A CLASS OF PIECEWISE LINEAR MAPS

FRANCO FINESCHI

Abstract. Piecewise linear functions defined by p-maps, linear only on a subset of r vectors and components, are introduced. Universal properties for this maps are proved. Spaces of extensions of differential forms by piecewise linear functions are considered.

1. Introduction

Piecewise linear functions are useful in several different contexts, piecewise linear manifolds, computer science or convex analysis are examples. A definition of a piecewise linear function is the following, see [8]. Let C a closed convex domain in $\Re^d$, a function $\Phi : C \to \Re$ is said to be piecewise linear if there is a finite family Q of closed domains such that $C = \cup Q$ and Φ is linear on every domain in Q. A linear function ϕ on $\Re^d$ which coincides with Φ on some $Q_i \in Q$ is said to be a component of Φ. In this paper is considered a more general class of piecewise linear functions. It is defined the set of maps $SW(E^m, T)$ which are linear only on a subset of r vectors and components. Then an exponential functor F is defined from linear spaces to the set $SW(E^m, T)$. It is proved the uniqueness and existence of a function $\circledast$ as universal element for the functor F. It is defined a r-subsetwise linear skewsymmetric $\Phi = \sum_{\mu,\nu} \lambda^\mu_\nu \phi$ map and it is proved that this is completely determined by its values for λ^μ_ν and on a basis of E. A r-determinant function is defined as a r-subsetwise linear skewsymmetric map $\Phi : E^m \to \Gamma$, where Γ is an arbitrary field of characteristic 0. Some properties of r-determinant maps are considered. It is defined the adjoint for a linear map $\psi \in L(E, F)$, where E and F are linear spaces, and the development of a r-determinant function by r- cofactors. Extensions of differential forms are defined by r-subsetwise skewsymmetric maps. Basis and spaces of generalized differential forms are studied.

2. R-Subsetwise Linear Mappings

Some properties of linear functions are extended to mappings which are linear only on subsets of r variables. Γ denotes an arbitrarily chosen field such that $char\,\Gamma = 0$. The multindex I^n_r of lenght r is defined by

$$I^n_r = \{(i_1, \dots, i_r) : \quad 1 \le i_1 < i_2 < \cdots < i_r \le n\}$$

besides, for a fixed natural k

$$\begin{aligned}(I^n_r)_k = \quad & \{(i_1, \dots, i_p, \dots, i_r) : 1 \le i_1 < \cdots < i_p = k < \cdots \le i_r \le n, \\ & \text{where } 1 \le k \le n\}\end{aligned}$$

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for the indices $j_1, \dots, j_k \in I^n_k$

$$(I^n_r)_{j_1,\dots,j_k} = \{(i_1, \dots, i_{p_1}, \dots, i_{p_k}, \dots, i_r) : \\ 1 \le i_1 < \dots < i_{p_1} = j_1 < \dots < i_{p_k} = j_k < \dots \le i_r \le n\}$$

Let $\{e_\nu\}$ be a basis of an n-dimensional vector space E and let $x^\mu = \sum_{\nu=1}^n x^\mu_\nu e_\nu$ be vectors of E, $n \ge 1$.

Definition 2.1. Let $L(E^r, T)$ be the space of linear mappings of E^r into the vector space T. Consider a mapping

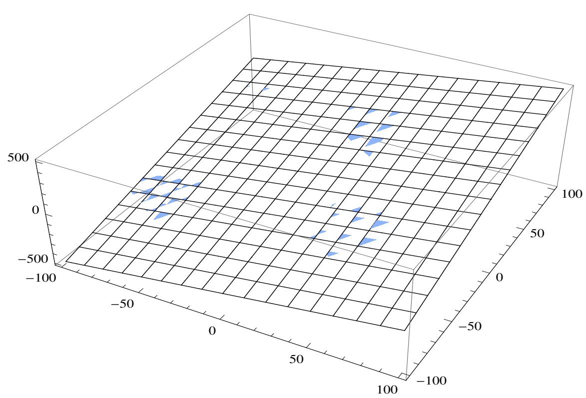
$$\begin{cases} \Phi : & E^m \to T \\ \Phi : & (x_1, \dots, x_m) \mapsto \sum_{\mu,\nu} \lambda^\mu_\nu \phi(x^{\mu_1}_\nu e_\nu, \dots, x^{\mu_r}_\nu e_\nu) \quad 1 \le r \le n,\ 1 \le r \le m,\ \ \lambda^\mu_\nu \in \Gamma \end{cases}$$

where the sum is over every system of indices $\mu = \mu_1, \dots, \mu_r \in I^m_r$, $\nu = \nu_1, \dots, \nu_r \in I^n_r$. If $n = m$ then $r < n = m$. The sum $(x^{\mu_i}_{\nu_1} e_{\nu_1} + \dots + x^{\mu_i}_{\nu_r} e_{\nu_r})$ is denoted in short by $x^{\mu_i}_\nu e_\nu$, and $\phi : E^r \to T$ is an r-linear mapping. Then Φ is said to be r-linear with respect to the r-subsets of vectors and components, that is, an r-subsetwise linear mapping. The linear mappings ϕ are the components of Φ.

Example 2.1. The function $\Phi : \Re^{1\times 2} \to \Re$ defined by

$$\Phi(x, y) = 2x + 3y$$

is an 1-subsetwise linear function.



Graph of the function Φ. (Obtained by Mathematica).

Example 2.2. The map $\Phi : (\Re^2)^3 \to \Re^{2\times 2}$ defined by

$$\Phi[(x_{11}, x_{21}), (x_{12}, x_{22}), (x_{13}, x_{23})] = \lambda^{12} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} + \lambda^{13} \begin{pmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{pmatrix} + \\ + \lambda^{23} \begin{pmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{pmatrix} \qquad \lambda^\mu \in \Re$$

is an 2-subsetwise linear map.

Example 2.3. Let $f_1, \dots, f_r$ be a linearly independent set of the space $L(E^r, T)$, a r-subsetwise linear map is defined by

$$\Phi(x_1, \dots, x_m) = \sum_{\mu,\nu} \lambda_\nu^\mu (f_1(x_\nu^{\mu_1} e_\nu) \cdot f_2(x_\nu^{\mu_2} e_\nu) \cdots f_r(x_\nu^{\mu_r} e_\nu)) \qquad \lambda_\nu^\mu \in \Gamma$$

Theorem 2.1. An r-subsetwise linear mapping Φ, with $r < m$, is not linear

Proof. For any r-subsetwise linear mapping Φ, $r < m$,

$$\begin{aligned}\Phi(x_1, \dots, x_i + y_i, \dots, x_m) &= \sum_{\mu,\nu} \lambda_\nu^\mu \phi(x_\nu^{\mu_1} e_\nu, \dots, x_\nu^i e_\nu, \dots, x_\nu^{\mu_r} e_\nu) \\ &+ \sum_{\mu,\nu} \lambda_\nu^\mu \phi(x_\nu^{\mu_1} e_\nu, \dots, y_\nu^i e_\nu, \dots, x_\nu^{\mu_r} e_\nu) \\ &\neq \Phi(x_1, \dots, x_i, \dots, x_m) + \Phi(x_1, \dots, y_i, \dots, x_m)\end{aligned}$$

in the first sum on the right side $\mu = \mu_1, \dots, i, \dots, \mu_r \in I_r^m$. Unlike, in the second sum $\mu = \mu_1, \dots, i, \dots, \mu_r \in (I_r^m)_i$, so this sum cannot be $\Phi(x_1, \dots, y_i, \dots, x_m)$. □

As a special case, if $r = m$ then Φ is linear.
If $t : T \to H$ is linear and Φ is r-swlin (subsetwise linear) map, then

$$t \circ \Phi = t(\sum \lambda_\nu^\mu \phi) = \sum \lambda_\nu^\mu t \circ \phi$$

and $t \circ \Phi$ is a r-swlin map.
By the set $SW(E^m, T)$ of the r-swlin maps, the following exponential functor F, from linear spaces to sets, is defined by

$$F(T) = SW(E^m, T) \qquad \text{for any linear space } T$$

$$\begin{cases} F(t) : F(T) \to F(H) \\ F(t) : \Phi \mapsto t \circ \Phi \end{cases} \qquad \text{for any linear } t : T \to H$$

Theorem 2.2. For any r-swlin mapping $\Psi : E^m \to H$ there exists a unique linear mapping $f : E \circledast \cdots \circledast E \to H$ such that

$$f(x_1 \circledast \cdots \circledast x_m) = \Psi(x_1, \dots, x_m)$$

That is, the mapping $\circledast : E^m \to T$ is an universal element for the functor F.

Proof. The proof generalizes to swlin maps the classical proof of universality of the tensor product, see [4], [6].
Uniqueness. Suppose that $\circledast : E^m \to T$ and $\tilde{\circledast} : E^m \to \tilde{T}$ are universal elements for the functor F, then, there exist linear maps

$$f : T \to \tilde{T} \qquad \text{and} \qquad g : \tilde{T} \to T$$

such that

$$f(x_1 \circledast \cdots \circledast x_m) = x_1 \tilde{\circledast} \cdots \tilde{\circledast} x_m \qquad \text{and} \qquad g(x_1 \tilde{\circledast} \cdots \tilde{\circledast} x_m) = x_1 \circledast \cdots \circledast x_m$$

that is

$$g \circ f(x_1 \circledast \cdots \circledast x_m) = x_1 \circledast \cdots \circledast x_m \qquad \text{and} \qquad f \circ g(x_1 \tilde{\circledast} \cdots \tilde{\circledast} x_m) = x_1 \tilde{\circledast} \cdots \tilde{\circledast} x_m$$

by the universality of $\circledast$ and $\tilde{\circledast}$ it follows, respectively

$$1_T = g \circ f \qquad \text{and} \qquad 1_{\tilde{T}} = f \circ g$$

thus f and g are inverse linear isomorphisms.
Existence. Consider the free vector space $C(E^r)$ generated by the space E^r. Denote by $N(E^r)$ the subspace of $C(E^r)$ spanned by the vectors

$$\begin{aligned}&(x_\nu^{\mu_1}e_\nu,\dots,\delta_1 y_1+\delta_2 y_2,\dots,x_\nu^{\mu_r}e_\nu)-\delta_1(x_\nu^{\mu_1}e_\nu,\dots,y_1,\dots,x_\nu^{\mu_r}e_\nu)\\&-\delta_2(x_\nu^{\mu_1}e_\nu,\dots y_2,\dots,x_\nu^{\mu_r}e_\nu)\end{aligned}$$

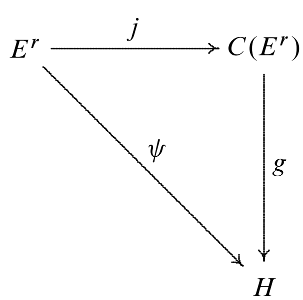
for $\mu=\mu_1,\dots,\mu_r\in I_r^m,\ \ \nu=\nu_1,\dots,\nu_r\in I_r^n,\ \ \delta_i\in\Gamma$ and $x_\nu^{\mu_r}e_\nu,y_1,y_2\in E^r$.
Set $S=C(E^r)/N(E^r)$ and let $\pi:C(E^r)\to S$ be the canonical projection. Define the map

$$\begin{cases}\circledast: & E^m\to S\\ \circledast: & (x_1,\dots,x_m)\mapsto\sum_{\mu,\nu}\lambda_\nu^\mu\pi(x_\nu^{\mu_1}e_\nu,\dots,x_\nu^{\mu_r}e_\nu)\end{cases}$$

Since π is a homomorphism, it follows that $\circledast$ is an r-swlin map.
If $z\in S$, then it is a finite sum

$$\begin{aligned}z&=\sum_\tau\delta^\tau(\sum_{\mu,\nu}\lambda_\nu^\mu\pi(x_\nu^{\mu_1}e_\nu,\dots,x_\nu^{\mu_r}e_\nu))_\tau\\&=\sum_\tau\delta^\tau(x_1\circledast\cdots\circledast x_m)_\tau\end{aligned}$$

so $\forall z\in S$, z is spanned by the products $x_1\circledast\cdots\circledast x_m$ and $I_m\circledast=S$.
Moreover let $\psi:\mathfrak{E}^r\to H$ be a r-linear map. Since $C(E^r)$ is a free vector space, there exists an unique linear map g such that the following diagram commutes



where j is the insertion of E^r in $C(E^r)$. So

$$g(x_\nu^{\mu_1}e_\nu,\dots,x_\nu^{\mu_r}e_\nu)=\psi(x_\nu^{\mu_1}e_\nu,\dots,x_\nu^{\mu_r}e_\nu)$$

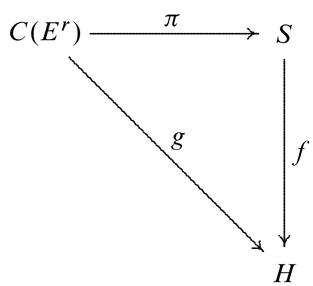
If

$$\begin{aligned}z=&(x_\nu^{\mu_1}e_\nu,\dots,\delta_1 y_1+\delta_2 y_2,\dots,x_\nu^{\mu_r}e_\nu)-\delta_1(x_\nu^{\mu_1}e_\nu,\dots,y_1,\dots,x_\nu^{\mu_r}e_\nu)\\&-\delta_2(x_\nu^{\mu_1}e_\nu,\dots y_2,\dots,x_\nu^{\mu_r}e_\nu)\end{aligned}$$

is a generator of $N(E^r)$, then

$$\begin{aligned}g(z)=\psi(z)=&\psi(x_\nu^{\mu_1}e_\nu,\dots,\delta_1 y_1+\delta_2 y_2,\dots,x_\nu^{\mu_r}e_\nu)-\delta_1\psi(x_\nu^{\mu_1}e_\nu,\dots,y_1,\dots,x_\nu^{\mu_r}e_\nu)\\&-\delta_2\psi(x_\nu^{\mu_1}e_\nu,\dots y_2,\dots,x_\nu^{\mu_r}e_\nu)\\=&0\end{aligned}$$

then $N(E^r)\subseteq Ker\, g$. For the principal theorem on factor spaces, see [5], there exists an unique linear map f such that the following diagram commutes



that is, π is an universal element. So

$$\begin{aligned}(f \circ \circledast)(x_1, \ldots, x_m) &= f(\sum_{\mu,\nu} \lambda_\nu^\mu \pi(x_\nu^{\mu_1} e_\nu, \ldots, x_\nu^{\mu_r} e_\nu)) \\ &= \sum_{\mu,\nu} \lambda_\nu^\mu f \circ \pi(x_\nu^{\mu_1} e_\nu, \ldots, x_\nu^{\mu_r} e_\nu) \\ &= \sum_{\mu,\nu} \lambda_\nu^\mu g(x_\nu^{\mu_1} e_\nu, \ldots, x_\nu^{\mu_r} e_\nu) \\ &= \sum_{\mu,\nu} \lambda_\nu^\mu \psi(x_\nu^{\mu_1} e_\nu, \ldots, x_\nu^{\mu_r} e_\nu) \\ &= \Psi(x_1, \ldots, x_m)\end{aligned}$$

□

Example 2.4. Consider the 2-swlin function Φ defined by

$$\begin{cases} \Phi : & (\Re^2)^3 \to \Re \\ \Phi : & (x_1, x_2, x_3) \mapsto \lambda^{12}(x_1, x_2) + \lambda^{13}(x_1, x_3) + \lambda^{23}(x_2, x_3) \qquad \lambda^{12}, \lambda^{13}, \lambda^{23} \in \Re \end{cases}$$

where the bilinear function $(-,-)$, on the right side, is the inner product in $\Re^2$. By the theorem 2.2, the map $\circledast : (\Re^2)^3 \to \Re^2 \circledast \Re^2 \circledast \Re^2$ is universal, so an unique linear function $f : \Re^2 \circledast \Re^2 \circledast \Re^2 \to \Re$ exists such that $f(x_1 \circledast x_2 \circledast x_3) = \Phi(x_1, x_2, x_3)$. Since $\Re^2 \circledast \Re^2 \circledast \Re^2$ is free, the function f is determined by its values $f(x_1 \circledast x_2 \circledast x_3)$ on the free generators $x_1 \circledast x_2 \circledast x_3$.

Corollary 2.1. For any r-swlin map $\Phi : E^m \to T$

$$\circledast(x_1, \ldots, x_m) = \sum_{\mu,\nu} \lambda_\nu^\mu \,(x_\nu^{\mu_1} e_\nu \otimes \cdots \otimes x_\nu^{\mu_r} e_\nu)$$

Proof. Since $\pi(x_\nu^{\mu_1} e_\nu, \cdots, x_\nu^{\mu_r} e_\nu) = x_\nu^{\mu_1} e_\nu \otimes \cdots \otimes x_\nu^{\mu_r} e_\nu$, by the theorem 2.2

$$\Phi(x_1, \ldots, x_m) = (f \circ \circledast)((x_1, \ldots, x_m) = f(\sum_{\mu,\nu} \lambda_\nu^\mu \,(x_\nu^{\mu_1} e_\nu \otimes \cdots \otimes x_\nu^{\mu_r} e_\nu)$$

□

Example 2.5. Let $\Phi : (\Gamma^n)^n \to T$ be a 2-swlin map. The tensor product $\otimes : \Gamma^n \times \Gamma^n \to M^{n \times n}$ is defined by $x_{i_1} \otimes x_{i_2} = x_{i_1} x'_{i_2}$, $x_i \in \Gamma^n$, see [4], then $\circledast$:

$(\Gamma^n)^n \to \Gamma^n \circledast \cdots \circledast \Gamma^n$ is given by

$$x_1 \circledast \cdots \circledast x_n = \sum_{(i_1,i_2)\in I_2^n} \lambda^{(i_1,i_2)} x_{i_1} \otimes x_{i_2}$$
$$= \begin{pmatrix} \sum_{(i_1,i_2)\in I_2^n} \lambda^{(i_1,i_2)} x_{1i_1} x_{1i_2} & \cdots & \sum_{(i_1,i_2)\in I_2^n} \lambda^{(i_1,i_2)} x_{1i_1} x_{ni_2} \\ \cdots & \cdots & \cdots \\ \sum_{(i_1,i_2)\in I_2^n} \lambda^{(i_1,i_2)} x_{ni_1} x_{1i_2} & \cdots & \sum_{(i_1,i_2)\in I_2^n} \lambda^{(i_1,i_2)} x_{ni_1} x_{ni_2} \end{pmatrix}$$

3. $\{r, \lambda\}$- DETERMINANT

If σ is a permutation, $\sigma \in S_r$, then the mapping $\sigma\phi : \mathfrak{E}^r \to F$ is defined by $\sigma\phi(x_1, \dots, x_r) = \phi(x_{\sigma_1}, \dots, x_{\sigma_r})$. More generally

Definition 3.1. Let $\Phi(x_1, \dots, x_m)$ be an r-swlin map, for any permutation $\sigma \in S_r$, the mapping $\sigma\Phi : E^m \to T$, is defined by

$$\sigma\Phi(x_1, \dots, x_m) = \sum_{\mu,\nu} \lambda_\nu^\mu \sigma\phi(x_\nu^{\mu_1} e_\nu, \dots, x_\nu^{\mu_r} e_\nu) = \sum_{\mu,\nu} \lambda_\nu^\mu \phi(x_\nu^{\sigma(\mu_1)} e_\nu, \dots, x_\nu^{\sigma(\mu_r)} e_\nu)$$

Definition 3.2. An r-swlin map $\Phi(x_1, \dots, x_m)$ is said skewsymmetric if for any $\sigma \in S_r$ is $\sigma\Phi = \epsilon_\sigma \Phi$ where $\epsilon_\sigma = 1$ ($\epsilon_\sigma = -1$) for any even (odd) permutation σ.

Theorem 3.1. An r-swlin map $\Phi = \sum \lambda_\nu^\mu \phi$ is skewsymmetric if and only if ϕ is skewsymmetric.

Proof. Suppose ϕ skewsymmetric, then

$$\sigma\Phi = \sum_{\mu,\nu} \lambda_\nu^\mu \sigma\phi(x_\nu^{\mu_1} e_\nu, \dots, x_\nu^{\mu_r} e_\nu) = \sum_{\mu,\nu} \lambda_\nu^\mu \epsilon_\sigma \phi(x_\nu^{\mu_1} e_\nu, \dots, x_\nu^{\mu_r} e_\nu) = \epsilon_\sigma \Phi$$

Conversely, $\sigma\Phi = \epsilon_\sigma \Phi$ implies

$$\sum_{\mu,\nu} \lambda_\nu^\mu \sigma\phi = \sum_{\mu,\nu} \lambda_\nu^\mu \epsilon_\sigma \phi$$

so $\sum_{\mu,\nu} \lambda_\nu^\mu (\sigma\phi - \epsilon_\sigma \phi) = 0$ for all $x_\nu^{\mu_1} e_\nu, \dots, x_\nu^{\mu_r} e_\nu$, then $\sigma\phi = \epsilon_\sigma \phi$. □

Theorem 3.2. Every r-swlin map $\Phi(x_1, \dots, x_m)$ determines an r-swlin skewsymmetric map Ψ, given by

$$\Psi = \sum_\sigma \epsilon_\sigma \sigma\Phi = \sum_{\mu,\nu} \sum_\sigma \lambda_\nu^\mu \epsilon_\sigma \; \sigma\phi(x_\nu^{\mu_1} e_\nu, \dots, x_\nu^{\mu_r} e_\nu)$$

where the second sum on right side is over all permutations $\sigma \in S_r$.

Proof. For any $\tau \in S_r$

$$\tau\Psi = \sum_{\mu,\nu} \tau(\sum_\sigma \lambda_\nu^\mu \epsilon_\sigma \sigma\phi) = \sum_{\mu,\nu} \epsilon_\tau(\sum_\sigma \lambda_\nu^\mu \epsilon_\sigma \sigma\phi) = \epsilon_\tau(\sum_{\mu,\nu} \sum_\sigma \lambda_\nu^\mu \epsilon_\sigma \sigma\phi) = \epsilon_\tau \Psi.$$

□

Theorem 3.3. Let $\Phi = \sum_{\mu,\nu} \lambda_\nu^\mu \phi : E^m \to F$ be an r-swlin skewsymmetric map, then Φ is completely determined by its values on a basis of E and by the constants λ_ν^μ.

Proof. Let $\{e_\nu\}$ be a basis of E. Let $x^i = \sum_{\xi=1}^n x_\xi^i e_\xi,\ i = 1,\dots,m$ be vectors in E and $X = (x_\xi^i)$, then

$$\begin{aligned}
&\Phi(x_1,\dots,x_m) = \Phi(\textstyle\sum_{\xi=1}^n x_\xi^1 e_\xi,\dots,\sum_{\xi=1}^n x_\xi^m e_\xi)\\
&= \textstyle\sum_{\mu,\nu} \lambda_\nu^\mu \phi((\sum_{\xi=1}^n x_\xi^{\mu_1} e_\xi)_\nu,\dots,(\sum_{\xi=1}^n x_\xi^{\mu_r} e_\xi)_\nu) \qquad \nu \in I_r^n,\ \mu \in I_r^m\\
&= \textstyle\sum_{\mu,\nu} \lambda_\nu^\mu (\sum_{\rho=\rho_1,\dots,\rho_r} \epsilon_\rho x_{\nu_{\rho_1}}^{\mu_1}\cdots x_{\nu_{\rho_r}}^{\mu_r} \phi(e_{\nu_{\rho_1}},\dots,e_{\nu_{\rho_1}})) \qquad \rho \in S_r\\
&= \textstyle\sum_{\mu,\nu} \lambda_\nu^\mu |X_\nu^\mu| \phi(e_{\nu_1},\dots,e_{\nu_r})
\end{aligned}$$

where X_ν^μ is the square submatrix of X determined by rows indexed by ν and columns indexed by μ. □

Example 3.1. Let $\Phi : (\Re^3)^3 \to \Re^3$ be a 2-swlin skewsymmetric map defined by

$$\Phi(x_1,x_2,x_3) = \sum_{(i_1,i_2),(j_1,j_2)\in I_2^3} \lambda_{i_1,i_2}^{j_1,j_2} \phi \begin{pmatrix} x_{i_1,j_1} & x_{i_1,j_2}\\ x_{i_2,j_1} & x_{i_2,j_2}\end{pmatrix}$$

where $x_i = \sum_{k=1}^3 x_{k,i} e_k \quad \in \Re^3$. Then

$$\begin{aligned}
\Phi(x_1,x_2,x_3) &= \sum_{(i_1,i_2),(j_1,j_2)\in I_2^3} \lambda_{i_1,i_2}^{j_1,j_2} \phi(x_{i_1 j_1} e_{i_1} + x_{i_2 j_1} e_{i_2}, x_{i_1 j_2} e_{i_1} + x_{i_2 j_2} e_{i_2})\\
&= \sum_{(i_1,i_2),(j_1,j_2)\in I_2^3} \lambda_{i_1,i_2}^{j_1,j_2} \phi \begin{vmatrix} x_{i_1,j_1} & x_{i_1,j_2}\\ x_{i_2,j_1} & x_{i_2,j_2}\end{vmatrix} \phi(e_{i_1}, e_{i_2})
\end{aligned}$$

Definition 3.3. Let $\{e_\nu\}$ be a basis of E, then an r-swlin skewsymmetric map $\Delta_E(x_1,\dots,x_m) \ :\ E^m \to \Gamma$ such that $\phi(e_{\nu_1},\dots,e_{\nu_r}) = 1, \quad \nu \in I_r^n$, is said an r-determinant function.

The scalar $det_{r,\lambda} X = \sum_{\mu,\nu} \lambda_\nu^\mu |X_\nu^\mu|$ will be said the (r,λ)-determinant of $X = (x_\xi^i)$, relative to the basis $\{e_\nu\}$. If $\lambda_\nu^\mu = |X_\nu^\mu|$ we denote $det_r X = |X|_r = \sum_{\mu,\nu} |X_\nu^\mu|^2$, see [2].

Example 3.2. In order to obtain a non-trivial example of r-determinant function, consider a 2-swlin function $\Phi = \sum_{\mu,\nu} \lambda_\nu^\mu \phi$ defined by

$$\Phi(x_1,\dots,x_m) = \sum_{\mu,\nu} \lambda_\nu^\mu \langle e^{*\mu_1}, x_\nu^{\mu_1} e_\nu\rangle \cdots \langle e^{*\mu_r}, x_\nu^{\mu_r} e_\nu\rangle$$

that is

$$\phi(x_\nu^{\mu_1} e_\nu,\dots,x_\nu^{\mu_r} e_\nu) = \langle e^{*\mu_1}, x_\nu^{\mu_1} e_\nu\rangle \cdots \langle e^{*\mu_r}, x_\nu^{\mu_r} e_\nu\rangle$$

where $\{e_\nu\}$, $\{e^{*\nu}\}$ are a pair of dual bases in E and $E^* = L(E) = \{f : f : E \to \Gamma,\ f \text{ linear}\}$ respectively, with $dim\, E = dim\, E^* \geq r$. The bilinear function $\langle , \rangle$ is non-degenerate and it is defined by

$$\langle e^{*\mu_i}, x_\nu^{\mu_i} e_\nu\rangle = e^{*\mu_i}(x_\nu^{\mu_i} e_\nu)$$

then

$$\begin{aligned}
\Phi(x_1,\dots,x_m) &= \sum_\mu \lambda_\mu^\mu \langle e^{*\mu_1}, x_{\mu_1}^{\mu_1} e_{\mu_1}\rangle \cdots \langle e^{*\mu_r}, x_{\mu_r}^{\mu_r} e_{\mu_r}\rangle\\
&= \sum_\mu^\mu \lambda_\mu^\mu x_{\mu_1}^{\mu_1} \cdots x_{\mu_r}^{\mu_r}
\end{aligned}$$

The set of the r-swlin maps is denoted by $SW(E^m, T)$. The exponential functor F, from linear spaces to sets, is defined by

$$F(T) = SW(E^m, T) \qquad \text{for any linear space } T$$

$$\begin{cases} F(t) : F(T) \to F(H) \\ F(t) : \Phi \mapsto t \circ \Phi \end{cases} \qquad \text{for any linear } t : T \to H$$

The following proposition states the universality of the r-determinant function.

Theorem 3.4. Let $\Delta_E = \sum_{\mu,\nu} \lambda^\mu_\nu \phi : E^m \to \Gamma$ be an r-determinant function in E, then for any r-swlin skewsymmetric mapping $\Theta = \sum_{\mu,\nu} \lambda^\mu_\nu \theta : E^m \to F$, there is an unique vector $f \in F$ such that

$$\Theta(x_1, \dots, x_m) = (\Delta_E(x_1, \dots, x_m)(f) = \sum_{\mu,\nu} \lambda^\mu_\nu |X^\mu_\nu| f_\nu \qquad \mu \in I^m_r, \ \ \nu \in I^n_r, \ \ x_i \in E$$

where f_ν are the components of the vector

$$f = (\theta(e_{\nu^1_1}, \dots, e_{\nu^1_r}), \dots, \theta(e_{\nu_1^{\binom{n}{r}}}, \dots, e_{\nu_r^{\binom{n}{r}}}))$$

and ν^i are the $\binom{n}{r}$ elements of I^n_r.

Proof. Let $\{e_i\}$, $i = 1, \dots, n$ be a basis of E such that

$$\Delta_E(x_1, \dots, x_m) = \sum_{\mu,\nu} \lambda^\mu_\nu |X^\mu_\nu| \phi(e_{\nu_1}, \dots, e_{\nu_r}) = \sum_{\mu,\nu} \lambda^\mu_\nu |X^\mu_\nu|$$

that is , $\phi(e_{\nu_1}, \dots, e_{\nu_r}) = 1$.

Then, for any r-swlin skewsymmetric map

$$\Psi(x_1, \dots, x_m) = \sum_{\mu,\nu} \lambda^\mu_\nu |X^\mu_\nu| \psi = (\Delta_E(x_1, \dots, x_m))(f)$$

it follows

$$\begin{aligned} \psi(e_{\nu_1}, \dots, e_{\nu_r}) &= \phi(e_{\nu_1}, \dots, e_{\nu_r}) \theta(e_{\nu_1}, \dots, e_{\nu_r}) \\ &= 1 \cdot \theta(e_{\nu_1}, \dots, e_{\nu_r}) \end{aligned}$$

so Θ and Ψ have the same values on the basis $\{e_\nu\}$ and by theorem 3.3 it follows $\Theta = \Psi$. □

If Δ_E and Δ'_E are two r-determinant functions in E, then $\eta\Delta_E + \theta\Delta'_E$, $\eta, \ \theta \in \Gamma$, is a r-determinant function too.

Let Δ_F be an r-determinant function in F and let $\psi : E \to F$ be a linear mapping of vector spaces, where $dim\, E = n$, $dim\, F = t$, then $\Delta_\psi : E^m \to \Gamma$, defined by

$$\Delta_\psi(x_1, \dots, x_m) = \Delta_F(\psi x_1, \dots, \psi x_m) = \sum_{\mu,\tau} \lambda^\mu_\tau \phi_F((\psi x^{\mu_1})_\tau, \dots, (\psi x^{\mu_r})_\tau)$$

is an r-determinant function in E, where $\phi_F : F^r \to \Gamma$ is an r-linear mapping on F, $\mu \in I^m_r$, $\tau \in I^t_r$.

By theorem 3.4, $\Delta_\psi = \Delta_F(f) = \sum_{\mu,\nu,\tau} \lambda^\mu_\tau |X^\tau_\nu| f_\nu$ for an unique vector $f = (f_\nu)$.

Let Δ'_F be another nonnull swilin skewsymmetric map, then

$$\Delta'_F = \Delta_F(g) = \sum_{\mu,\nu,\tau} \lambda^\mu_\tau |X^\tau_\nu| g_\nu$$

and

$$\Delta'_\psi = \Delta_\psi(g) = (\Delta_F(f))(g) = \sum_{\mu,\nu,\tau} \lambda^\mu_\tau |X^\tau_\nu| f_\nu g_\nu = \Delta'_F(f_\nu)$$

so the vector f does not depend on the choise of Δ_F and it is determined by the map ψ, then the notation $f = det\,\psi$.

Example 3.3. Let ψ and A_ψ be a linear map and its matrix respectively, defined by

$$\begin{cases} \psi : \Re^2 \to \Re^3 \\ \psi : (x, y) \mapsto (x, y, x + y) \end{cases} \qquad A_\psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

besides let $\Delta_{\Re^3} : (\Re^3)^3 \to \Re$ be a 2-determinant function and $x_i \in \Re^2$, then

$$\begin{aligned}
\Delta_\psi &= \Delta_{\Re^3}(\psi x_1, \psi x_2, \psi x_3) = \lambda^{12}\phi(\psi x_1, \psi x_2) + \lambda^{13}\phi(\psi x_1, \psi x_3) + \lambda^{23}\phi(\psi x_2, \psi x_3) \\
&= \lambda^{12}\phi(\sum_{i=1}^{2} x_{i1}\psi e_i, \sum_{i=1}^{2} x_{i2}\psi e_i) + \lambda^{13}\phi(\sum_{i=1}^{2} x_{i1}\psi e_i, \sum_{i=1}^{2} x_{i3}\psi e_i) \\
&\quad + \lambda^{23}\phi(\sum_{i=1}^{2} x_{i2}\psi e_i, \sum_{i=1}^{2} x_{i3}\psi e_i) \\
&= \lambda^{12}|X^{12}|\phi(\psi e_1, \psi e_2) + \lambda^{13}|X^{13}|\phi(\psi e_1, \psi e_2) + \lambda^{23}|X^{23}|\phi(\psi e_1, \psi e_2)
\end{aligned}$$

where $|X^{ij}| = \begin{vmatrix} x_{1i} & x_{1j} \\ x_{2i} & x_{2j} \end{vmatrix}$. Since

$$\begin{aligned}
\phi(\psi e_1, \psi e_2) = \phi((1,0,1),(0,1,1)) &= \lambda_{12}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \lambda_{13}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \lambda_{23}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \\
&= \lambda_{12} + \lambda_{13} - \lambda_{23}
\end{aligned}$$

then

$$\begin{aligned}
\Delta_\psi &= \lambda^{12}|X^{12}|det_{2,\lambda}\,\psi + \lambda^{13}|X^{13}|det_{2,\lambda}\,\psi + \lambda^{23}|X^{23}|det_{2,\lambda}\,\psi \\
&= \Delta_{\Re^3}(det_{2,\lambda}\,\psi)
\end{aligned}$$

The expression for $det\,\psi$ may be obtained immediately by the matrix A_ψ, see [2]

$$det_{2,\lambda} A_\psi = det_{2,\lambda}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \lambda_{12}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \lambda_{13}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \lambda_{23}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = \lambda_{12} + \lambda_{13} - \lambda_{23}$$

Theorem 3.5. Let $\psi : E \to F$ be a linear mapping and $A_\psi = (\alpha^\tau_\nu)$ its matrix relative to the bases $\{e_\nu\}$, $\{f_\tau\}$, $\nu = 1, \dots, n$, $\tau = 1, \dots, t$. Let $\Delta_F = \sum_{\mu,\tau} \lambda^\mu_\tau \phi_F : F^m \to \Gamma$ be an r-determinant function. If $\phi_F(f^{\mu_1}_\tau, \dots, f^{\mu_r}_\tau) = 1$, then

i)

$$\Delta_\psi(x_1, \dots, x_m) = \sum_{\mu,\tau} \lambda^\mu_\tau (\sum_\nu |X^\mu_\nu||A^\tau_\nu|) \quad \mu \in I^m_r,\ \nu \in I^n_r,\ \tau \in I^t_r$$

ii)

$$\Delta_\psi(e_1, \dots, e_n) = \sum_{\nu,\tau} \lambda^\nu_\tau |A^\tau_\nu|$$

where A_ν^τ is the submatrix of A determined by rows indexed by ν and columns indexed by τ, for $\nu = \nu_1, \dots, \nu_r \in I_r^n$, $\tau = \tau_1, \dots, \tau_r \in I_r^t$. The vectors $x_1, \dots, x_m$, relative to the basis $\{e_\nu\}$, are expressed by $x^\mu = \sum_{\nu=1}^n x_\nu^\mu e_\nu$, $\mu = 1, \dots, m$ and $X = (x_\nu^\mu)$.

Proof. i)

$$\begin{aligned}
&\Delta_\psi(x_1, \dots, x_m) = \Delta_F(\psi x_1, \dots, \psi x_m) = \Delta_F(\textstyle\sum_{\nu=1}^n x_\nu^1 \psi e_\nu, \dots, \sum_{\nu=1}^n x_\nu^m \psi e_\nu)\\
&= \Delta_F(\textstyle\sum_{\nu=1}^n x_\nu^1 \sum_{\tau=1}^t \alpha_1^\tau f_\tau, \dots, \sum_{\nu=1}^n x_\nu^m \sum_{\tau=1}^t \alpha_m^\tau f_\tau)\\
&= \Delta_F(\textstyle\sum_{\tau=1}^t (\sum_{\nu=1}^n x_\nu^1 \alpha_\nu^\tau) f_\tau, \dots, \sum_{\tau=1}^t (\sum_{\nu=1}^n x_\nu^m \alpha_\nu^\tau) f_\tau)\\
&= \textstyle\sum_{\mu,\tau} \lambda_\tau^\mu \phi_F(((\sum_{\nu=1}^n x_\nu^{\mu_1} \alpha_\nu^\tau) f_\tau), \dots, ((\sum_{\nu=1}^n x_\nu^{\mu_r} \alpha_\nu^\tau) f_\tau) \qquad \tau \in I_r^t,\ \mu \in I_r^m\\
&= \textstyle\sum_{\mu,\tau} \lambda_\tau^\mu (\sum_{\rho=\rho_1,\dots,\rho_r} \epsilon_\rho (\sum_{\nu=1}^n x_\nu^{\mu_1} \alpha_\nu^{\tau_{\rho_1}}) \cdots (\sum_{\nu=1}^n x_\nu^{\mu_r} \alpha_\nu^{\tau_{\rho_r}})) \phi_F(f_\tau^{\rho_1}, \dots, f_\tau^{\rho_r})
\end{aligned}$$

$\rho \in S_r$, by

$$\sum_{\rho=\rho_1,\dots,\rho_r} \epsilon_\rho (\sum_{\nu=1}^n x_\nu^{\mu_1} \alpha_\nu^{\tau_{\rho_1}}) \cdots (\sum_{\nu=1}^n x_\nu^{\mu_r} \alpha_\nu^{\tau_{\rho_r}}) = \sum_\nu |X_\nu^\mu| |A_\nu^\tau|$$

it follows i).
ii) It is a special case of i) for $X = I_n$. □

The scalar $det_{r,\lambda}\psi = \sum_{\mu,\nu} \lambda_\nu^\mu |A_\nu^\mu|$ will be called the (r,λ)-determinant of ψ, relative to the bases $\{e_\nu\}, \{f_\mu\}$. If $\lambda_\nu^\mu = |A_\nu^\mu|$, then $\sum_{\mu,\nu} |A_\nu^\mu|^2$ will be denoted by $det_r\psi$ or $|\psi|_r$.

Theorem 3.6. Let $\psi : E \to F$ and $\theta : F \to G$ be linear mappings of vector spaces. Let Δ_F be a determinant function in F. If $x_1, \dots, x_m$ are vectors in E, then

$$\Delta_{\theta\circ\psi}(x_1, \dots, x_m) = \Delta_\theta \circ \Delta_\psi(x_1, \dots, x_m)$$

Proof.

$$\begin{aligned}
\Delta_{\theta\circ\psi}(x_1, \dots, x_m) &= \Delta_G(\theta \circ \psi(x_1, \dots, x_m))\\
&= \Delta_\theta(\psi(x_1, \dots, \psi x_m))\\
&= \Delta_\theta \circ \Delta_\psi(x_1, \dots, x_m)
\end{aligned}$$

□

4. The (t,k)-forms

Let $\Re_p^n$ be the tangent space of $\Re^n$ at the point p and let $(\Re_p^n)^*$ be the dual space. Let $\Lambda^k(\Re_p^n)^*$ be the linear space of the k-linear alternating maps $\phi : (\Re_p^n)^k \to \Re$, then denote by $\Lambda_t^k(\Re_p^n)^*$, with $k \le t \le n$, the set of all k-linear alternating maps $\phi : (\Re_p^n)^t \to \Re$. The set $\Lambda_t^k(\Re_p^n)^*$, by the usual operations of functions, is a linear space.
If $\phi_1, \dots, \phi_t$ belong to $(\Re_p^n)^*$, then an element $\phi_1 \wedge \dots \wedge \phi_t \in \Lambda_t^k(\Re_p^n)^*$ is obtained by setting

$$(\phi_1 \wedge \dots \wedge \phi_t)(v_1, \dots, v_k) = det_{k,\lambda}\phi_i(v_j) = \begin{vmatrix} \phi_1(v_1) & \cdots & \phi_1(v_k)\\ \cdots & \cdots & \cdots\\ \phi_t(v_1) & \cdots & \phi_t(v_k) \end{vmatrix}$$

where $i = 1, \dots, t,\ j = 1, \dots, k$ and $v_j \in \Re^n$.
Observe that $\phi_1 \wedge \dots \wedge \phi_t$ is k-linear and alternate.

Example 4.1. When ϕ_1, ϕ_2, ϕ_3 belong to $(\Re^3_p)^*$, an element $\phi_1 \wedge \phi_2 \wedge \phi_3 \in \Lambda^2_3(\Re^3_p)^*$ is obtained by the 2-swlin skewsymmetric map

$$(\phi_1 \wedge \phi_2 \wedge \phi_3)(v_1, v_2) = det_{2,\lambda}\phi_i(v_j) = \begin{vmatrix} \phi_1(v_1) & \phi_1(v_2) \\ \phi_2(v_1) & \phi_2(v_2) \\ \phi_3(v_1) & \phi_3(v_2) \end{vmatrix}$$

$$= \sum_{i_1<i_2} \lambda_{i_1 i_2} \begin{vmatrix} \phi_{i_1}(v_1) & \phi_{i_1}(v_2) \\ \phi_{i_2}(v_1) & \phi_{i_2}(v_2) \end{vmatrix}$$

$$(i_1, i_2) \in I^3_2,\ \lambda_{i_1 i_2} \in \Re$$

and $\phi_1 \wedge \phi_2 \wedge \phi_3$ is a bilinear alternating map on the vectors v_1, v_2.

Let $x^i : \Re^n \to \Re$ be the function which assigns to each point of $\Re^n$ its i^{th}-coordinate. Then $(dx^i)_p$ is a linear map in $(\Re^n)^*$ and the set $\{(dx^i)_p;\ i = 1, \dots, n\}$ is the dual basis of the standard $\{(e_i)_p\}$. The element $(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_t})_p$ is denoted by $(dx^{i_1} \wedge \cdots \wedge dx^{i_t})_p$ and belongs to $\Lambda^k_t(\Re^n_p)^*$.

Theorem 4.1. The set $\{(dx^{i_1} \wedge \cdots \wedge dx^{i_t})_p\}$, $i_1, \dots, i_t \in I^n_t$ is a basis for $\Lambda^k_t(\Re^n_p)^*$.

Proof. the elements of $\{(dx^{i_1} \wedge \cdots \wedge dx^{i_t})_p\}$ are linearly independent. In fact, suppose

$$\sum_{i_1,\dots,i_t \in I^n_t} a_{i_1,\dots,i_t} dx^{i_1} \wedge \cdots \wedge dx^{i_t} = 0$$

then, for any $(e_{j_1}, \dots, e_{j_k})$, with $j_1, \dots, j_k \in I^n_k$, it follows

$$\sum_{i_1,\dots,i_t \in I^n_t} a_{i_1,\dots,i_t} dx^{i_1} \wedge \cdots \wedge dx^{i_t}(e_{j_1}, \dots, e_{j_k})$$

$$= \sum_{i_1,\dots,i_t \in I^n_t} a_{i_1,\dots,i_t} \begin{vmatrix} dx^{i_1}e_{j_1} & \cdots & dx^{i_1}e_{j_k} \\ \cdots & \cdots & \cdots \\ dx^{i_t}e_{j_1} & \cdots & dx^{i_t}e_{j_k} \end{vmatrix}$$

$$= \sum_{i_1,\dots,i_t \in I^n_t} a_{i_1,\dots,i_t} \begin{vmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_k} \\ \cdots & \cdots & \cdots \\ \delta^{i_t}_{j_1} & \cdots & \delta^{i_t}_{j_k} \end{vmatrix}$$

$$= \sum_{r_1,\dots,r_t} \lambda_{r_1,\dots,r_t} a_{r_1,\dots,r_t} \qquad r_1, \dots, r_t \in (I^n_t)_{j_1,\dots,j_k}$$

$$= 0$$

Without loss of generality, suppose $\lambda_{r_1,\dots,r_t}$ all equal, then the $\binom{n}{k}$ equations $\sum_{r_1,\dots,r_t} a_{r_1,\dots,r_t} = 0$, $r_1, \dots, r_t \in (I^n_t)_{j_1,\dots,j_k}$, $j_1, \dots, j_k \in I^n_k$, are a linear omogeneous full rank system, so it has only the trivial solution. That is $a_{i_1,\dots,i_t} = 0$.
The set $\{(dx^{i_1} \wedge \cdots \wedge dx^{i_t})_p\}$ spans $\Lambda^k_t(\Re^n_p)^*$, in other words any $\phi \in \Lambda^k_t(\Re^n_p)^*$ may be written

$$\phi = \sum_{i_1,\dots,i_t \in I^n_t} a_{i_1,\dots,i_t} dx^{i_1} \wedge \cdots \wedge dx^{i_t} \qquad i_1, \dots, i_t \in I^n_t$$

in fact, if

$$\psi = \sum_{i_1,\dots,i_t \in I^n_t} \phi(e_{i_1}, \dots, e_{i_t}) dx^{i_1} \wedge \cdots \wedge dx^{i_t}$$

then $\psi(e_{i_1},\ldots,e_{i_t}) = \phi(e_{i_1},\ldots,e_{i_t})$ for all $i_1,\ldots,i_t \in I_t^n$, so $\psi = \phi$. Setting $\psi(e_{i_1},\ldots,e_{i_t}) = a_{i_1,\ldots,i_t}$, it follows the expression of ϕ. □

The above proposition generalizes the known theorem about the basis $\{dx^{i_1} \wedge \cdots \wedge dx^{i_k}\}$ of the space $\Lambda^k(\Re_p^n)^*$, see [1].

Theorem 4.2. The linear spaces $\Lambda_t^k(\Re_p^n)^*$ and $\Lambda^k(\Re_p^n)^*$ coincide.

Proof. Let $\omega = (\phi_1 \wedge \cdots \wedge \phi_t)(v_1,\ldots,v_k) \in \Lambda_t^k(\Re_p^n)^*$, then

$$\begin{aligned}\omega &= \sum_{i_1,\ldots,i_k \in I_k^n} \lambda_{i_1,\ldots,i_k} \begin{vmatrix} \phi_{i_1}(v_1) & \cdots & \phi_{i_1}(v_k) \\ \cdots & \cdots & \cdots \\ \phi_{i_k}(v_1) & \cdots & \phi_{i_k}(v_k) \end{vmatrix} \\ &= \sum_{i_1,\ldots,i_k \in I_k^n} \lambda_{i_1,\ldots,i_k} (\phi_1 \wedge \cdots \wedge \phi_k)(v_1,\ldots,v_k)\end{aligned}$$

so $\omega \in \Lambda^k(\Re_p^n)^*$. Conversely, let 0 be the null function in $(\Re_p^n)^*$, then any $\psi \in \Lambda^k(\Re_p^n)^*$ may be written as

$$\psi = (\psi_1 \wedge \cdots \wedge \psi_k)(v_1,\ldots,v_k) = (\psi_1 \wedge \cdots \wedge \psi_k \wedge 0 \wedge \ldots \wedge 0)(v_1,\ldots,v_k)$$

so $\psi \in \Lambda_t^k(\Re_p^n)^*$. □

If $\omega \in \Lambda_t^k(\Re_p^n)^*$, then ω may be decomposed by elements of $\Lambda_{t-j}^k(\Re_p^n)^*$, where $k \le t-j \le t$, in fact

Theorem 4.3. Let $\omega = (\phi_1 \wedge \ldots \wedge \phi_t)(v_1,\ldots,v_k) \in \Lambda_t^k(\Re_p^n)^*$, then

$$\omega = \frac{\lambda_{i_1,\ldots,i_{t-j}}}{(t-k)\cdots(t-k-j+1)} \sum_{I_{t-j}^t} (\phi_{i_1} \wedge \ldots \wedge \phi_{i_{t-j}})(v_1,\ldots,v_k)$$

Proof.

$$\begin{aligned}\omega &= \frac{\lambda_{i_1,\ldots,i_{t-1}}}{(t-k)} \sum_{I_{t-1}^t} (\phi_{i_1} \wedge \ldots \wedge \phi_{i_{t-1}})(v_1,\ldots,v_k) \\ &= \cdots \qquad \cdots \qquad \cdots \\ &= \frac{\lambda_{i_1,\ldots,i_{t-j}}}{(t-k)\cdots(t-k-j+1)} \sum_{I_{t-j}^t} (\phi_{i_1} \wedge \ldots \wedge \phi_{i_{t-j}})(v_1,\ldots,v_k)\end{aligned}$$

indeed ω is the sum of $\binom{t}{k}$ determinants, the last right side has the same number

$$\frac{t\cdots(t-j+2)}{(t-k)\cdots(t-k-j+1)} \binom{t-j}{k} \binom{t-j+1}{t-j}$$

□

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Department DISAG, University of Siena, Piazza S.Francesco, 53100 Siena, ITALY
E-mail address: fineschi@unisi.it