Uncertainty and information

Alberto Baccini

Forthcoming in *Handbook of the History of Economic Analysis*, G. Faccarello and H.D. Kurz (eds.).

Uncertainty and information are ideas that have a central role in contemporary economics in the domain of decision theory. The expected utility hypothesis is a powerful instruments widely used in theoretical and empirical analysis. Contemporary economics considers the theme of uncertainty as a branch of decision theory. This is not a necessary result of a linear history, rather of an intellectual path that was not easily foreseeable at the end of Nineteenth Century. The leading role in this story is not played by utility, as is usual in the traditional reconstructions of historians of economic thought, but by probability. Uncertainty is in fact a multifaceted concept. It refers to a subjective condition or a mental status of an agent not knowing for certain the consequences of a present or a future event (subjective uncertainty). It refers also to an objective status of things that may results in different outcomes (frequency of occurences), or are knowable only through careful measurements subjected to errors (objective uncertainty). Because the modern developments could happen, two conditions were required: both objective and subjective uncertainty needed to be treated with the device of probability. Both these recognitions slowly emerged between 17th and 20th century.

1. From the origins to the St. Petersburg paradox

The idea of probability originates with gambling. The so called "classical theory of probability was developed in order deal with the particular objective uncertainty in the games of chance. The history of its emergence was masterfully reconstructed by Ian Hacking (1975). Mathematical theory of probability is generally taken to begin in 1654 with a correspondence between Blaise Pascal and Pierre de Fermat where some gambling problems were analyzed. In 1657 Christian Huygens published a *Libellus de Ratiociniis in Ludo Aleae* (*On Calculations in the Game of Dice*) that "for nearly half a century … was the unique introduction to the theory of probability" (David 1962: 115). In this *Libellus* the notion of expectation in gambling was clearly defined in reference to the problem of the fair price for a gamble. The notion of fair price or of a fair entry fee for a game of chance was of practical importance for gamblers deciding to participate in a bet; or for when it was necessary to divide a stake because the gamble was interrupted before its conclusion; or again for when a gambler was asked by another to sell his/her place in the gamble. The mixture of money and chance was considered a natural idea, and the basic tenet of "equipossibility", as named later by Pierre-Simon de Laplace, was based on the symmetry (fairness) of the device used for a gamble. It was therefore possible to consider only games with even chance (even lay). The interest for these problems most likely originated from a robust economic incentive. The rough technology used for the construction of the gambling machine (dice, roulette wheels etc.) allowed that professional gamblers might systematically gain if they made their bets after having discovered the odds realized in a single device. The alternative reference point is the law of chance governing an ideally fair device. The notion of the expected value of a game of chance is the value of a game in a fair device. The fair price is then equated to the expected value of a fair game. This is Huygens' postulate:

> "That my Chance or Expectation to win any thing is worth just such a Sum, as wou'd procure me in the same Chance an Expectation at a fair Lay. As for Example, if any one shou'd put 3 Shillings in one Hand, without letting me know which, and 7 in the other, and give me Choice of either of them; I say, it is the same thing as if he shou'd give me 5 Shillings; because with 5

> Shillings I can, at a fair Lay, procure the same even Chance or Expectation to win 3 or 7 Shillings" (Huygens 1657: 1).

At around the same time Blaise Pascal formulated a problem now considered as the first appearance of decision theory under uncertainty. The question "Either God is or he is not" is considered similar to the tossing of a (fair) coin which will come down head or tail; and "you must wager":

> "Let us weigh up the gain and the loss involved in calling heads that God exists. Let us assess the two cases: if you win you win everything, if you lose you lose nothing. Do not hesitate then; wager that he does exist … here there is an infinity of infinitely happy life to be won" (Pascal 1670).

Pascal's wager testified for the first time that it is possible in the calculation of the expected value of a game to exchange monetary value with the happiness derived from winning the game. This is the first conceptual step toward Daniel Bernoulli's solution of the St. Petersburg paradox.

In 1738 Daniel Bernoulli published in Latin his *Specimen Theoria Nova de Mensura Sortis* [Exposition of a New Theory on the Measurement of Risk]. In fact, the general problem tackled by Bernoulli is the classical one about the fair price of a gamble. Contrary to Huygens' approach –the fair price of a game is its expected value-, Bernoulli proposed to replace money with the advantage (Latin word: *emolumentum*) derived from money. So, the fair price of the game became the *emolumentum medium* of the game. In the first English translation (1954) the Latin word "emolumentum" became "utility", and "emolumentum medium" became "moral expectation" giving a modern flavor to Bernoulli's intuition:

> "If the utility of each possible profit expectation is multiplied by the number of ways in which it can occur, and we then divide the sum of these products by the total number of possible cases, a mean utility [moral expectation] will be obtained, and the profit which corresponds to this utility will equal the value of the risk in question"(Bernoulli 1954: 24).

All of the three cited authors lacked the explicit reference to the notion of probability. This is not surprising because they used the more basic idea of odds ratio. They had in mind something corresponding to the aleatory notion of possibility: in the ideal model of the game of chance it is possible to list all possible outcomes and to count the favorable outcomes. In the definition of expectation there was not probability, but the ratio of favourable to total possible outcomes. According to Hacking (1975), the idea of possible outcome was interpreted in terms of a physical possibility, or the physical propensity of each outcome to happen. The application of this kind of analysis was therefore confined to cases where it was possible to list all possible outcomes; each outcome having the same physical propensity to happen.

The explicit treatment of these questions in terms of probability was definitively cleared by Laplace (Laplace 1812: 432-45). It was through Laplace's work that the distinction between *fortune physique* (physical advantage) and *fortune morale* (moral advantage) became common in 19th century probability treatises (Todhunter 1865). Since then, the notions of the expected value of a game and of expected utility may be written in its modern notation, replacing odds ratios with probability. Let x_i be the monetary value -the Laplacean *fortune physique*- of the *i*-th outcome of a gamble and p_i its probability; then the expected value of the gamble is $E = \sum_{i=1}^{n} p_i x_i$. If we let $u(x_i)$ be the moral advantage derived –that is the subjective evaluation in terms of utility or satisfaction – from the monetary value x_i, it is possible to calculate the expected utility of the gamble as $EU = \sum_{i=1}^{n} p_i\, u(x_i)$.

Bernoulli's memoir is universally known for containing the St. Petersburg paradox – so called because it was published in the *Commentarii* of the St. Petersburg Academy. When the fair price of a gamble is equated to its expected value, and a particular gamble is considered, the following paradox emerges (the seminal contribution on this is Samuelson 1977): the particular gamble consists in repeatedly tossing a coin in the air; if heads comes up on the first toss, the gambler will receive two shillings; if it does not come up until the second toss, he will receive four shillings; heads only on the third toss will pay eight shillings, and so forth and so on. The expected value for the *n*-th toss is $\frac{1}{2^n}2^n$ shillings; if the game continues indefinitely, the expected value of the game is infinite, given that $E = \sum_{i=1}^{\infty}\frac{1}{2^n}2^n = \infty$. The fair price of the game is then infinite, but simple observation says that "no one would be willing to purchase it at a moderately high price" (Bernoulli 1954: 31). If, according to Bernoulli, the expected value is replaced by a bounded expected utility, the paradox vanishes. Bernoulli proposed the following model where a person possessing a certain quantity of money w, receives an additional quantity dw. Bernoulli maintained that the relative value of this increase is directly proportional to dw and inversely proportional to w, that is $du = k\frac{dw}{w}$ where k is a constant. From this derives the logarithmic function of the *fortune morale,* that is $u = a + k\log w$. When this formula is used, the fair price of the St. Peterburg game becomes finite; a reasonable result for thought experiments with gambling agents.

2. Subjective probability and mercantile speculations

Despite Laplace's advancement, the applicability of expectations (mathematical or moral) continued to be limited to gambling or to situations where it was possible to define probability as in games of chance. In fact the classical theory of probability was developed from the systematic study of gambling and it turned out to be impossible to use it for a theory of choice applied domains different from gambling. The limits of probability semantics narrowed the boundaries of its practical application. The passage from this domain to the application of probability to more general problems of choice under uncertainty -the extensions of Pascal's wager to everyday problems - happened sometime in the first part of 19th century. And this development was determined by a change in the philosophical interpretation of probabilities.

Antoine A. Cournot (1843) in the *Exposition de la Théorie des Chances et des Probabilités* developed a pluralist view of probabilities. He maintained the notion of "mathematical probability", the one traditionally associated with gambling, defined as the ratio of favourable outcomes to total outcomes, and put it side by side to a strictly subjective interpretation of probability, depending on the status of imperfect knowledge of an individual (Cournot 1843: 438). This kind of probability governs the rules regarding the stakes in a "marché aléatoire" (aleatory market). He explicitly introduced, perhaps for the first time, the modern notion of lottery as the general device to treat problems of choice under uncertainty (Cournot 1843: 89-90). He suggested that in a lottery where the prize is a generic good, every ticket may be considered as an "eventual right" over the good. Every ticket obviously may be sold in a market and its fair price ("valeur vénale") is its mathematical expectation, where probability is defined in reference to the subjective status of knowledge of agents. At the same time, Cournot, having rejected utility in the theory of demand (Fry and Ekelund 1971), also strongly rejected the idea of expected utility, considering it as "arbitrary" and "without real applications". Considering utility and not probability as the main problem for the development of expected utility framework is an idea that emerged probably for the first time with Cournot. It is the story that, from this time on, as we have anticipated, characterizes all historical reconstructions of this problem made by economists.

In the same time period in England Augustus de Morgan explicitly used mathematical expectation as a device to model "mercantile speculations" and "every species of affair in which no absolute certainty exists." (De Morgan 1838: 98). For de Morgan the nature of probability was also very different from the one proposed in classical theory: "Probability is the feeling of the mind, not the inherent property of a set of circumstances"(De Morgan 1838: 7), and this probability is different for different persons depending on the status of their knowledge and "impressions". This enlargement of the notion of probability paved the way to rigorous application of it to all "questions involving loss and gain" (De Morgan 1838: 103): that is, not only to gambling but also to matters of "commercial speculation" (De Morgan 1847: 404) and principally to problems regarding insurance offices. The passage from the notion of mathematical expectation to that of "moral expectation" involves, according to De Morgan, a supplementary problem concerning "the temperament of the individual", that is, in modern jargon, the question of risk propensity and aversion:

> "different persons will look forward in the same circumstances with different degrees of hope. One man will consider himself better off than before when he as bartered one pound certain for an even chance of two; a second will contemplate loss more strongly than gain, and will consider himself damnified by the exchange"(De Morgan 1847: 409).

Indeed, according to De Morgan the main problem in practical applications arise not from the side of utility, but from the side of probability: when the probability of an outcome is very small, and benefits depend upon this vanishing probability, then "the mathematical expectation is not a sufficient approximation to the actual phenomenon of the mind, even when the fortune of the player forms no part of the consideration" (De Morgan 1847: 409). In the Saint Petersburg paradox for example, neither mathematical expectation nor moral expectation may be considered as the fair price of the game, because both depend on very small probabilities. This intuition is similar to the notion of weight of probability in prospect theory (see below).

3. Marginalist expected utility and the frequentist dead end

Cournot and De Morgan laid the basis for modern choice under uncertainty, and both underlined the conceptual problems related to the use of utility and probability for this purpose. Schlee (1992) documented that during the Marginalist revolution, the expected utility was used to analyze a variety of decisions made under uncertainty, a result ascribed to the definition of a sound notion of marginal (decreasing) utility. From the point of view adopted here this explanation is only partial. A more correct one requires a distinction between scholars that in the last thirty years of 19th century discussed these questions in reference to treatises on probability, ethics, psychology and philosophy in general, and those who considered the problem as exclusively belonging, so to speak, to the economic profession. William S. Jevons and Francis Ysidro Edgeworth are examples of the first kind of approach, Alfred Marshall and Arthur Cecil Pigou of the second.

Bernoulli's hypothesis was "self-evident" for Jevons who took its application not only to gambling but also to commerce for granted (Jevons 1879: 173-74). The basic structure of human reasoning "in selecting a course of action which depends on uncertain events, as, in fact, does everything in life", consists in multiplying

> "the quantity of feeling attaching to every future event by the fraction denoting its probability. A great casualty, which is very unlikely to happen, may not be so important as a slight casualty which is nearly sure to happen. Almost unconsciously we make calculations of this kind more or less accurately in all the ordinary affairs of life ; and in systems of life, fire, marine, or other insurance, we carry out the

calculations to great perfection. In all industry directed to future purposes, we must take similar account of our want of knowledge of what is to be."(Jevons 1879: 36)

Behind all this there was not only a new idea of utility, but also a logical theory of probability developed in *The Principles of Science* (Jevons 1874). According to Jevons, probability is the "noblest creation of the intellect" (Jevons 1874: 200), in that it "deals with quantity of knowledge, an expression of which a precise explanation and measure can presently be given. An event is only probable when our knowledge of it is diluted with ignorance, and exact calculation is needed to discriminate how much we do and do not know"(Jevons 1874: 199). According to Jevons, the idea that probability "belongs wholly to the mind"(Jevons 1874: 198) does not signify that it is personalistic, as in De Morgan. On the contrary, it is the basis of "rational expectation" obtained "by measuring the comparative amounts of knowledge and ignorance" (Jevons 1874: 200). The principal value of the theory is normative, consisting "in correcting and guiding our belief, and rendering our states of mind and consequent actions harmonious with our knowledge of exterior conditions" (Jevons 1874: 199) .

In Marshall 's *Principles of Economics* (Marshall 1961 [1890]), the question of choice under uncertainty was made completely professional; it was treated in a footnote in Chapter 6 of Book 3 and in note IX of the Mathematical appendix. Marshall cited the introductory chapter of Jevons as his main reference. He used expected utility to solve truly economic problems. In particular

> "to measure numerically the present value of a future pleasure, on the supposition that we know, (i) its amount, (ii) the date at which it will come, if it comes at all, (iii) the chance that it will come, and (iv) the rate at which the person in question discounts future pleasures" (Marshall 1961 [1890]: 135).

Its present value is the value of future expected utility (ibid.: 840). He then discussed the problem that the maximization of the expected utility of wealth is inconsistent with gambling at fair odds, introducing the idea that gambling may be explained if the pleasure of gambling is separately considered (Schlee 1992).

Marshall's treatment paved the way to the (moderate) diffusion of expected utility hypothesis in economics – almost always associated with the assumption of diminishing marginal utility. Scholars applied it in order to explain advantages of fair insurance or the necessity of a reward for risk bearing (Schlee 1992: 737). Pigou (1924) treated risk or "uncertainty bearing" as a factor of production. On the whole, uncertainty in post-Marshallian tradition was treated routinely and in a soft manner, probably following the Marshallian statement that the measurement of expected utility "belongs to Hedonics, and not properly to Economics" (Marshall 1961 [1890]: 840).

Francis Y. Edgeworth adopted a radically divergent view since he was completely engaged in the traditional vision according to which probability theory was the main problem (Baccini 2009, 2011). His reference point was the frequentist approach to probability , presented in the epoch-making work of John Venn *The Logic of Chance* (1888). Both Edgeworth and Venn agreed that a coherent frequentist theory prevented the cogent use of probability for choice under uncertainty for logical and philosophical reasons. From a logical point of view, frequentist probability is objective, it is defined as relative frequency, and may be applied properly only to a particular class of things: series. Probability serves to construct assertions such as, "The relative frequency of individuals with property *R* in population *P* is *x*." If probability is not a property of individual events constituting the series, then it is invalid to infer from the preceding one an assertion such as "The probability that individual *i*, who belongs to population *P*, enjoys property *R* is *x*." This has relevant effects for decision theory: if choices are exercised essentially around single events, probability can be of no help in the choice of actions, being unable to say anything at all about those events. As a consequence, probability cannot be used as a decision-making instrument, if not in extremely limited cases where some very peculiar conditions hold (Baccini 2001). The second point in the argumentation was related to the nineteenth-century debate on the theory of human action. According to the philosopher Alexander Bain (1859), belief is preparedness to act, a necessary condition for human action. In order to apply probability to decision theory, the relations between belief and probability had to be defined properly. At the end of different strategies of reasoning, Venn and Edgeworth agreed in concluding that probability is not a direct measurement of belief, that it does not necessarily give rise to belief, and, as a consequence, it is not useful for theory of human choice (Baccini 1997, 2001, 2007).

At the turn of the Century two roads are clearly delineated for uncertainty in economics. The first one shuts out uncertainty and expected utility in the professional toolbox of economists, limiting its use to the narrow problems of gambling and insurance. The second road, inherited by the secular tradition of probabilists and philosophers, encountered the frequentist dead end.

4. Keynes' *Treatise on Probability*

This is the context in which John Maynard Keynes wrote his doctoral dissertation that he discussed in 1908 and published much later in a revised form as *A Treatise on Probability* (Keynes 1921). Bradley W. Bateman (1996) argues that Keynes's *Treatise* found its origin in reflections on George E. Moore's *Principia Ethica* and, in particular, in his intuition that the weak point of Moore's argument lay in the use of a naive version of frequentist probability. When Keynes began to study probability theory, he discovered the logical and practical limitations of the frequentist tradition. Keynes's problem became, therefore, not so much that of criticising Moore's probability, but rather of finding *an alternative theory* that made it possible to consider probability as a guide to action (Baccini 2004). Indeed, according to Keynes,

> "the importance of probability can only be derived from the judgement that it is rational to be guided by it in action ; and a practical dependence on it can only be justified by a judgement that in action we *ought* to act to take some account of it. It is for this reason that probability is to us the 'guide of life'" (Keynes 1921: 323).

The *Treatise* contains the development of a new logical theory of probability: it is concerned with the "degree of belief" which is *rational* to entertain in given conditions of knowledge, and not, as in De Morgan, merely with the actual beliefs of particular individuals "which may or may not be rational" (Keynes 1921: 4). According to Keynes, there is a direct connection between probability, rational belief and action: to have a belief signifies being disposed to act on the basis of it: "the probable is the hypothesis on which it is rational for us to act" (Keynes 1921: 307).

But the question is not so simple, "for the obvious reason that of two hypotheses it may be rational to act on the less probable if it leads to the greater good" (Keynes 1921: 307). Mathematical expectation and expected utility maximization are not the right tools to solve these kinds of problems. Keynes raised three formidable objections to the theory of mathematical expectation, all from the side of probability. The first was that probability is not fully measurable; the second that in mathematical expectation the weight of the argument, that is the amount of evidence upon which probability is based, is not considered; the third that the element of "risk" is completely ignored, assuming that "an even chance of heaven or hell is precisely as much to be desired as the certain attainment of a state of mediocrity" (Keynes 1921: 312). To overcome these objections, Keynes proposed a "conventional coefficient", substituting the probability value in the mathematical expectation formula, and considering together the probability of "goodness", the risk associated with it and the weight of evidence, that is the degree of unreliability or ambiguity of the information on which the probability value is based (Brady 1993). The Keynesian coefficient may be formulated as $c = \frac{2pw}{(1+q)(1+w)}$ where p is probability, $q = 1 - p$ is the risk coefficient and $0 \leq w \leq 1$ the weight of the argument. The introduction of this coefficient generalized the expected value formula by transforming it, according to Keynes, into an useful tool guiding actions in conditions of uncertainty. In the Keynesian formulation, the standard case –when an agent makes her choice using the expected value of different outcomes- becomes a very particular case. This case happens when the agent retains that the information at her disposal is unambiguous and certain, that is $w = 1$; and at the same she is risk neutral, in the particular sense that she prefers to not consider the risk value of her choice, and then she drops q from the Keynesian coefficient.

Keynes added to this exposition a paragraph containing a famous disclaimer regarding the usefulness of mathematics in moral sciences, that attracted the attention of posterity. As a consequence, Keynes refusal of expected utility was categorized, starting at least from Shackle (1952) as a by-product of a general scepticism about the possibility to apply mathematical tools to economic problems. Instead, his contribution may be better understood as an escape from both utilitarianism (the refusal of the use of utility value) and frequentism, and as an anticipation of a modern approach to choice under uncertainty focused on the problem of weighting probability values with the measures of the reliability of information on which probability values are based.

5. Von Neumann and Morgenstern's utility

According to Ellsberg, "it was the feeling that the emphasis on mathematical expectation was arbitrary and unrealistic which led to the decline of the concept even before doubt arose that a measurable utility could be discovered to make it meaningful" (1954: 537). The decline of the Bernoulli-Marshallian tradition (Arrow 1951; Friedman and Savage 1948; Marschak 1938; Tintner 1942) was stopped by the appearance of Von Neumann and Morgenstern's book (1944), containing a complete and to professional economists satisfactory axiomatic treatment of choice under uncertainty. It represented a dramatic break in continuity and opened the way to a central field in 20th century economic analysis.

Von Neumann's and Morgestern's discussion of the question of utility was "mainly opportunistic" (Von Neumann and Morgenstern 1953: 8). They needed a cardinal measure of utility to be considered "identical" to "money or a single monetary commodity" (unrestrictedly divisible, substitutable and transferable) representing the aim of all participants in the economic system (Von Neumann and Morgenstern 1953: 8). Their goal was to define a cardinal notion of utility without following the unsatisfactory patterns of their predecessors. In particular they were interested in a cardinality not based on a more or less introspective measure of pleasure or satisfaction derived from goods. They tried also to divorce their approach from cardinality based on to the comparability of preference differences (Fishburn 1989: 131). The solution consisted in the definition of numerical utility "as being that thing for which the calculus of mathematical expectation is legitimate" (Von Neumann and Morgenstern 1953: 28). The basic operation in deriving a cardinal utility index is the analysis of a situation where an agent is choosing between a sure outcome and two possible outcomes with given probabilities. In order to fix the origin and unit of the utility index, arbitrary numbers are assigned to the two outcomes *A* and *B*, with an order respecting the order of preference of the agent. For the sake of simplicity, set the utility of the worst outcome *A* to $U_A = 0$, and of the best *B to* $U_B = 1$. Consider now a third outcome *C* which the agent ranks between the first two. The utility index of *C* is the probability *p* at which the agent is indifferent between having *C* with certainty or participating in a lottery with a probability p of winning *B*, and $(1-p)$ of winning *A*. Thus $U_C = pU_B + (1-p)U_A = p$. The same reasoning may be applied to define other utility numbers reflecting other choices consistent with this last one for other intermediate outcomes. If, for example, an alternative outcome *D* which the agent ranks between *C* and *B* is considered, the utility index is $U_D = q$ where q is the probability for which the agent is indifferent between having *D* with certainty or participating in a lottery with a probability q of winning *B* and $(1-q)$ of winning *C*. The only consistency requirement is that $U_B > U_D > U_C$. It is therefore possible to elicit the entire utility function of the agent (for modern expositions see Binmore 2009; Itzhak Gilboa 2009); the utility index so defined is unique up to a linear transformation, that is a cardinal measure of utility. It is defined directly through the observation of choices of agents in risk situations, and it allows us to describe agents with different patterns of behaviors in risk situations. Von Neumann and Morgenstern derived this operational result from an "axiomatic treatment of numerical utilities". A "controversial" axiom is also introduced, according to which the agent is indifferent between two possible outcomes which are derivable from each other according to the rules of probability. On this basis it is possible to reduce systematically to a simple lottery, lotteries in which prizes are other lotteries. The controversial nature of this axiom rests on the fact that it constrains the agent to be indifferent about the number of steps of the gamble, since the interest lies only in the final possible outcome, and not in intermediate possible winnings (D. Ellsberg 1954: 543). Now it is finally possible to calculate the expected utility of different possible outcomes as $EU = \sum p_i U(x_i)$, where $U(x_i)$ are the utility values constructed with the operations described above regarding possible outcomes (x_i), and not the utility values of riskless outcome, as in the Bernoulli-Marshallian tradition. Consider two even lotteries with prizes respectively *A* and *D,* and *C* and *D*, as defined above. The expected utility of the first one is $EU_1 = 0.5 \times U_A + 0.5 \times U_D = 0.5p$, and of the second one $EU_2 = 0.5 \times U_C + 0.5 \times U_D = 0.5p + 0.5q$; the two lotteries can be ordered in relation to their expected utilities $EU_2 > EU_1$; the rational agents chooses respecting this order. The choice of agents can be described as they are maximizing their expected utility.

From our point of view, the nature of probability must still be discussed. Von Neumann and Morgenstern construction used probability as an individual numerical estimate of utility. They therefore stated clearly that a "subjective concept" of probability "would not serve" their purpose. They insisted "upon the alternative, perfectly well founded interpretation of probability as frequency in long runs. This gives directly the necessary numerical foothold"(Von Neumann and Morgenstern 1953: 19). Moreover they did not cite any authors about this point. It is therefore possible to conjecture that they were unaware of, or considered irrelevant the discussion about the proper domain and applicability of frequentist probability. They evidently considered the discussion on the nature of probability involved in their theory as a minor point. This interpretation is supported also by a footnote in which they suggested that it is possible to make a joint axiomatization of probability and utility (Von Neumann and Morgenstern 1953: 19, footnote 2), without resorting to statistical probability.

Von Neumann and Morgenstern's book opened a lively discussion among economists in which it emerged the complete professionalization of the theme. The first question obviously was that of cardinal utility: Von Neumann and Morgenstern were accused of binging economics back to the pre-Pareto and Hicks's era (Baumol 1951). The second one concerned the kind of probability assumed. This probability, as underlined by Savage, "can apply fruitfully only to repetitive events" and cannot be used to elicit "which of several actions is the most promising" because probability is not assigned to the truth of propositions (Savage 1972 [1954]: 4). It was through searching for a solution to this last problem that Savage generalized the structure of the Von Neumann and Morgenstern utility.

6. The emergence of subjective expected utility as the mainstream paradigm

As we have seen, Von Neumann and Morgenstern derived utility *given* the notion of statistical probability. Some years earlier Bruno de Finetti (1937) developed a notion of subjective probability starting from the choice problem of an individual maximizing an expected monetary value. Frank P. Ramsey (1931 [1926]), in discussing Keynes's probability, developed a subjective view of probability and utility without assuming any of these concepts as primitive. This strategy was adopted by Savage. He "virtually copied" (Savage 1972 [1954]: 97) the treatment reserved by Von Neumann and Morgenstern to utility by developing the idea of a subjective probability. Savage's model includes two primitive concepts: outcomes, as usual, and states, as the list of all scenarios that may happen. The outcome is the conjoined result of the agent's choice of an act and of the unfolding of a state of the world. When the agent makes her choice over acts, she does not reason in terms of numbers, utility and probabilities; but on a very simple framework containing the description of possible states and the unique outcome resulting by the choice of every act in every state (i.e. when an act has been chosen by an agent, and a state of the world has been unfolded, only an outcome is necessarily verified). With this simple structure and seven axioms, Savage demonstrated that both a (bounded) utility function and probability measure exist, in such a way that decisions are made as if the agent is maximizing the expectation of the utility relative to the probability measure (for a presentation see Itzhak Gilboa 2009: 94-112). More precisely, the subjective expected utility hypothesis is equivalent to the joint hypothesis that the agent possesses a subjective probabilistic belief about the states of the world $\mu(s_i)$, and a Von Neumann and Morgenstern utility function over outcomes $U(x_i)$; and she evaluates acts according to a preference function (for a finite state space) of the form $W = \sum_{i=1}^{n} U(x_i)\,\mu(s_i)$. Savage restricted the applicability of his theory to what he called "small world". In a small world it is always possible for an agent to "look … before you leap" given that it is possible to have a description so complete that the consequences of every action would be known. This idea can be expressed from another point of view: an agent in a small world can take account in advance of the impact of all future possible information on her subjective beliefs about the state of the world. The consistency of an agent's choices are guaranteed if her personal degrees of belief are coherent in such a way that a Dutch book - a system of bets which guarantee that anyone who takes them all on will lose no matter what happens - could not be made against her.

Savage's results reinforced the Von Neumann and Morgenstern construction, and the subjective expected utility became the major paradigm in decision making in the second half of the 20th century. According to this view the expected utility maximization with respect to a subjective probability is the only rational way of behaviour suitable for cogently treating every kind of economic problem characterized by the presence of uncertainty. Arrow-Pratt measure of risk aversion provided a powerful operative tool (Arrow 1965; Pratt 1964). Since then applications of the expected utility model flowered in problems of optimal savings, international trade, portfolio selection, environmental economics, and economic analysis of law.

7. Contemporary developments and prospect theory

The success of the Von Neumann and Morgenstern's model was not overshadowed by research results documenting that expected utility maximization is not a good predictor of real choices. Maurice Allais (1953) provided the first evidence that people tend to violate axioms of Von Neumann and Morgenstern, by placing more weight on certainty than the standard theory predicts. Daniel Ellsberg (1961) showed that people behave in a way that cannot be described by Savage's subjective probability. In particular people tend to be "ambiguity averse", that is people prefer situations where probabilities are known to situations where probabilities are unknown, thus violating one of the Savage's axioms, the so called sure thing principle. (An excellent review of this literature may be found again in Schoemaker 1982).

Contemporary developments try to cope with these kinds of problems by adopting different strategies. The first one is probably the more conservative, because it maintains the subjective notion of probability. It consists in generalizations of the expected utility model by removing linearity in the probabilities and by positing nonlinear functional forms for the preference function. Several such forms have been formally proposed and axiomatized, most are capable of generating well known features, such as risk aversion and violations of independence axioms (Machina 2008). A second strategy consists in replacing probability with alternative notions of belief able to describe the way that people make decisions, or the way they can be convinced to make rational decisions when standard probabilities cannot be defined. The basic intuition may be grasped by the following example. Consider two coins. The experience of repeated flips of the first coin suggest that it is a fair one; therefore it is reasonable to give probability 0.5 to head and 0.5 to tail. Suppose that the other coin is completely unknown; you have no reason to prefer one side to the other (symmetric information). If one decides to give probability 0.5 to both head and tail, then this assignment is very different from the preceding one based on frequency experience. This last kind of assignment does not necessarily respect the additive rule for probability. For example: one can legitimately assigns, respecting symmetric lack of information, a nonadditive probability $v(H) = v(T) = 0.4$; nonetheless it is true that $v(H \cup T) = 1$. Building on this intuition, in the so called Choquet expected utility the standard probability is replaced by a *capacity* or a *nonadditive* probability. These models explain most of the observed paradoxes but they also offer simple but flexible representations, and allow for more diversified patterns of behavior under uncertainty (Schmeidler 1989). A third strategy consists in considering explicitly how beliefs are constructed. In the case-based decision theory (I. Gilboa and Schmeidler 1995), cases are considered primitive and a system of axiom was construed permitting the choice of the best act based on its past performance in similar cases. Each act is evaluated by the sum of the utility levels that resulted from using this act in past cases, each weighted by the similarity of that past case to the problem at hand.

Probably, the most innovative contributions to this stream of literature were authored by Daniel Kahneman and Amos Tversky. From the early 1960 they developed the systematic study of several violations of the standard assumptions of stability of preferences, and of invariance of choices with respect to the particular kind of description of risky prospects. Their laboratory experiments and those of their followers appear to be "a knockdown refutation of the claim that the Von Neumann and Morgenstern theory is usually a good predictor of how ordinary people behave" (Binmore 2009: 58). In their approach, the so called *prospect theory*, the Von Neumann and Morgenstern's utility is replaced by the psychological values of gain and loss. This psychological value is similar to the Edgeworthian notion of utility as experienced pleasure, objectively measurable by means of a technical device called "hedonimeter" (Baccini 2011). Also "the decision weights that people assign to outcomes are not identical to the probabilities of these outcomes, contrary to the expectation principle. … The expectation principle, by which values are weighted by their probability, is poor psychology" (Kahneman 2011). In this case the line of reasoning recalls Keynes's notion of the weight of an argument, discussed above. In particular Kahneman and Tversky documented (Kahneman and Tversky 1979): 1. a psycophysics of value according to which people are risk averse in the domain of gains and risk seeking in the domain of losses; 2. a psycophisics of chance in which people overweight sure things and improbable events, relative to events of moderate probability; 3. that decision problems can be described or *framed* in multiple ways that give rise to different preferences, contrary to the invariance criterion of rational choice.

Prospect theory is considered as the best available description of how people evaluate risk in experimental settings. Despite this, there are relatively few broadly accepted application to economics (for a review see (Barberis 2013)). A probable explanation may be that prospect theory represents a paradigm shift in the theory of choice under uncertainty and therefore it is not easily adaptable to the problems discussed in the domain of normal economics.

REFERENCES

Allais, M. (1953), 'Le Comportement de l'Homme Rationnel devant le Risque: Critique des Postulats et Axiomes de l'Ecole Americaine', *Econometrica,* 21 (4), 503-46.

Arrow, Kenneth J. (1951), 'Alternative Approaches to the Theory of Choice in Risk-Taking Situations', *Econometrica,* 19 (4), 404-37.

--- (1965), *Aspects of the theory of risk-bearing* (Helsinki: Yrjö Jahnssonin Säätiö).

Baccini, Alberto (1997), 'Edgeworth on the Fundamentals of Choice under Uncertainty', *History of Economic Ideas,* 5 (2), 27-51.

--- (2001), 'Frequentist Probability and Choice under Uncertainty', *History of Political Economy,* 33 (4), 743-72.

--- (2004), 'High Pressure and Black Clouds: Keynes and the Frequentist Theory of Probability', *Cambridge Journal of Economics,* 28 (5), 653-66.

--- (2007), 'Edgeworth on the Foundations of Ethics and Probability', *European Journal of the History of Economic Thought,* 14 (1), 79-96.

--- (2009), 'F. Y. Edgeworth's Treatise on Probabilities', *History of Political Economy,* 41 (1), 143-62.

--- (2011), 'Francis Ysidro Edgeworth on the Regularity of Law and the Impartiality of Chance', in T. Boylan, R. Prendergast, and J. Turner (eds.), *A History of Irish Economic Thought* (London: Routledge), 233-48.

Bain, Alexander (1859), *The emotions and the will* (London,: J. W. Parker and son) xxviii, 649, l p.

Barberis, Nicholas C. (2013), 'Thirty Years of Prospect Theory in Economics: A Review and Assessment', *The Journal of Economic Perspectives,* 27 (1), 173-95.

Bateman, Bradley W. (1996), *Keynes's uncertain revolution* (Ann Arbor: University of Michigan Press) xi, 183 p.

Baumol, William J. (1951), 'The Neumann-Morgenstern Utility Index--An Ordinalist View', *The Journal of Political Economy,* 59 (1), 61-66.

Bernoulli, Daniel (1954), 'Exposition of a New Theory on the Measurement of Risk', *Econometrica,* 22 (1), 23-36.

Binmore, K. G. (2009), *Rational decisions* (The Gorman lectures in economics; Princeton: Princeton University Press) x, 200 p.

Brady, Michael Emmett (1993), 'J. M. Keynes's Theoretical Approach to Decision-making Under Conditions of Risk and Uncertainty', *The British Journal for the Philosophy of Science,* 44 (2), 357-76.

Cournot, Antoine Augustin (1843), *Exposition de la Théorie des Chances et des Probabilités* (Paris: Hachette).

David, F.N. (1962), *Games, Gods and Gambling. A History of Probability and Statistical Ideas* (London: Charles Griffin & Co.).

De Finetti, Bruno (1937), 'La prévision: ses lois logiques, ses sources subjectives', *Annales de l'Institute Henry Poincaré,* 7 (1), 1-68.

De Morgan, Augustus (1838), *An Essay on Probabilities, and Their Application to Life Contingencies and Insurance Offices* (A new edn., The Cabinet Cyclopædia v 13; London,: Printed for Longman, Orme, Brown, Greens & Longmans) 2 p. l., xviii, 306, xl p.

--- (1847), 'Theory of Probabilities', *The Encyclopaedia of Pure Mathematics forming part of the Encyclopaedia Metropolitana* (Glasgow: Griffin and Company), 393-490.
Ellsberg, D. (1954), 'Classic and Current Notions of "Measurable Utility"', *The Economic Journal,* 64 (255), 528-56.
Ellsberg, Daniel (1961), 'Risk, Ambiguity, and the Savage Axioms', *The Quarterly Journal of Economics,* 75 (4), 643-69.
Fishburn, Peter C. (1989), 'Retrospective on the utility theory of von Neumann and Morgenstern', *Journal of Risk and Uncertainty,* 2 (2), 127-57.
Friedman, Milton and Savage, L. J. (1948), 'The Utility Analysis of Choices Involving Risk', *The Journal of Political Economy,* 56 (4), 279-304.
Fry, Clifford L. and Ekelund, Robert B., Jr. (1971), 'Cournot's Demand Theory: A Reassessment', *History of Political Economy,* 3 (1), 190-97.
Gilboa, I. and Schmeidler, D. (1995), 'Case-based decision theory', *Quarterly Journal of Economics,* 110, 605-39.
Gilboa, Itzhak (2009), *Theory of decision under uncertainty* (Econometric Society monographs; Cambridge ; New York: Cambridge University Press) xiv, 214 p.
Hacking, Ian (1975), *The emergence of probability : a philosophical study of early ideas about probability, induction and statistical inference* (London ; New York: Cambridge University Press) 209 p.
Huygens, Christiaan (1657), *De Ratiociniis in Ludo Aleae [The Value of all Chances in Games of Fortune; Cards, Dice, Wagers, Lotteries & C. Mathematically Demonstrated]*, trans. [Engl. Transl.] (London: Keimer and Woodward [1714]).
Jevons, William Stanley (1874), *The principles of science : a treatise on logic and scientific method*, 2 vols. (London: Macmillan and Co.).
--- (1879), *The Theory of Political Economy* (3d edn.; London: MacMillan).
Kahneman, D. (2011), *Thinking, Fast and Slow* (New York: Farrar, Straus and Giroux).
Kahneman, D. and Tversky, A (1979), 'Prospect theory: an analysis of decision under risk', *Econometrica,* 47, 262-91.
Keynes, John Maynard (1921), *A treatise on probability* (London,: Macmillan and co., limited) xi, 466 p.
Laplace, Pierre Simon (1812), *Théorie analytique des probabilités* (Paris,

Bruxelles,: Culture et civilisation) 464 p.
Machina, Mark J. (2008), 'Non-expected utility theory', in Steven N. Durlauf and Lawrence E. Blume (eds.), *The New Palgrave Dictionary of Economics* (Basingstoke: Palgrave Macmillan).
Marschak, J. (1938), 'Money and the Theory of Assets', *Econometrica,* 6 (4), 311-25.
Marshall, Alfred (1961 [1890]), *Principles of economics*, 2 vols. (9th (variorum) edn.; London, New York,: Macmillan for the Royal Economic Society).
Pascal, Blaise (1670), *Pensées* ([English Translation] Penguin).
Pigou, A. C. (1924), *The economics of welfare* (2d edn.; London: Macmillan) xxviii, 783 p.
Pratt, John W. (1964), 'Risk Aversion in the Small and in the Large', *Econometrica,* 32 (1/2), 122-36.
Ramsey, F.P. (1931 [1926]), 'Truth and Probability', *The Foundations of Mathematics and other Logical Essays* (London: Kegan, Paul), 156-98.
Samuelson, Paul A. (1977), 'St. Petersburg Paradoxes: Defanged, Dissected, and Historically Described', *Journal of Economic Literature,* 15 (1), 24-55.
Savage, Leonard J. (1972 [1954]), *The foundations of statistics* (2d rev. edn.; New York,: Dover Publications) xv, 310 p.
Schlee, Edward E. (1992), 'Marshall, Jevons, and the Development of the Expected Utility Hypothesis', *History of Political Economy,* 24 (3), 729-44.
Schmeidler, D. (1989), 'Subjective Probability and Expected Utility without Additivity', *Econometrica,* 57 (3), 571-87.
Schoemaker, Paul J. H. (1982), 'The Expected Utility Model: Its Variants, Purposes, Evidence and Limitations', *Journal of Economic Literature,* 20 (2), 529-63.
Shackle, G. L. S. (1952), *Expectation in economics* (2nd edn.; Cambridge: University Press) xvi, 144 p.

Tintner, Gerhard (1942), 'A Contribution to the Non-Static Theory of Choice', *The Quarterly Journal of Economics,* 56 (2), 274-306.
Todhunter, I. (1865), *History of the mathematical theory of probability from the time of Pascal to that of Laplace* (Cambridge, London,: Macmillan and co.) xvi, 624 p.
Venn, John (1888), *The logic of chance. An essay on the foundations and province of the theory of probability, with especial reference to its application to moral and social science* (3d. First edition 1866 edn.; London, Cambridge,: Macmillan) xxvii, 370 p.
Von Neumann, John and Morgenstern, Oskar (1944), *Theory of games and economic behavior* (Princeton,: Princeton university press) xviii, 625 p.
--- (1953), *Theory of games and economic behavior* ([3d edn.; Princeton,: Princeton University Press) 641 p.