



# Comparing the degrees of enumerability and the closed Medvedev degrees

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Received: 26 March 2018 / Accepted: 15 September 2018  
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## Abstract

We compare the degrees of enumerability and the closed Medvedev degrees and find that many situations occur. There are nonzero closed degrees that do not bound nonzero degrees of enumerability, there are nonzero degrees of enumerability that do not bound nonzero closed degrees, and there are degrees that are nontrivially both degrees of enumerability and closed degrees. We also show that the compact degrees of enumerability exactly correspond to the cototal enumeration degrees.

**Keywords** Medvedev degrees · Enumeration degrees

**Mathematics Subject Classification** 03D30

## 1 Introduction

The purpose of this work is to explore the distribution of the so-called *degrees of enumerability* with respect to the closed degrees within the Medvedev degrees. Both the enumeration degrees and the Turing degrees embed into the Medvedev degrees. The Medvedev degrees corresponding to enumeration degrees are called *degrees of enumerability*, and the Medvedev degrees corresponding to Turing degrees are called

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The first author was supported by EPSRC Overseas Travel Grant No. EP/R006458/1.  
The second author is a member of INDAM-GNSAGA.

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*degrees of solvability*. The embedding of the Turing degrees into the Medvedev degrees is particularly nice. The degrees of solvability are all closed (being the degrees of singleton sets), and the collection of all degrees of solvability is definable in the Medvedev degrees. On the other hand, whether the degrees of enumerability are definable in the Medvedev degrees is a longstanding open question of Rogers [15,16].

In light of Roger’s question and the nice definability and topological properties of the degrees of solvability, we find it natural to investigate the behavior of the degrees of enumerability with respect to the closed degrees. Together, our main results show that the relation between the degrees of enumerability and the closed degrees is considerably more nuanced than the relation between the degrees of solvability and the closed degrees.

- There are nonzero closed degrees that do not bound nonzero degrees of enumerability. In fact, there are nonzero degrees that are closed, uncountable, and meet-irreducible that do not bound nonzero degrees of enumerability (Proposition 6).
- There are nonzero closed (indeed, compact) degrees of enumerability that do not bound nonzero degrees of solvability (Theorem 19). Moreover, the compact degrees of enumerability exactly correspond to the cototal enumeration degrees (Theorem 17).
- There are nonzero degrees of enumerability that do not bound nonzero closed degrees (Theorem 22).

We work in Baire space and interpret an arbitrary set  $\mathcal{A} \subseteq \omega^\omega$  as representing an abstract mathematical problem, namely the problem of finding (or, computing) a member of  $\mathcal{A}$ . For this reason, we refer to subsets of Baire space as *mass problems*. For sets  $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$ , we say that  $\mathcal{A}$  *Medvedev (or strongly) reduces to*  $\mathcal{B}$ , and we write  $\mathcal{A} \leq_s \mathcal{B}$ , if there is a Turing functional  $\Phi$  such that  $\Phi(\mathcal{B}) \subseteq \mathcal{A}$ , meaning that  $\Phi(f)$  is total and is in  $\mathcal{A}$  for every  $f \in \mathcal{B}$ . Under the interpretation of subsets of Baire space as mathematical problems,  $\mathcal{A} \leq_s \mathcal{B}$  means that problem  $\mathcal{B}$  is at least as hard as problem  $\mathcal{A}$  in a computational sense because every solution to problem  $\mathcal{B}$  can be converted into a solution to problem  $\mathcal{A}$  by a uniform computational procedure.

Medvedev reducibility induces an equivalence relation called *Medvedev (or strong) equivalence* in the usual way:  $\mathcal{A} \equiv_s \mathcal{B}$  if and only if  $\mathcal{A} \leq_s \mathcal{B}$  and  $\mathcal{B} \leq_s \mathcal{A}$ . The  $\equiv_s$ -equivalence class  $\text{deg}_s(\mathcal{A}) = \{\mathcal{B} : \mathcal{B} \equiv_s \mathcal{A}\}$  of a mass problem  $\mathcal{A}$  is called its *Medvedev (or strong) degree*, and the collection of all such equivalence classes, ordered by Medvedev reducibility, is a structure called the *Medvedev degrees*. The Medvedev degrees form a bounded distributive lattice (in fact, a Brouwer algebra), with least element  $\mathbf{0} = \{\mathcal{A} : \mathcal{A} \text{ has a recursive member}\}$  and greatest element  $\mathbf{1} = \{\emptyset\}$ . Joins and meets in the Medvedev degrees are computed as follows:

$$\begin{aligned} \text{deg}_s(\mathcal{A}) \vee \text{deg}_s(\mathcal{B}) &= \text{deg}_s(\mathcal{A} \oplus \mathcal{B}) \\ \text{deg}_s(\mathcal{A}) \wedge \text{deg}_s(\mathcal{B}) &= \text{deg}_s(0 \smallfrown \mathcal{A} \cup 1 \smallfrown \mathcal{B}). \end{aligned}$$

For joins,  $\mathcal{A} \oplus \mathcal{B} = \{f \oplus g : f \in \mathcal{A} \text{ and } g \in \mathcal{B}\}$ , where  $f \oplus g$  is the usual Turing join of  $f$  and  $g$ :  $(f \oplus g)(2n) = f(n)$  and  $(f \oplus g)(2n + 1) = g(n)$ . For meets,  $0 \smallfrown \mathcal{A} \cup 1 \smallfrown \mathcal{B}$

is the set obtained by prepending 0 to every function in  $\mathcal{A}$ , prepending 1 to every function in  $\mathcal{B}$ , and taking the union of the resulting sets.

Under the interpretation of mass problems as mathematical problems, problem  $\mathcal{A} \oplus \mathcal{B}$  corresponds to the problem of solving problem  $\mathcal{A}$  and solving problem  $\mathcal{B}$ , and problem  $0 \frown \mathcal{A} \cup 1 \frown \mathcal{B}$  corresponds to the problem of solving problem  $\mathcal{A}$  or solving problem  $\mathcal{B}$ . Medvedev introduced the structure that now bears his name in [11]. For an introduction to the Medvedev degrees, including its origins and motivation, see [16, Chapter 13.7]. For surveys on the Medvedev degrees and related topics, see [6,22]. For recursive aspects of the Medvedev degrees, see [5]. For algebraic aspects of the Medvedev degrees and applications to intermediate logics, see for instance [8,18,21].

**1.1 Notation**

We use the following notation and terminology regarding strings and trees. Denote by  $\omega^{<\omega}$  the set of all finite strings of natural numbers, and denote by  $2^{<\omega}$  the set of all finite binary strings. For  $\sigma \in \omega^{<\omega}$ ,  $|\sigma|$  denotes the length of  $\sigma$ . We denote the empty string by  $\emptyset$ . For  $\sigma, \tau \in \omega^{<\omega}$ ,  $\sigma \subseteq \tau$  means that  $\sigma$  is an initial segment of  $\tau$ , and  $\sigma \frown \tau$  denotes the concatenation of  $\sigma$  and  $\tau$ . Similarly, for  $\sigma \in \omega^{<\omega}$  and  $f \in \omega^\omega$ ,  $\sigma \subset f$  means that  $\sigma$  is an initial segment of  $f$ , i.e.,  $(\forall n < |\sigma|)(f(n) = \sigma(n))$ , and  $\sigma \frown f$  denotes the concatenation of  $\sigma$  and  $f$ :

$$(\sigma \frown f)(n) = \begin{cases} \sigma(n) & \text{if } n < |\sigma| \\ f(n - |\sigma|) & \text{if } n \geq |\sigma|. \end{cases}$$

If  $\sigma \in \omega^{<\omega}$  and  $f \in \omega^\omega$ ,  $\sigma \# f$  denotes the result of replacing the initial segment of  $f$  of length  $|\sigma|$  by  $\sigma$ :

$$(\sigma \# f)(n) = \begin{cases} \sigma(n) & \text{if } n < |\sigma| \\ f(n) & \text{if } n \geq |\sigma|. \end{cases}$$

For  $\sigma \in \omega^{<\omega}$  and  $\mathcal{A} \subseteq \omega^\omega$ , we define  $\sigma \frown \mathcal{A} = \{\sigma \frown f : f \in \mathcal{A}\}$  and  $\sigma \# \mathcal{A} = \{\sigma \# f : f \in \mathcal{A}\}$ . Finally, for  $\sigma \in \omega^{<\omega}$  and  $n \leq |\sigma|$ ,  $\sigma \upharpoonright n$  denotes the initial segment  $\langle \sigma(0), \dots, \sigma(n - 1) \rangle$  of  $\sigma$  of length  $n$ . Similarly, for  $f \in \omega^\omega$  and  $n \in \omega$ ,  $f \upharpoonright n$  denotes the initial segment  $\langle f(0), \dots, f(n - 1) \rangle$  of  $f$  of length  $n$ .

A tree is a set  $T \subseteq \omega^{<\omega}$  that is closed under initial segments:  $(\forall \sigma, \tau \in \omega^{<\omega})(\sigma \subseteq \tau \text{ and } \tau \in T) \rightarrow \sigma \in T$ . A node  $\sigma$  in a tree  $T$  is a leaf if there is no  $\tau \supset \sigma$  with  $\tau \in T$ . A tree  $T$  is finitely branching if for every  $\sigma \in T$  there are at most finitely many strings  $\tau \in T$  with  $|\tau| = |\sigma| + 1$ . A string  $\sigma \in \omega^{<\omega}$  is bounded by an  $h \in \omega^\omega$  (or  $h$ -bounded) if  $(\forall n < |\sigma|)(\sigma(n) < h(n))$ . Likewise, a tree  $T$  is  $h$ -bounded if  $(\forall \sigma \in T)(\sigma \text{ is } h\text{-bounded})$ . For  $b \in \omega$ ,  $b$ -bounded means bounded by the function that is constantly  $b$ . An  $f \in \omega^\omega$  is an infinite path through a tree  $T$  if  $\forall n(f \upharpoonright n \in T)$ . The subset of Baire space consisting of all infinite paths through a tree  $T$  is denoted by  $[T]$ . The closed subsets of Baire space are exactly those of the form  $[T]$  for a tree  $T$ ,

and the compact subsets of Baire space are exactly those of the form  $[T]$  for a finitely branching tree  $T$ .

Throughout, we refer to a standard listing  $(\Phi_e : e \in \omega)$  of all Turing functionals on Baire space. If  $\Phi$  is a Turing functional and  $\sigma$  is a finite string of natural numbers, then  $\Phi(\sigma)$  denotes the longest string  $\tau$  such that  $(\forall m < |\tau|)(\tau(m) = \Phi(\sigma)(m) \downarrow)$ . We also refer to a standard listing  $(\Psi_e : e \in \omega)$  of all enumeration operators. If  $\Psi$  is an enumeration operator and  $\phi$  is a partial function, then  $\Psi(\phi)$  stands for  $\Psi(\text{graph}(\phi))$ . Recall that  $\langle \cdot, \cdot \rangle : \omega^2 \rightarrow \omega$  is the usual recursive Cantor pairing function and that  $\text{graph}(\phi) = \{\langle n, y \rangle : n \in \text{dom}(\phi) \text{ and } \phi(n) = y\}$ .

For further background concerning recursion theory, trees, and the topology of Baire space, we refer the reader to standard textbooks such as [14,16].

## 1.2 Degrees of solvability and degrees of enumerability

As discussed in the introduction, part of the interest in the Medvedev degrees comes from the fact that the structure embeds both the Turing degrees and the enumeration degrees. Singleton subsets of Baire space are called *problems of solvability*, and their corresponding Medvedev degrees are called *degrees of solvability*. It is easy to see that the assignment  $\text{deg}_T(f) \mapsto \text{deg}_s(\{f\})$  embeds the Turing degrees into the Medvedev degrees, preserving joins and the least element, and that the range of this embedding is exactly the degrees of solvability. Moreover, the degrees of solvability are definable in the Medvedev degrees [5,11] (see also [16,22]).

To embed the enumeration degrees into the Medvedev degrees, given a nonempty  $A \subseteq \omega$ , let

$$\mathcal{E}_A = \{f : \text{ran}(f) = A\}.$$

$\mathcal{E}_A$  is called the *problem of enumerability of A*, and it represents the problem of enumerating the set  $A$ . The corresponding Medvedev degree  $\mathbf{E}_A = \text{deg}_s(\mathcal{E}_A)$  is called the *degree of enumerability of A*. For nonempty  $A, B \subseteq \omega$ , it is easy to see that  $A \leq_e B$  if and only if  $\mathcal{E}_A \leq_s \mathcal{E}_B$ . This gives rise to an embedding  $\text{deg}_e(A) \mapsto \mathbf{E}_A$  of the enumeration degrees into the Medvedev degrees. The embedding preserves joins and the least element, and the range of the embedding is exactly the degrees of enumerability [11] (see also [16,22]). Again we mention that, contrary to definability of the degrees of solvability, it is still an open question (see Rogers [15,16]) whether the degrees of enumerability are definable, or at least invariant under automorphisms, in the Medvedev degrees.

The following lemma (which we state and prove for later reference) is well-known. It corresponds to the fact that the Turing degrees embed (again via an embedding that preserves joins and the least element) into the enumeration degrees of total functions.

**Lemma 1** *If  $f : \omega \rightarrow \omega$  is total, then  $\mathcal{E}_{\text{graph}(f)} \equiv_s \{f\}$ .*

**Proof** Clearly  $\mathcal{E}_{\text{graph}(f)} \leq_s \{f\}$  via the Turing functional  $\Phi(f)(n) = \langle n, f(n) \rangle$ . To see that  $\{f\} \leq_s \mathcal{E}_{\text{graph}(f)}$ , let  $\Gamma$  be the Turing functional such that, for every total  $g : \omega \rightarrow \omega$  and  $n \in \omega$ ,  $\Gamma(g)(n)$  searches for the least  $k$  such that  $g(k) = \langle n, y \rangle$  for some  $y$ , and outputs  $y$ . Then  $\Gamma(g) = f$  whenever  $\text{ran}(g) = \text{graph}(f)$ , so  $\{f\} \leq_s \mathcal{E}_{\text{graph}(f)}$ .  $\square$

In analogy with the common terminology used in the enumeration degrees, we say that a problem of enumerability  $\mathcal{E}$  is *total* if  $\mathcal{E} \equiv_s \{f\}$  for some total  $f$ . That is, a problem of enumerability is total if it is Medvedev-equivalent to a problem of solvability. Likewise, we say that a degree of enumerability is *total* if it is the Medvedev degree of a total problem of enumerability. Now recall that an  $A \subseteq \omega$  is *quasiminimal* if  $A$  is not r.e. and there is no nonrecursive total  $f$  with  $f \leq_e A$  (meaning, as usual, that there is no nonrecursive total  $f$  with  $\text{graph}(f) \leq_e A$ ). We say that a problem of enumerability  $\mathcal{E}$  is *quasiminimal* if  $\mathcal{E} \equiv_s \mathcal{E}_A$  for a quasiminimal  $A$ . Likewise, we say that a degree of enumerability is *quasiminimal* if it is the Medvedev degree of a quasiminimal problem of enumerability. Lemma 1 implies the following lemma.

**Lemma 2** *If  $\mathcal{E}_A$  is a quasiminimal problem of enumerability, then  $\mathbf{0} <_s \mathbf{E}_A$  and  $\mathcal{E}_A \not\equiv_s \{f\}$  for every total  $f$  (in fact  $\{f\} \not\leq_s \mathcal{E}_A$  for every nonrecursive total  $f$ ).*

Both the degrees of solvability and the degrees of enumerability enjoy the algebraic property of *meet-irreducibility*. Recall that an element  $a$  of a lattice  $L$  is called *meet-reducible* if it is the meet of a pair of strictly larger elements:  $(\exists b, c \in L)(b > a \text{ and } c > a \text{ and } a = b \wedge c)$ . An element of a lattice is called *meet-irreducible* if it is not meet-reducible. It is well-known that, in a distributive lattice  $L$  such as the Medvedev degrees, an element  $a$  is meet-irreducible if and only if  $(\forall b, c \in L)((a \geq b \wedge c) \rightarrow (a \geq b \text{ or } a \geq c))$  (see [2, Section III.2]).

We now recall some helpful terminology and a lemma before proving that the degrees of solvability and the degrees of enumerability are meet-irreducible. These facts are known in the literature, but we include proofs for the sake of completeness. For a mass problem  $\mathcal{A}$  and a  $\sigma \in \omega^{<\omega}$ , let  $\mathcal{A}_\sigma = \{f \in \mathcal{A} : \sigma \subset f\}$ . Call a mass problem  $\mathcal{A}$  *uniform* if  $\mathcal{A}_\sigma \leq_s \mathcal{A}$  whenever  $\sigma \in \omega^{<\omega}$  is such that  $\sigma \subset f$  for some  $f \in \mathcal{A}$ .

**Lemma 3** [5, Corollary 2.8] *Every uniform mass problem has meet-irreducible Medvedev degree.*

**Proof** Suppose that  $\mathcal{A}$  is a uniform mass problem and that  $\mathcal{B}$  and  $\mathcal{C}$  are arbitrary mass problems such that  $0 \wedge \mathcal{B} \cup 1 \wedge \mathcal{C} \leq_s \mathcal{A}$ . We may assume that  $\mathcal{A} \neq \emptyset$  as clearly  $\mathbf{1}$  is meet-irreducible. Let  $\Phi$  be such that  $\Phi(\mathcal{A}) \subseteq 0 \wedge \mathcal{B} \cup 1 \wedge \mathcal{C}$ . Choose any  $f \in \mathcal{A}$ , and let  $\sigma \subset f$  be such that  $\Phi(\sigma)(0) \downarrow$ . Let  $b = \Phi(\sigma)(0)$ , and observe that  $b \in \{0, 1\}$ . Suppose for the sake of argument that  $b = 0$ . Then, as every  $f \in \mathcal{A}_\sigma$  begins with  $\sigma$  and is in  $\mathcal{A}$ , we have that  $\Phi(f)(0) = 0$  for every  $f \in \mathcal{A}_\sigma$ , thus yielding  $\mathcal{B} \leq_s 0 \wedge \mathcal{B} \leq_s \mathcal{A}_\sigma \leq_s \mathcal{A}$ . Similarly, if  $b = 1$ , then  $\mathcal{C} \leq_s \mathcal{A}$ . Thus either  $\mathcal{B} \leq_s \mathcal{A}$  or  $\mathcal{C} \leq_s \mathcal{A}$ . So  $\mathcal{A}$  has meet-irreducible degree. □

**Proposition 4** [11,20] *In the Medvedev degrees, every degree of solvability is meet-irreducible, and every degree of enumerability is meet-irreducible.*

**Proof** It suffices to prove that the degrees of enumerability are meet-irreducible (see [20, Theorem 4.5]) because every degree of solvability is also a degree of enumerability. (It is also easy to simply observe that if  $0 \wedge \mathcal{B} \cup 1 \wedge \mathcal{C} \leq_s \{f\}$ , then either  $\mathcal{B} \leq_s \{f\}$  or  $\mathcal{C} \leq_s \{f\}$ .)

Let  $\mathbf{E}_A$  be the degree of enumerability of  $A \subseteq \omega$ . The proposition follows from Lemma 3, as it is easy to see that  $\mathcal{A} = \mathcal{E}_A$  is uniform: if  $\sigma \subset f$  for some  $f \in \mathcal{E}_A$ , just consider the reduction  $\mathcal{A}_\sigma \leq_s \mathcal{A}$  given by  $\Phi(f) = \sigma \hat{\ } f$ .  $\square$

In a similar spirit, Dymont proved that if  $\mathcal{B}$  is a countable (or finite) mass problem, if  $\mathcal{E}_A$  is the problem of enumerability of  $A \subseteq \omega$ , and if  $\mathcal{B} \leq_s \mathcal{E}_A$ , then there is a  $g \in \mathcal{B}$  such that  $g \leq_c A$  [5, Theorem 3.4]. Call a Medvedev degree *countable* if it is the degree of a countable (or finite) mass problem, and call it *uncountable* otherwise. Dymont’s result implies that if  $\mathbf{E}$  is a nontotal degree of enumerability, then  $\mathbf{E}$  is uncountable [5, Corollary 3.14].

## 2 Comparing degrees of enumerability and closed degrees

A Medvedev degree is called *closed* if it is of the form  $\text{deg}_s(\mathcal{C})$  for a closed  $\mathcal{C} \subseteq \omega^\omega$ . Every degree of solvability is closed because singletons are closed. Thus there are closed degrees of enumerability because every degree of solvability is also a degree of enumerability. It is, however, easy to produce examples of closed degrees that are not degrees of enumerability. Let  $f, g \in \omega^\omega$  be such that  $f \upharpoonright_T g$ . Then  $\text{deg}_s(\{f, g\})$  is closed, but it is not a degree of enumerability because it is meet-reducible (as  $\text{deg}_s(\{f, g\}) = \text{deg}_s(\{f\}) \wedge \text{deg}_s(\{g\})$ ), whereas all degrees of enumerability are meet-irreducible by Proposition 4. In fact, by the discussion following Proposition 4, we know that a degree of enumerability must be meet-irreducible and either total (i.e., a degree of solvability) or uncountable. This begs the question of whether there are Medvedev degrees that are closed, meet-irreducible, and uncountable, yet not degrees of enumerability. We show that the Medvedev degree of the  $\{0, 1\}$ -valued diagonally nonrecursive functions is such a degree.

Recall that  $f \in \omega^\omega$  is *diagonally nonrecursive* (DNR for short) if  $\forall e(\Phi_e(e) \downarrow \rightarrow f(e) \neq \Phi_e(e))$ . Let  $\text{DNR}_2 = \{f \in 2^\omega : f \text{ is DNR}\}$ .

**Lemma 5** *Let  $T \subseteq \omega^{<\omega}$  be an infinite  $h$ -bounded tree for some  $h \in \omega^\omega$ . If  $A \subseteq \omega$  is such that  $\mathcal{E}_A \leq_s [T]$ , then  $A$  is r.e. in  $T \oplus h$ .*

**Proof** Let  $\Phi$  be such that  $\Phi([T]) \subseteq \mathcal{E}_A$ . Using  $T \oplus h$  as an oracle, enumerate the set

$$B = \{n : \exists k(\forall h\text{-bounded } \sigma \text{ with } |\sigma| = k)(\sigma \in T \rightarrow n \in \text{ran}(\Phi(\sigma)))\}.$$

We show that  $B = A$ , thus showing that  $A$  is r.e. in  $T \oplus h$ .

Suppose that  $n \in B$ . Let  $k$  be such that  $n \in \text{ran}(\Phi(\sigma))$  whenever  $\sigma \in T$  has length  $k$ . Let  $f \in [T]$ . Then  $f \upharpoonright k \in T$ , so  $n \in \text{ran}(\Phi(f \upharpoonright k))$ . However,  $\text{ran}(\Phi(f)) = A$  because  $\Phi(f) \in \mathcal{E}_A$ , so it must be that  $n \in A$ . Hence  $B \subseteq A$ .

Now suppose that  $n \notin B$ . Then for every  $k$  there is an  $h$ -bounded  $\sigma$  of length  $k$  with  $\sigma \in T$  but  $n \notin \text{ran}(\Phi(\sigma))$ . So the subtree  $S \subseteq T$  given by  $S = \{\sigma \in T : n \notin \text{ran}(\Phi(\sigma))\}$  is infinite. By König’s lemma, there is a path  $f \in [S] \subseteq [T]$ . However,  $n \notin \text{ran}(\Phi(f)) = A$ , giving  $n \notin A$  as desired.  $\square$

In the next proposition, our proof that  $\text{deg}_s(\text{DNR}_2)$  is uncountable relies on the following fact. If  $\mathcal{A} \subseteq 2^\omega$  is a nonempty  $\Pi_1^0$  class with no recursive member and  $\mathcal{B}$  is

a countable mass problem with no recursive member, then  $\mathcal{B} \not\leq_s \mathcal{A}$ . This fact follows immediately from [7, Theorem 2.5], which essentially states that such an  $\mathcal{A}$  must in fact have continuum-many members that are all pairwise Turing incomparable and also all Turing incomparable with all members of  $\mathcal{B}$ . So in fact  $\mathcal{B} \not\leq_w \mathcal{A}$ , where  $\leq_w$  is *Muchnik reducibility*:  $\mathcal{X} \leq_w \mathcal{Y}$  if  $(\forall g \in \mathcal{Y})(\exists f \in \mathcal{X})(f \leq_T g)$ . That  $\mathcal{B} \not\leq_s \mathcal{A}$  can also be deduced from the well-known fact that the image of a recursively bounded  $\Pi_1^0$  class under a Turing functional is another recursively bounded  $\Pi_1^0$  class (see [17, Theorem 4.7]), which is easier to prove than [7, Theorem 2.5]. Suppose for a contradiction that  $\mathcal{B} \leq_s \mathcal{A}$  via the Turing functional  $\Phi$ . Then  $\mathcal{B}_0 = \Phi(\mathcal{A}) \subseteq \mathcal{B}$  is a countable recursively bounded  $\Pi_1^0$  class and therefore must have a recursive member, contradicting that  $\mathcal{B}$  has no recursive member. This argument can also be used to show that  $\mathcal{B} \not\leq_w \mathcal{A}$  because if  $\mathcal{B} \leq_w \mathcal{A}$ , then (see [17, Lemma 6.9]) there is a nonempty  $\Pi_1^0$  class  $\mathcal{A}_0 \subseteq \mathcal{A}$  such that  $\mathcal{B} \leq_s \mathcal{A}_0$ , and then the argument can be repeated with  $\mathcal{A}_0$  in place of  $\mathcal{A}$ .

**Proposition 6** *The Medvedev degree  $\text{deg}_s(\text{DNR}_2)$  is closed, meet-irreducible, and uncountable, yet also not a degree of enumerability (in fact, it does not bound any nonzero degree of enumerability).*

**Proof** It is well-known that  $\text{DNR}_2$  is a  $\Pi_1^0$  class because  $\text{DNR}_2 = [T]$  for the recursive tree

$$T = \{\sigma \in 2^{<\omega} : (\forall e < |\sigma|)(\Phi_e(e) \text{ halts within } |\sigma| \text{ steps} \rightarrow \sigma(e) \neq \Phi_e(e))\}.$$

By the above discussion, if  $\mathcal{B}$  is a countable mass problem with no recursive member, then  $\mathcal{B} \not\leq_s \text{DNR}_2$ . Hence  $\text{deg}_s(\text{DNR}_2)$  is uncountable. That  $\text{deg}_s(\text{DNR}_2)$  is meet-irreducible follows from Lemma 3, as it is easy to see that  $\mathcal{A} = \text{DNR}_2$  is uniform. If  $\sigma \subset f$  for an  $f$  in  $\text{DNR}_2$ , consider the reduction procedure  $\mathcal{A}_\sigma \leq_s \mathcal{A}$  given by  $\Phi(f) = \sigma \# f$ .

That  $\text{deg}_s(\text{DNR}_2)$  is not a degree of enumerability follows from Lemma 5. We know that  $\text{DNR}_2 = [T]$  for a recursive tree  $T \subseteq 2^{<\omega}$ . Thus if  $\mathcal{E}_A \leq_s \text{DNR}_2$  for some  $A \subseteq \omega$ , then  $A$  would have to be r.e. by Lemma 5. However, if  $A$  is r.e., then  $\mathcal{E}_A$  would have a recursive member, in which case  $\text{DNR}_2 \not\leq_s \mathcal{E}_A$ . Thus there is no  $A$  such that  $\text{DNR}_2 \equiv_s \mathcal{E}_A$ . In fact  $\text{DNR}_2$  does not bound any nonzero degree of enumerability.  $\square$

If  $\mathbf{E}_A \neq \mathbf{0}$  and  $\mathbf{E}_A$  is not quasiminimal, then there are nonrecursive functions  $f$  such that  $\{f\} \leq_s \mathcal{E}_A$ , so  $\mathbf{E}_A$  bounds some nonzero closed degree. As observed in [9], there are also quasiminimal degrees of enumerability  $\mathbf{E}_A$  that bound nonzero closed degrees. Given an infinite set  $A$ , consider the mass problem

$$\mathcal{C}_A = \{f : f \text{ is one-to-one and } \text{ran}(f) \subseteq A\}.$$

As observed in [3],  $\mathcal{C}_A$  is closed,  $\text{deg}_s(\mathcal{C}_A) \leq_s \mathbf{E}_A$ , and, if  $A$  is immune (meaning that  $A$  has no infinite r.e. subset), then  $\text{deg}_s(\mathcal{C}_A) \neq \mathbf{0}$ . So, if  $A$  is immune and of quasiminimal e-degree (which is the case, for instance, if  $A$  is a 1-generic set, see [4]), then we have a quasiminimal degree of enumerability which bounds a nonzero closed degree. On the other hand, if  $A$  contains an infinite set  $B$  such that  $A \not\leq_e B$ , then

$\mathcal{E}_A \not\leq_s \text{deg}_s(\mathcal{C}_A)$  because in this case  $\mathcal{C}_A \leq_s \mathcal{E}_B$  (as  $\mathcal{C}_B \subseteq \mathcal{C}_A$ ) but  $\mathcal{E}_A \not\leq_s \mathcal{E}_B$ . This gives examples of sets  $A$ , even of total e-degree, for which  $\mathbf{0} <_s \text{deg}_s(\mathcal{C}_A) <_s \mathbf{E}_A$ .

**Proposition 7** *There is a total  $f: \omega \rightarrow \omega$  such that  $\text{deg}_s(\mathcal{C}_{\text{graph}(f)}) \neq \mathbf{0}$  and  $\mathcal{C}_{\text{graph}(f)} <_s \mathcal{E}_{\text{graph}(f)}$ .*

**Proof** By the above remarks and by Lemma 1, consider two biimmune sets  $A, B$  with  $A \upharpoonright_T B$ , and let  $f = \chi_A \oplus \chi_B$  (where  $\chi_Z$  denotes the characteristic function of  $Z$ ). Then  $\text{graph}(f)$  is immune, and it contains an infinite subset (for instance  $\{ \langle 2x, f(2x) \rangle : x \in \omega \}$ ) to which it does not Turing-reduce, and hence, by totality, to which it does not e-reduce.  $\square$

However, if  $f$  is total, then  $\mathcal{C}_{\text{graph}(f)} \equiv_s \mathcal{E}_{\text{graph}(f)}$  is almost true, as argued in the following proposition.

**Proposition 8** *If  $f: \omega \rightarrow \omega$  is total, then there is a set  $B \equiv_e \text{graph}(f)$  such that  $\mathcal{C}_B \equiv_s \mathcal{E}_{\text{graph}(f)}$ .*

**Proof** Given  $f$  total, let  $B = \{ \sigma \in \omega^{<\omega} : \sigma \subset f \}$ . It is easy to see that  $f \equiv_e B$ , so  $\mathcal{E}_{\text{graph}(f)} \equiv_s \mathcal{E}_B$ . To see that  $\mathcal{E}_B \leq_s \mathcal{C}_B$ , let  $\Phi$  be a Turing functional such that, for every  $g$  and  $n$ ,  $\Phi(g)(n)$  searches for an  $m$  such that  $g(m)$  is a string  $\sigma$  with  $|\sigma| \geq n$  and then outputs  $\sigma \upharpoonright n$ . Then  $\text{ran}(g)$  is an infinite subset of  $B$  whenever  $g \in \mathcal{C}_B$ , in which case  $\text{ran}(\Phi(g)) = B$ . Hence  $\Phi$  witnesses that  $\mathcal{E}_B \leq_s \mathcal{C}_B$ .  $\square$

While it is true that every total degree of enumerability bounds (and in fact is equivalent to) a closed mass problem, if we move away from totality, then all possibilities may occur. That is, there are nontotal (in fact quasiminimal) degrees of enumerability that are closed (in fact compact, see Theorem 19 below), and there are nonzero degrees of enumerability that do not bound nonzero closed degrees (see Theorem 22 below).

### 2.1 Compactness and cototality

We make use of *uniformly e-pointed trees*. This notion was originally introduced by Montalbán [13] in the context of computable structure theory (see also [12]), and it has since been studied by McCarthy in the context of the enumeration degrees [10]. Montalbán’s uniformly e-pointed trees are subtrees of  $2^{<\omega}$ , which we refer to as *uniformly e-pointed trees w.r.t. sets*. We find it convenient to work with finitely branching subtrees of  $\omega^{<\omega}$  instead, so we define *uniformly e-pointed trees w.r.t. functions*<sup>1</sup>.

**Definition 9** For a function  $g \in 2^\omega$ , let  $g^+ = \{ n : g(n) = 1 \}$  denote the set of which  $g$  is the characteristic function.

**Definition 10** • A *uniformly e-pointed tree with respect to sets* is a tree  $T \subseteq 2^{<\omega}$  with no leaves for which there is an enumeration operator  $\Psi$  such that  $(\forall g \in [T])(\Psi(g^+) = T)$ .

<sup>1</sup> The authors are thankful to Alexandra A. Soskova and Mariya I. Soskova for bringing to their attention, after a first draft of this paper was completed, the notion of uniformly e-pointed tree w.r.t. sets, called simply *uniformly e-pointed* by Montalbán and McCarthy.



- A *uniformly e-pointed tree with respect to functions* is a finitely branching tree  $T \subseteq \omega^{<\omega}$  with no leaves for which there is an enumeration operator  $\Psi$  such that  $(\forall g \in [T])(\Psi(g) = T)$ . (Recall that, as  $\Psi$  is an enumeration operator,  $\Psi(g)$  means  $\Psi(\text{graph}(g))$ .)

We show that the two notions of uniform e-pointedness coincide up to e-equivalence.

**Proposition 11** *Every uniformly e-pointed tree w.r.t. sets is a uniformly e-pointed tree w.r.t. functions.*

**Proof** Let  $T \subseteq 2^{<\omega}$  be a uniformly e-pointed tree w.r.t. sets. Let  $\Psi$  be an enumeration operator such that  $(\forall g \in [T])(\Psi(g^+) = T)$ . Fix an enumeration operator  $\Gamma$  such that  $(\forall A \subseteq \omega)(\Gamma(\chi_A) = A)$ . By composing  $\Psi$  and  $\Gamma$ , we get an enumeration operator  $\Theta$  such that  $(\forall g \in [T])(\Theta(g) = T)$ . Thus  $T$  is a uniformly e-pointed tree w.r.t. functions.  $\square$

**Proposition 12** *Let  $T \subseteq \omega^{<\omega}$  be a uniformly e-pointed tree w.r.t. functions. Then there is a uniformly e-pointed tree  $S \subseteq 2^{<\omega}$  w.r.t. sets such that  $S \equiv_e T$ . (In fact we may choose  $S$  so that  $[S]$  consists of exactly the characteristic functions of the graphs of elements of  $[T]$ .)*

**Proof** Let  $T \subseteq \omega^{<\omega}$  be a uniformly e-pointed tree w.r.t. functions. Say that  $\gamma \in 2^{<\omega}$  is *consistent with  $T$*  if there is a  $\sigma \in T$  such that

$$(\forall (i, n) \prec \langle \gamma \rangle)(i \prec \langle \sigma \rangle \text{ and } (\gamma \upharpoonright (i, n)) = 1 \leftrightarrow \sigma(i) = n).$$

Notice that if  $\eta \subseteq \gamma \in 2^{<\omega}$  and  $\gamma$  is consistent with  $T$ , then  $\eta$  is also consistent with  $T$ . Let

$$S = \{\gamma \in 2^{<\omega} : \gamma \text{ is consistent with } T\}.$$

Then  $S$  is a tree,  $S$  has no leaves because  $T$  has no leaves, and it is immediate to check that  $S \leq_e T$ . To see that  $T \leq_e S$ , observe that

$$T = \{\sigma \in \omega^{<\omega} : (\exists \gamma \in S)(\forall i \prec \langle \sigma \rangle)(\langle i, \sigma(i) \rangle \in \text{dom}(\gamma) \text{ and } \gamma(\langle i, \sigma(i) \rangle) = 1)\}.$$

Furthermore,  $[S] = \{\chi_{\text{graph}(f)} : f \in [T]\}$ . If  $f \in [T]$ , then  $\chi_{\text{graph}(f)} \upharpoonright n$  is consistent with  $T$  for every  $n$  (as witnessed by  $f \upharpoonright n$ ), thus  $\chi_{\text{graph}(f)} \upharpoonright n \in S$  for every  $n$ , thus  $\chi_{\text{graph}(f)} \in [S]$ . Conversely, suppose that  $f \notin [T]$ . Then there is an  $n$  such that  $f \upharpoonright n \notin T$ . We want to find an  $m$  such that  $\chi_{\text{graph}(f)} \upharpoonright m \notin S$  in order to conclude that  $\chi_{\text{graph}(f)} \notin [S]$ . By the fact that  $T$  is finitely branching, let  $k$  be large enough so that  $(\forall i \prec \langle \sigma \rangle)(\sigma(i) < k)$  whenever  $\sigma \in T$  has length  $\leq n$ . Let  $m \succ \langle n, k \rangle$ . Suppose for a contradiction that  $\chi_{\text{graph}(f)} \upharpoonright m$  is consistent with  $T$ , and let  $\sigma$  witness this. Then it must be that  $|\sigma| \geq n$  and  $(\forall i \prec n)(\sigma(i) = f(i))$ . Thus  $\sigma \supseteq f \upharpoonright n$ , contradicting that  $f \upharpoonright n \notin T$ . Thus  $\chi_{\text{graph}(f)} \upharpoonright m$  is not consistent with  $T$ , so  $\chi_{\text{graph}(f)} \upharpoonright m \notin S$ .

To finish, we need to find an enumeration operator  $\Psi$  such that  $(\forall g \in [S])(\Psi(g^+) = S)$ . So let  $\Theta$  be an enumeration operator such that  $(\forall f \in [T])(\Theta(f) = T)$ , and let

$\Gamma$  be an enumeration operator witnessing that  $S \leq_e T$ . By composing  $\Gamma$  and  $\Theta$ , we get an enumeration operator  $\Psi$  such that  $(\forall f \in [T])(\Psi(f) = S)$ . However, this is exactly what we want because we have shown that if  $g \in [S]$ , then  $g = \chi_{\text{graph}(f)}$  for some  $f \in [T]$  and therefore that if  $g \in [S]$  then  $g^+ = \text{graph}(f)$  for some  $f \in [T]$ . Thus  $(\forall g \in [S])(\Psi(g^+) = S)$ , as desired (recall that  $\Theta(f) = \Theta(\text{graph}(f))$ ).  $\square$

A set  $A$  is called *cototal* if  $A \leq_e \bar{A}$ , and an e-degree is called *cototal* if it contains a cototal set [1]. Every uniformly e-pointed tree w.r.t. sets is cototal by [10, Theorem 4.7], and, by [10, Corollary 4.9.1], an e-degree is cototal if and only if it contains a uniformly e-pointed tree w.r.t. sets.

**Proposition 13** *An enumeration degree is cototal if and only if it contains a uniformly e-pointed tree w.r.t. functions.*

**Proof** An e-degree is cototal if and only if it contains a uniformly e-pointed tree w.r.t. sets by [10, Theorem 4.7] if and only if it contains a uniformly e-pointed tree w.r.t. functions by Propositions 11 and 12.

We also find it interesting to give a more direct proof that every uniformly e-pointed tree w.r.t. functions has cototal enumeration degree. This can be accomplished via the easy characterization of the cototal enumeration degrees in terms of the *skip operator* from [1].

Recall that  $(\Psi_e : e \in \omega)$  is a standard list of all enumeration operators, and recall the following definitions.

- For an  $A \subseteq \omega$ ,  $K_A = \{\langle e, x \rangle : x \in \Psi_e(A)\}$ .
- For an  $A \subseteq \omega$ ,  $A^\diamond = \bar{K}_A$  is called the *skip* of  $A$ .

By [1, Proposition 1.1], a set  $A \subseteq \omega$  has cototal enumeration degree if and only if  $A \leq_e A^\diamond$ .

Let  $T$  be a uniformly e-pointed tree w.r.t. functions. We show that  $T \leq_e T^\diamond$  and therefore that  $T$  has cototal enumeration degree. Let  $\Psi$  be an enumeration operator such that  $(\forall f \in [T])(\Psi(f) = T)$ . For each  $n \in \omega$ , let  $T^n = \{\sigma \in T : |\sigma| = n\}$  denote level  $n$  of  $T$ . For  $b, n \in \omega$ , let  $b^n = \{\sigma \in \omega^{<\omega} : |\sigma| = n \text{ and } (\forall i < |\sigma|)(\sigma(i) < b)\}$  denote the set of all  $b$ -bounded strings of length  $n$ . Let  $B = \{\langle n, b \rangle : T^n \setminus b^n \neq \emptyset\}$ . That is,  $B$  is the set of all pairs  $\langle n, b \rangle$  where  $b$  is not big enough to bound every entry of every string in  $T^n$ . We have  $B \leq_e T$ , thus  $B \leq_1 K_T$ , and therefore  $\bar{B} \leq_e T^\diamond$ . The point is that if  $\langle n, b \rangle \in \bar{B}$ , then  $T^n \subseteq b^n$ , which allows us enumerate  $T$  from an enumeration of  $\bar{T} \oplus \bar{B}$ . Indeed,

$$T = \{\sigma : (\exists \langle n, b \rangle \in \bar{B})(\exists L \subseteq b^n \cap \bar{T})(\forall \tau \in b^n \setminus L)(\sigma \in \Psi(\tau))\}.$$

That is, we know that  $\sigma \in T$  when we see a bound  $T^n \subseteq b^n$  and a set of strings  $L \subseteq b^n$  that are *not* in  $T$  such that the remaining  $\tau \in b^n \setminus L$  all satisfy  $\sigma \in \Psi(\tau)$ . Thus  $T \leq_e \bar{T} \oplus \bar{B} \leq_e T^\diamond$ , so  $T$  has cototal enumeration degree.  $\square$

We extend the cototal terminology to the degrees of enumerability by saying that  $\mathbb{E}_A$  is *cototal* if  $A$  has cototal enumeration degree. To conclude this section, we show that cototality and compactness are equivalent properties of a degree of enumerability.

**Lemma 14** *Let  $A \subseteq \omega$  be nonempty, and let  $\mathcal{C} \subseteq \omega^\omega$  be closed such that  $\mathcal{C} \leq_s \mathcal{E}_A$ . Then there is a tree  $T \subseteq \omega^{<\omega}$  with no leaves such that  $T \leq_e A$  and  $[T] \subseteq \mathcal{C}$ . Furthermore, if  $\mathcal{C}$  is compact, then  $T$  is finitely branching.*

**Proof** Let  $\Phi$  be a Turing functional such that  $\Phi(\mathcal{E}_A) \subseteq \mathcal{C}$ , with  $\mathcal{C}$  closed. Let

$$T = \{\sigma : \exists \alpha(\text{ran}(\alpha) \subseteq A \text{ and } \sigma \subseteq \Phi(\alpha))\}.$$

Then  $T$  is a tree and  $T \leq_e A$ . To see that  $T$  has no leaves, let  $\sigma \in T$ , and let  $\alpha$  be such that  $\text{ran}(\alpha) \subseteq A$  and  $\sigma \subseteq \Phi(\alpha)$ . Let  $f: \omega \rightarrow \omega$  be such that  $\alpha \subset f$  and  $\text{ran}(f) = A$ . Let  $\beta$  be such that  $\alpha \subseteq \beta \subset f$  and  $\Phi(\beta)(|\sigma|) \downarrow$ . Then  $\sigma \subsetneq \Phi(\beta) \in T$ , so  $\sigma$  is not a leaf. To see that  $[T] \subseteq \mathcal{C}$ , we consider a  $g \in [T]$  and show that  $g$  is in the closure of  $\mathcal{C}$ . To this end, let  $n \in \omega$ , let  $\alpha$  be such that  $\text{ran}(\alpha) \subseteq A$  and  $g \upharpoonright n \subseteq \Phi(\alpha)$ , and let  $f: \omega \rightarrow \omega$  be such that  $\alpha \subset f$  and  $\text{ran}(f) = A$ . Then  $\Phi(f) \in \mathcal{C}$  and  $\Phi(f) \upharpoonright n = g \upharpoonright n$ . Hence  $g$  is in the closure of  $\mathcal{C}$ , so  $g \in \mathcal{C}$ .

Lastly, if  $\mathcal{C}$  is compact, then  $[T]$  is compact because  $[T] \subseteq \mathcal{C}$ . This means that  $T$  must be finitely branching because  $T$  has no leaves. □

**Lemma 15** *Let  $A \subseteq \omega$  be nonempty. Then  $\mathbf{E}_A$  is compact if and only if there is a uniformly e-pointed tree  $T \subseteq \omega^{<\omega}$  w.r.t. functions such that  $T \equiv_e A$ .*

**Proof** Suppose that  $\mathcal{E}_A \equiv_s \mathcal{C}$ , where  $\mathcal{C}$  is compact. Let  $\Phi$  be a Turing functional such that  $\Phi(\mathcal{C}) \subseteq \mathcal{E}_A$ .  $\mathcal{C}$  is compact, so its image  $\mathcal{D} = \Phi(\mathcal{C})$  is also compact by the continuity of the Turing functional  $\Phi$ . By Lemma 14, there is a finitely branching tree  $T \subseteq \omega^{<\omega}$  with no leaves such that  $T \leq_e A$  and  $[T] \subseteq \mathcal{D} \subseteq \mathcal{E}_A$ . Furthermore,  $A = \bigcup_{\sigma \in T} \text{ran}(\sigma)$  because  $[T] \subseteq \mathcal{E}_A$ , which implies that  $A \leq_e T$ . Hence  $T \equiv_e A$ . Also, if  $g \in [T]$ , then  $\text{ran}(g) = A$ , thus there is a uniform procedure enumerating  $A$  and hence  $T$  from any enumeration of  $g$ , which shows that  $T$  is uniformly e-pointed w.r.t. functions.

Conversely, suppose that there is a uniformly e-pointed tree  $T \subseteq \omega^{<\omega}$  w.r.t. functions such that  $T \equiv_e A$ . Then  $\mathcal{E}_A \leq_s [T]$ , as one can uniformly transform any function  $g \in [T]$  into a function that enumerates  $A$  because  $T \leq_e g$  uniformly and  $A \leq_e T$ . To see that  $[T] \leq_s \mathcal{E}_A$ , consider the Turing functional  $\Phi$  which, on an  $f \in \mathcal{E}_A$ , uses  $\text{ran}(f)$  to simultaneously enumerate  $T$  (via the reduction  $T \leq_e A$ ) and a path through  $T$  (which is possible because  $T$  has no leaves). Thus  $\mathcal{E}_A \equiv_s [T]$ , and  $[T]$  is compact because  $T$  is finitely branching. □

Observe that the proof of Lemma 15 also proves the following fact, which we record for posterity.

**Proposition 16** *Let  $A \subseteq \omega$  be nonempty. If  $T \subseteq \omega^{<\omega}$  is a finitely branching tree with no leaves such that  $T \leq_e A$  and  $[T] \subseteq \mathcal{E}_A$ , then  $T \equiv_e A$ ,  $[T] \equiv_s \mathcal{E}_A$ , and  $T$  is uniformly e-pointed w.r.t. functions.*

**Theorem 17**  *$\mathbf{E}_A$  is a compact degree of enumerability if and only if  $A$  has cototal enumeration degree. Hence a degree of enumerability is compact if and only if it is cototal.*

**Proof** The degree of enumerability  $\mathbf{E}_A$  is compact if and only if  $A \equiv_e T$  for some uniformly e-pointed tree  $T \subseteq \omega^{<\omega}$  w.r.t. functions by Lemma 15, which is the case if and only if  $A$  has cototal enumeration degree by Proposition 13.  $\square$

### 2.2 A quasiminimal degree of enumerability that is compact

The existence of quasiminimal problems of enumerability that are equivalent to compact mass problems is a consequence of Theorem 17 and the fact that there are cototal quasiminimal e-degrees [1]. We think, however, that it is instructive to directly construct a quasiminimal uniformly e-pointed tree w.r.t. functions. The corresponding degree of enumerability is then quasiminimal by definition and compact by Lemma 15.

Recall that  $\langle \cdot, \cdot \rangle: \omega^2 \rightarrow \omega$  is the recursive pairing function. Let  $\pi_0, \pi_1: \omega \rightarrow \omega$  denote the projection functions  $\pi_0(\langle m, n \rangle) = m$  and  $\pi_1(\langle m, n \rangle) = n$ .

**Lemma 18** *There is a finitely branching tree  $A \subseteq \omega^{<\omega}$  such that*

- $A$  has no leaves,
- $A$  is quasiminimal, and
- $\text{ran}(\pi_1 \circ f) = A$  for every  $f \in [A]$ .

*Notice that such a tree is uniformly e-pointed w.r.t. functions.*

**Proof** For the purposes of this proof, we make the following definitions for finite trees  $T, S \subseteq \omega^{<\omega}$ :

- $\text{leaves}(T) = \{\sigma \in T : \sigma \text{ is a leaf of } T\}$ .
- $S$  leaf-extends  $T$  if  $T \subseteq S$  and  $(\forall \tau \in S \setminus T)(\exists \sigma \in \text{leaves}(T))(\sigma \subseteq \tau)$ ;
- $S$  properly leaf-extends  $T$  if  $S$  leaf-extends  $T$  and  $(\forall \sigma \in T)(\exists \tau \in S)(\sigma \subsetneq \tau)$ .

We build a sequence of finite trees  $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ , where  $A_{s+1}$  properly leaf-extends  $A_s$  for each  $s \in \omega$ . This way,  $A = \bigcup_{s \in \omega} A_s$  has no leaves and is finitely branching. Furthermore, we build the sequence so that

$$(\forall s \in \omega)(\forall \sigma \in \text{leaves}(A_{s+1}))(A_s \subseteq \text{ran}(\pi_1 \circ \sigma) \subseteq A_{s+1}).$$

This ensures that  $\text{ran}(\pi_1 \circ f) = A$  for every  $f \in [A]$ . To help ensure that  $A >_e \emptyset$ , we also maintain a sequence of finite sets of strings  $O_0 \subseteq O_1 \subseteq O_2 \subseteq \dots$  such that  $\forall s(A_s \cap O_s = \emptyset)$ .

We satisfy the requirements

$$\mathcal{Q}_e: A \neq W_e$$

$$\mathcal{R}_e: \text{if } \Psi_e(A) \text{ is the graph of a total function } f, \text{ then } f \text{ is recursive.}$$

*Stage 0:* set  $A_0 = \{\emptyset\}$ , and set  $O_0 = \emptyset$ .

*Stage  $s + 1 = 2e + 1$ :* We satisfy  $\mathcal{Q}_e$ . If  $W_e$  is finite, then set  $O_{s+1} = O_s$ . If  $W_e$  is infinite, then choose any  $\sigma \in W_e \setminus A_s$ , and set  $O_{s+1} = O_s \cup \{\sigma\}$ . To extend  $A_s$  to  $A_{s+1}$ , first choose  $n$  greater than (the code of) every element in  $O_{s+1}$ . Then choose any

enumeration  $(\alpha_i)_{i < k}$  of  $A_s$ . Then let  $\beta$  be the string  $\langle \langle n, \alpha_0 \rangle, \langle n, \alpha_1 \rangle, \dots, \langle n, \alpha_{k-1} \rangle \rangle$ . Now let  $A_{s+1}$  be the tree obtained by extending each leaf of  $A_s$  by  $\beta$ :

$$A_{s+1} = \{ \sigma : (\exists \tau \in \text{leaves}(A_s))(\sigma \subseteq \tau \hat{\ } \beta) \}.$$

Having chosen  $n$  big enough, we have guaranteed that  $A_{s+1}$  is disjoint from  $O_{s+1}$ . Stage  $s + 1 = 2e + 2$ : We satisfy  $\mathcal{R}_e$ . Set  $O_{s+1} = O_s$ . For finite trees  $T, S \subseteq \omega^{<\omega}$ , call  $S$  a good extension of  $T$  if  $S$  leaf-extends  $T$ ,  $S \cap O_{s+1} = \emptyset$ , and  $(\forall \sigma \in S)(\text{ran}(\pi_1 \circ \sigma) \subseteq S)$ . Ask if there is a good extension  $R$  of  $A_s$  such that

$$(\exists m, n, o)(n \neq o \text{ and } \langle m, n \rangle \in \Psi_e(R) \text{ and } \langle m, o \rangle \in \Psi_e(R)).$$

If there is such an  $R$ , let  $\widehat{A}_s = R$ . Otherwise, let  $\widehat{A}_s = A_s$ . Now extend  $\widehat{A}_s$  to  $A_{s+1}$  the same way that we extend  $A_s$  to  $A_{s+1}$  during the odd stages. The fact that  $(\forall \sigma \in \widehat{A}_s)(\text{ran}(\pi_1 \circ \sigma) \subseteq \widehat{A}_s)$  ensures that  $(\forall \sigma \in A_{s+1})(\text{ran}(\pi_1 \circ \sigma) \subseteq A_{s+1})$ . This completes the construction.

Let  $A = \bigcup_{s \in \omega} A_s$ . We show that all requirements are satisfied.

For requirement  $\mathcal{Q}_e$ , consider stage  $s + 1 = 2e + 1$ . If  $W_e$  is finite, then  $A \neq W_e$  because  $A$  is infinite. If  $W_e$  is infinite, then at stage  $s + 1$  we chose a  $\sigma \in W_e \setminus A_s$  and put  $\sigma$  in  $O_{s+1}$ . Thus  $\forall t(\sigma \notin A_t)$ , so  $\sigma \notin A$ . Hence  $A \neq W_e$ .

For requirement  $\mathcal{R}_e$ , suppose that  $\Psi_e(A)$  is the graph of a total function  $f$ , and consider stage  $s + 1 = 2e + 2$ . We show that  $\text{graph}(f)$  is r.e., which implies that  $f$  is recursive. Let

$$X = \{ \langle m, n \rangle : \text{there is a good extension } B \text{ of } A_s \text{ with } \langle m, n \rangle \in \Psi_e(B) \}$$

(where here ‘good’ means with respect to the  $O_{s+1}$  at stage  $s + 1$ ). Clearly  $X$  is r.e. We show that  $X = \text{graph}(f)$ . For  $\text{graph}(f) \subseteq X$ , suppose that  $\langle m, n \rangle \in \text{graph}(f) = \Psi_e(A)$ . Let  $t \geq s + 1$  be such that  $\langle m, n \rangle \in \Psi_e(A_t)$ . Then  $A_t$  is a good extension of  $A_s$  with  $\langle m, n \rangle \in \Psi_e(A_t)$ , so  $\langle m, n \rangle \in X$ . Conversely, suppose that  $\langle m, n \rangle \in X$ , and let  $B$  be a good extension of  $A_s$  with  $\langle m, n \rangle \in \Psi_e(B)$ . If  $\langle m, n \rangle \notin \text{graph}(f)$ , then  $\langle m, o \rangle \in \text{graph}(f) = \Phi_e(A)$ , where  $o = f(m) \neq n$ . Let  $t \geq s + 1$  be such that  $\langle m, o \rangle \in \Psi_e(A_t)$ . Then  $A_t$  is a good extension of  $A_s$ , and, moreover,  $A_t \cup B$  is also a good extension of  $A_s$ . Thus there is a good extension  $R = A_t \cup B$  of  $A_s$  such that  $n \neq o$  and  $\langle m, n \rangle \in \Psi_e(R)$  and  $\langle m, o \rangle \in \Psi_e(R)$ , for some  $m, n, o \in \omega$ . Therefore, at stage  $s + 1$ , we extended  $A_s$  to an  $A_{s+1}$  such that

$$(\exists m, n, o)(n \neq o \text{ and } \langle m, n \rangle \in \Psi_e(A_{s+1}) \text{ and } \langle m, o \rangle \in \Psi_e(A_{s+1})).$$

This contradicts that  $\Psi_e(A)$  is the graph of a function.

All together, we have that  $A$  has no leaves, that  $\text{ran}(\pi_1 \circ f) = A$  for every  $f \in [A]$  by construction, and that  $A$  is quasiminimal by the  $\mathcal{Q}_e$  requirements and the  $\mathcal{R}_e$  requirements. □

**Theorem 19** *There is a degree of enumerability  $\mathbf{E}_A$  that is both quasiminimal and compact. Hence  $\mathbf{E}_A$  is closed, nonzero, and does not bound any nonzero degree of solvability.*

**Proof** Let  $A$  be the tree from Lemma 18. Then  $A$  has quasiminimal  $e$ -degree, so  $\mathbf{E}_A$  is quasiminimal by definition. Furthermore,  $A$  is uniformly  $e$ -pointed w.r.t. functions, so  $\mathbf{E}_A$  is compact by Lemma 15.  $\square$

**Remark 20** In Lemma 18, one can make the tree  $A$  be not cototal by a small modification to the proof. Thus although every uniformly  $e$ -pointed tree w.r.t. functions has cototal  $e$ -degree by Proposition 13, it is not the case that every such tree is cototal as a set.

To modify the proof, replace each old  $\mathcal{Q}_e$  requirement  $A \neq W_e$  with the new requirement  $A \neq \Psi_e(\bar{A})$ . (Notice that a set  $A$  satisfying all of the new  $\mathcal{Q}_e$  requirements is still not r.e., which is required in order for a set  $A$  to be quasiminimal.) To satisfy the new  $\mathcal{Q}_e$ , modify stage  $s + 1 = 2e + 1$  as follows. If there are a finite set  $D \subseteq \bar{A}_s$  and a string  $\sigma \in \bar{A}_s$  with  $\sigma \in \Psi_e(D)$ , then choose such a  $D$  and  $\sigma$ , and set  $O_{s+1} = O_s \cup D \cup \{\sigma\}$ . Otherwise simply set  $O_{s+1} = O_s$ . Then choose  $n$  greater than (the code of) every element in  $O_{s+1}$ , and extend  $A_s$  to  $A_{s+1}$  as before. To verify that  $\mathcal{Q}_e$  is satisfied, suppose for a contradiction that  $\Psi_e(\bar{A}) = A$ . As  $A$  is infinite and  $A_s$  is finite, fix some  $\sigma \in A \setminus A_s$ . Let  $D \subseteq \bar{A} \subseteq \bar{A}_s$  be a finite set such that  $\sigma \in \Phi_e(D)$ . Then, at stage  $s + 1$ , we were able to choose a  $D$  and  $\sigma$ , ensuring that  $\Phi_e(\bar{A}) \neq A$ . This is a contradiction.

### 2.3 A degree of enumerability that does not bound any nonzero closed degree

Finally, we show that there are examples of nonzero degrees of enumerability that do not bound nonzero closed degrees. Such examples are of course quasiminimal, and indeed the property of being nonzero but not above any nonzero closed degree can be viewed as an interesting generalization of quasiminimality. Theorem 22 below can also be phrased by saying that there are nonzero degrees of enumerability that do not lie in the filter generated by the nonzero closed degrees, which coincides with the collection of all Medvedev degrees bounding nonzero closed degrees (see [19]).

**Lemma 21** *There is a set  $A >_e \emptyset$  such that, for all  $T \leq_e A$ , if  $T$  is a subtree of  $\omega^{<\omega}$  with no leaves, then  $T$  has an r.e. subtree with no leaves.*

**Proof** For the purposes of this proof, we assume that if  $\Psi$  is an enumeration operator,  $X \subseteq \omega$ , and  $\Psi(X)$  enumerates some  $\sigma \in \omega^{<\omega}$  (i.e.,  $\sigma \in \Psi(X)$ ), then it also enumerates all  $\tau \subseteq \sigma$ . In fact, from any enumeration operator  $\Gamma$ , one can effectively produce an enumeration operator  $\Psi$  such that, for all  $X$ ,

- $\Psi(X)$  is a tree, and
- if  $\Gamma(X)$  is a tree, then  $\Psi(X) = \Gamma(X)$ .

To accomplish this, just take  $\Psi = \{ \langle \tau, D \rangle : (\exists \sigma)(\tau \subseteq \sigma \text{ and } \langle \sigma, D \rangle \in \Gamma) \}$ . Therefore, we can define an effective list  $(\Psi_e : e \in \omega)$  of enumeration operators such that

- $\Psi_e(X)$  is a tree for every  $e$  and  $X$ , and
- if  $T \leq_e X$  for a tree  $T$  and set a  $X$ , then there is an  $e$  such that  $\Psi_e(X) = T$ .

Also, recall the notation  $g^+ = \{n : g(n) = 1\}$  from Definition 9. We extend this notation to strings  $\alpha \in 2^{<\omega}$  by defining  $\alpha^+ = \{i < |\alpha| : \alpha(i) = 1\}$ . Additionally, if

$A \subseteq \omega$  and  $\alpha \in 2^{<\omega}$ , we write  $A \subseteq^+ \alpha^+$  to mean that  $(\forall n < |\alpha|)(n \in A \rightarrow \alpha(n) = 1)$  (i.e.,  $\{n \in A : n < |\alpha|\} \subseteq \alpha^+$ ).

We satisfy the requirements

$$\mathcal{Q}_e : A \neq W_e$$

$\mathcal{R}_e$  : either  $\Psi_e(A)$  contains a leaf or there is an r.e.  $T \subseteq \Psi_e(A)$  with no leaves.

We build a sequence of binary strings  $\alpha_0 \subseteq \alpha_1 \subseteq \alpha_2 \subseteq \dots$  along with sequences of recursive sets  $I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$  and  $J_0 \subseteq J_1 \subseteq J_2 \subseteq \dots$  such that, for every  $s \in \omega$ ,  $I_s \setminus J_s$  is infinite and  $J_s \subseteq^+ \alpha_s^+ \subseteq I_s$ . In the end, we let  $A = \bigcup_{s \in \omega} \alpha_s^+$ , and we have  $\bigcup_{s \in \omega} J_s \subseteq A \subseteq \bigcap_{s \in \omega} I_s$ .

Stage 0: Set  $\alpha_0 = \emptyset$ , set  $I_0 = \omega$ , and set  $J_0 = \emptyset$ .

Stage  $s + 1 = 2e + 1$ : We satisfy  $\mathcal{Q}_e$ . Let  $n \in I_s \setminus J_s$  be least such that  $n > |\alpha_s|$ . If  $n \in W_e$ , set  $I_{s+1} = I_s \setminus \{n\}$ , set  $J_{s+1} = J_s$ , and extend  $\alpha_s$  to an  $\alpha_{s+1}$  with  $J_s \subseteq^+ \alpha_{s+1}^+ \subseteq I_s$  and  $\alpha_{s+1}(n) = 0$ . If  $n \notin W_e$ , set  $I_{s+1} = I_s$ , set  $J_{s+1} = J_s$ , and extend  $\alpha_s$  to an  $\alpha_{s+1}$  with  $J_s \subseteq^+ \alpha_{s+1}^+ \subseteq I_s$  and  $\alpha_{s+1}(n) = 1$ .

Stage  $s + 1 = 2e + 2$ : We satisfy  $\mathcal{R}_e$ . Ask if there is a  $\beta \supseteq \alpha_s$  and a recursive set  $R$  such that

- $J_s \subseteq^+ \beta^+ \subseteq R \subseteq I_s$ ,
- $R \setminus J_s$  is infinite, and
- there is a  $\sigma \in \Psi_e(\beta^+)$  that is a leaf in  $\Psi_e(R)$ .

If there are such  $\beta$  and  $R$ , set  $\alpha_{s+1} = \beta$ , set  $I_{s+1} = R$ , and set  $J_{s+1} = J_s$ . If there are no such  $\beta$  and  $R$ , then set  $\alpha_{s+1} = \alpha_s$ , set  $I_{s+1} = I_s$ , and choose any recursive  $J_{s+1}$  whose characteristic function extends  $\alpha_{s+1}$ ,  $J_s \subseteq J_{s+1} \subseteq I_{s+1}$ , and  $J_{s+1} \setminus J_s$  and  $I_{s+1} \setminus J_{s+1}$  are both infinite. This completes the construction.

Let  $A = \bigcup_s \alpha_s^+$ . The  $\mathcal{Q}_e$  requirements are clearly satisfied, and together they ensure that  $A$  is not r.e. Hence  $A >_e \emptyset$ .

Now suppose that  $T \leq_e A$  is a tree with no leaves, and let  $\Psi_e$  be such that  $T = \Psi_e(A)$ . At stage  $s + 1 = 2e + 2$ , there must not have been a  $\beta$  and an  $R$  because if there were, then we would have  $\beta = \alpha_{s+1}$  and  $\beta^+ \subseteq A \subseteq R = I_{s+1}$ , so there would be a leaf  $\sigma \in \Psi_e(A) = T$ . It must therefore be that  $\Psi_e(J_{s+1})$  is a tree with no leaves. To see this, suppose instead that  $\Psi_e(J_{s+1})$  has a leaf  $\sigma$ . Let  $\beta$  be such that  $\alpha_{s+1} \subseteq \beta$ ,  $\beta^+ \subseteq J_{s+1}$ , and  $\sigma \in \Psi_e(\beta^+)$ . Then at stage  $s + 1$ , we could have taken  $\beta$  and  $R = J_{s+1}$ , which is a contradiction. This finishes the proof because  $\Psi_e(J_{s+1}) \subseteq T$  since  $J_{s+1} \subseteq A$ , and  $\Psi_e(J_{s+1})$  is r.e. since  $J_{s+1}$  is recursive.  $\square$

**Theorem 22** *There is a nonzero degree of enumerability that does not bound a nonzero closed degree.*

**Proof** Let  $A$  be as in Lemma 21. Consider a closed  $\mathcal{C} \leq_s \mathcal{E}_A$ . By Lemma 14, there is a tree  $T \leq_e A$  with no leaves such that  $[T] \subseteq \mathcal{C}$ . By Lemma 21,  $T$  has an r.e. subtree  $S$  with no leaves. Thus  $[S] \subseteq [T] \subseteq \mathcal{C}$ . However, being a tree with no leaf,  $S$  has a recursive path, so  $\mathcal{C}$  has a recursive member, so  $\text{deg}_s(\mathcal{C}) = \mathbf{0}$ .  $\square$

**Acknowledgements** We thank Douglas Cenzer, Antonio Montalbán, Alexandra A. Soskova, and Mariya I. Soskova for helpful comments and discussions.

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