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# EXTERNALITIES AGGREGATION IN NETWORK GAMES*

BY FRANCESCO FERI AND PAOLO PIN[1]

*Department of Economics, Royal Holloway University of London, United Kingdom, and Department of Economics, Università di Trieste, Italy; Department of Economics and Statistics, Università di Siena, Italy, and IGIER and BIDSA, Università Bocconi, Italy*

We generalize the results on the monotonicity of equilibria for network games with incomplete information. We show that not only the distinction between *strategic complements* and *strategic substitutes* is important in determining the nature of the Bayesian Nash equilibria, but the nature of the statistic itself is also determinant. We show that understanding the underlying forces behind people's choices may be of fundamental importance for a policymaker that wants to incentivize efficient behavior.

## 1. INTRODUCTION

Following the paper "Network Games" by Galeotti et al. (2010) (henceforth: NG), many recent models on games with local externalities assume that the agents playing the game are nodes of a network environment, and that they have to take an action that has local externalities channeled through the topology of the network. However, the agents have a limited ability to observe the structure of the network and even the identity of their peers. Essentially, the agents as nodes know only their own degree in the network and have some information about the general network formation process that generated the whole social network. The realization could be such that the degree of neighbors is independently and identically distributed, and is also independent on the degree of the node itself. More generally, it is possible to formalize any stochastic process for the realization of the network, as in the recent paper by Acemoglu et al. (2016), taking into account any possible source of correlation. In *network games* with limited observability, the nodes compute the expected payoff through a *statistic* over the sample of the actions of the neighbors that they will end up finding in the pool. From an applied point of view, such models are good tools for analyzing many complex social phenomena, such as peer effects, the spread of habits, marketing for goods with externalities, vaccination policies, and contributions to public goods, to name a few. In most of these cases, a policymaker who wants to increase or reduce a certain action of citizens, or a firm that wants to increase effort or consumption of clients, does typically also observe from surveys only a proxy of the degree of each player and the action that each player takes.

It must be noted, however, that currently, the theoretical predictions across the models based on network games are not always consistent when it comes to assigning some correlation between the degree of nodes and their actions: that is, who will endogenously have a

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greater tendency to vaccinate during a flu pandemic, those with many or those with few links? When a bad habit, such as smoking, spreads in a school and there are peer effects, who will be more likely to smoke, those with many or those with few links?[1] Since some of the theoretical predictions in these models differ from those of NG, from a theoretical point of view, it becomes important to study what differs within apparently similar assumptions.

Here, we point out that the distinctions made up to now are not sufficient to understand the possible outcomes of those games. Apart from the degree of correlation mentioned above, the literature has focused on the sign of the externality, distinguishing between *local public goods* and *local public bads*, on the cross derivative between the externalities and own actions, distinguishing between games of *substitutes* and games of *complements*,[2] and on the degree correlation of the network formation process. However, not much attention has been placed on the nature of the *statistic* on neighbors' actions that is affecting the payoff.

Although many statistics are expected to be constant with respect to the number of links that a node has,[3] we note that some statistics (like the maximum) are expected to increase with the number of links, whereas other statistics (like the minimum) are expected to decrease. We provide novel and nontrivial results for these type of statistics and show that knowledge of the nature of the statistics is not only crucial for the correct computation of the equilibria but also for the identification of the nature of the game (substitutability or complementarity).

The article is organized in the following way: next section provides introductory examples, and in Section 3, we describe the general model. Section 4 contains the main results. Section 5 discusses the main results and the relation with the previous literature. In particular, Subsection 5.2 discusses an example to show why our theory is useful for a policymaker, who could otherwise observe data from the real world, misinterpret the incentives of people and, because of that, implement a counterproductive policy. Section 6 concludes the discussion and provides possible directions for further research. Formal proofs, with some ancillary results, are found in the Appendix.

## 2. THREE SIMPLE SCENARIOS

This section presents and analyzes simple examples of games played on networks, to illustrate the effect of the type of the statistics (used to aggregate the neighbors' actions) on the individual decisions. As first two simple scenarios, we consider, respectively, a strategic vaccination model,[4] and a model of acquisition of information. These are both activities with positive externalities and the substitute property (i.e., in equilibrium, the more my neighbors contribute the less I will). However, they differ in terms of how the neighbors' decisions affect the payoffs. In the information acquisition case, an agent is influenced by the neighbor who knows more (this is a *best-shot* game), so that having more neighbors increases the probability of finding a well-informed one, and agents with a higher degree will be more likely to free ride and not acquire information themselves. In the strategic vaccination case, an agent is influenced by the minimum contribution in own neighborhood (this is a *weakest-link* game),[5] so that having more neighbors increases the probability of finding a nonvaccinated one, and agents with a higher degree will be more likely to vaccinate.

[1] On the vaccination example, see Goyal and Vigier (2014, 2015), Galeotti and Rogers (2015), and again Acemoglu et al. (2016). On the smoking example, see Currarini et al. (2013).

[2] On this, see also the discussion in Jackson and Zenou (2014) and Bramoullé and Kranton (2016) about *strategic complements* and *strategic substitutes*.

[3] This last category includes the mean and the mode, which are the statistics most used in the empirical literature on peer effects and reference groups (a good survey is given in Blume et al., 2010).

[4] In Subsection 5.2, we discuss an alternative assumption of incentives when modeling vaccination decisions.

[5] Minimum and maximum contributions from neighbors relate respectively to *weakest-link* and *best-shot* games, as introduced by Hirshleifer (1983) in a nonnetwork context. NG and Boncinelli and Pin (2012) discuss network best shot games, whereas classical *weakest-link* games are those related to contagion, as Galeotti and Rogers (2015).

Assume $n$ players, each one represented by a node in a directed network. A link in the network represents an interaction. Players do not know the network structure but are informed only about their own degree (i.e., the number of other players whose actions affect their payoff) and that each link of the network is formed by an independent probability $p \in (0,\ 1)$. In this way, beliefs about neighbors' degrees follow a binomial distribution, that is, for all nodes, the probability that a neighbor is of degree $k$ is given by

$$Q\,(k;\,p) = \binom{n-2}{k-1}\ p^{k-1}(1-p)^{n-k-1}.$$

Players take action $a \in \{0, 1\}$ in a one-shot game. If they choose 1, their payoff is 1 minus a cost $c \in (0,\ 1)$; otherwise, choosing 0 the payoff depends on the actions taken by neighbors in the following two ways: In the first scenario, the payoff is equal to 1 if at least one of the neighbors has chosen 1; otherwise, it is 0; in the second scenario, the payoff is equal to 1 if all neighbors have chosen 1; otherwise, it is 0. In the first case (which refers to the example of acquisition of information), the statistic (on the neighbor actions) affecting the individual payoff is the maximum, in the second case (which refers to the example of vaccination), the relevant statistic is the minimum. Because the expected payoff from action 0 is strictly related to the expected value of the statistic, agents with a higher expected value of the statistic have an incentive to take action 0.

Let us concentrate our attention on symmetric strategies, which means that all agents with the same degree play the same strategy. Let $\sigma_k$ be the probability that an agent with degree equal to $k$ chooses action 1. Then, the probability that a randomly selected neighbor is choosing action 1 is given by $p_\sigma = \sum_{h=1}^{n-1} Q(h;\,p) \cdot \sigma_h$.

We start considering the first scenario (information acquisition case) where the relevant statistic affecting the individual payoff is the maximum of the neighbors' actions. The expected payoff of playing 0 is

$$1 - (1 - p_\sigma)^k.$$

For all strategy profiles, this value is increasing in the degree $k$ of the agent. This is enough to state that an equilibrium cannot be constant (i.e., all players play the same strategy), and that every equilibrium is decreasing with respect to the agent degree $k$, that is, agents with degree above a certain threshold will play action 0 and agents with degree below a certain threshold will play action 1 (with agents at the threshold possibly randomizing).[6]

Now consider the second scenario (strategic vaccination case) where the relevant statistic affecting the individual payoff is the minimum of the neighbors' actions. The expected value of playing 0 now is ${p_\sigma}^k$, which is decreasing in the degree $k$ of an agent. This is enough to state that an equilibrium cannot be constant, and that every equilibrium is increasing with respect to the degree $k$.

Comparing these two scenarios, the relation between the equilibrium strategies depends on the characteristics of the statistic, even if we are always in the case of strategic substitutes.

We are aware that in many empirical applications, the relevant statistic is the average on the neighbors' actions, and therefore, we include a third scenario. As in the two previous examples, if an agent chooses action 1, his payoff is 1 minus a cost $c \in (0,\ 1)$; otherwise, choosing 0, his payoff is given by an increasing function of the average of the neighbors' actions. Let this function be equal to 0 when the average of the neighbors' action is 0 and equal to 1 when this average is equal to 1.

In this case, under symmetric strategies, the action that a randomly selected neighbor will play is a random variable with expected value $E(a) = p_\sigma$, and variance $\mathrm{var}(\sigma) = (1 - p_\sigma) \cdot$

[6] For the sake of clarity, we provide a direct proof of this statement in Appendix A.1, even if this can be proven also with the main result of this article.

$p_\sigma$. For a node with degree $k$, the expected value of the average on the neighbors actions is independent on the agent degree $k$, and is always just $E(a)$. But the variance of the statistic for a node of degree $k$ is $\frac{\mathrm{var}(\sigma)}{k}$, and this depends on the degree. For an agent with degree $k$, the variance of the expected value of the mean is decreasing in $k$. This implies that, for any strategy profile, individuals with many neighbors face less uncertainty about the realization of the statistic.

Therefore, if the payoff from playing action 0 is not linear with respect to the average of neighbors actions, we could again expect some relation between degree and equilibrium strategies. It is possible to prove (and we do so in Appendix A.1) that when the payoff from action 0 is a strictly concave function of the average action of neighbors, agents with high degree will have a higher expected payoff from action 0, and therefore, all equilibria will be decreasing with the degree. If instead the payoff from action 0 is a strictly convex function of the average action of neighbors, agents with low degree will have a higher expected payoff from action 0, and therefore, all equilibria will be increasing with degree.

In this third scenario, where the statistic is the mean, the relation between the equilibrium strategies depends on the second derivative of the expected payoff from playing 0, with respect to the mean of neighbors' actions. As a player compares the expected payoff from playing 0 with the payoff from playing 1 (which is sure in these examples), what becomes pivotal is the second-order derivative of the marginal payoff, which, in turn, can be seen as a third-order derivative of the payoff.

Note that it is possible to build similar examples for strategic complements simply assuming that if subjects choose 0 their payoff is 0; otherwise, choosing 1, the payoff is determined by a statistic $s$ (computed on the actions taken by neighbors) minus a cost $c \in (0, 1)$. It is easy to show that when the statistic is the minimum (maximum), all equilibria are not increasing (not decreasing) with respect to degree $k$. Therefore, it is clear from these examples that the simple observation that actions increase or decrease with respect to degree is not enough to identify the strategic nature of the game (which, as we stressed in the introduction, is crucial for the implementation of policies and marketing strategies). Once we have discussed the model and its results, we will come back to this point in Subsection 5.2.

## 3. THE GENERAL MODEL

Let $\mathcal{N} = \{1, 2, \ldots .n\}$ be a finite set of agents. Each agent $i \in \mathcal{N}$ obtains some partial information about the realization of a random network and then chooses an action $x_i \in \mathcal{X}$, where $\mathcal{X} \subseteq \mathbb{R}$ is a compact set. Payoffs are assigned in a way that depends on the realized network environment and, as we allow for mixed strategies, on the realized strategy profile. The structure is the one of a Bayesian game, and we follow the notation of NG, integrating it with some of the formalization from Acemoglu et al. (2016).

3.1. *The Network.* Our network environment is represented by a (possibly directed) *network* $g$, in which the set of nodes is the set of agents, and a link $ij \in g$ denotes that the action of agent $j$ affects $i$'s payoff. By $N_i(g) = \{j \in \mathcal{N} : ij \in g\}$, we denote the set of neighbors of $i$ in $g$ (excluding $i$) and by $k_{i,g} \in \mathcal{K}$, we mean the number of such neighbors (i.e., $i$'s out-degree in the network) where $\mathcal{K}$ is the set of natural numbers $\{0, 1, 2, \ldots, n-1\}$. We call $V_{k,g}$ the set of nodes that have degree $k$ in network $g$, and by $v_{k,g}$ the cardinality of this set. Network $g$ is obtained from a probability distribution $P$ over all the $2^{n(n-1)}$ possible networks. We call $P$ the *network formation process*.

Following NG, we call *degree independence* the lack of correlation between own degree and the neighbors' degree.[7] Instead, when knowing own degree $k$ changes the expectation on the

[7] The *configuration model* proposed by Bender and Canfield (1978) is an example of network formation process characterized by *degree independence*. This is a model of random network where a certain degree distribution is given, and as the number $n$ of nodes grows to infinity, knowing only own degree provides no additional information

neighbors' degree, we may have either degree *assortativity* or *disassortativity*.[8] More in detail, the assortativity is measured by the Pearson correlation coefficient of the degree between pairs of linked nodes. Positive values of this coefficient indicate that it is more likely that the relationships will be between nodes of similar degree, whereas negative values indicate that it is more likely that the relationships will be between nodes of opposed degree.[9]

3.2. *The Statistic.* In our games, player $i$ chooses $x_i$ and her $k_{i,g}$ neighbors choose the action profile $\vec{x}_{i,g} = (x_1, \dots, x_{k_i})$. The effects of local interaction are aggregated by the function $s$. Formally, $s$ is a different $k$-dimensional function for every $k \in \mathcal{K}$. So, $s$ is a family of $n$ functions[10] and each of these functions is anonymous on the arguments, which means that any permutation of the elements of $\vec{x}_{i,g}$ will give the same result.

Furthermore, we assume that functions $s$ are monotonically increasing in the actions of the neighbors. This general specification includes measures of central tendency as the mean, the median, and the mode, as well as minimum and maximum values. In the following, we refer to any measure $s$ as a *statistic*:

3.3. *Payoffs.* Payoffs are based on the realized network $g$. Player $i$'s payoff function when she chooses $x_i$ and her $k_{i,g}$ neighbors choose $\vec{x}_{i,g} = (x_1, \dots, x_{k_i})$ is as follows:

$$\Pi_{k_{i,g}}(x_i, \vec{x}_{i,g}) = f(x_i, s(\vec{x}_{i,g})) - c(x_i), \tag{1}$$

where $s$ is the *aggregator* computed on the set of the neighbors' actions, as defined above, and $f(x, s)$ is continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ with:

$$f_x > 0, \quad \text{and} \quad f_{xx} \leq 0.$$

We say that when $f_s > 0$, we have *positive externalities*, and when $f_s < 0$, we have *negative externalities* with respect to the statistic. Moreover, if $f_{xs} < 0$ or $f_{xs} > 0$, we say, respectively, that $f$ has the *substitutes* or the *complements* property with respect to the statistic. Finally, $c(x)$ is a convex cost function such that $c_x > 0$ and $c_{xx} \geq 0$.

REMARK 1. We are aware that our definition of externalities, and consequently, the definitions of the properties of complementarities and substitutabilities are not exactly the conventional one. Indeed (and following NG), in the literature, externalities are usually defined according to how the neighbors' actions directly affect the players' payoffs. Note, however, that our definition coincides with the standard definition because we assume that the statistic $s$ is monotonic increasing in its arguments as, for example, in the three scenarios of Section 2 (we analyze this issue with more details in Subsection 5.1 below).[11]

on the degree of neighbors, which can be supposed as being drawn uniformly and i.i.d. from that degree distribution. On this see also Pin and Rogers (2016).

[8] In NG, this notion is related to the function that rules the network externalities of the game (i.e., $f$), and they talk about *positive* or *negative neighbor affiliation*.

[9] This coefficient lies between $-1$ and 1. When it is equal to 1, the network is said to have perfect assortative mixing patterns, when it is equal to 0, the network displays degree independence, when it is equal to $-1$, the network is completely disassortative. For details, refer to Newman (2002)

[10] Formally, $s$ is a function from $\mathbb{R}_+^{k_{i,g}}$ to $\mathbb{R}_+$ and, if $k_{i,g} = 0$, then $s$ is a constant. In principle, we even allow them to be different functions for each $k$. We can write this dependence on $k$ as $s : \bigcup_{k \in \mathbb{N}} \{k\} \times \mathcal{X}^k \to \mathbb{R}$. In this way, $s(\cdot, \cdot)$ is a function of two arguments: the degree $k$ and a vector of dimension $k$. With this notation, the general expression for the payoff functions needs a small change and becomes: $\Pi_{k_{i,g}}(x_i, \vec{x}_{i,g}) = f(x_i, s(k_{i,g}, \vec{x}_{i,g})) - c(x_i)$.

[11] Indeed, it is directly verifiable that assuming $s(\vec{x}_{i,g}) \geq s(\vec{x}'_{i,g})$ for all $\vec{x}_{i,g} \geq \vec{x}'_{i,g}$, we have that for all $x_i \geq x'_i$ $f(x_i, s(\vec{x}_{i,g})) - f(x'_i, s(\vec{x}_{i,g})) \geq f(x_i, s(\vec{x}'_{i,g})) - f(x'_i, s(\vec{x}'_{i,g}))$ if $f_{xs} > 0$ and $f(x_i, s(\vec{x}_{i,g})) - f(x'_i, s(\vec{x}_{i,g})) \leq f(x_i, s(\vec{x}'_{i,g})) - f(x'_i, s(\vec{x}'_{i,g}))$ if $f_{xs} < 0$.

3.4. *Strategy Profiles.* A strategy for player $i$ is a mapping $\sigma_i : \mathcal{K} \to \Delta(\mathcal{X})$, where $\Delta(\mathcal{X})$ is the set of probability distributions on $\mathcal{X}$, that is, $\sigma_i = [\sigma_{ik}]_{k\in\mathcal{K}}$, where $\sigma_{ik}$ is the mixed strategy played by player $i$ of degree $k$. In this way, $\sigma_i$ indicates a vector of functions: for each possible degree $k$ that player $i$ observes before choosing her action, she chooses a probability distributions on $\mathcal{X}$.

Furthermore, $\vec{\sigma}_{ig}$ is the strategy profile of $i$'s neighbors in network $g$, $\vec{\sigma} = [\sigma_i]_{i\in\mathcal{N}}$ is the strategy profile of the game, and $\vec{\sigma}_{-i} = [\sigma_j]_{j\in\mathcal{N}/i}$ is the set of strategy profiles of all players, excluding $i$.

3.5. *Information.* The only piece of information that an agent $i$ obtains before deciding her action, on top of the common prior $P$, is her own degree $k_{i,g}$ in the realized network $g$. Then the players play a game of incomplete information described by the quadruple $(\mathcal{N}, \mathcal{X}, (\Pi_{k_{i,g}})_{k_{i,g}\in\mathcal{K}}, P)$.

In the following, we consider *symmetric Bayesian Nash equilibria* in which every agent with the same information and facing the same ex ante conditions (i.e., each agent $i$ with the same degree $k$) chooses the same strategy, that is, $\sigma_{ik} = \sigma_{jk}$ for any $k \in \mathcal{K}$ and for any $i, j \in \mathcal{N}$. Whenever this does not generate ambiguity, we start denoting only with subscript $k$ all the quantities that are common for all agents with degree $k$.

Here, we define the properties of strategies that will turn useful in the analysis.

DEFINITION 1. A symmetric strategy profile $\vec{\sigma}$ is *all equal* if $\sigma_k = \sigma_h$ for any two different $k, h \in \mathcal{K}$.

DEFINITION 2. A symmetric strategy profile $\vec{\sigma}$ is *first-order stochastic dominance (FOSD) increasing* if, for every $k \in \mathcal{K} \setminus \{n-1\}$, we have that either $\sigma_{k+1} = \sigma_k$ or $\sigma_{k+1}$ FOSD $\sigma_k$, and for at least one $k \in \mathcal{K} \setminus \{n-1\}$, we have that $\sigma_{k+1}$ FOSD $\sigma_k$.

Analogously, $\vec{\sigma}$ is FOSD decreasing if, for every $k \in \mathcal{K} \setminus \{n-1\}$, we have that either $\sigma_{k+1} = \sigma_k$ or $\sigma_k$ FOSD $\sigma_{k+1}$, and for at least one $k \in \mathcal{K} \setminus \{n-1\}$, we have that $\sigma_k$ FOSD $\sigma_{k+1}$.

It immediately follows that, in the context of pure strategies, FOSD increasing means that $x_{k+1} \geq x_k$, with strict inequality for at least one $k \in \mathcal{K} \setminus \{n-1\}$.

Given a realized network $g$ and a strategy profile $\vec{\sigma}$, the expected payoff of agent $i$ of degree $k$, if she knows the exact network realization, her own position, and the positions of all other nodes, is given by:[12]

$$\Pi^e_{i,g}(\sigma_i, \vec{\sigma}_{-i}) = \int_{\mathcal{X}^n} \Pi_{k_{i,g}}(x_i, \vec{x}_{i,g})\, d\vec{\sigma}. \tag{2}$$

In addition, we also have to include the uncertainty about the realization of the network. Adding to this, the expected payoff of agent $i$ of degree $k$ is

$$\Pi^e_k(\sigma_i, \vec{\sigma}_{-i}) = \frac{\sum_g P(g) \cdot \sum_{i\in V_{k,g}} \Pi^e_{i,g}(\sigma_i, \vec{\sigma}_{ig})}{\sum_g P(g) \cdot v_{k,g}}. \tag{3}$$

Describing this in words, an agent evaluates all possible nodes $i$ with degree $k$ in any possible realized network $g$, updating priors with the information that her degree is actually $k$.[13] For each such node $i$'s position and network $g$, and for each realization of $\vec{\sigma}$, there will be a vector $\vec{x}_{i,g}$ that lists each neighbor's action, depending on their degree in network $g$.

[12] With a slight but unambiguous abuse of notation, we denote with the integral notation also expected outcomes derived from discrete probabilities that, for example, are always the case when $\mathcal{X}$ is finite.

[13] Note that $\Pi^e_k(\sigma_i, \vec{\sigma}_{-i}) = \sum_g P(g|k) \cdot \frac{\sum_{i\in V_{k,g}} \Pi^e_{ig}(\sigma_i, \vec{\sigma}_{ig})}{v_{k,g}}$ where $P(g|k) = \frac{v_{k,g} \cdot P(g)}{\sum_g v_{k,g} \cdot P(g)}$ is the updated probability of network $g$.

The Bayesian Nash equilibria can be represented simply as a (mixed) strategy profile $\vec{\sigma}^*$, where every agent $i$, depending on her degree $k_i$, will choose an optimal strategy $\sigma_k^*$, which maximizes the individual expected payoff for agent $i$ from (3).

Let $\varphi_{ig}(s|\vec{\sigma}_{ig})$ be the probability density function of $s$ exactly for node $i$ in network $g$ when the strategy profile of the $i$'s neighbors is $\vec{\sigma}_{ig}$. For an agent observing only her own degree $k$, the posterior distribution for the statistic $s$ will be:

$$\varphi_k(s|\vec{\sigma}, P) \equiv \frac{\sum_g P(g) \cdot \sum_{i \in V_{k,g}} \varphi_{ig}(s|\vec{\sigma}_{ig})}{\sum_g P(g) \cdot v_{k,g}}. \tag{4}$$

In words, for a given strategy profile, the distribution of the statistic $s$, for an agent observing that her own degree is $k$, is simply the average of the distributions of $s$ for all nodes with degree $k$ across all possible networks, weighted by the probability that a specific network is realized. Therefore, since the Bayesian updating based on the network structure is linear, the expected value of $s$ for an agent of degree $k$ is

$$E_k(s|\vec{\sigma}, P) = \frac{\sum_g P(g) \cdot \sum_{i \in V_{k,g}} E_{ig}(s|\vec{\sigma}_{ig})}{\sum_g P(g) \cdot v_{k,g}}, \tag{5}$$

where $E_{ig}(s|\vec{\sigma}_{ig})$ is the expected value of $s$ for node $i$ in network $g$ and when the strategy profile of the $i$'s neighbors is $\vec{\sigma}_{ig}$. Then, for a given strategy profile, the expected value of $s$, for an agent observing that her own degree is $k$, is simply the average of the expected value of $s$ for all nodes with degree $k$ across all possible networks, weighted by the probability that a specific network is realized.

We call $\Phi_k(s|\vec{\sigma}, P)$ the cumulative probability distribution on $s$ from $\varphi_k(s|\vec{\sigma}, P)$. Then $\Phi_k(s|\vec{\sigma}, P)$ summarizes all the information provided by $P$ (the network formation process) and $\vec{\sigma}$ (the strategy profile), given $k$. Finally, by $\mathrm{Var}_k(s|\vec{\sigma}, P)$, we denote the variance of $s$ for an agent of degree $k$ when the strategy profile of the game is $\vec{\sigma}$ and the network formation process is $P$.

3.6. *Definitions.* Given that the type of statistic $s$ affects the individual payoff and the optimal individual behavior, through the network formation process $P$ and the equilibrium strategy $\vec{\sigma}$, we highlight its relevant characteristics. In Definitions 3–6, we consider nonnecessarily symmetric strategy profiles $\vec{\sigma}$.

DEFINITION 3. Given a network formation process $P$, a statistic $s$ is *degree-stable* with respect to $P$ if for every $\vec{\sigma}$ and $k \in \mathcal{K} \setminus \{n-1\}$, we have that $E_{k+1}(s|\vec{\sigma}, P) = E_k(s|\vec{\sigma}, P)$.

DEFINITION 4. Given a network formation process $P$, a statistic $s$ is *degree-increasing* (*degree-decreasing*) with respect to $P$ if for every $\vec{\sigma}$ and $k \in \mathcal{K} \setminus \{n-1\}$, we have that $E_{k+1}(s|\vec{\sigma}, P) > E_k(s|\vec{\sigma}, P)$ $(E_{k+1}(s|\vec{\sigma}, P) < E_k(s|\vec{\sigma}, P))$. A statistics $s$ is weakly degree-increasing (or degree-decreasing) when the conditions are satisfied only for every not *all equal* $\vec{\sigma}$.

Using the standard definitions of FOSD and second-order stochastic dominance (SOSD) for random variable, we can classify statistics $s$ according to the following criteria.[14]

DEFINITION 5. Given a network formation process $P$, a statistic $s$ is *FOSD degree-increasing* (or *FOSD degree-decreasing*) with respect to $P$ if for every $\vec{\sigma}$, $k \in \mathcal{K} \setminus \{n-1\}$, and $x \in \mathbb{R}$,

[14] Let be $\alpha$ and $\beta$ two random variables distributed on $\mathbb{R}$ with cumulative functions, respectively, of $\Phi_\alpha(x)$ and $\Phi_\beta(x)$. We say that $\alpha$ FOSD $\beta$ if, for all $x \in \mathbb{R}$, $\Phi_\alpha(x) \le \Phi_\beta(x)$ with strict inequality for some $x$. Furthermore, we say that $\alpha$ SOSD $\beta$ if, for all $y \in \mathbb{R}$, $\int_{-\infty}^{y} \Phi_\alpha(x)\, dx \le \int_{-\infty}^{y} \Phi_\beta(x)\, dx$ with strict inequality for some $y$.

TABLE 1
EXAMPLE OF THE MAIN STATISTICS AND THEIR PROPERTIES WHEN P SATISFIES DEGREE INDEPENDENCE

| Statistic | Properties |
|---|---|
| Average | Degree-stable and SOSD |
| Minimum | Weakly degree-decreasing and weakly FOSD degree-decreasing |
| Maximum | Weakly degree-increasing and weakly FOSD degree-increasing |
| Mode | Degree-stable and SOSD |
| Median | Degree-stable and SOSD |
| Range | Weakly degree-increasing and weakly FOSD degree-increasing |
| Sum | Degree-increasing and FOSD degree-increasing |

we have that $\Phi_{k+1}(x|\vec{\sigma}, P) \leq \Phi_k(x|\vec{\sigma}, P)$ ($\Phi_{k+1}(x|\vec{\sigma}, P) \geq \Phi_k(x|\vec{\sigma}, P)$) with strict inequality for some $x$. A statistics $s$ is weakly FOSD degree-increasing (or degree-decreasing) when the strict inequalities hold only for every not *all equal* $\vec{\sigma}$.

DEFINITION 6. Given a network formation process $P$, a statistic $s$ satisfies SOSD with respect to $P$ if for every $\vec{\sigma}$ and $y \in \mathbb{R}$, we have the following inequality:

$$\int_{-\infty}^{y} \Phi_{k+1}(x|\vec{\sigma}, P)\, dx \leq \int_{-\infty}^{y} \Phi_k(x|\vec{\sigma}, P)\, dx, \tag{6}$$

with strict inequality for some $y$. A statistic $s$ is weakly SOSD when the strict inequalities hold only for every not *all equal* $\vec{\sigma}$.

We remark that the difference between a statistic that is monotone increasing in its arguments (as we define a statistic, at page 7) and the notion of statistic that is degree-increasing with respect to $P$ is given in Definitions 4 and 5 (the latter two imply that statistic $s$ is increasing, in expectation, in the number of peers that are sampled). When $P$ is degree-independent, then all the examples discussed in Section 2 are monotone statistics in their arguments, the max and min are, respectively, degree-increasing and degree-decreasing. Table 1 shows these and also other cases.

The following two results are useful to understand the meaning of the assumptions made in our propositions:

(1) It is directly verifiable that an FOSD degree-increasing statistic (Definition 5) implies a degree-increasing one (Definition 4).
(2) If $s$ is degree-stable and satisfies SOSD (Definitions 3 and 6), then $s$ is *converging*, in the sense that for every $\sigma$ and $k \in \mathcal{K} \setminus \{n-1\}$, we have that $\mathrm{Var}_{k+1}(s|\vec{\sigma}, P) < \mathrm{Var}_k(s|\vec{\sigma}, P)$ (we prove this in Appendixz A.2 as a corollary to Lemma A.1).

When $P$ has degree independence, for any strategy profile $\vec{\sigma}$, many standard statistics as the mean, the median, or the sample variance, are both degree–stable and converging. Even so, under degree independence, examples of degree–increasing and degree–decreasing statistics are the maximum and the minimum, respectively (whenever the strategy profile $\vec{\sigma}$ is not all equal).

In general, when $P$ does not show degree-independence, the value of the statistics $s$ for an agent of given degree $k$ will depend on both $P$ and on the strategy profile $\vec{\sigma}$. Consider a degree-stable statistic under degree-independence, for example, the mean. Suppose that the players play a symmetric FOSD increasing strategy profile $\vec{\sigma}$ and that $P$ is characterized by degree assortativity. In this case, one should check that the expected value of the mean over the neighbors actions is increasing with the degree.

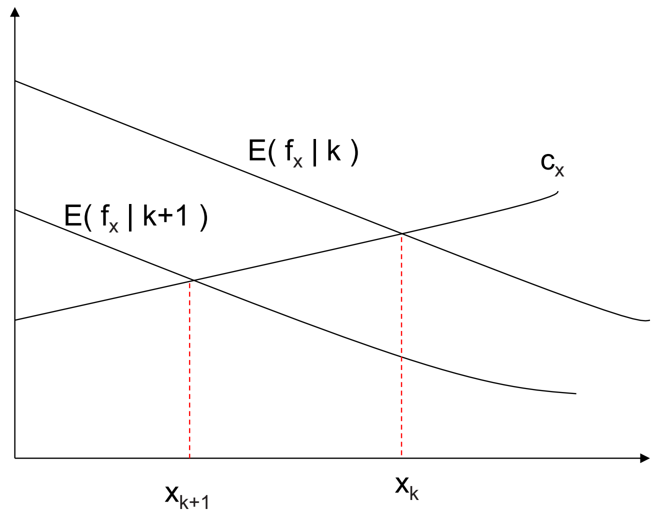


FIGURE 1

INTUITION FOR LEMMA 1 WHEN $x^*(k) \geq x^*(k+1)$ [COLOR FIGURE CAN BE VIEWED AT WILEYONLINELIBRARY.COM]

## 4. RESULTS

In this game, the existence of a symmetric Bayesian Nash equilibrium follows directly from the Kakutani fixed point theorem, as mixed equilibria on a compact set $\mathcal{X}$ form a convex compact set. So, we immediately start to characterize the equilibria.

### 4.1. *Characterization of Equilibria.*

Our first lemma provides a general result, in which we aggregate in a single expression the function $f$ that determines the payoffs together with all the information we have about the network structure and the statistics from $\varphi_k(y|\vec{\sigma}, P)$. This result provides a general check to determine whether there is monotonicity in the equilibrium of the game

and we use it as a lemma for the following results:

LEMMA 1. *Consider the expected marginal payoff given by quantity*

$$E_k(f_x|x, \vec{\sigma}, P) \equiv \int_{-\infty}^{\infty} f_x(x, y) \cdot \varphi_k(y|\vec{\sigma}, P)\, dy, \tag{7}$$

*where $f_x$ is the derivative of $f$ with respect to $x$. Then:*

*(1) if (7) is nonincreasing in $k$ for any $\vec{\sigma}$, then in every symmetric Bayesian Nash equilibrium of the network game, the equilibrium strategy $\sigma^*$ is either all equal or FOSD decreasing in $k$;*
*(2) if (7) is nondecreasing in $k$ for any $\vec{\sigma}$, then in every symmetric Bayesian Nash equilibrium of the network game, the equilibrium strategy $\sigma^*$ is either all equal or FOSD increasing in $k$.*

The formal proof is in Appendix A.3, as all the proofs of the following results. Figure 1 provides a visual interpretation of the argument in the proof, which is classical and is simply based on best responses: From the payoff function (1), the optimal response for a player is when the marginal costs (increasing by assumptions) intersect with the expected marginal revenues (decreasing by assumption). So, if an increase/decrease in $k$ has a monotonic effect on these expected marginal revenues, also the intersection point will move monotonically.

Our second result is about a characterization of Bayesian Nash equilibria that is based on the definitions provided in the previous section. The following proposition describes the equilibrium strategies and their relation with the characteristics of the statistics $s$:

PROPOSITION 1. *Consider some network formation process P, then:*

*1a. if s is FOSD degree-increasing (degree-decreasing) with respect to P, then for any symmetric Bayesian Nash equilibrium $\sigma^*$: (i) if $f_{xs} > 0$, then $\sigma^*$ is FOSD nondecreasing (nonincreasing), if instead (ii) $f_{xs} < 0$, then $\sigma^*$ is FOSD nonincreasing (nondecreasing);*
*1b. and if s only weakly satisfies FOSD, then $\sigma^*$ could also be all equal;*
*2a. if s is degree-stable and satisfies SOSD with respect to P, then for any symmetric Bayesian Nash equilibrium $\sigma^*$, (i) if $f_{xss} > 0$, then $\sigma^*$ is FOSD decreasing, (ii) if $f_{xss} < 0$, then $\sigma^*$ is FOSD increasing, and (iii) if $f_{xss} = 0$, then $\sigma^*$ is all equal;*
*2b. if instead s only weakly satisfies SOSD, then, in cases (i) and (ii) above, $\sigma^*$ could also be all equal.*

The proofs of points (1a) and (1b) are analogous to the proof of Proposition 2 in NG, points (2a) and (2b) follow directly from a result that we prove in Lemma A.1 in Appendix A.2.

Points (1a) and (1b) show that when the statistic $s$ is increasing (decreasing) with respect to the degree, then the equilibrium strategies are increasing (decreasing) with respect to the degree in the case of strategic complements and decreasing (increasing) with respect to the degree in the case of strategic substitute. The statistic given by the sum of neighbors actions, under degree independence, satisfies point (1a). Examples of statistics that satisfy the properties in point (1b) under degree independence are, respectively, the maximum (as in *one shot games*, see, e.g., in NG and Boncinelli and Pin 2012) and the minimum (as in *weakest links* or in minimum effort games). These are also the cases discussed in Section 2 above.

Points (2a) and (2b) of Proposition 1 show that when the statistic $s$ is degree-stable and satisfies *SOSD*, equilibrium strategies do not depend on the characteristics of complementarity or substitutability of the utility function but on the sign of the third partial derivative $f_{xss}$. The equilibrium strategies are FOSD increasing (decreasing) when $f_{xss} < 0$ ($f_{xss} > 0$) irrespective of the utility function's properties of complementarity or substitutability. This result arises from the fact that the assumptions of degree-stability and *SOSD* for the statistics $s$, together with $f_{xss} < 0$ ($f_{xss} > 0$), imply that the expected marginal utility of action $x_i$ is increasing (decreasing) in $k$. Furthermore, we want to stress that this is not only a theoretical and abstract case but an effective one.

Indeed, under degree independence, the mean represents a case of statistic satisfying the assumptions in point (2b)[15] and, as described in Blume et al. (2010), it is largely used in the empirical literature on peer effects and reference groups.

4.2. *The Mean and All Equal Strategies.* When the statistics is the mean, it is directly verifiable that SOSD is only weakly satisfied. Then, from points (2a) and (2b) of Proposition 1, equilibria could be *all equal*, as well as FOSD increasing or decreasing, according to the sign of $f_{xss}$. Then it is of interest to understand the conditions under which *all equal* equilibria arise or are excluded. Because of this, we provide an example, similar to those in Section 2, where all equal strategy profiles cannot be equilibria.

[15] It is not difficult to check that, under degree independence, the mean satisfies SOSD. The argument is, however, more subtle than it appears at a first sight. Even if it is easy to see that if we move from the expected outcome of a node with degree $k$ to the expected outcome of a node with degree $k+1$, we maintain the mean and we decrease the variance in the distribution of the realized mean, what we obtain is only a necessary condition for SOSD (if this was a sufficient condition, we would have the inverse logical implication of point 2 at page 11, which is instead not necessarily true). The actual sufficient condition for SOSD, which is satisfied by the mean under degree independence, is that when passing from the distribution faced by a node of degree $k+1$ to the distribution faced by a node of degree $k$ it is as if we were doing a *mean preserving spread*, and it is well known that this induces an SOSD shift (see, e.g., Stiglitz and Rothschild 1970).

EXAMPLE 1. Suppose that $P$ has degree independence and that $s$ is the mean. Assume that $\mathcal{X} = \{0, 1\}$, that

$$f(x,s) = x\left(-\alpha s^2 - \beta s + 1\right),$$

and that $c(x) = c \cdot x$, with $\alpha + \beta > 1 - c > 0$. We also do not impose separate conditions on the sign of $\alpha$ and $\beta$, and in principle, this does not tell us whether $f$ has the complements or the substitute property, because $f_{xs} = -2\alpha s - \beta$ is not defined—but this is not what actually determines the characterization of the equilibrium.

First, under these conditions, we cannot have an all equal equilibrium where all players play pure strategy 0 or pure strategy 1.

Now, suppose that there is an equilibrium in which all players mix between 0 and 1. Equilibrium conditions require that players are all indifferent between the two actions. Then, the expected payoffs from the two actions are equal if, for all $k$ with positive support from $P$, we have $E_k(-\alpha s^2 - \beta s + 1 - c) = 0$, that rewritten is:

$$\alpha \mathrm{Var}_k(s) + \alpha (E_k(s))^2 + \beta E_k(s) = 1 - c.$$

However, as $k$ increases, $\mathrm{Var}_k(s)$ decreases, and then we cannot have all players being indifferent, independently on their degree $k$. So, we exclude the all equal equilibrium and points (1a) and (1b) of Proposition 1 tell us that we have a monotonic equilibrium that will be strictly increasing or decreasing depending on the sign of $f_{xss} = 2\alpha$.

In general, and not only when the statistic $s$ is the mean, one concern that arises from Lemma 1 is that it allows for equilibria where all agents play the same strategy, independently of their degree (all equal equilibrium). Here below, we exclude this case whenever the equilibrium is in mixed strategies. Note that the following proposition does not require degree independence of the network formation process $P$:

PROPOSITION 2. *All equal equilibria in mixed strategies cannot exist in the following cases:*

*(1) If $s$ is* weakly FOSD increasing *or* weakly FOSD decreasing *with respect to $P$*
*(2) if $s$ is (stable), weakly satisfies (SOSD) with respect to $P$ and either $f_{xss} > 0$ or $f_{xss} < 0$*

When the conditions stated in the previous proposition are satisfied, all equal equilibria in pure strategies can still exist. But a direct consequence of the previous result is that these equilibria are not robust to small perturbations as stated in the following corollary:

COROLLARY 3. *Under the condition of Proposition 2, all equal equilibria in pure strategies are not* trembling hand perfect

Now we consider the case in which agents' strategy is perturbed by errors, that is, they make, with some positive probability $\varepsilon$, random errors and with probability $1 - \varepsilon$ that they play a best response to the perturbed strategy. We refer to this situation like that where agents are affected by *bounded rationality*. The next corollary shows that when the nodes are affected by *bounded rationality*, then only FOSD increasing or decreasing equilibria exist. [16]

COROLLARY 4. *Consider some network formation process $P$, and suppose that agents are affected by* bounded rationality. *Then, in every symmetric Bayesian Nash equilibrium, the equilibrium strategy, $\sigma_k^*$, is FOSD increasing (decreasing) if:*

[16] In this perturbed environment, a Bayesian Nash equilibrium is a strategy profile $\sigma^*$ such that, for every $k$, $\sigma_k^*$ is a best response to the perturbed $\sigma^*$, that is, to a strategy profile $\sigma = (1 - \varepsilon)\sigma^* + \varepsilon\sigma_\varepsilon$ where $\sigma_\varepsilon$ is the strategy profile played in the case of a random error

*(1) s is stable, and weakly satisfies SOSD and* $f_{xss} < 0$ *(*$f_{xss} > 0$*);*
*(2) s is weakly FOSD increasing and* $f_{xs} > 0$ *(*$f_{xs} < 0$*); and*
*(3) s is weakly FOSD decreasing and* $f_{xs} < 0$ *(*$f_{xs} > 0$*)*

The proof is omitted as this corollary is a direct application of the result stated in Proposition 2 to the case of a perturbed environment. The previous corollary has direct empirical implications on the equilibria we expect to observe in the real world. Indeed, in the behavioral literature, there is large evidence that individuals are characterized by bounded rationality that induces them to make mistakes. For example, Costa-Gomes et al. (2001) and Sutter et al. (2013) calibrate that when those errors happen, actions are chosen from a uniform distribution. From the point of view of opponents, this case is analogous to having agents playing mixed strategies. So, the result of this corollary is a straight application of Proposition 2. Moreover, it implies that with real subjects, we mainly expect to observe equilibria that are either FOSD increasing or decreasing.

## 5. DISCUSSION OF THE RESULTS

Lemma 1 and Proposition 1 state the conditions that allow for either increasing or decreasing equilibria for a given network formation process $P$. Although Lemma 1 is very general and uses simple best response arguments to establish a sufficient condition under which all equilibria are monotone either increasing or decreasing, Proposition 1 provides more strict but easier to check sufficient conditions, focusing on the properties of the statistic $s$. Therefore, it follows that when statistics $s$ does not meet the conditions of Proposition 1 there is still room to meet the condition of Lemma 1.

To prove the examples in Section 2 we used the arguments described by Lemma 1, that is, we prove that the expected marginal revenue is either increasing or decreasing with the degree. Under degree independence Proposition 1 allows straightforward conclusions for the most common statistics. Indeed, under degree independence, the properties of the statistics are the same that under random sampling with the degree replacing the size of the sample. It is directly verifiable that, in Section 2 , the statistics used in scenarios 1 and 2 (the maximum and the minimum) satisfy Definitions 4 and 5, whereas the statistic used in scenario 3, the average, satisfies Definitions 3 and 6. Therefore the conclusions of the three scenarios can be derived by a straight application of Proposition 1.

We note that the conditions of Proposition 1 could be difficult to prove when the network process $P$ does not display degree independence. Using some argument of continuity we could assume that the equilibria in the case of degree independence could be robust to some (small) amount of either negative or positive degree correlation. The next example shows actually a case where this happens and Lemma 1 can be directly applied to check for this.

EXAMPLE 2. Consider $P$ such that networks are undirected, the nodes can have only degree 1 or 2, and they face ex ante symmetric probability $0 < p < 1$ of finding all neighbors of the same degree, and $1 - p$ of finding all neighbors of the other degree (so, when $p \to 1$, we have a network made almost only of circles and disconnected couples, and as $p$ decreases, we have more and more triplets of nodes in a line). Suppose that the nodes play symmetric pure strategies $y_1, y_2 \in \mathcal{X} \subseteq \mathbb{R}_+$, and that the statistic $s$ is the sum of neighbors' actions.

Lemma 1 tells us that we need to consider the relation between

$$E_1(f_x|x, \vec{\sigma}, P) = pf_x(x, y_1) + (1-p)f_x(x, y_2)$$

and

$$E_2(f_x|x, \vec{\sigma}, P) = pf_x(x, 2y_2) + (1-p)f_x(x, 2y_1).$$

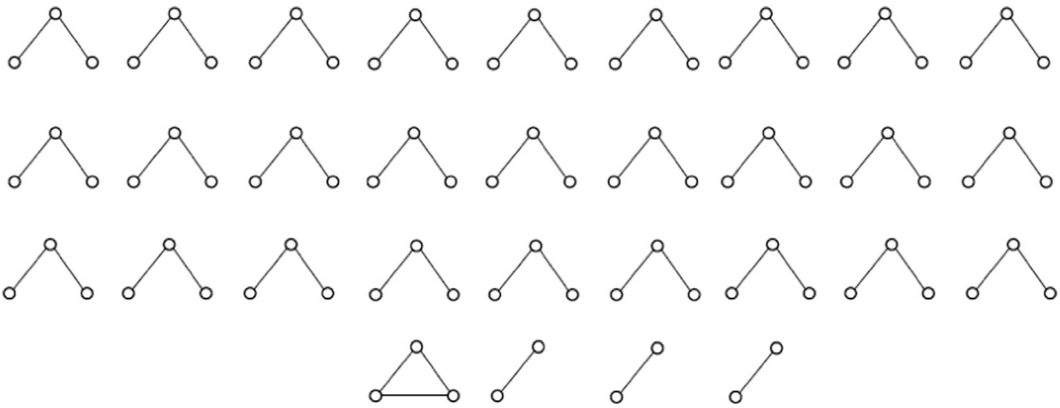

FIGURE 2

THE NETWORK CONSIDERED IN EXAMPLE 3

We note that $P$ displays degree independence when $p = 0.5$, then we assume that $1/3 < p < 2/3$, so that $2p > (1 - p)$ and $2(1 - p) > p$. Assume also that $f_x$ is increasing ($f_{xs} > 0$, complementarity between $x$ and $s$) and convex ($f_{xss} > 0$) in $s$, so that $f_x(x, 2s) - f_x(x, 0) > 2(f_x(x, s) - f_x(x, 0))$. Then, for any $y_1, y_2 \geq 0$, the following holds:

$$\begin{aligned} E_2(f_x|x, \vec{\sigma}, P) &= p(f_x(x, 2y_2) - f_x(x, 0)) + (1 - p)(f_x(x, 2y_1) - f_x(x, 0)) + f_x(x, 0) \\ &\geq 2p(f_x(x, y_2) - f_x(x, 0)) + 2(1 - p)(f_x(x, y_1) - f_x(x, 0)) + f_x(x, 0) \\ &> (1 - p)(f_x(x, y_2) - f_x(x, 0)) + p(f_x(x, y_1) - f_x(x, 0)) + f_x(x, 0) \\ &= E_1(f_x|x, \vec{\sigma}, P). \end{aligned}$$

So, according to Lemma 1, in every symmetric Bayesian Nash equilibrium of the network game, the optimal best responses are increasing, such that $x_1^* < x_2^*$. Note that this conclusion is valid for the interval $1/3 < p < 2/3$, that is, for degree independence as well as for some amount of degree correlation

Previous example suggests that in cases of strong degree correlation (either $p < \frac{1}{3}$ or $p > \frac{2}{3}$), the condition required by Lemma 1 could not be meet. So, a natural question is what happen in these cases. The next example, using a similar setup to the previous one, shows that a monotone equilibrium may still always exist, but both increasing and decreasing equilibria may coexist.

EXAMPLE 3. Consider the network process of Example 2 and assume that $p = 0.1$ so that it shows degree disassortativity. An example of this network is given in Figure 2 where an agent is assigned by uniform probability to any node and knows only its own degree and does not know in which of the disconnected components she is placed. The action space of the nodes is $\mathcal{X} = \{0, 1\}$, the statistic $s$ is just the sum of neighbors actions (a case of increasing statistic) and the payoff is

$$\Pi_{k_{i,g}}(x_i, s) = \frac{x_i}{1 + s} - c \cdot x_i,$$

with $c = 0.6$. This is a game of substitutes with negative externalities.

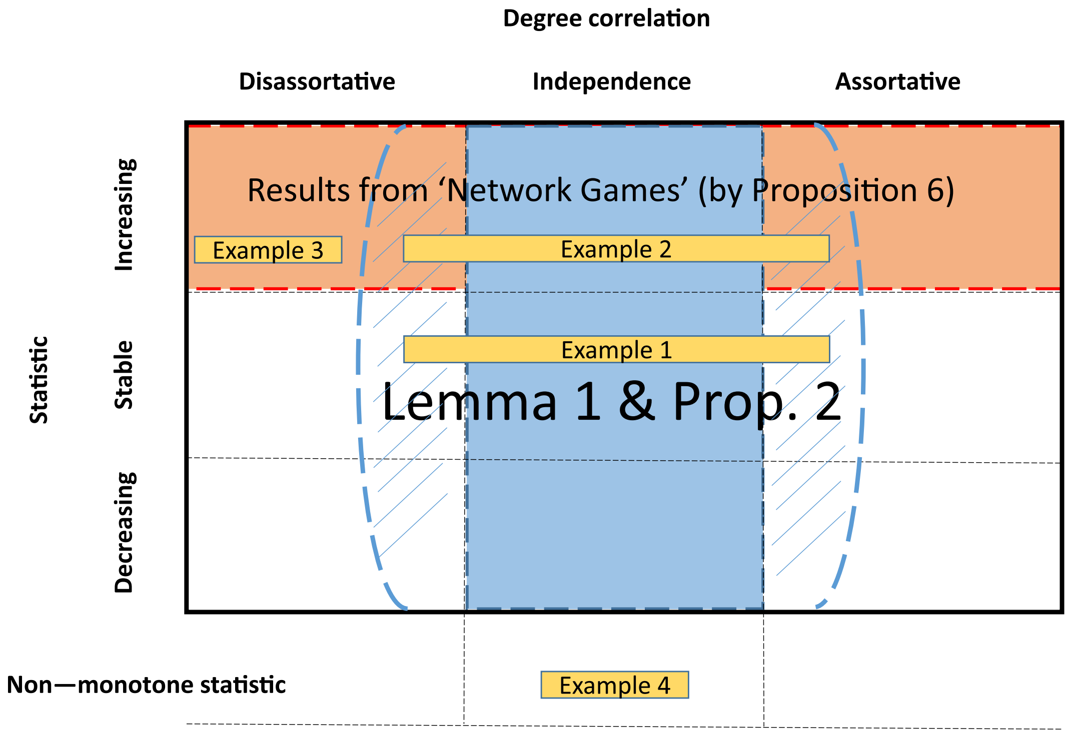


FIGURE 3

CLASSIFICATION OF THE STATISTIC (ROWS) ADDS A DIMENSION TO THE ANALYSIS OF NETWORK GAMES IN ADDITION TO THE DEGREE CORRELATION (COLUMNS), AND THE ONE ON COMPLEMENTS/SUBSTITUTES GAMES (NOT SHOWN HERE) [COLOR FIGURE CAN BE VIEWED AT WILEYONLINELIBRARY.COM]

This network game has the two following equilibria: a decreasing one in which degree-1 nodes play 1 and degree-2 nodes play 0, but also an increasing one in which degree-1 nodes play 0 and degree-2 nodes play 1.

The reason why the two equilibria coexist is that conditions 1 and 2 in Lemma 1 are not met. In detail, we observe that condition 1 is satisfied only for all $\vec{\sigma}$ that are decreasing, in the same way as condition 2 is satisfied only for all those $\vec{\sigma}$ that are increasing. However, statistic $s$ is not FOSD decreasing (or increasing) for all possible $\vec{\sigma}$.

5.1. *Relation to the Previous Literature.* Here, we discuss how the results provided in Lemma 1 and Proposition 1 generalize those of NG. First, our results depend on whatever *statistic* enters in the strategic interaction, and on the statistic's relation with the game structure. Second, the relation between monotonicity of equilibria and the strategic nature of the game (substitutability or complementarity) holds only when this statistic is naturally increasing or decreasing with the size of the sampling set of the players. Otherwise, we need to check for third cross-derivatives or for monotonicity of the expected marginal profits with respect to the degree.

So, the question here is: what are the analogies with NG? Most of the results in that paper (from their Proposition 2) are based on what they call Property A: the value of the payoff computed on a vector does not change when the vector size is increased by one, adding a null element. Then we can restate an analogous definition that fits our framework. The formal definition is easy if we remember that $s$ is actually a class of function from $\mathbb{R}^k$ to $\mathbb{R}$, for any $k \in \mathbb{N}$, and that these functions are all anonymous on the arguments.

DEFINITION 7. A statistic $s$ satisfies property A if for every $\vec{x} \in \mathbb{R}^k$ and $\vec{x}_{+0} = (0, \vec{x})' \in \mathbb{R}^{k+1}$, we have that $s(k, \vec{x}) = s(k+1, \vec{x}_{+0})$.

Common examples of statistics that satisfy property A are the sum or the maximum. Therefore, for these statistics, our results are coincident with those in NG. But our model covers all situations where the statistic does not satisfy property A as, for example, all sample moments (where the sample is represented by the set of actions of the neighbors). The easiest and more common example is the average.[17] The only restriction of our model is that statistic $s$ has to be increasing in its arguments. So, the natural question that arises is: what happens when this assumption is not satisfied? The following example shows that unexpected equilibria might arise when the statistic $s$ is not monotone in its arguments:

EXAMPLE 4. Consider the case in which $\mathcal{X} = \{0, 1\}$, with the statistic $s$ defined on every vector of at least two elements, as the difference between its two greatest elements. This statistic clearly satisfies Property A from NG, but it does not satisfy our definition of a statistic (given that is not monotone increasing in its arguments).[18] Since $\mathcal{X} = \{0, 1\}$, we have that $s$ is 1 if and only if there is one and only one element 1 in the vector. Otherwise it is 0. Consider the case of degree independence, so that the matching process is i.i.d.. So, if a fraction $p$ of the nodes plays 1, then the probability that $s$ is 1 is

$$p_k = k \cdot p(1-p)^{k-1},$$

which can be nonmonotonic in $k$. Imagine that the degree distribution is such that a fraction 0.15 of the nodes have degree 2, a fraction 0.7 have degree 3, and the remaining fraction 0.15 of nodes have degree 4. Payoff is $\Pi_{k_{i,g}}(x_i, s) = \sqrt{x+s} - c \cdot x_i$ (a case of substitutes) where $c = 0.75$.

In this case, there is an equilibrium in which the nodes with degree 2 and 4 contribute 1, whereas the nodes with degree 3 contribute 0. With this strategy profile $p = 0.3$, $p_2 = 0.42$, $p_3 = 0.441$, and $p_4 = 0.4116$. The expected net value of contributing is given by

$$\Delta_k = p_k\left(\sqrt{2} - 1\right) + (1 - p_k),$$

and this is above 0.75 for $k \in \{2, 4\}$, but not for $k = 3$, proving that this strategy profile is an equilibrium.

In the previous example, the statistic $s$ is not monotonically increasing. So, it produces a different ordering of vectors of the neighbors' actions with respect to the model in NG, where the definitions of *complements* and *substitutes* are implicitly based on the natural partial ordering between vectors.[19]

So, although according to our definition, the previous example is a case of strategic substitutes, it is undefined under the standard definition used in NG. Therefore, our definitions of strategic substitutes and strategic complements do not coincide with those of NG when the statistic does not respect the natural partial ordering of vectors. As we noted before, choosing a statistic $s$ that is monotonically increasing in all its arguments, that is, respecting the natural partial ordering,[20] the definitions of strategic substitutes/complements coincide in both our

[17] There are also other statistics that do not satisfy property A and that are not sample moments, for example, median, mode, and range.

[18] It also does not satisfy any of the definitions from Definitions 3–5.

[19] As an example, in the case of degree equal to 3, the ordering of the possible vectors of neighbors' actions (from the smaller to the larger) under statistics $s$ is: $(0,0,0) \sim (0,1,1) \sim (1,1,1) \prec (0,0,1)$. Using the criterion NG, the ordering would be: $(0,0,0) \prec (0,0,1) \prec (0,1,1) \prec (1,1,1)$.

[20] This naturally relates to standard utility theory and to the assumption of nonsatiation. Note that, except from Example 4, all our examples have statistics monotonically increasing in all their arguments.

framework and in the one of NG. Moreover, even if in NG's payoff function, there is not an explicit statistic $s$ but only a vector of the neighbors' actions, it is possible to check that the main results in NG are a specific case of our framework. Indeed, assuming that $s$ is increasing and satisfies property A, the resulting (compounded) payoff function has the same properties of the payoff function in NG.

The next proposition formalizes these considerations and provides a link between our formulation and the results from NG when the network displays degree independence.

PROPOSITION 5. *Suppose that* $\mathcal{X} \subseteq \mathbb{R}_+$, *and that* $0 \in \mathcal{X}$. *If* $s$ *satisfies Property A, then* $s$ *is FOSD increasing with respect to a network formation process* $P$ *with degree independence.*

Therefore, when $s$ satisfies property A and the network formation process $P$ is characterized by degree independence, then the characteristics of Bayesian Nash equilibria are described in point 1 of our Proposition 1. So, it is straightforward that all the results from NG that are consequences from their Proposition 2 are specific cases of our model.

5.2. *An Example on the Implications for Policy makers.* The following extended example is a good illustration of why our theoretical results could be of interest to policymakers. Imagine an authority facing a world of agents that decide endogenously upon a vaccination program. The decision of each agent $i$ is to vaccinate or not, and the policymaker observes $x_i \in \{0, 1\}$. The information that the policymaker has comes from a survey, where she observes how many interactions agents have and their choice. To keep things simple, imagine that agents can have degree 1, 2, or 3, with probabilities, respectively, $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$, and that the degree correlation is null.[21] The policymaker also observes only those agents with degree 2 or 3 adopt vaccination, whereas those with degree 1 do not. The policymaker also knows from the survey that the cost for vaccinating is $\frac{2}{3}$ for each agent. However, the policymaker does not know the statistic that affects the behavior of people and what is the function $f$ that relates own actions with peers' actions.

In fact, she is in doubt between two different models that are both consistent with the observed data. According to the first model, which is the same discussed in Section 2, the main force governing the choice of agents has the substitute property and is affected by the minimum of other's actions. So, $f(x, s)$ is equal to 1 if $x = 1$ and is equal to the minimum of peers actions (i.e., $s$) if $x = 0$. This first approach captures the typical free riding aspects of vaccinations.

The second model that the policymaker is considering explains people's behavior by a predominance of complementarity effects. In this case, people will tend to conform to what their peers are doing, either because there are peer effects in the payoffs, or because there is flow of information about the vaccination scheme.[22] In this second case, the statistic that people look at is the maximum, and $f(x, s)$ is equal to 0 if $x = 0$ and is equal to the maximum of peers actions (i.e., $s$) if $x = 1$.[23]

[21] Since the payoff comparisons that drive players decision are given by strict inequalities, even small perturbations to the probabilities of each agent to meet others agents, will not affect the outcome, as discussed in Example 2.

[22] Both types of externality effects are documented in the epidemiological literature that has analyzed individual incentives for vaccinating or not. For example, citing Bauch et al. (2003):

> History details numerous examples of vaccine refusal (Durbach, 2000; Streefland, 2001; Albert et al., 2001). Some of these examples embody the prisoner's dilemma effect. The grounds for refusal vary widely but often are related to perceived risks of vaccination.

More recently, Bodine-Baron et al. (2013) discuss how both complementarity and substitute effects could be at play in peer-based vaccination decisions. Rao et al. (2017) and Romley et al. (2016) provide evidence, respectively, for the substitute and for the complements effect.

[23] Note that since the strategy space $\mathcal{X}$ has only two elements, in both models, functions $f$ and $c$ trivially satisfy continuity and are at the same time concave and convex with respect to $\mathcal{X}$, as discussed in the model description in Section 3.

It is easy to check that for both these models what the policymaker observes is an equilibrium.

Now assume that she wants to implement a policy to maximize some objective function, as, for example, increasing the coverage of the vaccination scheme. Several problems arise if she is not able to identify which of the two models is at work. Imagine, in fact, that she believes that the second model is the right one. In this case of complements, there is also another equilibrium where everybody vaccinates and there is full coverage. So, the policymaker imagines that if she offers the vaccine to a share of the population of size $\frac{1}{5}$, among those not already vaccinated, she will obtain her desired result.

However, if she is wrong and the world is instead in the scenario of the first model, then the effect of her policy would be to (temporarily) disincentivize vaccination among all those that were originally vaccinated. Furthermore in this case (strategic substitutes), a policy can have only temporary effects as there exists only one equilibrium. This implies that to get a permanent higher rate of vaccination, it is necessary to implement a structural change, for example, a policy that changes the people perception of costs and benefits, and that produces a new (alternative) equilibrium (with higher rate of vaccination). By a straight computation, we can see that when the cost reduces to $\frac{1}{4}$ half of the agents with degree 1 vaccinate in (unique) equilibrium.

## 6. CONCLUSION

In many applications, externalities, peer effects, learning, and/or strategic interactions between individuals, can all be easily modeled as network games between agents of a social network. The neighbors of a node are in one to one correspondence with the peers of the individual, and the actions of those neighbors enter in that individual's payoff function. The existing literature addresses whether there is a complementarity or substitutability effect between own action and the statistic on the actions of neighbors, and points out the influence that the payoff function has on the correlation between the players' degree and the actions they take in equilibrium. Even if our model is a Bayesian game with ordered types and the existence of monotone equilibria derives from standard arguments (see, e.g., Milgrom and Shannon, 1994, Van Zandt and Vives, 2007, and Reny, 2011), there are some differences with this literature. In network games, the type (the degree) defines not only the agent's utility function, but also the number of interactions.

We note that the way in which the neighbors' actions affect the individual payoff can be channeled, for example, by the average, as in most of peer effects framings or non-Bayesian learning models. Individual payoff can also be the maximum of neighbors' actions, as in local public goods games, or the minimum, as in vaccination games that simulate the risk of pandemic contagion. More, in general, the neighbors' actions can affect some statistic that, in turn, affects the individual payoff. In this article, we have shown that to characterize the equilibria, it is also important to know the nature and the characteristics of the statistics aggregating the neighbors' action.

Figure 3 summarizes the classification that we apply and shows how our results integrate with those in the literature. Our Lemma 1 and Proposition 1 provide a characterization of the domain in which monotonic equilibria are the only possible ones, when the statistic is the mean, as in Example 1. or when there is some positive or negative degree correlation, as in Example 2. Example 3 shows instead that under too extreme correlation (a negative one in the example) both increasing and decreasing equilibria can coexist.[24] Finally, with Proposition 5, we have included all the results from NG in the domain of increasing statistics.

[24] It should be noted that Example 3 falls exactly within the assumptions of Proposition 4 in NG, which is about the existence of monotonic equilibria (decreasing in the case of the example), but does not claim that all equilibria are of that form.

Additional comments are the following: First, a statistic may not be stable or monotone, as in our Example 4. In such cases, equilibria that are nonmonotonic in the degree may exist. Finally, the case of a stable statistic is the most intuitive and is the most used in the empirical literature on peer effects and reference groups. In this case, we have shown that it is important to also look at third order derivatives of the payoff function. We have also shown that assuming some noise or a slight amount of bounded rationality provides a strict monotonic characterization of the equilibria (Corollary 4). However, we leave to future research the identification of necessary conditions that may be more easily identifiable in real data, and give a clearer economic interpretation.

Our results can be useful for empirical analysis. The monotonicity of equilibria provides theoretical restrictions on equilibria that can be used to identify the strategic nature of interactions (complementarity versus substitutability). Furthermore, these monotonicity results are derived between two characteristics that are easily observed and recorded (degree and action). Finally, we note that if only one of these two variable is observed, monotonicity can be used to infer the value of the other variable. For example, we can use it if we want to estimate the number of contacts of a subject in a context where we observe only her decisions (or actions). In our model agents differs only for the degree that they have in the network. An interesting extension of our model is a setting where agents differ also for some other observable/unobservable characteristics. In this new framework, it would be very useful for econometric analysis to find the conditions under which monotonicity results survive.

For these reasons, we believe that our results will turn out to be useful for both theorists who are studying specific models, and for applied researchers who are studying the interactions of economic agents.

## APPENDIX

A.1 *Proof for the Example in Section 2 Italy.* **Proof first and second scenario**: A strategy profile where all agents choose $a = 1$ is not an equilibrium. Suppose such strategy profile, each agent gets a payoff of $1 - c$. Any agent deviating to $a = 0$ can increase her payoff to 1. A strategy profile where all agents choose $a = 0$ is not an equilibrium. Suppose such strategy profile, each agent gets a payoff of 0. Any agent deviating to $a = 1$ can increase her payoff to $1 - c$. A strategy profile where agents with different degree play a mixed strategy cannot be an equilibrium. Suppose a strategy profile where agents with degree $k$ and $k'$ where $k \neq k'$ play a mixed strategy. The equilibrium conditions $1 - (1 - p_\sigma)^k = 1 - c$ and $1 - (1 - p_\sigma)^{k'} = 1 - c$ cannot be both satisfied because the left-hand side (LHS) is increasing in the degree and the right hard side is constant. Therefore, a strategy profile where all agents play the same strategy is not an equilibrium. It follows that in an equilibrium strategy profile, some agents of degree $k$ will play $a = 1$, and some agents of degree $k'$ will play $a = 0$, with the possibility that agents of a given degree $k''$ will play a mixed strategy. The equilibrium conditions are, respectively, $1 - (1 - p_\sigma)^k \leq 1 - c$, $1 - (1 - p_\sigma)^{k'} \geq 1 - c$ and $1 - (1 - p_\sigma)^{k''} = 1 - c$. Given that the LHS of the three conditions is increasing in the degree, in equilibrium has to be $k < k'' < k'$. This completes the proof. The proof of the second scenario uses similar arguments, and therefore is omitted.

**Proof Third Scenario**: Let $s_k$ be the average action of the neighbors of a node of degree $k$, $f(s_k)$ be a strictly concave function of $s_k$ and $p_\sigma$ be the probability that a randomly selected neighbor is choosing action 1. Now we claim that $E[f(s_k)] < E[f(s_{k+1})]$. This last inequality can be written as:

$$\sum_{n=0}^{k} \binom{k}{n} \cdot p_\sigma^n \cdot (1 - p_\sigma)^{k-n} \cdot f\left(\frac{n}{k}\right) < \sum_{n=0}^{k+1} \binom{k+1}{n} \cdot p_\sigma^n \cdot (1 - p_\sigma)^{k+1-n} \cdot f\left(\frac{n}{k+1}\right).$$

The LHS can be written as

$$\sum_{n=0}^{k} \binom{k}{n} \cdot (p_\sigma^{n+1} \cdot (1-p_\sigma)^{k-n} + p_\sigma^{n} \cdot (1-p_\sigma)^{k-n+1}) \cdot f\left(\frac{n}{k}\right),$$

and arranging the terms inside the summatory, this becomes

$$(1-p_\sigma)^{k+1} \cdot f(0) + \sum_{n=1}^{k} \left[ p_\sigma^{n} \cdot (1-p_\sigma)^{k-n-1} \cdot \left( \binom{k}{n-1} \cdot f\left(\frac{n-1}{k}\right) + \binom{k}{n} \cdot f\left(\frac{n}{k}\right)\right)\right] + p_\sigma^{k+1} \cdot f(1).$$

We replace by this expression the LHS of the inequality and we get:

$$\sum_{n=1}^{k} \left[ p_\sigma^{n} \cdot (1-p_\sigma)^{k-n-1} \cdot \left( \binom{k}{n-1} \cdot f\left(\frac{n-1}{k}\right) + \binom{k}{n} \cdot f\left(\frac{n}{k}\right)\right)\right] < \sum_{n=1}^{k} \binom{k+1}{n} \cdot p_\sigma^{n} \cdot (1-p_\sigma)^{k+1-n} \cdot f(\frac{n}{k+1}).$$

We note that $\binom{k}{n-1} \cdot \frac{n-1}{k} + \binom{k}{n} \cdot \frac{n}{k} = \frac{n}{k+1}$. Then, by the strict concavity of $f(\cdot)$, we can say that the $n$th element of the summatory on the left is smaller than the $n$-th element of the right summatory. This proves the claim. So, we can conclude that for all strategy profiles, the expected payoff from action 0 is increasing in the degree $k$ of the agent. Then, using similar arguments to those in the proof of the first scenario, we can state that an equilibrium cannot be constant (i.e., all players play the same strategy), and that every equilibrium is decreasing with respect to the agent degree $k$. The proof of the case of strictly convex function uses similar arguments, and therefore, is omitted

A.2 *Some Lemmas.* We extend the results from utility theory (see, e.g., section 4.2 in the notes from Levin 2006) to our context with the following corollary.

We define by $s$ a random variable distributed on $\mathbb{R}$ with density function $\varphi_k(s)$ that depends on $k \in \mathcal{K}$, and we call $\Phi_k(s)$ its cumulative function. Following Definition 6, we say that $s$ satisfies SOSD if for every $y \in \mathbb{R}$, we have:

$$\int_{-\infty}^{y} \Phi_{k+1}(x)\ dx \leq \int_{-\infty}^{y} \Phi_k(x)\ dx. \tag{A.1}$$

LEMMA A.1. *Let statistic s be stable and satisfying SOSD.*

*(1) If $u(\cdot)$ is a positive-valued concave function, then*

$$\int u(s) \cdot \varphi_{k+1}(s)\ ds \geq \int u(s) \cdot \varphi_k(s)\ ds. \tag{A.2}$$

*(2) If $u(\cdot)$ is a positive-valued convex function, then*

$$\int u(s) \cdot \varphi_{k+1}(s)\ ds \leq \int u(s) \cdot \varphi_k(s)\ ds. \tag{A.3}$$

PROOF. Let us start by assuming that $s$ is stable and satisfies SOSD, and that $u$ is positive valued and concave, that is, that $u > 0$ and $u_{ss} \leq 0$. Let us call $I(x) \equiv \int_{-\infty}^{x} \Phi_k(s)\, ds - \int_{-\infty}^{x} \Phi_{k+1}(s)\, ds$, which is nonnegative by inequality (A.1). Also, integrating by parts

$$\int_{-\infty}^{x} \Phi_k(s)\, ds = [s \cdot \Phi_k(s)]_{-\infty}^{x} - \int_{-\infty}^{x} s\, d\Phi_k(s).$$

Replacing into the expression for $I(x)$ and taking its limit to $\infty$, the stability of $s$ implies that

$$\lim_{x\to\infty} I(x) = \int_{-\infty}^{\infty} s\, d\Phi_{k+1}(s) - \int_{-\infty}^{\infty} s\, d\Phi_k(s) = 0. \tag{A.4}$$

Since $I(x)$ is nonnegative, also

$$-\int_{-\infty}^{\infty} u_{ss}(s) I(x) ds \geq 0.$$

Integrating by parts, expression (A.4) is equivalent to

$$[-u_s(s) \cdot I(x)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} u_s(s)\,(\Phi_k(s) - \Phi_{k+1}(s))\, ds \geq 0. \tag{A.5}$$

By (A.4), the first term is equal to 0. Then again integrating by parts, we get

$$[u(s)\,(\Phi_k(s) - \Phi_{k+1}(s))]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(s)\,(\varphi_k(s) - \varphi_{k+1}(s))\, ds \geq 0. \tag{A.6}$$

It is directly verifiable that the first term is equal to 0. Therefore, inequality (A.4) can be rewritten as:

$$-\int_{-\infty}^{\infty} u(s)\,(\varphi_k(s) - \varphi_{k+1}(s))\, ds \geq 0, \tag{A.7}$$

so $\int u(s)(\varphi_k(s) - \varphi_{k+1}(s))ds$ is non-positive, which proves the statement. With the same reasoning, the case in which $u$ is positive valued and convex leads to the reverse inequality. □

COROLLARY 6. *If statistic s is stable and satisfies SOSD, then* $\mathrm{Var}_{k+1}(s|\sigma, P) < \mathrm{Var}_k(s|\sigma, P)$.

PROOF. We have that

$$\mathrm{Var}_k(s|\sigma, P) = \int s^2 \cdot \varphi_k(s)\, ds - (E_k(s))^2.$$

So, when $s$ is stable, $(E(s))^2$ remains constant, and since $s^2$ is convex, we get the result from the previous Lemma A.1. □

LEMMA A.2. *If the statistic s is FOSD increasing, and* $u(\cdot)$ *is a positive valued nondecreasing (nonincreasing) function, then*

$$\int u(s) \cdot \varphi_{k+1}(s)\, ds \geq (\leq \text{, respectively})) \int u(s) \cdot \varphi_k(s))\, ds \tag{A.8}$$

*with strict inequality if $u(\cdot)$ is strictly increasing (decreasing). If the sample statistic s is FOSD decreasing, and $u(\cdot)$ is a positive nondecreasing (nonincreasing) function, then*

$$\int u(s)\cdot\varphi_{k+1}(s)\ ds \leq (\geq\ ,\ \textit{respectively}))\int u(s)\cdot\varphi_k(s))\ ds \tag{A.9}$$

*with strict inequality if $u(\cdot)$ is strictly increasing (decreasing).*

PROOF. Integrating by parts $\int_{-\infty}^{\infty} u_s(s)(\Phi_k(s)-\Phi_{k+1}(s))dy$. we have that:

$$\int_{-\infty}^{\infty} u_s(s)\,(\Phi_k(s)-\Phi_{k+1}(s))\,dy = -\int_{-\infty}^{\infty} u(s)\,(\varphi_k(s)-\varphi_{k+1}(s))\,ds$$

(see the proof of Lemma A.1, Equations (A.5) and (A.6)). So, a sufficient condition to determine the sign of

$$\int_{-\infty}^{\infty} u(s)\,(\varphi_{k+1}(s)-\varphi_k(s))\,ds$$

is the sign of the integral on theLHS. □

When statistic $s$ is FOSD increasing and $u(\cdot)$ is nondecreasing (nonincreasing), we have that $(\Phi_k(s)-\Phi_{k+1}(s)) \geq 0$ and $u_s(s)\geq 0$ ($u_s(s)\leq 0$) for every $s$ so the integral on the LHS is nonnegative (nonpositive). Moreover, if $u(\cdot)$ is strictly increasing (decreasing), we have that $u_s(s) > 0$ ($u_s(s) < 0$) for every $s$ so that the integral on the LHS is strictly positive (negative). The second part of the lemma is proved in a similar way and it is omitted

A.3 *Proof of the Propositions.* We first prove the technical results of Lemma 1, then we use it as a lemma to prove Proposition 1. Finally, we prove Proposition 5.

PROOF OF LEMMA 1 (PAGE 12). Suppose the quantity in (7) is nonincreasing in $k$. To compute $x_k^*$, we need to maximize the expectation of (1). Applying Leibniz's rule, the first-order conditions are:

$$E\left[\frac{\partial}{\partial x} f\,(x_k^*, s(\vec{x}_{i,g}))\right] = \frac{\partial}{\partial x}c(x_k^*),$$

or equivalently

$$\int_{-\infty}^{\infty} f_x\,(x_k^*, y)\cdot\varphi_k(y|\vec{\sigma}, P)\ dy = c_x(x_k^*).$$

Since $f_x$ and $c_x$ are both strictly positive, and they are both strictly monotone with different sign, there is a unique $x_k^* \in \mathbb{R}$ that satisfies the equality. If this $x_k^* \in \mathcal{X}$, then $\sigma_k^* = x_k^*$ is a pure strategy. However, this $x_k^*$ could not be an element of $\mathcal{X}$. In this last case, the optimal $\sigma_k^*$ should play one of the two (possibly both), leftmost $x_k^{*-}$ or rightmost $x_k^{*+}$, elements of $\mathcal{X}$ closest to $x_k^*$ in $\mathbb{R}$. If $x_k^{*-}$ and $x_k^{*+}$ give different expected payoffs, then $\sigma_k^*$ would be a pure strategy playing the best one of the two. Only in the case in which $x_k^{*-}$ and $x_k^{*+}$ give the same payoff, then any randomization $\sigma_k^*$ between these two points would be an optimal best response.

Using an equivalent argument, we can state that

$$\int_{-\infty}^{\infty} f_x\left(x_{k+1}^*, y\right)\cdot\varphi_{k+1}(y|\vec{\sigma}, P)\ dy = c_x(x_{k+1}^*).$$

Note that the two LHS of the FOCs are the expectation in (7) rewritten for agents with, respectively, degree $k$ and $k+1$. Given that expectation (7) is not increasing w.r.t. $k$, by assumption, it directly follows that $c_x(x^*_{k+1}) \leq c_x(x^*_k)$. By assumption on $c(x)$, since $x^*_{k+1}$ and $x^*_k$ are at the intersection of strictly monotone curves, it follows that $x^*_{k+1} \leq x^*_k$.

If $x^*_{k+1} = x^*_k$ for all $k$, then the strategy is *all equal*. If instead $x^*_{k+1} < x^*_k$ for some $k$, then there are four cases, depending on whether $x^*_{k+1}$ and $x^*_k$ are members of $x^*_k \notin \mathcal{X}$. Only if any of the two is not in $\mathcal{X}$ and the quantity in (7) is not locally strictly monotone, then we could have equality of best response strategies. This proves that if the quantity in (7) is nonincreasing in $k$, for every $x \in \mathcal{X}$, then in every symmetric Bayesian Nash equilibrium of the network game, the optimal action $\sigma^*$ is either all equal or FOSD nonincreasing in $k$. The reverse inequality can be proved analogously. □

PROOF OF PROPOSITION 1. Let s be FOSD increasing . If $f_{xs} > 0$ by Lemma A.2, we have that the quantity (7) is strictly increasing in $k$. Then, by Lemma 1, it directly follows that $\sigma^*$ is FOSD increasing or *all equal*. Note that as the quantity (7) is nondecreasing in $k$, FOCs require that $c_x(x^*_{k+1}) \geq c_x(x^*_k)$ that implies that $x^*_{k+1} \geq x^*_k$. If $c_{xx}(.) = 0$ again FOCs require $x^*_{k+1} \geq x^*_k$, because $f_x$ is not decreasing . This proves point (a). When $s$ is weakly FOSD increasing it is directly verifiable that quantity (7) is constant w.r.t. $k$ for *all equal* $\sigma$. Then FOCs require $x^*_{k+1} = x^*_k$. This proves point (b).

Point 2. Let s be stable and converging. The derivative with respect to $x_i$ of i's expected payoff $\frac{\partial}{\partial x_i}\Pi^e_k(x_i, \vec{\sigma}_{-i}) = \int_{-\infty}^{\infty} f_x(x, s) \cdot \varphi_k(s|\vec{\sigma}, P)\, ds$. If $f_{xss} > 0$ by Lemma A.1, we have that the derivative is decreasing in $k$. Then, by Lemma 1, it directly follows that $\sigma^*$ is FOSD decreasing. If $f_{xss} < 0$ by Lemma A.1, we have that the derivative is increasing in $k$. Then, by Lemma 1, it directly follows that $\sigma^*$ is FOSD increasing.

Finally, if $f_{xss} = 0$, again by Lemmas A.1 and 1, then $\sigma^*$ is stable in $k$. □

PROOF OF PROPOSITION 2. Assume an all equal strategy profile $\vec{\sigma}$. It can be represented by a linear combination of two nonequal strategy profiles, that is, $\vec{\sigma} = \frac{1}{2}\vec{\sigma}' + \frac{1}{2}\vec{\sigma}''$, where $\vec{\sigma}'$ and $\vec{\sigma}''$ are not all equal strategy profiles. Therefore, for an agent of degree $k$, the posterior distribution for the statistic $s$ will be:

$$\varphi_k(s|\vec{\sigma}, P) \equiv \frac{1}{2}\varphi_k(s|\vec{\sigma}', P) + \frac{1}{2}\varphi_k(s|\vec{\sigma}'', P),$$

and the derivative with respect to $x_i$ of i's expected payoff, $\int_{-\infty}^{\infty} f_x(x, s) \cdot \varphi_k(s|\vec{\sigma}, P)\, ds$, can be written as follows:

$$\frac{1}{2}\int_{-\infty}^{\infty} f_x(x, s) \cdot \varphi_k(s|\vec{\sigma}', P)\, ds + \frac{1}{2}\int_{-\infty}^{\infty} f_x(x, s) \cdot \varphi_k(s|\vec{\sigma}'', P)\, ds. \tag{A.10}$$

Part 1. Let $s$ be FOSD weakly increasing. Conditional on either $\vec{\sigma}'$ or $\vec{\sigma}''$ to be played, statistic $s$ is FOSD increasing. Then if $f_{xs} > 0$ by Lemma A.2, we know that both terms of (A.10) are increasing in $k$. By Lemma 1 directly follows that $\sigma^*$ is FOSD increasing in $k$. If $f_{xs} < 0$ by Lemma A.2 we know that both terms of (A.10) are decreasing in $k$. Then, by Lemma 1, it directly follows that $\sigma^*$ is FOSD decreasing. Following similar steps for the case of an FOSD weakly decreasing $s$ we can show that $\sigma^*$ is either FOSD increasing or FOSD decreasing for, respectively, $f_{xs} < 0$ and $f_{xs} > 0$. This eliminates the possibility to have an all equal equilibrium in mixed strategies

Part 2. Let $s$ be stable and weakly satisfying SOSD. Conditional on either $\vec{\sigma}'$ or $\vec{\sigma}''$ to be played statistic $s$ satisfies SOSD. If $f_{xss} > 0$ by Lemma A.1, we have that both terms of (A.10) are decreasing in $k$. Then, by Lemma 1, it directly follows that $\sigma^*$ is FOSD decreasing. If $f_{xss} < 0$, by Lemma A.1, we have both terms of (A.10) are increasing in $k$. Then, by Lemma 1, it directly follows that $\sigma^*$ is FOSD increasing. This eliminates the possibility to have an all equal equilibrium in mixed strategies □

PROOF OF COROLLARY 3. Let denote *a totally mixed strategy* a strategy profile in which every pure strategy is played with (eventually small) positive probability. Then following the proposition 8.F.1 in Mas-Collel et al. (1995) to prove this corollary, we need to show that an all equal equilibrium is never a best response to any sequence of *totally mixed strategy* converging to it. Therefore, it is enough to prove that an all equal equilibrium strategy never is a best response to any *totally mixed strategy* profile. Indeed, if the totally mixed strategy is not all equal, the proof of Proposition 1 shows as the expected marginal revenue is not constant with respect to $k$; otherwise, if the totally mixed strategy is all equal, the proof of proposition 2 shows that the expected marginal revenue is not constant with respect to $k$. These two results together with Lemma 1 are enough to state that an all equal strategy profile cannot be a best response to any totally mixed strategy for all $k$. □

Finally, we prove Proposition 5, that relates our result with those in NG.

PROOF OF PROPOSITION 5 (PAGE 22). For every $y$ in $\mathbb{R}$, we have that

$$\Phi[y|s,k] = Prob\left[\vec{x} \in \mathcal{X}^k : s(\vec{x}) \leq y\right],$$

and

$$\Phi[y|s,k+1] = Prob\left[\vec{x} \in \mathcal{X}^{k+1} : s(\vec{x}) \leq y\right].$$

Consider the operator $\sigma_0 : \mathcal{X}^{k+1} \to \mathcal{X}^{k+1}$ that takes a random element of $\vec{x}$ (with uniform probabilities) and puts it to 0. Then, also $E[s \circ \sigma_0(\cdot)]$ is a statistic (as it is anonymous), and by monotonicity of $s$, it is always the case that $s(\vec{x}) \geq E[s \circ \sigma_0(\vec{x})]$. Note also that it is probabilistically the same to extract with uniform probabilities $k$ elements, or to extract $k+1$ elements, and then, remove randomly one of them. So, we have that for every $y$

$$\Phi[y|s,k+1] \leq \Phi[y|E[s \circ \sigma_0(\cdot)], k+1] = \Phi[y|s,k],$$

which proves the statement. □

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