Weakly unimodal domains, anti-exchange properties, and coalitional strategy-proofness of aggregation rules

WEAKLY UNIMODAL DOMAINS, ANTI-EXCHANGE PROPERTIES, AND COALITIONAL STRATEGY-PROOFNESS OF AGGREGATION RULES

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Abstract. It is shown that simple and coalitional strategy-proofness of an aggregation rule on any rich weakly unimodal domain of a convex idempotent interval space are equivalent properties if that space satisfies *interval anti-exchange*, a basic property also shared by a large class of convex geometries including -but not reducing to- trees and Euclidean convex spaces. Therefore, *strategy-proof location problems in a vast class of networks* fall under the scope of that proposition.

It is also established that a much weaker *minimal anti-exchange* property is necessary to ensure equivalence of simple and coalitional strategy-proofness in that setting. An immediate corollary to that result is that such equivalence fails to hold both in certan median interval spaces including those induced by bounded distributive lattices that are not chains, and in certain non-median interval spaces including those induced by partial cubes that are not trees.

Thus, it turns out that anti-exchange properties of the relevant interval space provide a powerful *general* common principle that explains the varying relationship between simple and coalitional strategy-proofness of aggregation rules for rich weakly unimodal domains across different interval spaces, both median and non-median.

MSC 2010 Classification: 05C05, 52021, 52037

JEL Classification number : D71

Acknowledgment: Thanks are due to two anonymous reviewers for their helpful comments and criticisms. The usual disclaimer applies.

1. Introduction

Establishing under what circumstances a strategy-proof aggregation rule or decision mechanism is also coalitionally strategy-proof is a largely open issue of considerable interest that has in fact attracted some attention in the recent literature. In particular, Le Breton, Zaporozhets (2009) and Barberà, Berga, Moreno (2010) provide some quite interesting but very restrictive purely set-theoretic properties that ensure equivalence of strategy-proofness and coalitional strategy-proofness of social choice functions without making any reference whatsoever to the outcome space structure. The present paper will follow an entirely different route, focusing on a specific if vast class of outcome spaces sharing a certain geometric structure and on a strictly related class of large single peaked preference domains which rely precisely on that specific structure.

In that connection *single peaked*[1] *domains* are of special interest because it is well-known that on those domains and for certain outcome spaces where a *median operation* is well-defined, there exist non-dictatorial and non-constant median-based strategy-proof aggregation rules -including *simple majority* which is indeed the only anonymous, neutral and unanimity-respecting rule among them. Moreover, and more to the point, it is also known that in *some* of those single peaked domains *all the strategy-proof aggregation rules are coalitionally strategy-proof* as well (see e.g. Moulin (1980), Danilov (1994)) while *in other cases they are not, and the simple majority -or extended median-rule itself is not coalitionally strategy-proof* (see e.g. Nehring, Puppe (2007 (a),(b)), Savaglio, Vannucci (2014)).

What are then the factors of success for that particular class of robust mechanism design problems? It is crystal clear that existence of a well-defined median rule on the outcome space and restriction to a single peaked domain are key to ensure success. However, in view of the results mentioned above, this cannot be the whole story.

Clearly enough, some further properties of the outcome space must play a role, but which ones and in what combinations with the other requirements?

The present paper will show that entering explicit incidence-geometric considerations can contribute a considerable clarification to that matter, and provide some (partial) answers to the foregoing questions. Let us then briefly outline the approach to be proposed here.

To begin with, we start from a very general notion of single peakedness we label *weak unimodality*: a *weakly unimodal* domain embodies two basic requirements for each admissible preference: (i) existence of *a unique top outcome* and (ii) *consistency with a shared notion of 'compromise'* between every pair of outcomes comprising the top outcome (namely, a true 'compromise' between two such outcomes is never regarded

[1] In the present paper 'single peakedness' is used as a comprehensive non-technical term which is amenable to several specifications including weak unimodality, unimodality and locally strict unimodality as defined in the text below.

by a voter as worse than *both* of its 'extrema'[2]). Thus, a total preorder on a certain interval space is *weakly unimodal* if it has a unique 'top' a and is such that for any b, c of the underlying space, if c lies 'between' the 'top' a and b then its lower contour must include b.

Furthermore a domain of weakly unimodal total preorders is *rich* if for each pair of outcomes a, b there exists a weakly unimodal total preorder of that domain having a as its 'top', and whose upper contour at b is precisely the interval of a and b.

In particular, the *full* weakly unimodal domain is the rich domain consisting of *all* weakly unimodal preferences of the required type (e.g. total preorders, or linear orders).

Arguably, the most fitting environment to introduce the general notion of 'compromise' required by unimodality is perhaps provided by *interval spaces.* An interval space is a set X endowed with a suitable *interval function* $I : X^2 \to \mathcal{P}(X)$ mapping each pair of points of X into a subset of X denoting their (closed) 'interval' namely the set of points located 'between' them (see e.g. Sholander (1952, 1954), Mulder (1980), van de Vel (1993), Coppel (1998)). Then, the available compromises between two outcomes a, b consist precisely of the outcomes that belong to the interval of a and b. *Therefore, we shall henceforth identify the interval space with the compromise-structure of the outcome space which agents have been able to agree upon.*

The present paper addresses the issue of equivalence between simple and coalitional strategy-proofness of aggregation rules on rich weakly unimodal domains in a *general setting of minimally 'regular' interval spaces*, namely *convex* and *idempotent* interval spaces (an interval space is denoted here 'convex' if its intervals are convex in the obvious sense, and 'idempotent' if the degenerate interval between one point and itself reduces precisely to that point).

A sufficient condition for equivalence on any rich weakly unimodal domain (Theorem 1 below) is provided : it is shown that such *equivalence holds whenever the interval space satisfies a certain 'Interval Anti-Exchange' property.* Interval Anti-Exchange is a basic incidence-geometric property that is satisfied by standard Euclidean convex sets, and is shared by all trees[3].

The argument goes as follows:

[2]The subclass of *unimodal domains* obtain if requirement (ii) is extended to *every* ordered pair of outcomes. The alternative subclass of *locally strictly unimodal domains* (also sometimes labeled 'generalized single peaked' in the extant literature) obtain by further requiring the 'compromise' outcome to be *strictly better* than the non-top outcome among its 'extrema'. Unimodal and locally strictly unimodal domains are by far the more widely studied single peaked domains. One of the very few extensive studies of a weakly unimodal domain I am aware of is given in Berga (2002) where the underlying outcome space is a finite product of real bounded chains with the L_1 or *rectilinear metric.*

[3]It is easily checked that Interval Anti-Exchange is indeed independent of minimal 'regularity' of an interval space as defined above (see Coppel (1998) for a thorough discussion of the role of that property in convex geometry as a key property of the important subclass of so-called linear geometries that include Euclidean convex sets and trees).

(a) *strategy-proofness of an aggregation rule f on a rich weakly unimodal domain in a convex idempotent interval space plus Interval Anti-Exchange of that space jointly imply that two arbitrary profiles x_N,y_N of voters' choices result in distinct outcomes $u = f(x_N)$, $v = f(y_N)$ only if there exists at least one agent i among those that choose differently at x_N and y_N such that u is a compromise between v and her choice x_i at x_N;*

but then,

(b) *if x_i is the top outcome of agent i, weak unimodality implies that v cannot be strictly better than u for agent i, hence coalitional strategy-proofness of f follows.*

One significant implication of that result for *location problems in networks* is quite clear: *whenever the network is a tree or indeed any graph whose interval function is convex, idempotent and satisfies Interval Anti-Exchange, any strategy-proof aggregation rule for any corresponding rich weakly unimodal domain is also coalitionally strategy-proof on that domain.*

A much weaker *'Minimal Anti-Exchange' property is also shown to be a necessary condition for equivalence of simple and coalitional strategy-proofness of aggregation rules on rich weakly unimodal domains* (see Theorem 2 below).

It follows that, as a consequence, equivalence fails to hold in any median interval space induced by a bounded distributive lattice (or indeed by a bounded median graph) that is *not* a chain.

Such an equivalence failure is established by proving the existence of a non-trivial non-dictatorial strategy-proof aggregation rule on the relevant rich weakly unimodal domain that admits at least four distinct outcomes in its range and is not immune from coalitional manipulations. In that connection, it should also be emphasized that since constant and dictatorial rules are obviously coalitionally strategy-proof, it follows that **-**from a mechanism-design perspective**-** *equivalence failure on a certain weakly unimodal domain has also some positive, constructive implications because it implies the existence of non-trivial non-dictatorial strategy-proof aggregation rules on that domain.*

Summing up, the *main contributions* of the present paper may be described as follows.

First, the characterization via interval-monotonicity of strategy-proof voting rules for the full unimodal domain of linear orders on the interval space of a tree due to Danilov (1994) is *extended to any rich weakly unimodal domain of total preorders in an arbitrary convex interval space.*

Second, it is shown that *for any minimally 'regular' interval space Interval Anti-Exchange is sufficient to ensure equivalence of simple and coalitional strategy-proofness of aggregation rules on the corresponding rich weakly unimodal domain, both for median*[4] *and non-median interval spaces.*

[4]An interval space is said to be **median** if for any three points a, b, c, the intervals of their three pairs have precisely one point in common (their median). Notice that a convex idempotent interval space may or may not be median, but it is well-known that a median interval space is both convex and idempotent: see e.g. Mulder (1980).

Third, a considerably weaker anti-exchange property called *Minimal Anti-Exchange is shown to be necessary to ensure equivalence of simple and coalitional strategy-proofness of aggregation rules on the corresponding rich weakly unimodal domain:* thus, violation of Minimal Anti-Exchange explains equivalence failure for simple and coalitional strategy-proofness of aggregation rules on rich weakly unimodal domains in certain outcome spaces both median and non-median such as (the interval spaces of) distributive lattices and partial cubes (that are not trees) as defined below, respectively.

Finally, the *implications of the two foregoing results for full weakly unimodal equivalence in several interval spaces arising from outcome sets of special interest listed in Section 2 are pointed out* (and collected under Corollary 2 below). *Several known equivalence and inequivalence results and a few new ones are given a common geometric foundation.* The former include bounded chains and trees (equivalence), cliques ('trivial' i.e. impossibility induced equivalence), bounded distributive lattices other than chains (inequivalence). The latter include Euclidean convex sets, 'dual' learning spaces, networks consisting of joins of cliques (namely, complete graphs) and chains (equivalence), and the 'permutahedron' i.e. the network induced by linear orders on a finite set and their elementary permutations (inequivalence).

Apparently, an explicit consideration of incidence-geometric properties of the underlying outcome space offers a distinctive insight on the reasons underlying the respective success and failure of coalitional strategy-proofness of nice median-based aggregation rules for rich weakly unimodal domains in bounded chains and trees, and in bounded distributive lattices.

The remainder of this paper is organized as follows: Section 2 provides a list of unimodal domains of some interest, including both extensively studied and largely unexplored examples; Section 3 introduces the formal framework of the paper and presents its main results; Section 4 includes a discussion of related literature; Section 5 offers some short concluding remarks; all the proofs are collected in the Appendix.

2. Simple and coalitional strategy-proofness on rich weakly unimodal domains: equivalent properties or not?

This section is devoted to a detailed description of a few remarkable examples of outcome spaces that are covered by the results of the present paper.

As mentioned above, the simple-coalitional strategy-proofness equivalence issue for *some* rich weakly unimodal domains has been partially explored in some specific classes of outcome spaces, including some *median* interval spaces (recall that median interval spaces are those interval spaces such that the intervals of any three points have precisely one point in common, their *median)*.

Indeed, some facts about equivalence of simple and coalitional strategy-proofness (or its failure) on full unimodal domains in some specific *median* interval spaces are well-known. That is largely due to the circumstance that the structure of strategy-proof aggregation rules in those spaces is now well understood: in fact, it has been established that strategy-proof aggregation rules on unimodal domains in median interval spaces can be represented by iterated medians of projections (i.e. dictatorial rules) and constants (see e.g. Moulin (1980), Danilov (1994), Savaglio, Vannucci (2014)). Let us then start with a quick review of the best known classes of examples:

The outcome space is a bounded chain

If $(X, I(\leqslant))$ is the median interval space canonically induced by a bounded chain $(X, \leq)$ with $I(\leqslant)(x,y) = \{z \in X : x \leq z \leq y \text{ or } y \leq z \leq x\}$ for all $x, y \in X$, then the equivalence-issue is settled by the pioneering work of Moulin (1980), showing that (i) the strategy-proof rules for the full unimodal domain on $(X, I(\leqslant))$ are precisely those which can be represented as certain min-max lattice-polynomials , and (ii) all such strategy-proof rules are also coalitionally strategy-proof on the same domain. Thus, simple strategy-proofness and coalitional strategy-proofness are equivalent properties here. In particular, the simple majority rule is coalitionally strategy-proof.

The outcome space is a finite tree

If (X, I) is the median interval space canonically induced by a finite tree $G = (X, E)$ (i.e. a finite connected graph without cycles) [5]- namely $I = I^G$ with

$I^G(x,y) = \{z \in X : z$ lies on the unique shortest path joining x and $y\}$

for all $x, y \in X$-

the equivalence-issue is also settled by Danilov (1994), showing that (i) the strategy-proof rules for the full unimodal domain of *linear* orders on (X, I) are precisely those which can be represented as iterated medians of projections (i.e. dictatorial rules) and constants, and (ii) all such strategy-proof rules are also coalitionally strategy-proof on the same domain. Thus, simple strategy-proofness and coalitional strategy-proofness are equivalent properties for full weakly unimodal domains in finite trees. In particular, the extended median (or simple majority) rule is coalitionally strategy-proof.

[5]For any *graph* (or network) $G = (X, E)$ with vertex set X and edge set $E \subseteq \{\{x, y\} : x, y \in X\}$ the interval space $\mathcal{I} = (X, I^G)$ canonically induced by G is determined by defining the *distance* $d^G(x, y)$ of any outcomes/vertices x, y as the length of the *geodesics* or *shortest paths* connecting them, and including in the interval of two arbitrary outcomes or vertices x, y all the outcomes/vertices lying on one of such geodesics. Thus, for any $x, y \in X$:

$I^G(x,y) = \left\{z \in X : d^G(x,y) = d^G(x,z) + d^G(z,y)\right\}$.

The outcome space is a complete graph or clique

If $\mathcal{I} = (X, I^G)$ is the (non-median) interval space canonically induced by a complete graph or clique (namely a graph $G = (X, E)$ with $E = \{\{x, y\} : x, y \in X\}$) -hence $I^G(x, y) = \{x, y\}$ for all $x, y \in X$-

then the full weakly unimodal domain of total preorders on (X, I) amounts to the domain of all total preorders on X with a unique maximum. Therefore, the restriction of the so-called 'universal domain' to profiles of total preorders with a *unique* maximum may be regarded as *the full weakly unimodal domain over an outcome space without a shared nonempty compromise-structure.* In particular, if $|X| \geq 3$, it follows from the Gibbard-Satterthwaite Theorem (see e.g. Danilov, Sotskov (2002)) that dictatorial rules are the only strategy-proof aggregation rules for the full unimodal domain on (X, I^G) which admit at least *three* distinct outcomes in their range. Moreover, it can also be shown that if $\#X \geq 3$ there are no strategy-proof aggregation rules $f : X^N \rightarrow X$ having precisely *two* distinct outcomes (on this point, see the discussion following Corollary 2 below). Hence constant rules and dictatorial rules are the only strategy-proof aggregation rules on the full weakly unimodal domain of such (X, I^G). Since both constant and dictatorial rules are clearly coalitionally strategy-proof, it also follows that simple and coalitional strategy-proofness are equivalent properties here: the equivalence issue is quite easily and trivially settled for that domain.

The outcome space is a bounded distributive lattice

If $\mathcal{I}$ =(X, I^m) is the (median) interval space canonically induced by an arbitrary bounded distributive lattice $\mathcal{X} = (X, \leqslant, 0, 1)$ that is *not* a chain (as defined by the rule $I^m(x, y) = \{z : x \wedge y \leqslant z \leqslant x \vee y\}$, where $\wedge$ and $\vee$ denote the $\leqslant$-induced g.l.b. and l.u.b. operations[6]), the equivalence-issue is also already settled *in the negative* by Savaglio,Vannucci (2012) showing that (i) the strategy-proof rules for any rich weakly unimodal domain on (X, I^m) are precisely those which can be represented by certain equivalent classes of max-min and min-max lattice-polynomials or as iterated medians of projections (i.e. dictatorial rules) and constants, and (ii) if $(X, \leqslant, 0, 1)$ is a bounded distributive lattice but is not a chain, then there are strategy-proof voting rules on that domain that are not coalitionally strategy-proof.[7]

In particular, simple strategy-proofness and coalitional strategy-proofness are *not* equivalent properties for full weakly unimodal domains in the class of *all* median

[6]Note that $I^m = I(\leqslant)$ whenever $(X, \leqslant)$ is a chain. Another important subclass of (X, I^m) spaces obtain by taking X to be a finite product of bounded real chains as endowed with the 'rectlinear' or 'taxicab' metric L_1 and with corresponding interval function I canonically induced by geodesics i.e. shortest paths (that is precisely the case studied by Barberà, Gul, Stacchetti (1993) with reference to the full locally strictly unimodal domain, and by Berga (2002) with reference to the full weakly unimodal domain).

[7]Nehring, Puppe (2007 (a),(b)) do not address the equivalence-issue as such, but include results implying failure of coalitional strategy-proofness of the extended median rule on the domain of locally strictly unimodal linear orders in Boolean k-hypercubes $\mathbf{2}^k$ with $k \geq 3$.

interval spaces induced by some arbitrary bounded distributive lattice, or by some arbitrary median graphs. Moreover, it can be shown that the simple majority rule retains its strategy-proofness on such domains but may be *not coalitionally strategy-proof.* To check the last point, consider for instance the following example (see Bandelt, Barthélemy (1984) and Nehring, Puppe (2007 (b)) for similar examples on the Boolean cube) . Take the interval space induced by the Boolean square $\mathbf{2}^2 = (2^2, \leqslant)$ where

$\leqslant = \{(0,1),(x_1,1),(x_2,1),(1,1),(0,x_1),(0,x_2),(0,0),(x_1,x_1),(x_2,x_2)\}$

(we also posit for convenience of notation $1 = (1,1)$, $0 = (0,0)$, $x_1 = (1,0)$, $x_2 = (0,1)$.

Notice that such a (median) interval space $\mathcal{I} = (2^2, I^m)$ canonically induced by the Boolean square $\mathbf{2}^2$ is defined as follows:

$I^m(1,0) = I^m(x_1,x_2) = \{1,x_1,x_2,0\}$ and $I^m(x,y) = \{x,y\}$ otherwise.

Let $N = \{1,2,3\}$ and consider $\mathcal{I}$-unimodal and locally strictly $\mathcal{I}$-unimodal preference profiles

$(\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3), (\succcurlyeq'_1, \succcurlyeq'_2, \succcurlyeq'_3)$, defined respectively as follows:

$x_1 \succ_1 1 \succ_1 x_2 \sim_1 0$, $x_2 \succ_2 1 \succ_2 x_1 \sim_2 0$, $0 \succ_3 x_1 \sim_3 x_2 \sim_3 1$,

$x_1 \succ'_1 1 \succ'_1 x_2 \succ'_1 0$, $x_2 \succ'_2 1 \succ'_2 x_1 \succ'_2 0$, $0 \succ'_3 x_1 \succ'_3 x_2 \succ'_3 1$.

Now, it is immediately checked that the median of the top outcomes of both preference profiles is $\mu(x_1,x_2,0) = 0$ because $I^m(x_1,x_2) \cap I^m(x_2,0) \cap I^m(x_1,0) = \{0\}$. However, observe that e.g. $\mu(1,1,z) = 1$ for any $z \in 2^2$. It follows that the median rule μ is in fact manipulable by coalition $\{1,2\}$, hence it is clearly not coalitionally strategy-proof.

Let us now move on to a few interesting classes of networks/interval spaces where -to the best of the author's knowledge- very little is known about the structure of strategy-proof aggregation rules on the corresponding rich weakly unimodal domains.

To begin with, let us consider the class of (convex, idempotent) interval spaces as resulting from the following important class of outcome spaces.

The outcome space is a simplex in an Euclidean convex space

In that case $\mathcal{I} = (X, I^E)$ is the (convex, idempotent) interval space canonically induced by the standard closed m-simplex in an Euclidean convex space, namely $X = \left\{x \in \mathbb{R}_+^{m+1} : \sum_{i=0}^m x_i = 1\right\}$, and for all $x, y \in X$,

$I^E(x,y) = \{z \in X : z = \lambda x + (1-\lambda)y \text{ for some } \lambda \in [0,1]\}$.

That is clearly *not* a median interval space if $m \geq 2$: in fact, any nondegenerate triangle in X fails to admit a median as defined above. Thus, in particular, the simple majority rule (i.e. the extended n-ary median rule with n odd) is only available for $m = 1$. Some work has been devoted to the study of simple and coalitional strategy-proofness of aggregation rules on the domain of all affine total preference preorders on (X, I^E) (see e.g. Danilov, Sotskov (2002)), and of Euclidean preferences with a 'bliss point' on (X, I^E) (see Peters, van der Stael, Storcken (1992)). Notice that -for each agent- the latter Euclidean preferences may be construed as a subclass of locally strictly unimodal total preorders as defined in Note 1 above.[8] However, very little is apparently known about the class of all strategy-proof aggregation rules for rich weakly unimodal domains on (X, I^E), or the existence of non-trivial non-dictatorial strategy-proof aggregation rules on such domains. Proposition 1 and Corollary 2 of the present paper will settle in the negative the latter issue.

The outcome space is a partially ordered set.

Let $(X, \leqslant)$ be a partially ordered set (or poset), and $\mathcal{I} = (X, I(\leqslant))$ the interval space canonically induced by $(X, \leqslant)$, namely

$I(\leqslant)(x,y) = \{x,y\} \cup \{z \in X : x \leqslant z \leqslant y \text{ or } y \leqslant z \leqslant x\}$ for all $x, y \in X$.

Ordered sets are a pervasive structure and interval spaces of that kind encompass a massive collection of cases. The following example concerning *('dual') learning spaces* is worth mentioning (see e.g. Eppstein, Falmagne, Ovchinnikov (2008)). Consider a finite set Y of competences/tasks, and a set $K \subseteq \mathcal{P}(Y)$ (with $\varnothing, X \in K$) denoting (missing) *knowledge states* as represented through the *missing* competences such that for any $A, B \in K$, $A \subseteq B$:

(i) there exist $x_1, ..., x_m \in X$, and a chain $C_i \in K$, $i = 0, 1, ..., m$ with $B = C_0 \supset C_1 \supset ... \supset C_m = A$ such that $C_i = C_{i-1} \smallsetminus \{x_i\}$, $i = 1, ..., m$ denoting an admissible learning process as generated by adding competences one by one in a suitably feasible sequence[9];

(ii) if $x \in A$ and $B \smallsetminus \{x\} \in K$ then $A \smallsetminus \{x\} \in K$ (namely if competence x may be achieved starting from less advanced knowledge state B it may also be achieved starting from more advanced knowledge state A).

In that case the voting problem concerns choice of a target knowledge state: the relevant poset is $(K, \subseteq)$ hence $\mathcal{I} = (K, I(\subseteq))$.

Very little is known about the nature of strategy-proof aggregation rules for rich weakly unimodal domains on $(X, I(\leqslant))$, but the results of the present paper will enable us to predict that all of them -including of course $(K, I(\subseteq))$- are also coalitionally strategy-proof.

The next class of networks is also of considerable interest as models of location problems in a large collection of abstract spaces:

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[8]Namely , $x \succ y$ if $d(top(\succcurlyeq), x) < d(top(\succcurlyeq), y)$ where d denotes the Euclidean metric. Notice however that the corresponding interval spaces are agent-dependent. It follows that preferences of that sort cannot be construed as 'single peaked' in the usual sense, which requires a shared 'compromise-structure' on the outcome space.

[9]That sequence is an instance of a 'shelling process' i.e. decomposition of a structure by repeated elimination of some suitable elements.

The outcome space is a partial cube

Let $\mathcal{I}$ =(X, I^G) the interval space induced by a *partial cube*, namely a graph $G = (X, E)$ which an isometric subgraph [10] of a finite cube [11]. A remarkable example of that case obtains when the vertices of the given partial cube denote the *linear orders* on a finite set of cardinality m, which is precisely the relevant outcome space in strategy-proof preference aggregation problems when preferences are linear orders on a finite set (strategy-proof preference aggregation in a similar setting has been recently investigated by Bossert, Sprumont (2014)).[12] The partial cube having such linear orders as vertices -which is also denoted as the 'm-permutahedron' (see e.g. Eppstein, Falmagne, Ovchinnikov (2008))- is easily shown to be *not a tree* for $m \geq 3$.

Thus, the foregoing list includes examples of outcome spaces where the status of available information on the issue concerning equivalence of simple and coalitional strategy-proofness of aggregation rules for rich weakly unimodal domains is quite diverse. In a few of them, the 'weakly unimodal' equivalence-issue has been addressed and settled, at least for certain full single peaked domains (either affirmatively, for bounded chains, finite trees and complete graphs, or negatively, for bounded distributive lattices - and bounded median graphs - that are not chains). In other cases (e.g. Euclidean simplexes, partial cubes) no general results on the existence of non-trivial non-dictatorial strategy-proof aggregation rules for the full unimodal or weakly unimodal domains are available in earlier works.

It is therefore quite remarkable that the main results of the present paper (i.e. Theorems 1 and 2) jointly address and settle at once such 'weakly unimodal' equivalence-issue in all of the outcome spaces considered above (and several others).

Let us then eventually turn to the formal setting and the ensuing analysis.

3. Strategy-proofness and anti-exchange properties: model and results

Let $N = \{1, .., n\}$ denote the finite population of agents (with cardinality $|N| = n$), X an arbitrary nonempty set of alternative outcomes, and $\mathcal{I} = (X, I)$ the **interval space** of X , namely $I : X^2 \to \mathcal{P}(X)$ is an *interval function on* X i.e. it satisfies the following conditions:

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[10] A graph $G' = (Y, E')$ is an isometric subgraph of graph $G = (X, E)$ if $Y \subseteq X$, the edge set E' is the restriction of edge set E to pairs of vertices in Y, and the length of the shortest paths between any two vertices in Y is the same in G and G'.

[11] A finite cube on a finite set U is the graph having as vertices the subsets of U, with edges connecting any two subsets whose characteristic functions differ by just one value.

[12] The Bossert-Sprumont setting is just 'similar' to the one mentioned in the text in that the 'aggregated' preference relations of a profile of linear orders are allowed to be *non*-antisymmetric total preorders. That approach amounts to an aggregation problem with a *restricted* domain.

I-(i) (**Extension**): $\{x, y\} \subseteq I(x, y)$ for all $x, y \in X$,

I-(ii) (**Symmetry**): $I(x, y) = I(y, x)$ for all $x, y \in X$.

Notice that for any $Y \subseteq X$, an interval space $\mathcal{I} = (X, I)$ induces a natural interval space on Y, namely its *interval subspace* $\mathcal{I}_Y = (Y, I_Y)$ where I_Y denotes the restriction of I to Y^2.

In particular, we also assume $n \geq 2$ in order to avoid tedious qualifications, and will be mostly concerned with *idempotent* interval spaces i.e. with interval spaces whose interval function also satisfy the following conditions, namely

(**Idempotence**): $I(x, x) = \{x\}$ for all $x \in X$.

(**Convexity**): $I(u, v) \subseteq I(x, y)$ for all $x, y \in X$, and $u, v \in I(x, y)$.

Remark 1. Observe that Idempotence and Convexity are indeed mutually independent properties of interval spaces. To confirm that statement, consider interval spaces $\mathcal{I}_1 = (X, I_1)$, $\mathcal{I}_2 = (\{x, y, v, z\}, I_2)$ where $|X| > 1$, $|\{x, y, v, z\}| = 4$, $I_1(a, b) = X$ for all $a, b \in X$, while $I_2(x, y) = \{x, y, z\}$, $I_2(y, z) = \{y, v, z\}$, and $I_2(a, b) = \{a, b\}$ for all $a, b \in X$ such that $\{x, y\} \neq \{a, b\} \neq \{y, z\}$. It is immediately checked that $\mathcal{I}_1$ is convex but not idempotent, while $\mathcal{I}_2$ is idempotent but not convex since $\{y, z\} \subseteq I_2(x, y)$ and $v \in I_2(y, z) \setminus I_2(x, y)$.[13]

Finally, we should also mention that an interval space $\mathcal{I} = (X, I)$ is said to be a **median space** if I satisfies the following

(**Median Property**): for all $x, y, z \in X$, $|I(x, y) \cap I(y, z) \cap I(x, z)| = 1$.

The common point of the three intervals defined by each pair of any three points x, y, z in a median interval space (X, I) is said to be the *median* of those points, that therefore defines a ternary operation on X.

It is well-known that e.g. the interval spaces induced by trees or median semilattices (including distributive lattices) are median (see Sholander (1952), (1954)), and that any median interval space is also convex (see Mulder (1980), Theorem 3.1.4) and idempotent. It should also be emphasized that all the properties of an interval space considered above *are inherited by its interval subspaces.*

[13] An *idempotent* interval space (X, I) is said to be a *convex geometry* if it satisfies

(*Peano Convexity*) for all $x, y, v_1, v_2, z \in X$, if $y \in I(x, v_1)$ and $z \in I(y, v_2)$ then there exists $v \in I(v_1, v_2)$ such that $z \in I(x, v)$.

It can be quite easily shown that a convex geometry is in particular a convex interval space (see e.g. Coppel (1998), chpt.2, Proposition 1), but the converse however does not hold. Therefore all of the results of the present paper clearly hold in particular when restricting the statements to convex geometries.

Let $\succcurlyeq$ denote a total preorder i.e. a reflexive, connected and transitive binary relation on X (we shall denote by $\succ$ and $\sim$ its asymmetric and symmetric components, respectively. The following notation shall also be occasionally used to denote the upper and lower contours of $\succcurlyeq$: for any $x \in X$, $UC(\succcurlyeq, x) := \{y \in X : y \succcurlyeq x\}$, $LC(\succcurlyeq, x) := \{y \in X : x \succcurlyeq y\}$, $UC^*(\succcurlyeq, x) := \{y \in X : y \succ x\}$, $LC^*(\succcurlyeq, x) := \{y \in X : x \succ y\}$).

Then, the total preorder $\succcurlyeq$ is said to be **weakly unimodal** with respect to interval space $\mathcal{I} = (X, I)$ - or $\mathcal{I}$-**weakly unimodal**[14] - if and only if

U-(i) there exists a *unique maximum* of $\succcurlyeq$ in X, its *top* outcome -denoted $top(\succcurlyeq)$- and

U-(ii) for all $x, y, z \in X$, if $z \in I(x, y)$ and $x = top(\succcurlyeq)$ then $z \succcurlyeq y$.

We denote by $U_{\mathcal{I}}$ the set of all $\mathcal{I}$-*weakly unimodal* total preorders on X.

A set of weakly unimodal total preorders $D \subseteq U_{\mathcal{I}}$ is **rich** if for any $x, y \in X$ there exists $\succcurlyeq \in D$ such that $top(\succcurlyeq) = x$ and $UC(\succcurlyeq, y) = I(x, y)$.

Two important and widely studied subclasses of $U_{\mathcal{I}}$ are $\mathcal{I}$-**unimodal** and $\mathcal{I}$-**locally strictly unimodal** total preorders on X which obtain by combining U-(i) with

U'-(ii) for all $x, y, z \in X$, if $z \in I(x, y)$ then $\{u \in X : z \succcurlyeq u\} \cap \{x, y\} \neq \varnothing$

and

U''-(ii) for all $x, y, z \in X$, if $z \in I(x, y)$ and $x = top(\succcurlyeq)$ then $z \succ y$, respectively.

An N-profile of $\mathcal{I}$-weakly unimodal total preorders is a mapping from N into $U_{\mathcal{I}}$. We denote by $U_{\mathcal{I}}^N$ the **full I-weakly unimodal domain**, namely the set of *all* N-profiles of $\mathcal{I}$-unimodal total preorders (the **full $\mathcal{I}$-unimodal** and **full $\mathcal{I}$-locally strictly unimodal domains** are similarly defined in the obvious way).

An **aggregation rule** for (N, X) is a function $f : X^N \to X$. The following properties of an aggregation rule are key to the ensuing analysis:

(**Strategy-proofness**) An aggregation rule $f : X^N \to X$ is (simply) **strategy-proof** *on* $D^N \subseteq U_{\mathcal{I}}^N$ iff for all $(\succcurlyeq_i)_{i \in N} \in D^N$, and for all $i \in N$, $y_i \in X$, and $(x_j)_{j \in N} \in X^N$ such that $x_j = top(\succcurlyeq_j)$ for each $j \in N$, $f((x_j)_{j \in N}) \succcurlyeq_i f((y_i, (x_j)_{j \in N \smallsetminus \{i\}}))$.

(**Coalitional strategy-proofness**) An aggregation rule $f : X^N \to X$ is **coalitionally strategy-proof** *on* $D^N \subseteq U_{\mathcal{I}}^N$ iff for all $(\succcurlyeq_i)_{i \in N} \in U_{\mathcal{I}}^N$, and for all $C \subseteq N$, $(y_i)_{i \in C} \in X^C$, and $(x_j)_{j \in N} \in X^N$ such that $x_j = top(\succcurlyeq_j)$ for each $j \in N$, there exists $i \in C$ with $f((x_j)_{j \in N}) \succcurlyeq_i f((y_i)_{i \in C}, (x_j)_{j \in N \smallsetminus C}))$[15].

Clearly, *a* coalitionally strategy-proof aggregation rule is in particular strategy-proof, while the converse may not hold.

[14]The $\mathcal{I}$-qualifier will be typically omitted when the relevant $\mathcal{I}$ is unambiguously identified.

[15]Notice that coalitional strategy-proofness as defined here is only concerned with coalitional manipulations that are *strictly* advantageous for every member of the coalition. That version of coalitional strategy-proofness is the one adopted e.g. by Moulin (1980) and Danilov (1994). Other, stronger notions that are also considered in the extant literature will not concern us here.

(**Interval-monotonicity**) An aggregation rule $f : X^N \to X$ is $\mathcal{I}$-**monotonic** *(or interval-monotonic)* iff for all $i \in N$, $y_i \in X$, and $(x_j)_{j \in N} \in X^N$, $f((x_j)_{j \in N}) \in I(x_i, f(y_i, (x_j)_{j \in N \setminus \{i\}}))$.

We are now ready to state the main results of this paper concerning equivalence of strategy-proofness and coalitional strategy-proofness of aggregation rules *on any rich domain of weakly unimodal profiles.* Our results rely on the following proposition which establishes the equivalence between *monotonicity* with respect to an arbitrary convex interval space $\mathcal{I}$ and *strategy-proofness on any rich weakly unimodal domain* $D^N \subseteq U^N_{\mathcal{I}}$.

Proposition 1. *Let $\mathcal{I} = (X, I)$ be a convex interval space. Then, an aggregation rule $f : X^N \to X$ is strategy-proof on a rich weakly unimodal domain $D^N \subseteq U^N_{\mathcal{I}}$ iff it is $\mathcal{I}$-monotonic.*

Remark 2 Proposition 1 above provides a considerable generalization of Lemma 1 in Danilov (1994) which only refers to the median interval space induced by a finite tree. Moreover, it may also be regarded as a counterpart to -and an extension of-Proposition 3.2 in Nehring, Puppe (2007(a)), which concerns a quite specialized class of convex interval spaces (namely, those induced by 'property spaces' which also satisfy Antisymmetry as discussed below -hence Idempotence- and Separation as defined in Nehring, Puppe (2007) (a), p. 278, i.e. the S_3 property of van de Vel (1993), p. 53).

Actually, it transpires from the proof of Proposition 1 that $\mathcal{I}$-monotonicity of an aggregation rule $f : X^N \longrightarrow X$ implies its strategy-proofness on any rich domain $D^N \subseteq U^N_{\mathcal{I}}$ for *any* interval space $\mathcal{I}$, *whether convex or not.* However, in order to prove that strategy-proofness of f on D^N invariably implies $\mathcal{I}$-monotonicity of f, convexity of $\mathcal{I}$ cannot be dispensed with. To see this, consider the following simple example with $N = \{1, 2\}$, $\mathcal{I} = (X = \{x, y, u, v, z\}, I)$ such that $|X| = 5$, $I(x, y) = \{x, y, u, v\}$, and $I(a, b) = X$, $I(a, a) = \{a\}$ for all $a, b \in X$ such that $a \neq b$ and $\{a, b\} \neq \{x, y\}$. Clearly, $\mathcal{I}$ is by construction *not* convex since e.g. $u, v \in I(x, y)$ but $z \in I(u, v) \setminus I(x, y)$. Now, take $D^N = U^N_{\mathcal{I}}$ and consider an aggregation rule $f : X \times X \longrightarrow X$ such that $f(a, b) = f(a, c)$ for all $a, b, c \in X$, $f(x, x) = z$ and $f(a, x) = a$ for all $a \in X \setminus \{x\}$. By construction, $f(x, x) \notin I(x, f(y, x))$ hence f is *not* $\mathcal{I}$-monotonic. Now, notice that agent 2 is a dummy hence f -if it is not strategy-proof on $U^N_{\mathcal{I}}$ - can only be manipulated by agent 1. Let us now check that in fact agent 1 *cannot manipulate* f and therefore f is *strategy-proof* on $U^N_{\mathcal{I}}$. Indeed, suppose to the contrary and without loss of generality that there exists $\succcurlyeq \in U_{\mathcal{I}}$ such that $x = top(\succcurlyeq)$ and $x \succ y \succ z$. We may distinguish two possible cases: (i) there exists $a \in X \setminus \{x, y\}$ such that $a \succcurlyeq y$, and (ii) $y \succ a$ for all $a \in X \setminus \{x, y\}$. If (i) holds then $z \in X = I(y, a)$ whence weak unimodality is violated i.e. $\succcurlyeq \notin U_{\mathcal{I}}$, a contradiction. If (ii) holds, then $x \succ y \succ a \in I(x, y)$ hence again $\succcurlyeq \notin U_{\mathcal{I}}$, a contradiction, and strategy-proofness of f is therefore established.

The following property will play a pivotal role in the ensuing analysis

(**Interval Anti-Exchange (IAE)**): for all $x, y, v, z \in X$ such that $x \neq y$, if $x \in I(y,v)$ and $y \in I(x,z)$ then $x \in I(v,z)$.[16]

The next condition is a considerably weakened version of IAE:

(**Minimal Anti-Exchange** *(MAE)*): for all $x, y, v, z \in X$ such that $x \neq y$ and $v \neq z$ at least one of the following clauses is satisfied: (i) $I(y,v) \cap \{x,z\} \neq \{x,z\}$, (ii) $I(x,z) \cap \{y,v\} \neq \{y,v\}$, (iii) $I(v,z) \cap \{x,y\} \neq \varnothing$, (iv) $I(y,z) \cap \{x,v\} \neq \varnothing$.

Remark 3 It is easily checked that, for an arbitrary interval space $\mathcal{I} = (X, I)$, IAE does indeed entail MAE, while the converse statement does not generally hold. To see this, observe that by definition IAE amounts to requiring that for all $x, y, v, z \in X$ such that $x \neq y$, at least one of the following three clauses is satisfied: (i') $x \notin I(y,v)$, (ii') $y \notin I(x,z)$, (iii') $x \in I(v,z)$. Clearly, (i'),(ii') and (iii') entail (i),(ii) and (iii), respectively, whence MAE holds true whenever IAE does. On the other hand, consider interval space $\mathcal{I} = (X, I)$ with $X = \{x, y, v, z\}$, $|X| = 4$, and I as defined by the following rule: $I(x,z) = \{x,y,z\}$, $I(y,v) = \{x,y,v\}$, and $I(a,b) = \{a,b\}$ otherwise. Notice that, by construction, $\mathcal{I}$ is convex and idempotent. Moreover, $I(x,z) \cap \{y,v\} = \{y\} \neq \{y,v\}$ hence $\mathcal{I}$ satisfies MAE. However, $x \in I(y,v)$, $y \in I(x,z)$, and $x \notin I(v,z)$: therefore $\mathcal{I}$ fails to satisfy IAE.

The next proposition provides a remarkable property of $\mathcal{I}$-monotonic aggregation rules when $\mathcal{I}$ satisfies Interval Anti-Exchange:

Proposition 2. *Let $\mathcal{I} = (X, I)$ be an interval space that satisfies Interval Anti-Exchange, and $f : X^N \to X$ an $\mathcal{I}$-monotonic aggregation rule. Then, for all $x_N, y_N \in X^N$, $f(x_N) \neq f(y_N)$ entails that $f(x_N) \in I(x_i, y_i)$ for some $i \in N$.*

As suggested by the crucial role it plays in the proof of the foregoing proposition, Interval Anti-Exchange is definitely required to ensure that the property of $\mathcal{I}$-monotonic

[16] It should be noticed here that Interval Anti-Exchange, Idempotence and Convexity are mutually independent properties of an interval space. To check that statement, consider the following interval spaces: (i) $(X = \{x, y, u, v\}, I)$ with $I(x,y) = \{x,u,y\}$, $I(u,y) = \{u,v,y\}$ and $I(a,b) = \{a,b\}$ for all $\{a,b\} \notin \{\{x,y\},\{u,y\}\}$, which is by construction idempotent and can be easily shown to satisfy IAE, but is clearly not convex; (ii) $(X = \{x,y\}, I)$ with $I(x,x) = I(x,y) = \{x,y\}$, $I(y,y) = \{y\}$: that interval space is not idempotent but -as it is easily seen- it satisfies IAE and is obviously convex; (iii) $(X = \{x,y,z\}, I)$ with $I(x,z) = I(y,z) = \{x,y,z\}$, and $I(a,b) = \{a,b\}$ for all $\{a,b\} \notin \{\{x,z\},\{y,z\}\}$, which is by construction idempotent and convex but fails to satisfy IAE since $x \in I(y,z)$, $y \in I(x,z)$ and $x \notin I(z,z)$.

aggregation rules identified by Proposition 2 does indeed hold. To confirm that, consider again the median interval space $\mathcal{I} = (\mathbf{2}^3, I^m)$ induced by the Boolean cube and the *ternary* median operation μ on $\mathbf{2}^3$ as defined above in Section 2. It is easily checked that $\mathcal{I} = (\mathbf{2}^3, I^m)$ does *not* satisfy Interval Anti-Exchange: e.g. $x_1 \in I^m(x_5, 1)$, $x_5 \in I^m(x_1, x_3)$ but $x_1 \notin I^m(1, x_3) = \{1, x_3\}$. It can also be shown that μ is $\mathcal{I}$-monotonic, because projections and constants are obviously $\mathcal{I}$-monotonic, and the median preserves $\mathcal{I}$-monotonicity and can be represented as an iterated median of projections and constants (see Danilov, Sotskov (2002) and Savaglio, Vannucci (2014) for details). Next, take a $\mathcal{I}$-unimodal preference profile $(\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3)$ such that $top(\succcurlyeq_1) = x_1$, $top(\succcurlyeq_2) = x_2$, $top(\succcurlyeq_3) = 0$, and $x_3 \succ_i x_4$ for all $i \in \{1, 2, 3\}$ as previously considered in Section 2, and notice that $\mu(x_1, x_2, 0) = x_4$, while $\mu(x_3, x_3, 0) = x_3$. However, $I^m(x_1, x_3) = \{1, x_1, x_3, x_5\}$, $I^m(x_2, x_3) = \{1, x_2, x_3, x_6\}$, $I^m(0, 0) = \{0\}$ hence $x_4 \notin I^m(x_1, x_3) \cup I^m(x_2, x_3) \cup I^m(0, 0)$: thus, the thesis of Proposition 2 fails to hold for μ (for that choice of $\mathcal{I}$).

The next Theorem establishes that in convex idempotent interval spaces Interval Anti-Exchange ensures that simple (or individual) strategy-proofness and coalitional strategy-proofness of an aggregation rule are equivalent properties on any rich weakly unimodal domain.

Theorem 1. *Let $\mathcal{I} = (X, I)$ be a convex idempotent interval space that satisfies Interval Anti-Exchange (IAE), and $f : X^N \to X$ an aggregation which is strategy-proof on a rich weakly unimodal domain $D^N \subseteq U_{\mathcal{I}}^N$. Then, f is also coalitionally strategy-proof on D^N.*

Observe that, as mentioned in the Introduction, the argument underlying Theorem 1 may be summarized as follows: (a) *strategy-proofness of an aggregation rule f on a rich weakly unimodal domain in a convex idempotent interval space and Interval Anti-Exchange of that space jointly imply that two arbitrary outcome-profiles x_N,y_N result in distinct outcomes $u = f(x_N)$, $v = f(y_N)$ only if -for at least one agent i with $x_i \neq y_i$- u is a 'compromise' between v and i*'s *choice x_i at x_N*; but then, (b) *if x_i is the top outcome of voter i, weak unimodality implies that v cannot be strictly better than u for voter i, whence coalitional strategy-proofness of f follows*.

It should be noticed here that Propositions 1 and 2 and Theorem 1 extend and generalize some properties of the standard interval spaces induced by trees that are pointed out and exploited by Danilov (1994). Moreover, it turns out that *Theorem 1 implies at once that simple/individual and coalitional strategy-proofness on any rich weakly $\mathcal{I}$-unimodal domain are equivalent if $\mathcal{I} = (X, I)$ satisfies the following property, namely* :

(**Antisymmetry**): for all $x, y, z \in X$, if $x \in I(y,z)$ and $y \in I(x,z)$ then $x = y$.[17]

Indeed, the following Corollary to Theorem 1 obtains[18]

Corollary 1. *Let $\mathcal{I} = (X, I)$ be an antisymmetric interval space such that $|X| \leq 3$, and $f : X^N \to X$ a voting rule that is strategy-proof on the full weakly unimodal domain $U_{\mathcal{I}}^N$. Then, f is also coalitionally strategy-proof on $U_{\mathcal{I}}^N$.*

It should also be emphasized that Theorem 1 above amounts to a considerable generalization of two classic previous results on equivalence of simple and coalitional strategy-proofness due to Moulin (1980) and Danilov (1994), concerning bounded chains and finite trees, respectively.

We conclude with a partial converse result. Namely, a convex idempotent interval space $\mathcal{I}$ ensures equivalence of simple and coalitional strategy-proofness on the full weakly unimodal domain only if it also satisfies Minimal Anti-Exchange, as established by the following:

Theorem 2. *Let $\mathcal{I} = (X, I)$ be a convex and idempotent interval space such that every aggregation rule $f : X^N \to X$ which is strategy-proof on any rich weakly unimodal domain $D^N \subseteq U_{\mathcal{I}}^N$ is also coalitionally strategy-proof on D^N. Then, $\mathcal{I} = (X, I)$ satisfies Minimal Anti-Exchange (MAE).*

Notice that minor adaptations of the proof of the foregoing Theorem establish its counterparts concerning both the full unimodal domain and the full locally strictly unimodal domain for $\mathcal{I}$.[19] Notice that Theorem 2 holds for both median and nonmedian

[17]Observe that *Antisymmetry implies Idempotence* of an interval space $\mathcal{I} = (X, I)$: to see that, notice that since $x \in I(y, x)$ by Extension, $y \in I(x, x)$ for some $y \neq x$ entails a violation of Antisymmetry.

[18]To be sure, the present Corollary also follows from a result due to Barberà, Berga, Moreno (2010) establishing that coalitional strategy-proofness holds for any strategy-proof social choice function with an arbitrary domain of profiles of total preorders over an outcome set with at most three outcomes. The proof of the latter result, however, relies on a set-theoretic property called 'sequential inclusion' (see Section 4 below). Thus we 'almost' prove by incidence-geometric arguments the (stronger) proposition previously obtained by Barberà-Berga-Moreno through purely combinatorial arguments. We report here our Corollary 1 and its proof precisely to highlight this point. Notice that unfortunately we cannot ameliorate our result by IAE-based incidence-geometric arguments: to check that, just consider the three-point interval space

$(X = \{x, y, z\}, I)$ with $I(y, z) = I(z, y) = I(x, z) = I(z, x) = X$ and

$I(a, b) = \{a, b\}$ for any other pair $\{a, b\}$.

Clearly, (X, I) is not antisymmetric and violates IAE.

[19]Actually, the proof of Theorem 2 is immediately adapted to the unimodal case. A proof for rich locally strictly unimodal domains is easily produced by replacing preorders $\succcurlyeq^*$ and $\succcurlyeq'$ as defined in the proof of Theorem 2 with a suitable pair of linear orders refining them. More details are available from the author upon request.

interval spaces. One of the simplest examples of a convex idempotent space that fails to satisfy MAE is the median interval space (X, I^m) induced by the Boolean lattice $\mathbf{2^2} = (\{0,1,x,y\}, \vee, \wedge)$ by taking $X = \{0,1,x,y\}$ and defining I^m by the rule $I^m(a,b) = \{c \in X : a \wedge b \leqslant c \leqslant a \vee b\}$ where $u \leqslant v$ if and only if $u = u \wedge v$. Indeed, the results of Savaglio, Vannucci (2014) imply equivalence failure in such an interval space (and, more generally, in any interval space induced by a bounded distributive lattice that is not a chain). Moreover, a simple adaptation of the foregoing proof also shows that the median $\mu : \{0,1,x,y\}^3 \longrightarrow \{0,1,x,y\}$ as defined by the rule $\mu(z_1,z_2,z_3) = (z_1 \wedge z_2) \vee (z_1 \wedge z_3) \vee (z_2 \wedge z_3)$ is not coalitionally strategy-proof on $U_{\mathcal{I}}^N$ with $\mathcal{I} = (\{0,1,x,y\}, I^m)$: to see this, just consider a third total preorder $\succcurlyeq^\circ$such that $x \succ^\circ y \sim^\circ v \sim^\circ z$, and observe that $\succcurlyeq^\circ \in U_{\mathcal{I}}$, $\mu(v,y,x) = x$ and $\mu(z,z,x) = z$, hence coalition $\{1,2\}$ can successfully manipulate the 'sincere' median outcome at unimodal preference profile $(\succcurlyeq^*, \succcurlyeq', \succcurlyeq^\circ) \in U_{\mathcal{I}}^3$ as defined above in the proof of Theorem 2.

Now, as it is well-known, the interval spaces thus induced by distributive lattices are another prominent class of median interval spaces (along with the interval spaces induced by trees). Notice however that since partial cubes typically include cubes, their (typically non-median) interval spaces also violate MAE and therefore -precisely as the interval space of a Boolean distributive lattice $\mathbf{2^K}$ with $\mathbf{K}$ >1- admit nontrivial nondictatorial strategy-proof voting rules (such as rule f as defined in the proof of Theorem 2) that are *not* coalitionally strategy-proof. Indeed, in view of the proof of Theorem 2 (and as also suggested by Corollary 1 above), if a given convex idempotent interval space fails to satisfy MAE then there exists a non-trivial non-dictatorial strategy-proof voting rule on the full weakly unimodal domain of that space that admits at least *four* distinct outcomes, and is manipulable by some coalitions. Therefore Theorem 2 confirms that, generally speaking, 'weakly unimodal' equivalence of simple and coalitional strategy-proofness fails to hold in several important classes of interval spaces, both median and non-median.

In particular, relying on Theorems 1 and 2 we are now ready to provide a definite answer to the question concerning equivalence of simple and coalitional strategy-proofness of aggregation rules for rich weakly unimodal domains in the outcomes spaces considered in Section 2 above. That answer is detailed and included in the next Corollary where *'weakly unimodal equivalence'* is to be read as a shorthand for *equivalence of simple and coalitional strategy-proofness of aggregation rules for rich weakly unimodal domains* (in the interval space under consideration), and the inserted bibliography items single out results previously established -or implied- through alternative ad hoc arguments by other Authors (and address the reader to the original sources or Authors).[20]

Corollary 2. *Let $\mathcal{I}$ be a convex and idempotent interval space. Then*

[20] It should also be noticed that we could also easily obtain specialized versions of Theorems 1-2 and Corollary 1 for both full unimodal and full locally strictly unimodal domains.

(i) (Moulin (1980)) if $\mathcal{I}=(X, I(\leqslant))$ *is the median interval space canonically induced by a bounded chain* $\mathcal{X}=(X,\leqslant)$ *then weakly unimodal equivalence holds;*

(ii) (Danilov (1994)) if $\mathcal{I}=(X, I^G)$ *is the median interval space canonically induced by a finite tree* $G=(X,E)$ *(see notes 4 and 5) then weakly unimodal equivalence holds;*

(iii) (Gibbard, Satterthwaite and others) if $\mathcal{I}=(X, I^G)$ *is the interval space canonically induced by a clique* $G=(X,E)$ *then weakly unimodal equivalence holds;*

(iv) (Nehring, Puppe (2007 (b)), Savaglio, Vannucci (2014)) if $\mathcal{I}=(X, I^m)$ *is the median interval space canonically induced by a bounded distributive lattice* $\mathcal{X}=(X,\leqslant,0,1)$ *which is not a chain then weakly unimodal equivalence fails;*

(v) if $\mathcal{I}=(X, I^E)$ *is the interval space canonically induced by a simplex in an Euclidean convex space then weakly unimodal equivalence holds;*

(vi) if $\mathcal{I}=(X, I(\leqslant))$ *is the interval space canonically induced by a partially ordered set* $\mathcal{X}=(X,\leqslant)$ *then weakly unimodal equivalence holds;*

(vii) if $\mathcal{I}=(X, I^G)$ *is the interval space canonically induced by a partial cube* $G=(X,E)$ *which is not a tree then weakly unimodal equivalence fails;*

(viii) if $\mathcal{I}=(X, I^G)$ *is the interval space canonically induced by the graph* $G=(X,E)$ *resulting from the join of a clique and a bounded chain then weakly unimodal equivalence holds.*

Notice that point (i) is established by Moulin (1980) [21] through specific arguments based upon properties of medians in chains, and an explicit proof is only given for a subclass of anonymous strategy-proof social choice functions. Point (ii) is established by Danilov (1994) for the subdomain of unimodal *linear orders* in a tree by means of an argument that *implicitly* relies on Interval Anti-Exchange.

Point (iii) can also be regarded as a corollary to well-known results which can be established without any reference to Interval Anti-Exchange, but some care is needed to articulate a proper argument which substantiates that claim. Indeed, what follows from the Gibbard-Satterthwaite theorem for the 'universal' domain of linear orders is that if $|X|\geq 3$ then the only strategy-proof aggregation rules for the full unimodal domain are the constant rules, the dictatorial rules (which are both also coalitionally strategy-proof), and *possibly some other non-sovereign (i.e. non-surjective) aggregation rules with a two-valued range.* However, it can be easily shown that no such two-valued rule is strategy-proof on a rich weakly unimodal domain.[22] Moreover, if $|X|\leqslant 3$ then the combinatorial argument provided by Barberà, Berga, Moreno (2010) implies that any aggregation rule on X is strategy-proof on an arbitrary domain of total preorders if and only if it is also coalitionally strategy-proof on that domain.

[21]To be sure, Moulin proves a version of Corollary (ii) for a *restricted* unimodal domain where voters are not allowed to have the maximum or the minimum of the chain (or lattice) as their unique optimum. But Moulin's proof can be adapted to the full unimodal domain (and to the full weakly unimodal domain).

[22]Details are available from the author upon request.

Point (iv) concerning equivalence failure on any rich weakly unimodal domain in bounded distributive lattices was recently established by Savaglio, Vannucci (2014) by a direct argument that *implicitly* invokes necessity of Minimal Anti-Exchange, and is implied by Nehring, Puppe (2007 (b)) with reference to the locally strictly unimodal domain.

Thus, the proofs of points (i), (ii), (iii), (iv) of Corollary 2 proposed here offer an *alternative general argument* to establish those points by means of Interval Anti-Exchange and/or Minimal Anti-Exchange.

Points (v)-(vii) of Corollary 2 are -to the best of the author's knowledge- entirely novel results.

Finally, the outcome space of point (viii) is a special case of the space considered by Schummer,Vohra (2002) whose focus however is strategy-proofness on domains of metric-based total preorders with a unique top outcome.

Apparently, interval anti-exchange properties (through Theorems 1 and 2) provide a common unifying approach to 'weakly unimodal equivalence' issues which offers both a new proof of several important well-known theorems and some quite interesting novel results. Some comments on the practical significance of those results when combined with information on the median vs. non-median character of the relevant interval space are in order here.

We have already observed that when the interval space is median and weakly unimodal equivalence obtains (see cases (i),(ii), (vi) of Corollary 2), it follows that the simple majority rule is both well-defined and coalitionally strategy-proof, thus providing a successful solution to a most important mechanism design problem for aggregation rules.

If the relevant interval space is median and weakly unimodal equivalence fails (see case (iv) of Corollary 2) then the simple majority rule is well defined and strategy-proof but is not coalitionally strategy-proof. That fact typically signals that reasonably unbiased and coalitionally strategy-proof aggregation rules may not exist, and acceptable solutions (if they exist at all) have to embody trade-offs between those two requirements.

When weakly unimodal equivalence obtains and the interval space is not median (e.g. cases (iii), (v), (viii) of Corollary 2) then the median-related rules are not-well defined. Thus, strategy-proof aggregation rules may either reduce to dictatorial and constant rules (see case (iii), which is the situation typified by the Gibbard-Satterthwaite impossibility result) or include some minimally unbiased rule (see case (viii)). But in any case, of course, there is nothing to be gained by relaxing the requirement of coalitional strategy-proofness to simple strategy-proofness.

When the interval space is not median and weakly unimodal equivalence fails (e.g. case (vii) of Corollary 2) the simple majority rule is not well-defined, which typically implies that perfectly unbiased strategy-proof aggregation rules are not available. Further work will establishe if and when viable compromises between unbiasedness and (coalitional) strategy-proofness are in fact available

In that connection, point (viii) provides a somewhat positive result for outcome spaces consisting of the join of a bounded chain and a clique, whose interval space is non-median: namely, in that case the rule resulting from the combination of a local dictatorship and a local simple majority rule (along the lines proposed by Schummer, Vohra (2002) in their related but distinct framework) is coalitionally strategy-proof on any rich weakly unimodal domain.

4. Related literature

Since the relationships of our results to previous works concerning strategy-proof aggregation rules and their coalitional strategy-proofness properties have been repeatedly discussed in the text, culminating in Corollary 2, the present section will rather focus on previous attempts to provide *general conditions* under which equivalence between simple and coalitional strategy-proofness obtains.

Le Breton, Zaporozhets (2009) identifies a general *richness* condition on a set $\mathcal{D}$ of total preorders on an outcome set X which ensures that any social choice function $f : \mathcal{D}^N \rightarrow X$ that is *regular* (namely such that *every outcome in* $f(\mathcal{D}^N)$ *is the unique top outcome of some total preorder in* $\mathcal{D}$) *is strategy-proof if and only if it is also coalitionally strategy-proof.*[23] That richness condition is defined as follows: a set $\mathcal{D}$ of total preorders on X is said to be *rich* with respect to *strictly monotonic transformations* (or *SMT-rich)* if for every total preorder $\succcurlyeq$ in $\mathcal{D}$ and every pair of outcomes $x, y \in X$ such that $x \succ y$ and $x = top(\succcurlyeq')$ for some total preorder $\succcurlyeq'$ in $\mathcal{D}$ there exists a a total preorder $\succcurlyeq^*$ in $\mathcal{D}$ such that $x = top(\succcurlyeq^*)$ and $y \succ^* z$ for all $z \neq y$ with $y \succcurlyeq z$.

Le Breton, Zaporozhets (2009) only briefly addresses the case of 'single peaked' domains, claiming essentially that 'single peaked domains' large enough to support surjective social choice functions are SMT-rich: but that paper clearly refers to (what we denoted as) locally strictly unimodal preference profiles in a chain. When moving to general (full) unimodal domains, the verdict concerning SMT-richness is in fact much more blurred.

To begin with, observe that the social choice functions associated to aggregation rules on *full* weakly unimodal domains clearly satisfy by definition the general 'regularity' clause of the Le Breton-Zaporozhets proposition. However, the collection $U_{\mathcal{I}}$ of all weakly unimodal total preorders of a convex and idempotent (or even median) interval space $\mathcal{I} = (X, I)$ may be or may be *not* SMT-rich, depending on the choice of $\mathcal{I}$. For instance, if $\mathcal{I} = (X, I)$ is the interval space of a clique (which, recall, amounts to the

[23]Note that social choice functions whose outcomes only depend on profiles of top outcomes essentially amount to voting rules which inherit the strategy-proofness properties of the former: thus results of this type concerning social choice functions are *also relevant to voting rules.*

'universal' domain of all total preorders with a unique maximum) then $U_{\mathcal{I}}$ is clearly SMT-rich. On the contrary, if $\mathcal{I}=(X,I)$ is the interval space of a Boolean hypercube then $U_{\mathcal{I}}$ is *not* SMT-rich. To see that, consider for the sake of simplicity the Boolean square $2^2=\{(x,y):x,y\in\{0,1\}\}$ and its canonical interval space $\mathcal{I}$, unimodal total preorder $\succcurlyeq$ such that $(1,0)\succ(1,1)\sim(0,1)\sim(0,0)$, and outcomes $(1,0)$, $(0,1)$. SMT-richness of $U_{\mathcal{I}}$ would imply existence of a weakly unimodal total preorder $\succcurlyeq'$in $U_{\mathcal{I}}$ such that $(1,0)\succ'(0,1)\succ'(1,1)$ and $(1,0)\succ'(0,1)\succ'(0,0)$. Since by construction $(1,0)$, $(0,1)$ is a pair of 'opposite points' $I((1,0),(0,1))=2^2$: it follows that $\succcurlyeq'$ is not weakly unimodal since e.g. $L(\succcurlyeq',(1,1))\cap\{(1,0),(0,1)\}=\varnothing$ whence SMT-richness fails here. Concerning SMT-richness of (full) weakly unimodal domains in chains or trees, additional clarifications may be obtained by the discussion of Barberà, Berga, Moreno (2010) to follow.

Indeed, Barberà, Berga, Moreno (2010) also studies general sufficient conditions of a combinatorial nature ensuring equivalence of simple and coalitional strategy-proofness of social choice functions and aggregation rules on arbitrary preference domains. In particular, two such sufficient conditions for single profiles are identified and denoted as *Sequential Inclusion,* a requirement that applies to single preference profiles (i.e. an intraprofile condition), and *Indirect Sequential Inclusion* (an interprofile existence or 'richness' condition based upon Sequential Inclusion). Specifically, for each preference profile $(\succcurlyeq_i)_{i\in N}$ *Sequential Inclusion* relies on a family of binary relations $\succcurlyeq(S((\succcurlyeq_i)_{i\in N},y,z))$ as parameterized by ordered pairs (y,z) of outcomes and defined on $S((\succcurlyeq_i)_{i\in N},y,z)$, the set of agents who strictly prefer y to z at $(\succcurlyeq_i)_{i\in N}$: in particular, agent pair (i,j) is in $\succcurlyeq(S((\succcurlyeq_i)_{i\in N},y,z))$ if and only if i and j are in $S((\succcurlyeq_i)_{i\in N},y,z)$ and $L(z,\succcurlyeq_i)\subseteq L^*(y,\succcurlyeq_j)$. Of course any such $\succcurlyeq(S((\succcurlyeq_i)_{i\in N},y,z))$ is reflexive: Sequential Inclusion requires that all of them be also *connected and acyclic. Indirect Sequential Inclusion* is satisfied by a profile $(\succcurlyeq_i)_{i\in N}$ if *either* $(\succcurlyeq_i)_{i\in N}$ itself satisfy Sequential Inclusion *or* for each pair (y,z) of outcomes there exists a profile $(\succcurlyeq'_i: i\in S((\succcurlyeq_i)_{i\in N},y,z))$ such that: (i) $y\succ'_i z$ for each $i\in S((\succcurlyeq_i)_{i\in N},y,z)$, (ii) $z\succ'_i x$ for each $i\in S((\succcurlyeq_i)_{i\in N},y,z)$ and each outcome $x\neq z$ such that $z\succcurlyeq_i x$, and (iii) $\succcurlyeq(S((\succcurlyeq'_i: i\in S((\succcurlyeq_i)_{i\in N},y,z)))$ is connected and acyclic. A preference domain is then said to satisfy *Sequential Inclusion (Indirect Sequential Inclusion)* if each preference profile in that domain does satisfy it. Moreover, in Barberà, Berga, Moreno (2010) it is also shown that *if* the outcome set and the preference domain $\mathcal{D}^N$ of a social choice function meet a mild *consistency clause* (i.e. they may also be the outcome set and preference domain of a surjective and *regular* social choice function according to the definition by Le Breton and Zaporozhets as reported above) *then* $\mathcal{D}^N$ is SMT-rich only if it also satisfies Indirect Sequential Inclusion. Now, observe that our full weakly unimodal domains obviously satisfy that consistency clause. Hence, when it comes to full weakly unimodal domains $U_{\mathcal{I}}^N$ as discussed in the present paper, Indirect Sequential Inclusion qualifies as a more general sufficient condition than SMT-richness for equivalence of simple and coalitional strategy-proofness.

Under the label 'single peaked domains' Barberà, Berga, Moreno (2010) addresses the case of locally strictly unimodal domains in a chain, and points out that such domains satisfy Sequential Inclusion and therefore support equivalence of simple and coalitional strategy-proofness.

Thus, (Indirect) Sequential Inclusion and related properties seem to work quite well to assess equivalence of simple and coalitional strategy-proofness in (at least some) locally strictly unimodal domains.

However, rich weakly unimodal profiles of total preorders in a chain may well violate Sequential Inclusion, as established by the following example.

Let $\leqslant$ be a linear order on an outcome set $X = \{x, y, z, w\}$ such that $x < y < z < w$, and $N = \{1, 2\}$. Next, consider total preorders $\succcurlyeq_1$ and $\succcurlyeq_2$ on X defined as follows (using a most conventional notation, with $\succ_i$ and $\sim_i$ denoting of course the asymmetric and symmetric components of $\succcurlyeq_i$, $i = 1, 2$):

$x \succ_1 y \succ_1 w \sim_1 z$ and $z \succ_2 y \succ_2 w \sim_2 x$.

Notice that *unimodality* (hence weak unimodality) of $\succcurlyeq_1$ on $(X, I(\leqslant))$ holds because $I(\leqslant)(x, y) = \{x, y\}$ hence $I(\leqslant)(x, y) \cap \{w, z\} = \varnothing$ (and the second unimodality clause is also trivially satisfied), and *unimodality* of $\succcurlyeq_2$ on $(X, \leqslant)$ holds because $I(\leqslant)(z, y) = \{z, y\}$ hence $I(\leqslant)(z, y) \cap \{w, x\} = \varnothing$ (and the second unimodality clause is trivially satisfied, again).

Now, consider $S((R_1, R_2), (y, w))$ and $\succeq (S((R_1, R_2), (y, w)))$ as defined above. Clearly, $S((R_1, R_2), (y, w)) := \{i \in \{1, 2\} : w \in L^*(R_i, y)\} = \{1, 2\}$.

Moreover, $z \in L(R_1, w) \smallsetminus L^*(R_2, y)$, and $x \in L(R_2, w) \smallsetminus L^*(R_1, y)$.

Hence, by definition, neither $1 \succeq (S((R_1, R_2), (y, w)))2$

nor $2 \succeq (S((R_1, R_2), (y, w)))1$. Therefore, $\succeq (S((R_1, R_2), (y, w)))$ is *not connected* : it follows that profile $R_N = (R_1, R_2)$ -while being unimodal on chain $(X, \leqslant)$ - *does not satisfy Sequential Inclusion*.

It can also be easily checked by the reader that the foregoing profile *does not satisfy Indirect Sequential Inclusion either*, because the required strictly monotonic transformation $\succcurlyeq'_1$of unimodal total preorder $\succcurlyeq_1$ should be such that $y \succ'_1 w \succ'_1 z$ hence *not* unimodal on $(X, I(\leqslant))$ since $z \in I(\leqslant)(y, w)$. Furthermore, consider the same profile with $X = 2^2$, $x = (1, 1)$, $y = (1, 0)$, $w = (0, 1)$, $z = (0, 0)$ i.e. the Boolean square as endowed with its canonical interval space (X, I^m) that has been repeatedly considered in the present paper. It can be shown (an exercise left to the reader) that the given profile is unimodal in the Boolean square but does not satisfy Indirect Sequential Inclusion because e.g. the required strictly monotonic transformations of $\succcurlyeq_1$ would necessarily end up again in a total preorder $\succcurlyeq'_1$ such that $y \succ'_1 w \succ'_1 z$ while $z \in I^m(y, w)$ hence *not* unimodal. It follows that, as previously claimed in the Introduction, the results included in Barberà, Berga, Moreno (2010) are in fact quite inconclusive about equivalence of simple and coalitional strategy-proofness on full unimodal domains (let alone full weakly unimodal) in Boolean hypercubes and even in bounded chains (where equivalence is in fact well-established in the full unimodal domain thanks to Moulin (1980)).

Therefore, Indirect Sequential Inclusion and related properties definitely fail in the assessment of simple and coalitional strategy-proofness properties for aggregation rules on *full unimodal domains and therefore on the wider full weakly unimodal domain* as considered in the present paper. That apparent weakness of Indirect Sequential Inclusion should be contrasted with the remarkable effectiveness of Interval Anti-Exchange in the analysis of coalitional strategy-proofness properties on full weakly unimodal domains.

Indeed, a general comment on SMT-richness, Indirect Sequential Inclusion and Interval Anti-Exchange as alternative sufficient conditions for equivalence of simple and coalitional strategy-proofness of an aggregation rule on a rich weakly unimodal domain is in order here. SMT-richness and the even more general Indirect Sequential Inclusion property are meant to apply to *virtually arbitrary preference domains* hence are perforce *just set-theoretic* restrictions, while Interval Anti-Exchange takes full advantage of the *incidence-geometric structure embodied in the interval space underlying the relevant unimodal domain*. As a consequence, it is indeed not so difficult to devise *necessary conditions* for equivalence of simple and coalitional strategy-proofness on rich weakly unimodal domains *that are similar to (in fact a considerable weakening of)* Interval Anti-Exchange, as testified by Theorem 2 of the present paper. By contrast, identifying necessary conditions for such an equivalence by just relying on some weakening of SMT-richness or Indirect Sequential Inclusion is bound to be quite an hard task, that is very unlikely to be accomplished without introducing novel, specific restrictions more or less explicitly related to the relevant interval-space-theoretic structure. As a matter of fact, Le Breton, Zaporozhets (2009) does not address at all the issue of necessary conditions for equivalence, while Barberà, Berga, Moreno (2010) does include a result on necessary conditions for equivalence of simple and coalitional strategy-proofness that relies on Indirect Sequential Inclusion (see Barberà, Berga, Moreno (2010), Theorem 4). However, quite remarkably, the latter result combines Indirect Sequential Inclusion and closure of the preference domain with respect to preference inversions, a condition blatantly violated by full unimodal domains of total preorders, and therefore hardly helpful in the analysis of arbitrary rich weakly unimodal domains. In any case, it still remains to be explored the exact relationship of Interval Anti-Exchange properties to Minimal Anti-Exchange and the conditions -if any- under which the latter is also sufficient to ensure equivalence of simple and coalitional strategy-proofness.

5. Concluding remarks

It should be emphasized again that the sufficient condition for equivalence of simple and coalitional strategy-proofness of aggregation rules on rich weakly unimodal domains which has been established in the present paper is in fact quite general. As repeatedly mentioned above, *Interval Anti-Exchange* (*IAE*) is shared by all trees and indeed by all linear geometries[24] including Euclidean convex spaces but is characteristic of a much larger class of convex idempotent interval spaces. Therefore, our results provide significant information concerning problems of *strategy-proof location in a vast class of networks,* as testified by Corollary 2. Moreover, it can be shown that Theorem 1 also implies that the 'median voter theorem' for committee-decisions - establishing that the extended n-ary median over an 'odd' unimodal domain invariably selects a Condorcet majority winner - holds whenever the underlying median interval space satisfies Interval Anti-Exchange (the details will be spelled out elsewhere).

We have also shown that any convex idempotent interval space where the foregoing 'unimodal' equivalence obtains must satisfy *Minimal Anti-Exchange (MAE),* which in turn implies that such equivalence fails to hold in certain convex idempotent interval spaces, both median and non-median (and that such spaces admit the existence of non-trivial non-dictatorial strategy-proof aggregation rules with at least four distinct outcomes on their full unimodal domains). It remains to be seen whether or not some convex, idempotent interval spaces that satisfy MAE while violating IAE do also support such an equivalence of simple and coalitional strategy-proofness of aggregation rules on rich weakly unimodal domains. For any such interval space (both median and non-median), and for all convex, idempotent interval spaces that satisfy IAE but are not median the search for reasonably unbiased and coalitionally strategy-proof voting rules on large single peaked domains is a somewhat intriguing and possibly challenging open problem for future research.

6. Appendix. Proofs

Proof of Proposition 1.

Let us assume that $f : X^N \to X$ is *not* $\mathcal{I}$-monotonic: thus, there exist $i \in N$, $x'_i \in X$ and $x_N = (x_i)_{i\in N} \in X^N$ such that $f(x_N) \notin I(x_i, f(x'_i, x_{N\smallsetminus\{i\}}))$. Thus, by Extension of I, $x_i \neq f(x_N) \neq f(x'_i, x_{N\smallsetminus\{i\}})$.To begin with, observe that for any $x \in X$, if there exists $\succcurlyeq \in_{\mathcal{I}}$ such that $x = top(\succcurlyeq)$ then $I(x,x) = \{x\}$. Indeed, suppose that there exists $y \neq x$ such that $y \in I(x,x)$: then by definition $\succcurlyeq$ is not weakly unimodal with respect to $\mathcal{I}$ hence $\succcurlyeq \notin D$, a contradiction. Next, consider a total preorder $\succcurlyeq^* \in D$ such that $x_i = top(\succcurlyeq^*)$ and $UC(\succcurlyeq, f(x'_i, x_{N\smallsetminus\{i\}})) = I(x_i, f(x'_i, x_{N\smallsetminus\{i\}}))$. Such a preorder exists since D is rich. Now, by assumption $f(x_N) \in X \smallsetminus I(x_i, f(x'_i, x_{N\smallsetminus\{i\}}))$ while

[24] Once again, the reader is referred to Coppel (1998) for the relevant definitions.

$f(x'_i, x_{N \smallsetminus \{i\}}) \in I(x_i, f(x'_i, x_{N \smallsetminus \{i\}}))$ by Extension, hence by construction $f(x'_i, x_{N \smallsetminus \{i\}}) \succ^* f(x_N)$. Finally, consider $(\succcurlyeq_j)_{j \in N} \in D^N$ such that $x_j = top(\succcurlyeq_j)$ for all $j \in N$ and $\succcurlyeq_i = \succcurlyeq^*$: then, f is *not* strategy-proof on D^N.

Conversely, let f be monotonic with respect to $\mathcal{I}$. Next, consider any weakly unimodal profile $\succcurlyeq = (\succcurlyeq_j)_{j \in N} \in D^N$ and any $i \in N$. By definition of $\mathcal{I}$-monotonicity $f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}}) \in I(top(\succcurlyeq_i), f(x_i, x_{N \smallsetminus \{i\}}))$ for all $x_{N \smallsetminus \{i\}} \in X^{N \smallsetminus \{i\}}$ and $x_i \in X$. But then, since clearly by definition $top(\succcurlyeq_i) \succcurlyeq_i f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}})$, either $f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}}) = top(\succcurlyeq_i)$ or $f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}}) \succcurlyeq_i f(x_i, x_{N \smallsetminus \{i\}})$ by weak unimodality (and totality) of $\succcurlyeq_i$.

Hence, $f(top(\succcurlyeq_i), x_{N \smallsetminus \{i\}}) \succcurlyeq_i f(x_i, x_{N \smallsetminus \{i\}})$ in any case. It follows that f is indeed strategy-proof on D^N.

Proof of Proposition 2. Let $x_N, y_N \in X^N$, and $f(x_N) \neq f(y_N)$. Then, by $\mathcal{I}$-monotonicity of f,

$f(x_N) \in I(x_i, f(y_i, x_{N \smallsetminus \{i\}}))$ and $f(y_i, x_{N \smallsetminus \{i\}}) \in I(y_i, f(x_N))$ for each $i \in N$.

Now, take $i = 1$. If $f(x_N) \neq f(y_1, x_{N \smallsetminus \{1\}})$ then, thanks to Symmetry of I, Interval Anti-Exchange applies, whence $f(x_N) \in I(x_1, y_1)$, and the thesis immediately follows. Let us then suppose that, on the contrary, $f(x_N) = f(y_1, x_{N \smallsetminus \{1\}})$. Next, consider $f(y_1, y_2, x_{N \smallsetminus \{1,2\}})$.

By $\mathcal{I}$-monotonicity of f, $f(y_1, x_{N \smallsetminus \{1\}}) \in I(x_2, f(y_1, y_2, x_{N \smallsetminus \{1,2\}}))$ and $f(y_1, y_2, x_{N \smallsetminus \{1,2\}}) \in I(y_2, f(y_1, x_{N \smallsetminus \{1\}}))$.

If $f(x_N) = f(y_1, x_2, x_{N \smallsetminus \{1,2\}}) \neq f(y_1, y_2, x_{N \smallsetminus \{1,2\}})$ then again, by Interval Anti-Exchange of $\mathcal{I}$, it follows that $f(x_N) = f(y_1, x_2, x_{N \smallsetminus \{1,2\}}) \in I(x_2, y_2)$ as required by the thesis. Thus, assume again that on the contrary $f(x_N) = f(y_1, x_2, x_{N \smallsetminus \{1,2\}}) = f(y_1, y_2, x_{N \smallsetminus \{1,2\}})$. A suitable iteration of the previous argument allows us to establish that either $f(x_N) \in I(x_i, y_i)$ for some $i \in \{1, ..., n-1\}$ or $f(x_N) = f(y_{N \smallsetminus \{n\}}, x_n)$. In the former case the thesis holds. In the latter case, by $\mathcal{I}$-monotonicity of f, $f(x_N) = f(y_{N \smallsetminus \{n\}}, x_n) \in I(x_n, f(y_N))$ and $f(y_N) \in I(y_n, f(y_{N \smallsetminus \{n\}}, x_n))$. Since by hypothesis $f(x_N) \neq f(y_N)$ it follows, by Interval Anti-Exchange of $\mathcal{I}$, that $f(x_N) = f(y_{N \smallsetminus \{n\}}, x_n) \in I(x_n, y_n)$ and the thesis is therefore established.

Proof of Theorem 1. Indeed, suppose that f is not coalitionally strategy-proof on D^N. Then, there exist $S \subseteq N$, $(\succcurlyeq_i)_{i \in N} \in D^N$, $x_N \in X^N$ and $x'_S \in X^S$ such that for all $i \in S$, $top(\succcurlyeq_i) = x_i$ and $f(x'_S, x_{N \smallsetminus S}) \succ_i f(x_N)$ (where $\succ_i$denotes the asymmetric component of $\succcurlyeq_i$).

Notice that it may be assumed without loss of generality that $S = N$: to see this, consider $f_{x_{N \smallsetminus S}} : X^S \to X$ as defined by the rule $f_{x_{N \smallsetminus S}}(y_S) = f(y_S, x_{N \smallsetminus S})$ for all $y_S \in X^S$ and observe that, by construction, $f_{x_{N \smallsetminus S}}$ is both strategy-proof and *not* coalitionally strategy-proof. Let us then posit $f(x_N) = f(x_S) = u$, and $f(x'_N) = f(x'_S) = v$: by construction, $v \succ_i u$ for all $i \in N$. By Proposition 1 above, f is $\mathcal{I}$-monotonic: therefore, $f(v, x'_{N \smallsetminus \{1\}}) \in I(v, f(x'_N)) = I(v, v)$, whence $f(v, x'_{N \smallsetminus \{1\}}) = v$, by idempotence of $\mathcal{I}$.

Similarly, by $\mathcal{I}$-monotonicity of f again, $f(v, v, x'_{N \smallsetminus \{1,2\}}) \in I(v, f(v, x'_2, x'_{N \smallsetminus \{1,2\}})) = I(v, v)$: thus, by idempotence of $\mathcal{I}$ again,

$f(v, v, x'_{N \smallsetminus \{1,2\}}) = v$. A suitable iteration of the same argument establishes that $f(v, v, ..., v) = f(x'_N) = v$.

Now, suppose that there exists $i \in N$, such that $f(x_N) = u \in I(x_i, v)$: since $x_i = top(\succcurlyeq_i)$ and $v \succ_i u$ by assumption, then $\succcurlyeq_i \notin D$ because is not weakly unimodal with respect to $\mathcal{I}$, a contradiction. Therefore, $f(x_N) \notin I(x_i, v)$ for each $i \in N$. By Proposition 2 above it follows that $f(x_N) = f(v, ..., v) = f(x'_N)$, a contradiction again, whence the thesis is established.

*Proof of Corollary 1.*To begin with, notice that if $|X| \leq 3$, then any interval space (X, I) is convex: indeed, recall that in order to be *not* convex an interval space has to include at least two points x, y and two points u, v such that $\{u, v\} \subseteq I(x, y)$ but $I(u, v) \nsubseteq I(x, y)$ whence at least *four* points are needed. It is also immediately checked that *every* antisymmetric interval space $\mathcal{I} = (X, I)$ with $|X| \leq 3$ does satisfy Interval Anti-Exchange: to see that, take $X = \{x, y, z\}$ and assume that on the contrary there exist $a, b, c, d \in X$ such that $a \neq b$, $a \in I(b, c)$, $b \in I(a, d)$, and $a \notin I(c, d)$. Now, $a \notin I(c, d)$ implies $a \notin \{c, d\}$ hence either $c = d$ or $c = b$ or else $d = b$. If $c = d$ then by antisymmetry $a = b$, a contradiction. If $c = b$ then $a \in I(b, b)$ hence by idempotence $a = b$, a contradiction again (recall that, as already observed above, an antisymmetric interval space is also idempotent). Then, it must be the case that $d = b$ whence $a \in I(d, c) = I(c, d)$, a contradiction again. But then, Theorem 1 applies and the proof is complete.

Proof of Theorem 2. Indeed, suppose $\mathcal{I}$ does not satisfy MAE. Then, there exist $x, y, v, z \in X$ such that $x \neq y$, $v \neq z$, $x \in I(y, v)$, $y \in I(x, z)$, $v \in I(x, z)$, $z \in I(y, v)$, $I(v, z) \cap \{x, y\} = \varnothing$ and $I(y, z) \cap \{x, v\} = \varnothing$ (notice that by definition of I it follows at once that $|X| \geq 4$). But then, consider the following total preorder $\succcurlyeq^*$ on $Y = \{x, y, v, z\}$:

$\succcurlyeq^* = \{(v, z), (v, x), (v, y), (z, x), (z, y), (x, y), (y, x), (x, x), (y, y), (v, v), (z, z)\}$,

namely $v \succ^* z \succ^* x \sim^* y$.

Notice that we can assume without loss of generality that $X = Y = \{x, y, v, z\}$: (otherwise, we might apply the following proof to subspace $\mathcal{I}_Y = (Y, I_Y)$, to the same effect).

To begin with, observe that by construction weak $\mathcal{I}$-unimodality of $\succcurlyeq^*$ only requires that both $x \notin I(v, z)$ and $y \notin I(v, z)$. Thus, $\succcurlyeq^*$ is weakly $\mathcal{I}$-unimodal.

Next, consider another total preorder $\succcurlyeq'$on $\{x, y, v, z\}$:

$\succcurlyeq' = \{(y, z), (y, x), (y, v), (z, x), (z, v), (x, v), (v, x), (x, x), (y, y), (v, v), (z, z)\}$,

namely $y \succ' z \succ' x \sim' v$. Clearly, weak $\mathcal{I}$-unimodality of $\succcurlyeq'$ only requires that $x \notin I(y, z)$ and $v \notin I(y, z)$. Thus, $\succcurlyeq'$is also weakly $\mathcal{I}$-unimodal.

Then, consider the class of all voting rules $f' : X^N \to X$ such that for all $\mathbf{u} = u_{N \smallsetminus \{1,2\}} \in X^{N \smallsetminus \{1,2\}}$

$f'(v, y, \mathbf{u}) = x$, and $f'(z, z, \mathbf{u}) = z$.

Let us now show that there exists a voting rule f in that class which is $\mathcal{I}$-monotonic. To see that, observe that $\mathcal{I}$-monotonicity of f amounts precisely to conditions (a)-(l) as listed below: for all $\mathbf{u} \in X^{N \smallsetminus \{1,2\}}$,

(a) $f(x, x, \mathbf{u}) \in I(x, f(y, x, \mathbf{u})) \cap I(x, f(v, x, \mathbf{u})) \cap I(x, f(z, x, \mathbf{u})) \cap$
$\cap I(x, f(x, y, \mathbf{u})) \cap I(x, f(x, v, \mathbf{u})) \cap I(x, f(x, z, \mathbf{u}))$ hence
positing $f(x, x, \mathbf{u}) = x$ is clearly consistent with (a);

(b) $f(y, y, \mathbf{u}) \in I(y, f(x, y, \mathbf{u})) \cap I(y, f(v, y, \mathbf{u})) \cap I(y, f(z, y, \mathbf{u})) \cap$
$\cap\ I(y, f(y, x, \mathbf{u})) \cap I(y, f(y, v, \mathbf{u})) \cap I(y, f(z, y, \mathbf{u}))$ hence positing $f(y, y, \mathbf{u}) = y$ is clearly consistent with (b) (and (a));

(c) $f(v, v, \mathbf{u}) \in I(v, f(x, v, \mathbf{u})) \cap I(v, f(y, v, \mathbf{u})) \cap I(v, f(z, v, \mathbf{u})) \cap$
$\cap\ I(v, f(v, x, \mathbf{u})) \cap I(v, f(y, v, \mathbf{u})) \cap I(y, f(z, y, \mathbf{u}))$ hence positing $f(v, v, \mathbf{u}) = v$ is similarly consistent with the whole of (a),(b) and (c);

(d) $f(x, y, \mathbf{u}) \in I(x, f(y, y, \mathbf{u})) \cap I(x, f(v, y, \mathbf{u})) \cap I(x, f(z, y, \mathbf{u})) \cap$
$\cap\ I(y, f(x, x, \mathbf{u})) \cap I(y, f(x, v, \mathbf{u})) \cap I(y, f(x, z, \mathbf{u}))$ hence it must be the case that $f(x, y, \mathbf{u}) = x$ since by construction $I(x, f(v, y, \mathbf{u})) = I(x, x) = \{x\}$ (also notice that since by construction $x \in I(y, v)$ that value is certainly consistent with the whole of (a),(b),(c),(d) if $\{f(x, v, \mathbf{u}), f(x, z, \mathbf{u})\} \subseteq \{x, v\}$: so let us assume the latter inclusion as well);

(e) $f(v, x, \mathbf{u}) \in I(v, f(x, x, \mathbf{u})) \cap I(v, f(y, x, \mathbf{u})) \cap I(v, f(z, x, \mathbf{u})) \cap$
$\cap\, I(x, f(v, y, \mathbf{u})) \cap I(x, f(v, v, \mathbf{u})) \cap I(x, f(v, z, \mathbf{u}))$ hence $f(v, x, \mathbf{u}) = x$ since $I(x, f(v, y, \mathbf{u})) = I(x, x) = \{x\}$ (notice that that value is certainly consistent with the whole of (a),(b),(c)(d),(e) if
$\{f(y, x, \mathbf{u}), f(z, x, \mathbf{u})\} \subseteq \{x, y\}$ as well: then, let us also assume that inclusion);

(f) $f(y, v, \mathbf{u}) \in I(y, f(x, v, \mathbf{u})) \cap I(y, f(v, v, \mathbf{u})) \cap I(y, f(z, v, \mathbf{u})) \cap$
$\cap\ I(v, f(y, x, \mathbf{u})) \cap I(v, f(y, y, \mathbf{u})) \cap I(v, f(y, z, \mathbf{u}))$ (notice that, therefore, positing $f(y, v, \mathbf{u}) = f(x, v, \mathbf{u}) = f(z, v, \mathbf{u}) = v$ is consistent with (a),(b),(c),(d),(e),(f) as introduced above);

(g) $f(y, z, \mathbf{u}) \in I(y, f(x, z, \mathbf{u})) \cap I(y, f(v, z, \mathbf{u})) \cap I(y, f(z, z, \mathbf{u})) \cap$
$\cap\, I(z, f(y, x, \mathbf{u})) \cap I(z, f(y, y, \mathbf{u})) \cap I(z, f(y, v, \mathbf{u}))$ hence, $f(y, z, \mathbf{u}) = z$ and $f(x, z, \mathbf{u}) = v$ are jointly consistent with (a),(b),(c),(d),(e),(f),(g) since by assumption $z \in I(y, v)$.

(h) $f(v, z, \mathbf{u}) \in I(v, f(x, z, \mathbf{u})) \cap I(v, f(y, z, \mathbf{u})) \cap I(v, f(z, z, \mathbf{u})) \cap$
$\cap\ I(z, f(v, x, \mathbf{u})) \cap I(z, f(v, y, \mathbf{u})) \cap I(z, f(v, v, \mathbf{u}))$: observe that, since $v \in I(x, z)$, $f(v, z, \mathbf{u}) = v$ is indeed consistent with (a),(b),(c),(d),(e),(f),
(g),(h) as introduced above;

(i) $f(z, y, \mathbf{u}) \in I(z, f(x, y, \mathbf{u})) \cap I(z, f(y, y, \mathbf{u})) \cap I(z, f(v, y, \mathbf{u})) \cap$
$\cap\ I(y, f(z, x, \mathbf{u})) \cap I(y, f(z, v, \mathbf{u})) \cap I(y, f(z, z, \mathbf{u}))$ hence $f(z, y, \mathbf{u}) = y$ is consistent with (a),(b),(c),(d),(e),(f),(g),(h),(i)

since $y \in I(x,z) = I(z, f(x,y,\mathbf{u})) = I(z, f(v,y,\mathbf{u}))$;

(l) $f(z,v,\mathbf{u}) \in I(z, f(x,v,\mathbf{u})) \cap I(z, f(y,v,\mathbf{u})) \cap I(z, f(v,v,\mathbf{u})) \cap$
$\cap\, I(v, f(z,x,\mathbf{u})) \cap I(v, f(z,y,\mathbf{u})) \cap I(v, f(z,z,\mathbf{u}))$ hence in view of (e) $f(z,v,\mathbf{u}) = z$ and $f(z,x,\mathbf{u}) = y$ are jointly consistent with (a),(b),(c),(d),(e),
(f),(g),(h),(i),(l) since $z \in I(y,v) = I(v, f(z,x,\mathbf{u})) = I(v, f(z,y,\mathbf{u}))$;

Thus, we have just shown that there indeed exists a voting rule f that satisfies all of the requirements (a)-(l) above, and is therefore $\mathcal{I}$-monotonic, while at the same time being such that for all $\mathbf{u} = u_{N \smallsetminus \{1,2\}} \in X^{N \smallsetminus \{1,2\}}$

$f(v,y,\mathbf{u}) = x$, and $f(z,z,\mathbf{u}) = z$.

Now, take any profile $(\succcurlyeq_i)_{i \in N} \in U_{\mathcal{I}}^N$ of weakly $\mathcal{I}$-unimodal total preorders on X such that $\succcurlyeq_1 = \succcurlyeq^*$and $\succcurlyeq_2 = \succcurlyeq'$.

Then, by construction, $top(\succcurlyeq_1) = v$, $top(\succcurlyeq_2) = y$, $z \succ_1 x$, $z \succ_2 x$, $f(v,y,(top(\succcurlyeq_i)_{i \in N \smallsetminus \{1,2\}})) = x$, and $f(z,z,(top(\succcurlyeq_i)_{i \in N \smallsetminus \{1,2\}})) = z$. It follows that f is *not* coalitionally strategy-proof, yet in view of Proposition 1 f is (individually) strategy-proof, a statement that contradicts our general hypothesis.

Proof of Corollary 2. (i) It follows immediately from Theorem 1 and the proof of point (vi) of the present Corollary as reported below;

(ii) Recall that for any graph $G = (X,E)$, and any $x,y,z \in X$, $x \in I^G(y,z)$ if and only if

$$d^G(y,z) = d^G(y,x) + d^G(x,z).$$

Now, suppose that $x \neq y$, $d^G(y,z) = d^G(y,x) + d^G(x,z)$ and
$d^G(x,v) = d^G(x,y) + d^G(y,v)$.

Since $x \in I^G(y,z)$ and $y \in I^G(x,v)$, x lies on the unique path joining y and z, and y lies on the unique path joining x and v. Hence there exists a path joining z first to x, then x to y, and finally y to v. Since G is a tree, that is in fact the unique path joining z to v. Thus, by construction,

$d^G(z,v) = d^G(z,x) + d^G(x,v)$, and $x \in I^G(z,v)$ i.e. Interval Anti-Exchange holds, and the thesis follows immediately from Theorem 1.

(iii) Indeed, suppose that $x \neq y$, $x \in I^G(y,z)$ and $y \in I^G(x,v)$. Then, since G is a clique, it follows that $x = z$ and $y = v$. But then, $I^G(z,v) = I^G(x,y)$ hence clearly $x \in I^G(z,v)$ and Interval Anti-Exchange holds. Therefore, the thesis follows immediately from Theorem 1.

(iv) Let $\mathcal{X} = (X, \leqslant, 0, 1)$ be a bounded distributive lattice that is not a chain. Thus, there exist $x,y \in X$ such that $x \nleqslant y$ and $y \nleqslant x$. Hence, there also exist $x \wedge y \notin \{x,y\}$, $x \vee y \notin \{x,y\}$: clearly, by construction, $x \wedge y < x \vee y$. Now, consider $Y = \{x,y,x \wedge y, x \vee y\}$, $I^m(a,b)$ with $a,b \in Y$: indeed, by construction, $I^m(x,y) \cap I^m(x \wedge y, x \vee y) \supseteq Y$ while $I^m(a,b) \cap Y = \{a,b\}$ for any $a,b \in Y$ such that $\{a,b\} \notin \{\{x,y\}, \{x \wedge y, x \vee y\}\}$. Then, $I^m(x,y) \cap \{x \wedge y, x \vee y\} = \{x \wedge y, x \vee y\}$, $I^m(x \wedge y, x \vee y) \cap \{x,y\} = \{x,y\}$, $I^m(y, x \wedge y) \cap \{x \vee y, x\} = I^m(x, x \wedge y) \cap \{x \vee y, y\} = \varnothing$

whence Minimal Anti-Exchange does not hold. Therefore, the thesis follows immediately from Theorem 2.

(v) It is well-known that Euclidean convex spaces do satisfy Interval Anti-Exchange (see Coppel (1998)) . A direct proof of that claim is provided here just for the sake of completeness. Let us then suppose without loss of generality that $x \neq y$, $x \in I^E(y,v)$, and $y \in I^E(x,z)$ i.e. there exist real numbers $\lambda_1, \lambda_2 \in (0,1)$ such that $x = \lambda_1 y + (1-\lambda_1)v$ and $y = \lambda_2 x + (1-\lambda_2)z$ (if $\{\lambda_1, \lambda_2\} \cap \{0,1\} \neq \varnothing$ the thesis follows immediately). Then, $x = \lambda_1(\lambda_2 x + (1-\lambda_2)z) + (1-\lambda_1)v$ i.e.

$x = \frac{\lambda_1(1-\lambda_2)}{1-\lambda_1\lambda_2} z + \frac{1-\lambda_1}{1-\lambda_1\lambda_2} v$

where $1 \geq \frac{\lambda_1(1-\lambda_2)}{1-\lambda_1\lambda_2} \geq 0$, $1 \geq \frac{1-\lambda_1}{1-\lambda_1\lambda_2} \geq 0$ and $\frac{\lambda_1(1-\lambda_2)}{1-\lambda_1\lambda_2} + \frac{1-\lambda_1}{1-\lambda_1\lambda_2} = 1$ hence $x \in I^E(v,z)$ and Interval Anti-Exchange holds. Then, the thesis follows immediately from Theorem 1.

(vi) Let $x \neq y$, $x \in I(\leqslant)(y,v)$, $y \in I(\leqslant)(x,z)$. Hence, from $x \neq y$, $x \in I(\leqslant)(y,v)$ it follows that either $x = v$, or $y < x < v$, or else $v < x < y$; moreover, from $x \neq y$, $y \in I(\leqslant)(x,z)$ it follows that either $y = z$, or $x < y < z$, or else $z < y < x$. If $x = v$ or $y = z$ then clearly $x \in I(\leqslant)(v,z)$, while neither ($y < x < v$ and $x < y < z$) nor ($v < x < y$ and $z < y < x$) can possibly hold. If both $y < x < v$ and $z < y < x$ hold then clearly $z < y < x < v$ hence $x \in I(\leqslant)(v,z)$, and if both $v < x < y$ and $x < y < z$ hold clearly $v < x < y < z$ hence $x \in I(\leqslant)(v,z)$. Thus, in any case $x \in I(\leqslant)(v,z)$, Interval Anti-Exchange holds, and the thesis follows immediately from Theorem 1.

(vii) It follows from the observation that a partial cube $G = (X,E)$ which is not a tree must include a 'square' $\{x,y,v,z\}$ with $I^G(x,z) = I^G(y,v) \supseteq \{x,y,v,z\}$ and $I^G(a,b) \cap \{c,d\} = \varnothing$ for any a,b,c,d such that $\{a,b,c,d\} = \{x,y,v,z\}$ and $\{a,b\} \notin \{\{x,z\},\{y,v\}\}$: to such a 'square' the same argument provided in the proof of point (iv) above applies, whence the thesis follows from Theorem 2.

(viii) First, let $X = Y \cup Z$ where Y denotes a chain and Z denote the vertex set of a clique, $G = (X,E)$ with $E = \{\{u,v\} : (u,v) \in Y^2 \cup Z^2$ or $\{u,v\} = \{y^*, z^*\}\}$, $z^* \in Z$ is the (unique) 'hub' that lies on any path connecting an arbitrary element of Y to an arbitrary element of Z, and y^* is the unique element of Y which is adjacent to z^*. Now, observe that by construction there is precisely one shortest path joining any two vertices/outcomes (and z^*,y^* both lie on the path joining one vertex in Y to one vertex in Z). Therefore the same argument previously used for trees under point (ii) applies, Interval Anti-Exchange holds, and the thesis follows from Theorem 1.

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