

The function g is clearly computable and

$$\mathcal{C}_{g(e)} \text{ has degree of b.e. categoricity } \mathbf{0}' \Leftrightarrow e \in \overline{P}.$$

This proves that the index set is Σ_4^0 hard. □

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