Acta Sci. Math. (*Szeged*)
64/3–4 (1998), 397–429

Ternary deduction terms in residuated structures

Paolo Agliano

Communicated by G. Czédli

Abstract. In this paper we thoroughly investigate several kinds of residuated ordered structures, connected with propositional logics. In particular we give ternary deduction terms for several classes of algebras, that are the equivalent algebraic semantics of deductive systems, coming from logics not necessarily satisfying the structural rules.

0. Introduction

This paper can be viewed as a sequel to [7]; in the latter the authors investigate a large class of varieties with definable principal congruences that have a *ternary deduction term* (TD-term), a natural generalization (to non semisimple varieties) of the ternary discriminator term. The existence of such a term appears to be the characteristic feature of many varieties of algebras that arise naturally in algebraic logic. That paper, together with its sequel [8], are part of a program to develop a unified theory for the algebras arising from logic, within a context of universal algebra. The specific purpose of [7] is to explore to what extent the existence of a TD-term of a particular kind might serve as one of the basic concepts on which such theory can be based. For this purpose it is desirable to consider as wide a class as possible of algebras that are known to arise from logic: in [7] the variety of *hoops with compatible operators* is chosen for this purpose. Since the authors in [7] did not claim in any way that the chosen class is the most general, one cannot help wondering if the same theory can be exported to wider classes of varieties. This is exactly the purpose of this paper; armed with the additional insight afforded by the recent amount of work on substructural logic we extend the

Received December 27, 1996, in revised form April 20, 1998 and in final form May 18, 1998.
AMS Subject Classification (1991): 08A99, 06F05.

theory to a much wider (and perhaps more natural) class of algebras than hoops. More specifically we consider varieties of *semilattice-ordered residuated commutative monoids* (possibly with compatible operations). These algebras arise naturally from the algebrization of linear logic and generalize hoops in two different ways: first the ordering of hoops (the so-called *inverse divisibility ordering*) is replaced by an arbitrary semilattice ordering; secondly we do not require the *integrality* of the monoid, which in metalogical terms amounts to abandoning the structural rule of weakening. It turns out that almost the whole theory of TD-terms, and the filter theory on which is based, extends without much problems to these structures.

This paper is divided into four sections. In Section 1 we collect the basic definitions and the main results we are going to use in the paper. In Section 2 we investigate several classes residuated semilattices and monoids and we start developing a theory of filters for them. Section 3 is devoted to the main theorems about TD-terms. In Section 4 we discuss the relationships that our investigations have with a more general topic: the algebrization of logical systems in the Blok-Pigozzi's fashion [5].

Finally two words on general notation. Our usage is more or less the standard one for general algebra (see [22]) and abstract algebraic logic (see [6]), possibly with two exceptions. If $\mathbf{A}$ is an algebra and $X \subseteq A$, then we denote by $\vartheta_{\mathbf{A}}(X)$ the congruence of $\mathbf{A}$ generated by all the pairs of elements from X. Likewise $\mathrm{Sub}_{\mathbf{A}}(X)$ denotes the subalgebra of $\mathbf{A}$ generated by X. If $X = \{a, b\}$ then we will write $\vartheta_{\mathbf{A}}(a, b)$ and $\mathrm{Sub}_{\mathbf{A}}(a, b)$.

1. Preliminaries

In their sequence of papers [4],[3],[7] and [8], Blok and Pigozzi studied varieties with *equationally definable principal congruences*. These are varieties in which the principal congruences of any algebra are uniformly definable via a finite set of equations. In such varieties the compact congruences of any algebra form a (dual) Brouwerian semilattice, hence the variety mirrors in its congruence structure a significant fragment of intuitionistic logic. The connection can be made stricter is several cases; what follows is a brief account of the case we interested in.

A *ternary deduction term* (TD-term) for a class $\mathcal{K}$ of algebras is a ternary term such that, for any $\mathbf{A} \in \mathcal{K}$ and $a, b, c, d \in A$

$$p(a, a, b) = b,$$
$$p(a, b, c) = p(a, b, d) \qquad \text{if} \qquad (c, d) \in \vartheta(a, b).$$

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Any variety having a ternary deduction term has equationally definable principal congruences:

$$(c,d) \in \vartheta(a,b) \qquad \text{iff} \qquad p(a,b,c) = p(a,b,d).$$

In fact one implication is part of the definition and if $p(a,b,c) = p(a,b,d)$, then

$$c = p(a,a,c)\ \vartheta(a,b)\ p(a,b,c) = p(a,b,d)\ \vartheta(a,b)\ p(a,a,d) = d.$$

We will use the following sufficient condition for the existence of TD-term (the proof can be found in [7] pp. 26–27):

Theorem 1.1. *Let $\mathcal{V}$ be a variety; $p(x,y,z)$ is a TD-term for $\mathcal{V}$ if and only if*

1. *$p(x,x,y) \approx y$ hold in $\mathcal{V}$;*
2. *for any $\mathbf{A} \in \mathcal{V}$, $a,b \in A$ and any unary polynomial $q(x)$ of $\mathbf{A}$*

$$p(a,b,q(a)) = p(a,b,q(b)).$$

A TD-term is *commutative* if for any $a,b,c,a',b' \in A$

$$p(a,b,p(a',b',c)) = p(a',b',p(a,b,c));$$

it is *regular* with respect to a constant 1 if

$$p(p(a,b,1),1,a) = p(p(a,b,1),1,b).$$

A commutative, regular TD-term for $\mathcal{V}$ can be used to construct operations that reflect faithfully the conjunction and the (dual) relative pseudocomplementation of the Brouwerian semilattice of compact congruences of algebras in $\mathcal{V}$ (see [3] and [7]).

In order to justify our choice of names for the structures we are going to consider in Section 2 we anticipate here part of the discussion about the relationships of our investigation with the algebrization of logical systems. Let $\mathcal{L}$ be a propositional language containing a binary connective $\to$. Among the possible axioms involving only $\to$ some have received a particular attention (here p,q,r are propositional variables):

(I)	$p \to p$	$(reflexivity)$
(B)	$(q \to r) \to ((p \to q) \to (p \to r))$	$(transitivity)$
(C)	$(p \to (q \to r)) \to (q \to (p \to r))$	$(exchange)$
(K)	$p \to (q \to p)$	$(weakening)$
(W)	$(p \to (p \to q)) \to (p \to q)$	$(contraction)$,

The names (I),(B),(S),(W) and (K) are taken from combinatory logic; they sometimes contrast with their "long" names but they are very traditional. If a logic on $\mathcal{L}$ has at least *modus ponens* as inference rule, then these axioms are well-known not to be independent. Any logic over $\mathcal{L}$ can be classified according to the axioms satisfied by $\rightarrow$. For instance classical and intuitionistic propositional logics are KS-logics, Relevance Logic is a (distributive) BCW-logic, some of the logics studied by Ono and Komori in [23] are BCK-logics; Linear Logic is a BCI-logic. In contrast with this usage a logic satisfying (B), (K) and (I) is traditionally called a BCC-logic. The axioms (C),(W) and (K) (exchange, contraction and weakening) are usually called *structural rules* (for $\rightarrow$) and any logic lacking at least one of them is usually called a *substructural logic*. The motivations for abandoning the structural rules for $\rightarrow$ are usually extralogical. Their nature can be set-theoretic (as for BCC and BCK-logics), philosophical (as for Relevance Logic and Comparative Logics) or even proof-theoretical (as for Linear Logic). When an (at least) BI-logic is algebraized in the Blok–Pigozzi style [5], the implication induces on the algebraic structure a partial order; if more connectives are present in the language this may become a semilattice or a lattice order. In many interesting substructural logics two more binary connectives $\cdot$ and $\wedge$ stand in the forefront and the resulting structures are *semilattice ordered residuated semigroups*. If the logic has a constant $\top$, meaning universal truth, then the resulting structures are monoids; if exchange is present then the monoids are commutative and if weakening is present the monoids are *integral*, i.e. $\top$ is the largest element in the order. In classical and intuitionistic logic the two connectives $\cdot$ and $\wedge$ turn out to be equal and the resulting structures are in fact relatively pseudocomplemented semilattices, i.e. *Brouwerian semilattices*; these semilattices turn out to be *distributive* in the usual sense. The metalogical point of this paper is that many varieties of algebras that arise from the algebrization of BCI-logics have in fact regular, commutative TD-terms.

The idea of conducting such an investigation came to the author he was working on the algebraic semantics for Linear Logic [2]; he wishes to thank A. Ursini for drawing his attention to the subject.

2. Filters in residuated structures

2.1. Semilattices. The idea behind the structures in this subsection is to consider "residuals without residuation". Similar structures appear in [25] (Sec. 10) and especially in [14] under various names. Our choice of names is designed to emphasize the properties of the "implication" involved.

A *BI-semilattice* is a structure $\langle A, \rightarrow, \wedge, 1\rangle$ such that $\langle A, \wedge\rangle$ is a semilattice and for any $a, b, c \in A$ the following hold:

(2.1) $$1 \rightarrow a = a$$

(2.2) $$a \rightarrow a \geq 1$$

(2.3) $$(a \rightarrow b) \wedge (a \rightarrow c) = a \rightarrow (b \wedge c)$$

(2.4) $$a \rightarrow b \leq (c \rightarrow a) \rightarrow (c \rightarrow b)$$

(2.5) $$a \rightarrow b \geq 1 \text{ implies } a \leq b$$

(2.6) $$a \leq b \text{ implies } b \rightarrow c \leq a \rightarrow c.$$

It is clear from the presentation that the class of BI-semilattices is a quasivariety (and we suspect it is a *proper* quasivariety). First we note that in a BI-semilattice

(2.7) $$a \leq b \qquad \text{iff} \qquad a \rightarrow b \geq 1.$$

In fact, if $a \leq b$ then $a \wedge b = a$ and by (2.3)

$$(a \rightarrow a) \wedge (a \rightarrow b) = a \rightarrow (a \wedge b) = a \rightarrow a;$$

hence $a \rightarrow b \geq a \rightarrow a \geq 1$ by (2.2). For any BI-semilattice $\mathbf{A}$ set $\nabla_A = \{a : a \geq 1\}$. An (implicative) *filter* of a BI-semilattice is a semilattice filter F containing 1 (hence $\nabla_A \subseteq F$) and closed under *modus ponens*: if $a, a \rightarrow b \in F$, then $b \in F$.

An *enriched BI-semilattice* is an algebra $\mathbf{A}$ of the right type, whose $\{\rightarrow, \wedge, 1\}$-reduct $\bar{\mathbf{A}}$ is a BI-semilattice. By a *filter* of an enriched BI-semilattice we will always mean a filter of the underlying BI-semilattice structure.

If $\mathbf{A}$ is an (enriched) BI-semilattice and $\theta \in \mathrm{Con}(\mathbf{A})$, then

$$F_\theta = \nabla_A/\theta = \bigcup\{a/\theta : a \geq 1\}$$

is always a filter. In fact $1 \in F_\theta$; if $u \in F_\theta$ and $u \leq v$, then $u \rightarrow v \geq 1$. But $u \;\theta\; a$ for some $a \geq 1$ and so $u \wedge 1 \;\theta\; a \wedge 1 = 1$; therefore

$$v = 1 \rightarrow v \;\theta\; u \wedge 1 \rightarrow v$$

and

$$1 \leq u \rightarrow v \leq u \wedge 1 \rightarrow v,$$

implying $v \in F_\theta$. Moreover if $u, u \rightarrow v \in F_\theta$, then again $u \wedge 1 \;\theta\; 1$ and $u \rightarrow v \leq (u \wedge 1) \rightarrow v \in F_\theta$. Hence

$$v = 1 \rightarrow v \;\theta\; (u \wedge 1) \rightarrow v$$

and $v \in F_\theta$. A filter F of $\mathbf{A}$ that is F_θ for some $\theta \in \mathrm{Con}(\mathbf{A})$, will be called a *congruence filter*; it is easily seen that the congruence filters form a complete lattice under inclusion, which will be denoted by $\mathrm{CFil}(\mathbf{A})$. For $B \subseteq A$ the congruence filter generated by B will be denoted by $\mathrm{CF}_{\mathbf{A}}(B)$.

Theorem 2.1. *Let* $\mathbf{A}$ *be an enriched BI-semilattice such that* $\bar{\mathbf{A}}/\alpha$ *is a BI-semilattice for any* $\alpha \in \mathrm{Con}(\mathbf{A})$. *Then* $\mathrm{CFil}(\mathbf{A})$ *and* $\mathrm{Con}(\mathbf{A})$ *are isomorphic via the mapping*

$$F \longmapsto \theta_F = \{(a,b) : a \to b, b \to a \in F\}$$

and the inverse mapping is

$$\theta \longmapsto F_\theta = \nabla_A/\theta.$$

Proof. First suppose that $F \in \mathrm{CFil}(\mathbf{A})$, i.e. $F = F_\alpha$ for some $\alpha \in \mathrm{Con}(\mathbf{A})$. If $a \to b, b \to a \in F_\alpha$, then $a \to b \ \alpha \ u$ with $u \geq 1$. Then $(a \to b) \wedge 1 \ \alpha \ 1$ and hence (2.7) gives $a/\alpha = b/\alpha$, i.e. $(a,b) \in \alpha$. On the other hand if $(a,b) \in \alpha$, then $a \to b, b \to a \ \alpha \ a \to a \geq 1$ hence $a \to b, b \to a \in F_\alpha$. We have thus proved that $\theta_{F_\alpha} = \alpha$ and hence the mapping is well defined and onto.

Assume now $\theta_F \in \mathrm{Con}(\mathbf{A})$ for a filter F; we will show that $F = F_{\theta_F}$, thus proving that the mapping is one-to-one. If $a \in F_{\theta_F}$, then there is a $u \geq 1$ such that $a \to u, u \to a \in F$; but $u \in F$ and by modus ponens $a \in F$. Suppose now that $a \in F$; then $1 \to a \wedge 1 = a \wedge 1 \in F$ and by (2.6), $1 = 1 \to 1 \leq a \wedge 1 \to 1 \in F$. So $(a \wedge 1, 1) \in \theta_F$ and so $(a \wedge 1 \to a, a) \in \theta_F$. Since (2.6) gives

$$1 \leq a \to a \leq a \wedge 1 \to a.$$

we get $a \in F_{\theta_F}$ and so $F = F_{\theta_F}$. An obvious modification of the above gives the fact that the mapping and its inverse are order preserving. But it is folklore that this implies that the map is a complete lattice isomorphism. ∎

Corollary 2.2. *Let* $\mathbf{A}$ *be an enriched BI-semilattice satisfying the hypothesis of Theorem* 2.1. *Then for any* $c, d \in A$ *and* $B \subseteq A$, $(c,d) \in \vartheta(B)$ *iff* $c \leftrightarrow d \in \mathrm{CF}_{\mathbf{A}}(\{a \leftrightarrow b : a, b, \in B\})$, *where as usual* $x \leftrightarrow y := (x \to y) \wedge (y \to x)$.

In particular $(c,d) \in \vartheta(a,b)$ *iff* $c \leftrightarrow d \in \mathrm{CF}_{\mathbf{A}}(a \leftrightarrow b)$

Proof. By Theorem 2.1, $(c,d) \in \vartheta(B)$ iff $c \leftrightarrow d \in F_{\vartheta(B)}$; moreover

$$\begin{aligned} F_{\vartheta(B)} &= F_{\bigcap\{\alpha \in \mathrm{Con}(\mathbf{A}) : (a,b) \in \alpha,\ a,b \in B\}} \\ &= \bigcap\{F_\alpha : (a,b) \in \alpha,\ a,b \in B\} \\ &= \bigcap\{F : a \leftrightarrow b \in F,\ a,b \in B\} \\ &= \mathrm{CF}_{\mathbf{A}}(\{a \leftrightarrow b : a,b \in B\}) \end{aligned}$$

∎

Throughout this paper section we deal with many kinds of enriched BI-semilattices. In each case the key question is always: what are the congruence filters? Let us answer this question first for varieties of BI-semilattices.

Proposition 2.3. *Let $\mathcal{V}$ be a variety of BI-semilattices and $\mathbf{A} \in \mathcal{V}$. A filter F of $\mathbf{A}$ is a congruence filter iff*

$$(*) \quad a \to b,\ b \to a,\ c \to d,\ d \to c \in F \qquad \textit{implies} \qquad (a \to c) \to (b \to d) \in F.$$

Proof. $\mathbf{A}$ satisfies the hypothesis of Theorem 2.1. Hence for any filter F of $\mathbf{A}$, if θ_F is a congruence, then $F = F_{\theta_F}$ and so it is a congruence filter. Assume now that $(*)$ holds for a filter F. We show that θ_F is a congruence. It is symmetric by definition, reflexive since $1 \in F$ and $a \to a \geq 1$, and transitive by (2.4). The fact that is compatible with $\to$ is just $(*)$ while compatibility with $\wedge$ is a straightforward exercise using (2.3) and (2.7).

Conversely if $F = F_\alpha$ for some $\alpha \in \mathrm{Con}(\mathbf{A})$ and $a \to b, b \to a, c \to d, d \to a \in F_\alpha$, then by Theorem 2.1 $(a,b),(c,d) \in \alpha$. Hence $(a \to c, b \to d) \in \alpha$ and so $(a \to c) \to (b \to d) \in F_\alpha$.
■

In view of the previous proposition it is natural to ask whether there are "interesting" varieties of BI-semilattices. Of course the answer is yes and we proceed to introduce one of them.

A *BCI-semilattice* is a BI-semilattice satisfying the equations

$$x \to (y \to z) = y \to (x \to z) \tag{2.8}$$

$$x \leq ((x \to y) \wedge 1) \to y. \tag{2.9}$$

Proposition 2.4. *The class of BCI-semilattices is a variety, axiomatized by* (2.1)–(2.4)*,* (2.8) *and* (2.9)*.*

Proof. Any BCI-semilattice must satisfy those equations. Let $\mathbf{A}$ be an algebra (of the right type) satisfying (2.1)–(2.4), (2.8) and (2.9). It is sufficient to show that it satisfies (2.5) and (2.6). Let $a, b \in A$ with $a \to b \geq 1$. Then by (2.9)

$$a \leq ((a \to b) \wedge 1) \to b = 1 \to b = b$$

hence (2.5) holds. Now observe that $\mathbf{A}$ must satisfy also (2.7), that is a consequence of (2.2), (2.3) and (2.5). Also observe that (2.4), (2.7) and (2.8) imply the equation

$$x \to y \leq (y \to z) \to (x \to z).$$

Hence if $a, b, c \in A$ and $a \leq b$ then

$$1 \leq a \to b \leq (b \to c) \to (a \to c)$$

and so $b \to c \le a \to c$. ∎

Note that if $\mathbf{A}$ is an enriched BCI-semilattice, then $\mathbf{A}$ satisfies the hypotheses of Theorem 2.1, hence $\mathrm{CFil}(\mathbf{A})$ and $\mathrm{Con}(\mathbf{A})$ are isomorphic. In BCI-semilattices congruence filters are very friendly.

Proposition 2.5. *In a BCI-semilattice every filter is a congruence filter.*

Proof. Let $\mathbf{A}$ a BCI-semilattice and let F be a filter. In view of Proposition 2.3 it is enough to show that $(*)$ holds for F. Let then $a \to b, b \to a, c \to d, d \to c \in F$. Then by (2.4)

$$c \to d \le (a \to c) \to (a \to d) \in F$$

and by (2.4), (2.7) and (2.8)

$$b \to a \le (a \to d) \to (b \to d) \in F.$$

By the same token

$$(a \to c) \to (a \to d) \le ((a \to d) \to (b \to d)) \to ((a \to c) \to (b \to d)) \in F$$

and by *modus ponens* we get

$$(a \to c) \to (b \to d) \in F.$$

∎

Congruences also behave well in BCI-semilattices.

Proposition 2.6. *The variety of BCI-semilattices is congruence permutable with Mal'cev term*

$$m(x,y,z) = [((x \to y) \wedge 1) \to z] \wedge [((z \to y) \wedge 1) \to x]$$

Proof. Recall (2.9) and compute

$$\begin{aligned} m(x,y,y) &= [((x \to y) \wedge 1) \to y] \wedge [((y \to y) \wedge 1) \to x] \\ &= [((x \to y) \wedge 1) \to y] \wedge x = x \end{aligned}$$

and similarly $m(x,x,y) = y$. ∎

A *BI-monoid* is a *semilattice ordered left-residuated monoid* i.e. an algebra $\langle A, \to, \wedge, \cdot, 1 \rangle$ where

1. $\langle A, \wedge \rangle$ is a semilattice;
2. $\langle A, \cdot, 1 \rangle$ is a monoid;
3. $a \le b$ implies $ac \le bc$ and $ca \le ca$, i.e. the multiplication respects the meet ordering;
4. $a \le b \to c$ iff $ab \le c$, i.e. $\to$ is a left residuation w.r.t. the meet ordering.

Proposition 2.7. *In a BI-monoid the following hold:*

1. $x \leq y$ *iff* $x \to y \geq 1$*;*
2. $1 \to x = x$*;*
3. $(y \wedge z)x \leq yx \wedge zx$*;*
4. $x(y \wedge z) \leq xy \wedge xz$*;*
5. $x \to y \leq (z \to x) \to (z \to y)$*;*
6. $x \leq y$ *implies* $z \to x \leq z \to y$ *and* $y \to z \leq x \to z$*.*

Proof. 1. is obvious by left residuation and the fact that 1 is a two-sided unit. For 2. from $x \leq x$ we get $x \leq 1 \to x$ and from $1 \to x \leq 1 \to x$ we get $1 \to x \leq x$. 3. and 4. hold, since the multiplication respects the meet ordering. For 5. we have

$$(x \to y)(z \to x)z \leq (x \to y)x \leq y$$

and left residuation does the trick.

Finally to prove 6., assume $x \leq y$. Then from $z \to x \leq z \to x$ one gets $(z \to x)z \leq x$ by left residuation, hence $(z \to x)z \leq y$ and so $z \to x \leq z \to y$. On the other hand from $x \to y \geq 1$ and 4.

$$y \to z \leq (y \to z)(x \to y) \leq x \to z.$$

■

The class of BI-monoids is in fact a variety, as the following proposition shows.

Proposition 2.8. *The class of BI-monoids is a variety* $\mathcal{L}$*.* $\langle A, \to, \wedge, \cdot, 1 \rangle \in \mathcal{L}$ *iff*

1. $\langle A, \wedge \rangle$ *is a semilattice;*
2. $\langle A, \cdot, 1 \rangle$ *is a monoid;*
3. *the following hold in* $\mathbf{A}$:

$$x \to x \geq 1 \tag{2.10}$$

$$xy \to z = x \to (y \to z) \tag{2.11}$$

$$(x \to y) \wedge (x \to z) = x \to (y \wedge z) \tag{2.12}$$

$$(x \to y)x \leq y \tag{2.13}$$

$$(y \wedge z)x \leq yx \wedge zx \tag{2.14}$$

$$x \to y \leq (z \to x) \to (z \to y). \tag{2.15}$$

Proof. First we show that (2.10)–(2.15) hold in a BI-monoid. In Proposition 2.7 we have shown that (2.13), (2.14) and (2.15) hold in any BI-monoid, while (2.10) is a trivial consequence of left residuation. For (2.11) start from $x \to (y \to z) \leq x \to (y \to z)$ and apply residuation twice to get $x \to (y \to z) \leq xy \to z$. For the other inequality use the same technique starting from $xy \to z \leq xy \to z$. Finally $(x \to y) \wedge (x \to z) \geq x \to (y \wedge z)$ follows from Proposition 2.7(4). On the other hand

$$((x \to y) \wedge (x \to z))x \leq (x \to y)x \wedge (x \to z)x \leq y \wedge z$$

hence by left residuation $(x \to y) \wedge (x \to z) \leq x \to (y \wedge z)$.

Assume now that an algebra $\mathbf{A}$ satisfies 1.,2. and 3. above. Left monotonicity of the multiplication follows from (2.14). Then let us show that for $a, b \in A$

$$a \leq b \qquad \text{iff} \qquad a \to b \geq 1.$$

If $a \leq b$ then $a \wedge b = a$ and by (2.12)

$$(a \to a) \wedge (a \to b) = a \to (a \wedge b) = a \to a;$$

hence $a \to b \geq a \to a \geq 1$ by (2.10). Conversely if $a \to b \geq 1$, then by (2.13) and left monotonicity

$$a \leq (a \to b)a \leq b.$$

Hence $ab \leq c$ iff $1 \leq ab \to c$ iff (by (2.11)) $1 \leq a \to (b \to c)$ iff $a \leq b \to c$. Hence $\mathbf{A}$ is residuated with respect to $\leq$.

Finally assume $a \leq b$. From $bc \leq bc$ we get $b \leq c \to bc$ and hence $a \leq c \to bc$ so $ac \leq bc$. Hence the multiplication is monotonic in both arguments and $\mathbf{A}$ is a BI-monoid. ∎

Note that along the way we have also shown that BI-monoids are enriched BI-semilattices, i.e. the class of $\{\to, \wedge, 1\}$-subreducts of BI-monoids consists entirely of BI-semilattices. Accordingly a filter of a BI-monoid is simply a filter of the underlying BI-semilattice structure. It is an easy exercise to show that filters in this case are simply semilattice filters containing 1 and closed under multiplication. We will now describe the congruence filters of a BI-monoid.

Theorem 2.9. *The congruence filters of a BI-monoid* $\mathbf{A}$ *coincide with the congruence filters of the underlying BI-semilattice structure.*

Proof. Since the converse is obvious, it is sufficient to show that if a filter F satisfies $(*)$, then it is a congruence filter. In Proposition 2.3 we have shown that in this case

$$\theta_F = \{(a,b) : a \to b, b \to a \ \in F\}$$

is a BI-semilattice congruence. We will prove that it is compatible with $\cdot$ as well, thus proving that $\theta_F \in \mathrm{Con}(\mathbf{A})$. Suppose then that $(a,b) \in \theta_F$ and let $c \in A$; then $1 \le a \to (c \to ac) \in F$ and $b \to a \in F$. By (2.5) and the fact that F is closed under *modus ponens* we get that

$$b \to (c \to ac) = bc \to ac \in F.$$

Since the argument is symmetric we get $ac \to bc \in F$ and hence $(ac, bc) \in \theta_F$.

On the other hand $c \to (b \to cb) \in F$ and $(b \to cb) \to (a \to cb) \in F$ because of $(*)$, hence again by (2.5) and *modus ponens* we get $c \to (a \to cb) \in F$ and hence $c \to (a \to ca) \wedge c \to (a \to cb) \in F$. But

$$\begin{aligned} c \to (a \to ca) \wedge c \to (a \to cb) &= c \to ((a \to ca) \wedge (a \to cb)) \\ &= c \to (a \to (ca \wedge cb)) = ca \to (ca \wedge cb) \\ &= ca \to ca \wedge ca \to cb \le ca \to cb \in F. \end{aligned}$$

Again by symmetricity we get $cb \to ca \in F$ and so $(ca, cb) \in \theta_F$. So if $(a,b),(c,d) \in \theta_F$ then

$$ac \; \theta_F \; bc \; \theta_F \; bd,$$

hence θ_F is compatible with $\cdot$. By Theorem 2.1 we conclude that F is a congruence filter.

■

A BI-monoid in which the multiplication is commutative is a *BCI-monoid*. Alternatively a BCI-monoid can be defined as a BI-monoid satisfying (2.8), namely $x \to (y \to z) = y \to (x \to z)$. In fact if the multiplication is commutative then (2.8) is an immediate consequence of (2.11). Conversely if (2.8) holds, from $yx \to yx \ge 1$ we get $y \to (x \to yx) \ge 1$ and by (2.8) $x \to (y \to xy) \ge 1$, so eventually $xy \le yx$. Moreover from $(x \to y) \wedge 1 \le x \to y$, residuation and (2.8) we get (2.9). We have thus shown:

Proposition 2.10. *A BCI-monoid is an enriched BCI-semilattice. Hence the variety of BCI-monoids is congruence permutable.*

A more interesting and useful fact is the following.

Lemma 2.11. *Let* $\mathbf{A}$ *a BCI-monoid and let* a *be an idempotent of* $\mathbf{A}$*, i.e.* $a^2 = a$*. Then for any* $b, c \in A$

$$a \to (b \to c) \leq (a \to b) \to (a \to c). \tag{2.16}$$

Proof. We have

$$\begin{aligned}(a \to (b \to c))(a \to b)a &= (a \to (b \to c))(a \to b)a^2 \\ &= (a \to (b \to c))a(a \to b)a \\ &\leq (b \to c)b \leq c.\end{aligned}$$

Thus applying residuation twice we get the thesis. ■

Not surprisingly, congruence filters of BCI-monoids are very well-behaved.

Theorem 2.12. *In a BCI-monoid every filter is a congruence filter.*

Proof. Let $\mathbf{A} \in \mathcal{C}$. In view of Theorem 2.9 it is enough to show that any filter F of $\mathbf{A}$ satisfies $(*)$. A BCI-monoid is an enriched BCI-semilattice, hence the equation

$$x \to y \leq (y \to z) \to (x \to z) \tag{2.17}$$

holds in $\mathbf{A}$, since it is an easy consequence of (2.4) and (2.8). Let then $a \to b, b \to a, c \to d, d \to c \in F$. Then by (2.15)

$$c \to d \leq (a \to c) \to (a \to d) \in F$$

and by (2.17)

$$b \to a \leq (a \to d) \to (b \to d) \in F.$$

Again by (2.17)

$$(a \to c) \to (a \to d) \leq ((a \to d) \to (b \to d)) \to ((a \to c) \to (b \to d)) \in F$$

and by applying *modus ponens* we get

$$(a \to c) \to (b \to d) \in F.$$

■

Hence in a BCI-monoid a congruence filter is simply a semilattice filter containing 1 and closed under multiplication. This allows us to obtain a very good description of the congruence filter generated by a set. For any BCI-monoid $\mathbf{A}$ and $a \in A$ let $[a)$ be the semilattice filter generated by a and let

$$\widehat{a^n} = \overbrace{(a \wedge 1) \cdot (a \wedge 1) \cdot \ldots \cdot (a \wedge 1)}^{n \text{ times}}.$$

Proposition 2.13. *Let* $\mathbf{A}$ *be a BCI-monoid.*

1. *For any* $B \subseteq A$*;*

$$\mathrm{CF}_{\mathbf{A}}(B) = \bigcup\{[(b_1 \wedge 1) \ldots (b_n \wedge 1)) : n \in \omega,\ b_1, \ldots, b_n \in B\}.$$

2. *For any* $a \in A$

$$\mathrm{CF}_{\mathbf{A}}(a) = \bigcup_{n \in \omega} [\widehat{a^n})$$

Hence $b \in \mathrm{CF}_{\mathbf{A}}(a)$ *iff* $\widehat{a^n} \leq b$ *for some* $n \in \omega$*. It follows that a principal filter* $\mathrm{CF}_{\mathbf{A}}(a)$ *is principal as a semilattice filter iff there is an* $n \in \omega$ *with* $\widehat{a^n} \leq \widehat{a^{n+1}}$ *iff* $\mathrm{CF}_{\mathbf{A}}(a) = [\widehat{a^n})$.

Proof. (1) Let F be the right side of the above equality; F is clearly a semilattice filter. Moreover $1 \in F$ and if $b, c \in F$ then there are $b_1, \ldots, b_m, c_1, \ldots, c_k \in B$ such that $b \geq (b_1 \wedge 1) \ldots (b_m \wedge 1)$ and $c \geq (c_1 \wedge 1) \ldots (c_k \wedge 1)$; hence $bc \geq (b_1 \wedge 1) \ldots (b_m \wedge 1)(c_1 \wedge 1) \ldots (c_k \wedge 1)$ and so $bc \in F$. We conclude that F is a filter containing B. Now any filter containing B must contain $(b_1 \wedge 1) \ldots (b_n \wedge 1)$ for any n and $b_1, \ldots, b_n \in B$, hence it contains F.

(2) The first part follows from (1). If there is an n with $\widehat{a^n} \leq \widehat{a^{n+1}}$ then in fact the two are equal and so $F = [\widehat{a^n})$. On the other hand if $\mathrm{CF}_{\mathbf{A}}(a) = [b)$, then $[b) = \bigcup_{n \in \omega} [\widehat{a^n})$ and by compactness in the semilattice filter lattice we get $[b) = [\widehat{a^n})$ for some n. It then follows that $b = \widehat{a^n}$ and hence $\widehat{a^n} \leq \widehat{a^{n+1}}$ since the latter belongs to $\mathrm{CF}_{\mathbf{A}}(a)$. ■

Let $\mathbf{A}$ be a BCI-monoid and $a, b, c, d \in A$. Let also $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by $\{a, b, c, d\}$. Then by Proposition 2.13 and Corollary 2.2,

$$(c, d) \in \vartheta_{\mathbf{A}}(a, b) \qquad \text{iff} \qquad \exists n\ \widehat{(a \leftrightarrow b)^n} \leq c \leftrightarrow d.$$

It follows that

$$(c, d) \in \vartheta_{\mathbf{A}}(a, b) \qquad \text{iff} \qquad (c, d) \in \vartheta_{\mathbf{B}}(a, b)$$

and so any variety of BCI-monoids has the *congruence extension property.*

A unary operation $f(x)$, on a BCI-monoid $\mathbf{A}$, is *compatible* if for any $a, b \in A$ there is an n

$$\widehat{(a \leftrightarrow b)^n} \leq f(a) \rightarrow f(b).$$

An k-ary operation $f(x_1, \ldots, x_k)$ is *compatible* if

$$f_i(x) = f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_k)$$

is compatible for any $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k \in A$ and $i = 1, \dots, k$. Any constant operation is clearly compatible. The compatible operations on a BCI-monoid are exactly the operations having the substitution property with respect to any congruence. Suppose f is a compatible operation for $\mathbf{A}$ and suppose without loss of generality that f is unary. If $(a, b) \in \theta$, then $a \to b, b \to a \in F_\theta$, hence $(a \leftrightarrow b)^{\widehat{n}} \in F_\theta$ for any n. By definition of compatibility we deduce $f(a) \to f(b), f(b) \to f(a) \in F_\theta$, i.e. $(f(a), f(b)) \in \theta$. Conversely if $f(x)$ has the substitution property, then $(f(a), f(b)) \in \vartheta(a, b)$ and so $f(a) \leftrightarrow f(b) \in \mathrm{CF}_{\mathbf{A}}((a \leftrightarrow b)^{\widehat{n}})$. By Proposition 2.13 there is an m with

$$(a \leftrightarrow b)^{\widehat{m}} \le f(a) \to f(b).$$

A BCI-monoid *with compatible operations* is an algebra $\mathbf{A} = \langle A, \to, \wedge, \cdot, 1, f_i \rangle_{i \in I}$ where $\langle A, \to, \wedge, \cdot, 1 \rangle$ is a BCI-monoid and any f_i is compatible. It is clear that any congruence $\theta \in \mathrm{Con}(\mathbf{A})$ is completely determined by the BCI-monoid filter F_θ and that in fact $\mathrm{Con}(\mathbf{A})$ coincides with the congruence lattice of the underlying BCI-monoid structure. In what follows, unless otherwise specified, by a BCI-monoid we will always mean a BCI-monoid *with or without* compatible operations.

2.2. A digression. We have seen that the class of $\{\to, \wedge, 1\}$-subreducts of BCI-monoids (BI-monoids), consists entirely of BCI-semilattices (BI-semilattices). Is this class a variety? Does this class coincide with the class of BCI-semilattices (BI-semilattices)? The first question is easily solved in the case of BCI-monoids.

In Proposition 2.13(1) we have shown that, in a BCI-monoid, $a \in \mathrm{CF}_{\mathbf{A}}(B)$ iff there are $b_1, \dots, b_n \in B$ with

$$(b_1 \wedge 1) \dots (b_n \wedge 1) \le a.$$

This, by taking enough residuations, is equivalent to

$$(b_1 \wedge 1) \to ((b_2 \wedge 1) \to \dots \to ((b_n \wedge 1) \to a) \dots) \ge 1. \tag{2.18}$$

Let now $\mathcal{V}$ be a variety of BCI-monoids and let $S\mathcal{V}^{\to,\wedge}$ the class of its $\{\to, \wedge, 1\}$-subreducts.

Theorem 2.14. *For any variety $\mathcal{V}$ of BCI-monoids, the class $S\mathcal{V}^{\to,\wedge}$ is a congruence permutable variety.*

Proof. Since subreducts of BCI-monoids are BCI-semilattices, the theorem will follow once we show that $S\mathcal{V}^{\to,\wedge}$ is a variety. $S\mathcal{V}^{\to,\wedge}$ is clearly closed under subalgebras and direct products. Let then $\mathbf{A} \in S\mathcal{V}^{\to,\wedge}$ and let $\mathbf{B} \in \mathcal{V}$ such that $\mathbf{A} \leq \langle B, \to, \wedge, 1\rangle$. Let $\theta \in \mathrm{Con}(\mathbf{A})$ and let $F = F_\theta$, let $G = \mathrm{CF}_{\mathbf{B}}(F)$ and consider

$$\theta_G = \{(u,v) : u \to v, v \to u \in G\}.$$

Now $u \to v \in G = \mathrm{CF}_{\mathbf{B}}(F)$ iff there are $b_1, \ldots, b_n \in F$ with

$$(b_1 \wedge 1) \to ((b_2 \wedge 1) \to \ldots \to (b_n \to (u \to v)) \ldots) \geq 1.$$

If $u, v \in A$, then $u \to v \in F$, since F is closed under *modus ponens*, and similarly $v \to u \in F$. Therefore

$$\theta_G \cap A \times A \subseteq \{(u,v) : u \to v, v \to u \in F\} = \theta_F = \theta.$$

On the other hand, if $(u,v) \in \theta$, then $u \to v \;\theta\; u \to u \geq 1$, hence $u \to v \in F$; similarly $v \to u \in F$ and hence $\theta \subseteq \theta_G \cap A \times A$.

We conclude that $\mathbf{A}/\theta$ is isomorphic to a subalgebra of the $\{\to, \wedge, 1\}$-reduct of $\mathbf{B}/\theta$. Hence $\mathbf{A}/\theta \in S\mathcal{V}^{\to,\wedge}$ and the theorem is proved. ∎

The second question is harder, but luckily most of the work has already been done by Dunn, Meyer and Routley in a different context (see [14] and the bibliography therein). From now to the end of this section we will proceed more informally: the details can be filled with some (long) calculations. For more information consult [13] and [14].

Let $\mathbf{A}$ be a BI-semilattice and let Γ be the set of the *semilattice filters* of $\mathbf{A}$. Let U be the set of the *hereditary* subsets of Γ, i.e. those $X \subseteq \Gamma$ such that $F \in X$ and $F \subseteq G$ implies $G \in X$. Let $\mathbf{1} = \{F \in \Gamma : 1 \in F\}$. Clearly $\mathbf{1}$ is hereditary and the intersections of two hereditary subsets of Γ is hereditary. In Γ we define the following ternary relation (the *canonical accessibility relation* [14]):

$$R(F,G,H) \qquad \text{iff} \qquad \forall a,b,\; a \in F \text{ and } a \to b \in G \text{ implies } b \in H$$

Finally if $X, Y \in U$ we define

$$X \circ Y = \{H : \exists F \in Y,\ \exists G \in X,\ R(F,G,H)\}$$
$$X \to Y = \{H : \forall F,G,\ \text{if } R(F,H,G) \text{ and } F \in X,\ \text{then } G \in Y\}.$$

Proposition 2.15. *If* $\mathbf{A}$ *is a BI-semilattice, then* $\langle U, \rightarrow, \cap, \circ, \mathbf{1}\rangle$ *is a BI-monoid.*

Proof. First we have to check that U is closed under the operations introduced above. Consider $X, Y \in U$ and let us show that $X \rightarrow Y$ is hereditary. Let $H \in X \rightarrow Y$ and $H \subseteq H' \in \Gamma$. If $R(F, H', G)$, then $R(F, H, G)$ just by the definition. Hence if $F \in X$ we have $G \in X$ and so $H' \in X \rightarrow Y$. Similarly we show that U is closed under $\circ$.

Associativity of $\circ$ follows from a lengthy calculation in which equation (2.4) is used critically, while monotonicity of $\circ$ follows straight from the definition. Let us show that $\mathbf{1}$ is two sided identity. Note that for any $F \in \Gamma$, $R(F, \nabla_A, F)$ holds and, since $\nabla_A \in \mathbf{1}$, we get at once that $X = \mathbf{1} \circ X$. Conversely, let $H \in \mathbf{1} \circ X$; then there is an $F \in X$ and a $G \in \mathbf{1}$ with $R(F, G, H)$. If $a \in F$, then $a \rightarrow a \geq 1 \in G$ and hence $a \in H$, so that $F \subseteq H$. But X is hereditary and $F \in X$, so $H \in X$ and eventually $\mathbf{1} \circ X = X$. A very similar argument shows that $X \circ \mathbf{1} = X$. Finally residuation is shown to hold just using the definitions (and some patience). ∎

Theorem 2.16. *Any BI-semilattice can be embedded in a reduct of a BI-monoid. Hence BI-semilattices are exactly subreducts of BI-monoids.*

Proof. Let $\mathbf{A}$ be a BI-semilattice and let $\mathbf{U} = \langle U, \rightarrow, \cap, \circ, \mathbf{1}\rangle$ be the BI-monoid constructed as above. Define a mapping $h : \mathbf{A} \longrightarrow \mathbf{U}$ by setting

$$h(a) = \{F \in \Gamma : a \in F\}.$$

Clearly $h(a) \in U$ for all $a \in A$, $h(1) = \mathbf{1}$ and $h(a) \cap h(b) = h(a \wedge b)$. We now show that h preserves $\rightarrow$, i.e. that for all $H \in \Gamma$, $H \in h(a \rightarrow b)$ iff $H \in h(a) \rightarrow h(b)$. Via their definitions this is equivalent to show

$$a \rightarrow b \in H \qquad \text{iff} \qquad \forall F, G \text{ if } R(F, H, G) \text{ and } a \in F, \text{ then } b \in G.$$

The left-to-right implication follows from the definition of $R(F, H, G)$. To prove the other, assume that $a \rightarrow b \notin H$. We show that there F, G with $a \in F$, $R(F, H, G)$, but $b \notin G$. Let $F = [a)$ and $G = A - \{y : y \leq b\}$ (clearly a semilattice filter), so that $a \in F$ and $b \notin G$. If $x \in F$, $x \rightarrow y \in G$, but $y \notin G$, then $a \leq x$ and $y \leq b$. Using (2.4) and (2.6) one readily gets $x \rightarrow y \leq a \rightarrow b$, and so $a \rightarrow b \in H$, contrary to the hypothesis. It follows that $y \in G$ and hence $R(F, H, G)$.

To conclude the proof it remains to show that h is one-to-one. Suppose then that $h(a) = h(b)$; this implies $[a) \in h(b)$ and $[b) \in h(a)$, hence $b \in [a)$ and $a \in [b)$, hence $a = b$. ∎

Let now $\mathbf{A}$ be a BCI semilattice and let $\mathbf{U}$ as above. It is easily seen, using (2.1), (2.4) and (2.8), that the equation

$$x \leq (x \to y) \to y \tag{2.19}$$

holds in $\mathbf{A}$. This equation forces the multiplication on U to be commutative, i.e. $\mathbf{U}$ is a BCI-monoid. Really it is enough to show that for any F, G, H, $R(F, G, H)$ implies $R(G, F, H)$. Assume the former and let $a \in G$ and $a \to b \in F$. Then (2.19) yields $(a \to b) \to b \in G$. Since $a \to b \in F$ and $R(F, G, H)$ we conclude $b \in H$ and hence $R(G, F, H)$. Thus we have shown:

Theorem 2.17. *Any BCI-semilattice can be embedded in a reduct of a BCI-monoid. Hence BCI-semilattices are exactly subreducts of BCI-monoids.*

3. The TD-terms

We are now ready to state and prove our first main theorem.

Theorem 3.1. *Let $\mathcal{V}$ be a variety of BCI-monoids; then the following are equivalent.*

1. *For any $\mathbf{A} \in \mathcal{V}$ and $a, b \in A$, $CF(a \leftrightarrow b)$ is a principal semilattice filter.*
2. *There is an n such that the equation*

$$(x \leftrightarrow y)^{\widehat{n}} \leq (x \leftrightarrow y)^{\widehat{n+1}}$$

holds in $\mathcal{V}$.

3. *$p(x, y, z) = (x \leftrightarrow y)^{\widehat{n}} z$ is a TD-term for $\mathcal{V}$.*
4. *$\mathcal{V}$ has equationally definable principal congruences: for $\mathbf{A} \in \mathcal{V}$ and $a, b, c, d \in A$*

$$(c, d) \in \vartheta(a, b) \qquad \text{iff} \qquad (a \leftrightarrow b)^{\widehat{n}} \leq c \leftrightarrow d.$$

5. *$\mathcal{V}$ has (first order) definable principal congruences.*

Proof. 1. implies 2. via a standard Mal'cev argument. If we assume 2., by Proposition 2.13, for $a, b \in A$ we have $\mathrm{CF}_{\mathbf{A}}(a \leftrightarrow b) = [(a \leftrightarrow b)^{\widehat{n}})$. Hence 1. and 2. are equivalent.

Assume now 2. and note that $(x \leftrightarrow y)^{\widehat{n}}$ is idempotent. It is also clear that $(x \leftrightarrow x)^{\widehat{n}} y \approx y$ holds in $\mathcal{V}$. Let $\mathbf{A} \in \mathcal{V}$, $a, b \in A$ and $q(x)$ a unary polynomial: from $q(a) \to q(b)\ \vartheta(a,b)\ q(a) \to q(a) \geq 1$ we get $q(a) \to q(b) \in F_{\vartheta(a,b)} = \mathrm{CF}_{\mathbf{A}}(a \leftrightarrow b)$ and so $(a \leftrightarrow b)^{\widehat{n}} \leq q(a) \to q(b)$. By residuation we get $(a \leftrightarrow b)^{\widehat{n}} q(a) \leq q(b)$ and using the fact that $(a \leftrightarrow b)^{\widehat{n}}$ is an idempotent we get

$$(a \leftrightarrow b)^{\widehat{n}} q(a) \leq (a \leftrightarrow b)^{\widehat{n}} q(b).$$

The reverse inclusion is obtained via a symmetric argument. So Theorem 1.1 yields that $(x \leftrightarrow y)^{\widehat{n}} z)$ is a TD-term for $\mathcal{V}$.

Assume now 3. and let $\mathbf{A} \in \mathcal{V}$, $a, b, c, d \in A$. We first show that $(a \leftrightarrow b)^{\widehat{n}}$ is idempotent. Let $q(x) = (x \leftrightarrow b)^{\widehat{n}}$. Then by Theorem 1.1

$$(a \leftrightarrow b)^{\widehat{n}} (a \leftrightarrow b)^{\widehat{n}} = p(a, b, q(a)) = p(a, b, q(b)) = (a \leftrightarrow b)^{\widehat{n}} (b \leftrightarrow b)^{\widehat{n}} = (a \leftrightarrow b)^{\widehat{n}}.$$

Suppose now that $(c, d) \in \vartheta(a, b)$. Then

$$(a \leftrightarrow b)^{\widehat{n}} c = (a \leftrightarrow b)^{\widehat{n}} d.$$

By residuation and Proposition 2.7(6) we get

$$(a \leftrightarrow b)^{\widehat{n}} \leq c \to (a \leftrightarrow b)^{\widehat{n}} d \leq c \to d.$$

A symmetric argument yields $(a \leftrightarrow b)^{\widehat{n}} \leq d \to c$ and hence

$$(a \leftrightarrow b)^{\widehat{n}} \leq c \leftrightarrow d.$$

Assume now the latter. Then $(a \leftrightarrow b)^{\widehat{n}} \leq c \to d$ and by residuation $(a \leftrightarrow b)^{\widehat{n}} c \leq d$. Idempotence of $(a \leftrightarrow b)^{\widehat{n}}$ gives $(a \leftrightarrow b)^{\widehat{n}} c \leq (a \leftrightarrow b)^{\widehat{n}} d$ and a symmetric argument gives the reverse inclusion. We conclude that $p(a, b, c) = p(a, b, d)$ and so $(c, d) \in \vartheta(a, b)$.

4. obviously implies 5. If we assume 5. there is a first order formula $\varphi(x, y, z, w)$ such that for any $\mathbf{A} \in \mathcal{V}$ and $a, b, c, d \in A$, $(c, d) \in \vartheta(a, b)$ iff $\mathbf{A} \models \varphi(a, b, c, d)$. By Proposition 2.13 this implies

$$\mathcal{V} \models \forall xyzw(\varphi(x, y, z, w) \Longleftrightarrow \bigvee_{m \in \omega} ((x \leftrightarrow y)^{\widehat{m}} \leq (z \leftrightarrow w))).$$

By the compactness theorem there is an n with

$$\mathcal{V} \models \forall xyzw(\varphi(x, y, z, w) \Longleftrightarrow (x \leftrightarrow y)^{\widehat{n}} \leq (z \leftrightarrow w)).$$

It follows that $\mathrm{CF}_{\mathbf{A}}(a \leftrightarrow b) = [(a \leftrightarrow b)^{\widehat{n}})$ and the proof is complete. ■

Corollary 3.2. *The TD-term $p(x,y,z) = (x \leftrightarrow y)^{\widehat{n}} z$ is commutative and regular. Hence for a variety $\mathcal{V}$ of BCI-monoids the following are equivalent.*

1. *$\mathcal{V}$ has a regular, commutative TD-term;*
2. *$\mathcal{V}$ has a commutative TD-term;*
3. *$\mathcal{V}$ has a TD-term.*

Proof. Let $\mathbf{A} \in V$ and $a, b, c, a', b' \in A$; then

$$\begin{aligned} p(a,b,p(a',b',c)) &= (a \leftrightarrow b)^{\widehat{n}}(a' \leftrightarrow b')^{\widehat{n}} c = (a' \leftrightarrow b')^{\widehat{n}}(a \leftrightarrow b)^{\widehat{n}} c \\ &= p(a',b',p(a,b,c)), \end{aligned}$$

hence $p(x,y,z)$ is commutative.

To prove regularity observe first that if $u \in A$ and $u \leq 1$ then

$$(u \leftrightarrow 1)^{\widehat{n}} = (u \wedge u \to 1 \wedge 1)^n = u^n.$$

Hence

$$\begin{aligned} p(p(a,b,1),1,a) &= ((a \leftrightarrow b)^{\widehat{n}} \leftrightarrow 1)^{\widehat{n}} a = (a \leftrightarrow b)^{\widehat{n}} a = p(a,b,a) \\ &= p(a,b,b) = (a \leftrightarrow b)^{\widehat{n}} b = ((a \leftrightarrow b)^{\widehat{n}} \leftrightarrow 1)^{\widehat{n}} b = p(p(a,b,1),1,b) \end{aligned}$$

and $p(x,y,z)$ is regular. ■

To deal with the $\{\to, \wedge\}$ subreducts we need a technical definition. In a BCI-monoid define inductively $x \to^{\widehat{n}} y$ by

$$x \to^{\widehat{1}} y = (x \wedge 1) \to y \qquad x \to^{\widehat{n+1}} y = (x \wedge 1) \to (x \to^{\widehat{n}} y);$$

using the identity $xy \to z = x \to (y \to z)$ we see at once that

$$x \to^{\widehat{n}} y = x^{\widehat{n}} \to y.$$

Corollary 3.3. *Let $\mathcal{V}$ be a variety of BCI-monoids satisfying*

$$(x \leftrightarrow y)^{\widehat{n}} \leq (x \leftrightarrow y)^{\widehat{n+1}}$$

and let $\mathcal{W} = S\mathcal{V}^{\to,\wedge}$. Then the term

$$p(x,y,z) = (x \leftrightarrow y) \to^{\widehat{n}} z$$

is a commutative, regular TD-term for $\mathcal{W}$.

Proof. By what we said above it is enough to show that $t(x,y,z) = \widehat{(x \leftrightarrow y)^n} \to z$ is a commutative, regular TD-term for $\mathcal{V}$. In fact $t(x,y,z) = p(x,y,z)$ and the latter contains only $\to$ and $\wedge$. Since being a commutative, regular deduction term can be expressed via equations we will conclude that $p(x,y,z)$ is a commutative, regular TD-term for $\mathcal{W}$.

Note that $t(x,x,y) \approx y$ holds in $\mathcal{V}$. Let $\mathbf{A} \in \mathcal{V}$, $a,b \in A$ and $q(x)$ be a unary polynomial. Once again $q(a) \to q(b) \in \mathrm{CF}_{\mathbf{A}}(a \leftrightarrow b)$ and hence (via the previous theorem) $\widehat{(a \leftrightarrow b)^n} \leq q(a) \to q(b)$. Since $\widehat{(a \leftrightarrow b)^n}$ is idempotent, from Lemma 2.11 we get

$$((\widehat{a \leftrightarrow b)^n} \to q(a)) \to ((\widehat{a \leftrightarrow b)^n} \to q(b)) \geq ((\widehat{a \leftrightarrow b)^n} \to (q(a) \to q(b)) \geq 1$$

and hence $((\widehat{a \leftrightarrow b)^n} \to q(a)) \leq (\widehat{a \leftrightarrow b)^n} \to q(b))$. The opposite inclusion is obtained by symmetry, hence we conclude that $t(a,b,q(a)) = t(a,b,q(b))$; via Theorem 1.1 $t(x,y,z)$ is a TD-term. To check commutativity we use (2.8). For $a,b,c,a',b' \in A$:

$$\begin{aligned} t(a,b,t(a',b',c)) &= \widehat{(a \leftrightarrow b)^n} \to (\widehat{(a' \leftrightarrow b')^n} \to c) \\ &= \widehat{(a' \leftrightarrow b')^n} \to (\widehat{(a \leftrightarrow b)^n} \to c) = t(a',b',t(a,b,c)). \end{aligned}$$

The proof that $t(x,y,z)$ is regular is very similar to the one for $p(x,y,z)$ in Corollary 3.2. ∎

In view of the connections with algebraic logic, it is natural to ask what we can prove in case we deal with structures in which more operations are present. It turns out that the implication from 1. to 2. in Theorem 3.1 still holds, provided we have a *uniform* way of defining certain principal filters.

Corollary 3.4. *Let $\mathcal{V}$ be a variety of enriched BCI-monoids and suppose that there is a unary term $t(x)$ such that, for any $\mathbf{A} \in \mathcal{V}$ and for any $a,b \in B$, $\mathrm{CF}_{\mathbf{A}}(a \leftrightarrow b) = [t(a \leftrightarrow b))$. Then*

$$p(x,y,z) = t(x \leftrightarrow y)z \qquad\qquad p'(x,y,z) = t(x \leftrightarrow y) \to z$$

are commutative, regular TD-terms for $\mathcal{V}$.

Proof. First note that, for $a \in A$, $[t(a \leftrightarrow a)) = \mathrm{CF}_{\mathbf{A}}(a \leftrightarrow a) = F_{\vartheta_{\mathbf{A}}(a,a)} = \nabla_A$, hence $t(a \leftrightarrow a) = 1$. It follows that the equations $p(x,x,y) = p'(x,x,y) = y$ hold in $\mathcal{V}$. Next note that for any $a, b \in A$, $t(a \leftrightarrow b) \leq 1$ and so $t(a \leftrightarrow b)t(a \leftrightarrow b) \leq t(a \leftrightarrow b)$. On the other hand $t(a \leftrightarrow b)t(a \leftrightarrow b) \in [t(a \leftrightarrow b))$, hence they must coincide. We have thus proved that $t(a \leftrightarrow b)$ is idempotent. From here just follow the proofs of Theorem 3.1, and Corollary 3.2. ∎

If we want a full analog of Theorem 3.1 we must place more conditions on the additional operations. A non constant unary operation f on a BCI-monoid $\mathbf{A}$ is *normal* if $f(1) = 1$ and $f(a \wedge 1) = f(a) \wedge 1$, it is *increasing* if $x \leq y$ implies $f(x) \leq f(y)$ and it is *multiplicative* if $f(x) \cdot f(y) \leq f(xy)$. An operation f of arity $n \geq 1$ is *normal*, *increasing* or *multiplicative* if

$$f_i(x) = f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$$

is normal, increasing or multiplicative for any $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$ and $i = 1, \ldots, n$. We do not need to consider constant operations. In fact a constant operation on a BCI-monoid is always compatible in our sense so it does not change the behavior of filters and congruences.

If f is increasing and multiplicative then

$$f_i(x \to y)f_i(x) \leq f_i((x \to y)x) \leq f_i(y)$$

and so

$$f_i(x \to y) \leq f_i(x) \to f_i(y).$$

A *BCI-monoid with normal operators* is an algebra $\langle A, \to, \wedge, \cdot, 1, f_\lambda \rangle_{\lambda \in \Lambda}$ such that $\langle A, \to, \wedge, \cdot, 1 \rangle$ is a BCI-monoid and each f_λ is normal, increasing and multiplicative. As usual we start determining the congruence filters of a BCI-monoid with normal operators.

Proposition 3.5. *A filter G of $\mathbf{A}$ is a congruence filter iff it is closed under any unary polynomial $f_i(x)$ for any normal operator f of $\mathbf{A}$.*

Proof. Suppose G is a congruence filter of $\mathbf{A}$ and let $\theta \in \mathrm{Con}(\mathbf{A})$ such that $G = \nabla_A/\theta$. For any normal operator f, if $a \in G$, then $a \ \theta \ v \geq 1$ and hence $f_i(a) \ \theta \ f_i(v) \geq f_i(1) \geq 1$. Therefore $f_i(a) \in G$.

Suppose that G is a filter closed under all the polynomials f_i. Let

$$\theta_G = \{(a,b) : a \to b, b \to a \in G\}.$$

Since G is a filter of the underlying BCI-monoid structure, then θ_G is a BCI-monoid congruence and $G = \nabla_A/\theta_G$. It is enough to show that $\theta_G \in \mathrm{Con}(\mathbf{A})$. Let f be a normal operator of $\mathbf{A}$; then if $(a, b) \in \theta_G$, then $a \to b, b \to a \in G$. Hence

$$f_i(a \to b) \leq f_i(a) \to f_i(b) \in G \qquad f_i(b \to a) \leq f_i(b) \to f_i(a) \in G$$

and so $(f_i(a), f_i(b)) \in G$. From here the compatibility of f is straightforward. ∎

Now we try to get a good description of the congruence filter generated by a set. Let $\mathrm{P}(\mathbf{A})$ be the set of all unary polynomials of $\mathbf{A}$ involving only the normal operators and $\mathrm{T}(\mathbf{A})$ be the set of all terms in $\mathrm{P}(\mathbf{A})$ (i.e. those without any parameter, except for 1 and the constant operations). Note that if all operators are unary then $\mathrm{P}(\mathbf{A}) = \mathrm{T}(\mathbf{A})$.

Proposition 3.6. *Let $\mathbf{A}$ be a BCI-monoid with normal operators.*

1. *If $B \subseteq A$, then $a \in \mathrm{CF}_{\mathbf{A}}(B)$ iff there are $b_1, \ldots, b_n \in B$ and $p_1, \ldots, p_n \in \mathrm{P}(\mathbf{A})$ such that*
$$(p_1(b_1) \wedge 1) \ldots (p_n(b_n) \wedge 1) \leq a.$$
2. *If $F, G \in \mathrm{CFil}(\mathbf{A})$,*
$$F \vee G = \{c : ab \leq c, \text{ for some } a \in F,\ b \in G\}.$$
3. *For any $a \in A$, $\mathrm{CF}_{\mathbf{A}}(a)$ is principal as a semilattice filter iff there is a unary polynomial $p \in \mathrm{P}(\mathbf{A})$ with $\mathrm{CF}_{\mathbf{A}}(a) = [p(a))$.*

Proof. For 1., it is clear that the set of elements satisfying the hypothesis is a filter. If $(p_1(b_1) \wedge 1) \ldots (p_n(b_n) \wedge 1) \leq a$ and g is a unary polynomial coming from an operator then, using the fact that g is normal, increasing and multiplicative we get

$$\begin{aligned} g(a) &\geq g((p_1(b_1) \wedge 1) \ldots (p_n(b_n) \wedge 1)) \\ &\geq g(p_1(b_1) \wedge 1) \ldots g(p_n(b_n) \wedge 1) = (gp_1(b_1) \wedge 1) \ldots (gp_n(b_n) \wedge 1) \end{aligned}$$

hence it is also a congruence filter. On the other hand any congruence filter containing B must contain any $(p_1(b_1) \wedge 1) \cdots (p_n(b_n) \wedge 1)$, hence 1. holds.

For 3., the right-hand-side of the displayed equality is clearly a congruence filter containing F and G. On the other hand any element in $G \vee H = \mathrm{CF}_{\mathbf{A}}(G \cup H)$ must be of that form by 1.

Consider now 4. By 1.

$$\mathrm{CF}_{\mathbf{A}}(a) = \bigcup\{[(p_1(a_1)\wedge 1)\ldots(p_n(a_n)\wedge 1)) : p_1,\ldots,p_n \in \mathrm{P}(\mathbf{A}),\ n\in\omega\}.$$

If $\mathrm{CF}_{\mathbf{A}}(a) = [b)$, then by compactness in the semilattice filter lattice we have that $\mathrm{CF}_{\mathbf{A}}(a) = [(p_1(a)\wedge 1)\ldots(p_k(a)\wedge 1))$ for some $p_1,\ldots,p_k \in \mathrm{P}(\mathbf{A})$. Then take $p(x) = (p_1(x)\wedge 1)\cdots(p_k(x)\wedge 1)$.

■

Corollary 3.7. *Let $\mathcal{V}$ be a variety of BCI-monoids with normal operators and let $\tau = \{f_1, f_1, f_2, \ldots\}$ be the set of normal operators symbols. Then the class $S\mathcal{V}^{\rightarrow,\wedge,\tau}$ is a variety.*

To see this just follow the proof of Proposition 3.3 and use 3.6(1). Now we can prove the analog of Theorem 3.1 for BCI-monoids with normal operators.

Theorem 3.8. *Let $\mathcal{V}$ be a variety of BCI-monoids with normal operators and suppose that each operator is unary. Then the following are equivalent.*

1. *For any $\mathbf{A}\in\mathcal{V}$, $a,b\in A$, $\mathrm{CF}_{\mathbf{A}}(a\leftrightarrow b)$ is principal as a semilattice filter.*
2. *$\mathcal{V}$ has a commutative, regular TD-term.*
3. *$\mathcal{V}$ has a commutative TD-term.*
4. *$\mathcal{V}$ has a TD-term.*
5. *$\mathcal{V}$ has equationally definable principal congruences.*
6. *$\mathcal{V}$ has (first order) definable principal congruences.*

Proof. To show that 1. implies 2., it is sufficient to prove that there is a unary term t such that $\mathrm{CF}_{\mathbf{A}}(a\leftrightarrow b) = [t(a\leftrightarrow b))$ for any $\mathbf{A}\in\mathcal{V}$, $a,b\in A$ (see Corollary 3.4). Really, it is enough to show it only for the two-generated algebras in $\mathcal{V}$, since in this case

$$\mathrm{CF}_{\mathbf{A}}(a) = \bigcup_{b\in A}\mathrm{CF}_{\mathrm{Sub}_{\mathbf{A}}(a,b)}(a).$$

We will make also use of the following fact, whose proof can be found in [21]:

Fact. If $h:\mathbf{A}\longmapsto\mathbf{B}$ is a onto homomorphism and $a,b\in A$, then

1. $h(\vartheta_{\mathbf{A}}(a,b))\subseteq\vartheta_{\mathbf{B}}(h(a),h(b))$;
2. $h^{-1}(\vartheta_{\mathbf{B}}(h(a),h(b))) = \vartheta_{\mathbf{A}}(a,b)\vee\mathrm{Ker}h$.

Let then $\mathbf{F}$ be the algebra freely generated by x,y. By hypothesis $\mathrm{CF}_{\mathbf{F}}(x\leftrightarrow y)$ is principal as a semilattice filter. By Proposition 3.6(3) there is a unary term t

such that $\mathrm{CF}_{\mathbf{F}}(x \leftrightarrow y) = [t(x \leftrightarrow y))$. Let now $\mathbf{A} \in \mathcal{V}$ be generated by a, b and let $h : \mathbf{F} \longmapsto \mathbf{A}$ be the onto homomorphism defined by $h(x) = a$, $h(y) = b$. Then $h(t(x \leftrightarrow y)) = t(a \leftrightarrow b)$ and clearly $t(x \leftrightarrow y) \in \mathrm{CF}_{\mathbf{F}}(x \leftrightarrow y) = F_{\vartheta_{\mathbf{F}}(x,y)}$. By 1. in the fact above we get that $t(a \leftrightarrow b) \in \mathrm{CF}_{\mathbf{A}}(a \leftrightarrow b)$.

On the other hand suppose $c \in \mathrm{CF}_{\mathbf{A}}(a \leftrightarrow b)$ and let $z \in F$ with $h(z) = c$. Note that $h(z) = c\ \vartheta_{\mathbf{A}}(a,b)\ u \geq 1$ and so $z \in F_{h^{-1}(\vartheta_{\mathbf{A}}(a,b))}$. By 2. in the fact above $z \in F_{\vartheta(x,y)} \vee F_{\mathrm{Ker}h}$ and by Proposition 3.6(2) there is a $u \in F_{\mathrm{Ker}h}$ with $t(x \leftrightarrow y)u \leq z$. Note that u Ker$h$ $v \geq 1$ and so $h(u) \geq 1$. We compute

$$\begin{aligned} t(a \leftrightarrow b) &= t(a \leftrightarrow b) \cdot 1 \\ &\leq h(t(x \leftrightarrow b)) \cdot h(u) \\ &= h(t(x \leftrightarrow y)u) \\ &\leq h(z) = c. \end{aligned}$$

So $[t(a \leftrightarrow b)) = \mathrm{CF}_{\mathbf{A}}(t(a \leftrightarrow b))$.

The implications from 2. to 3., from 3. to 4., from 4. to 5. and from 5. to 6. are obvious. Assume then 6. Then there is a first order formula $\varphi(x, y, z, w)$ such that for any $\mathbf{A} \in \mathcal{V}$ and $a, b, c, d \in A$

$$(c, d) \in \vartheta_{\mathbf{A}}(a, b) \qquad \text{iff} \qquad \mathbf{A} \models \varphi(a, b, c, d).$$

Consider now $\mathrm{P}(\mathbf{A}) = \mathrm{T}(\mathbf{A})$ and let Γ be the set of finite subsets of $\mathrm{T}(\mathbf{A})$. Using Proposition 3.6 we get

$$\mathcal{V} \models \forall xyzw \left(\varphi(x, y, z, w) \Longleftrightarrow \bigvee_{\gamma \in \Gamma} \left(\prod_{t \in \gamma} (t(x \leftrightarrow y) \wedge 1) \leq z \leftrightarrow w \right) \right).$$

By the compactness theorem of equational logic, there are $t_1, \ldots, t_n \in \mathrm{T}(\mathbf{A})$ with

$$\mathcal{V} \models \forall xyzw \left(\varphi(x, y, z, w) \Longleftrightarrow \prod_{i=1}^{n} (t_i(x \leftrightarrow y) \wedge 1) \leq z \leftrightarrow w \right).$$

Letting now $t(x) = \prod_{i=1}^{n} t_i(x)$, we have that in any algebra $[t(a \leftrightarrow b)) = \mathrm{CF}_{\mathbf{A}}(a \leftrightarrow b)$. Thus 1. holds.

■

Corollary 3.9. *Let $\mathcal{V}$ be a variety of BCI-monoids with unary normal operators and let $\mathcal{W} = S\mathcal{V}^{\rightarrow,\wedge,\tau}$. If $\mathcal{V}$ satisfies Theorem* 3.8 *then also $\mathcal{W}$ satisfies Theorem* 3.8.

Proof. The only thing to check is that the implication from 1. to 2. of Theorem 3.8 still holds. Let $t(x)$, h, and a, b, c, x, y, z, u as in Theorem 3.8. Since $z \in F_{\vartheta(x,y)} \vee F_{\mathrm{Ker}h}$, by Proposition 3.6(2) there is a $u \in F_{\mathrm{Ker}h}$ with $u \leq t(x \leftrightarrow y) \rightarrow z$. Since h is a homomorphism

$$1 \leq h(u) \leq t(a \leftrightarrow b) \rightarrow c$$

so $t(a \leftrightarrow b) \leq c$. The rest follows easily. ∎

Remarks. (1) The alert reader will certainly observe that Theorem 3.1 is a corollary of Theorem 3.8. We could have then presented Theorem 3.8 directly, deduce Theorem 3.1 as a corollary and spare a tree of the Brazilian forest. However we feel that sometimes elegance is second to clarity.

(2) We have an obvious metastatement: an analog of Theorem 3.8 holds for any variety of enriched BCI-monoids, for which we have a “good” description of the congruence filter generated by a set.

4. The logics

In this section we discuss some varieties of BCI-semilattices and monoids with normal operators and the logics that are naturally associated to them. We stress that by the word logic, we always mean a propositional logic regarded as a *deductive system*. The basic definitions below are taken from [5].

Let $\mathcal{L}$ be a propositional language. The set $\mathrm{FM}_{\mathcal{L}}$ of formulas is built in the usual way using propositional variables. An *assignment* is any mapping from the set of variables to formulas; clearly such assignment extends naturally to a map $\sigma\colon \mathrm{FM}_{\mathcal{L}} \longrightarrow \mathrm{FM}_{\mathcal{L}}$. Such a map will be called a *substitution*.

A deductive system S on $\mathcal{L}$ is a pair $\langle \mathcal{L}, \vdash_{\mathrm{S}} \rangle$ where $\vdash_{\mathrm{S}} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}) \times \mathrm{Fm}_{\mathcal{L}}$ and moreover for any $\Gamma, \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$

(i) $\varphi \in \Gamma$ implies $\Gamma \vdash_{\mathrm{S}} \varphi$;
(ii) $\Gamma \vdash_{\mathrm{S}} \varphi$ and $\Gamma \subseteq \Delta$ implies $\Delta \vdash_{\mathrm{S}} \varphi$;
(iii) $\Gamma \vdash_{\mathrm{S}}$ and $\Delta \vdash_{\mathrm{S}} \psi$ for any $\psi \in \Gamma$ implies $\Delta \vdash_{\mathrm{S}} \varphi$;
(iv) $\Gamma \vdash_{\mathrm{S}} \varphi$ implies $\Gamma' \vdash_{\mathrm{S}} \varphi$ for some finite $\Gamma' \subseteq \Gamma$;
(v) $\Gamma \vdash_{\mathrm{S}} \varphi$ implies $\sigma(\Gamma) \vdash_{\mathrm{S}} \sigma(\varphi)$ for every substitution σ.

The relation $\vdash_{\mathrm{S}}$ will be called the *entailment* (or *consequence*) *relation*. A deductive system can be given also in terms of axioms and rules of inference. The entailment is then defined as: $\Gamma \vdash_{S} \varphi$ iff φ is contained in the smallest set of formulas that includes Γ together with all substitution instances of axioms and closed under all inference rules. A suitable class $\mathcal{K}$ of algebras (of the same type as $\mathcal{L}$) can then be associated to the deductive system (and $\mathcal{K}$ is called an *algebraic semantics* for S). When the connection is "the best possible" then $\mathcal{K}$ is called an *equivalent algebraic semantic* for S. We will not give any formal definition of this concept and the reader is advised to keep at hand at least [5]. An informal hint is the following: an *equivalent algebraic semantics* (EAS) for a deductive system S is a class of algebras that is to S *exactly what* the variety of Boolean algebras is to Classical Propositional Calculus, i.e. is a class of algebras of the same type as the Lindenbaum algebra of S which satisfies certain (strong) conditions.

Note that the procedure of transforming a logic into a deductive system, when applied to substructural logics, may conflict with the extralogical motivations of the logic themselves. In fact the entailment $\vdash_{\mathrm{S}}$ is always "structural", in that it satisfies the structural rules when applied to sequents. Since this paper is mainly devoted to explore the "algebraic" consequences of algebrization we shall not be concerned with this point.

4.1. Linear Logic. Linear Logic has been introduced by J.-Y. Girard [17]. It can be loosely described as a "resource-sensitive" logic, which keeps track of the number of times data of given types are used. Its implicational fragment is a BCI-logic, i.e. it does not satisfy weakening and contraction. Nevertheless Linear Logic cannot be considered a substructural logic, in that we have two unary operators (the *exponentials*) that serve to introduce weakening and contraction in a controlled way on individual formulas. The propositional language of classical linear logic consists of four family of connectives:

1. the *multiplicative* connectives: $\otimes$ (the tensor product), $\wp$ (the par, i.e. the parallel "or"), $\rightarrow$ (the linear implication, Girard's $\multimap$), $\perp$ (the bottom) and $\mathbf{1}$;
2. the *additive* connectives: $\vee$, $\wedge$, $\top$ and $\mathbf{0}$;
3. the linear negation $\urcorner$ (Girard's $(\ )^{\perp}$), which is a De Morgan involution w.r.t. $\vee$ and $\wedge$;
4. the *exponentials*: ! and ?.

A suggestive way of thinking of how this connectives work is to think at formulas as data types. For instance $A \wedge B$ is a datum from which we can extract once either a datum of type A and a datum of type B; $A \otimes B$ is just a pair of data; $A \rightarrow B$ is a method of transforming a single datum of type A into the datum of type B; $!A$ indicates that we can extract as many data of type A as we like (weakening and contraction on the left side of a sequent) and so on.

In [2] it is shown that any fragment of Linear Logic containing $\rightarrow$, $\wedge$ and $\mathbf{1}$ is algebraizable. The EAS of the $\{\rightarrow, \wedge, \mathbf{1}\}$-fragment is just the variety of BCI-semilattices, and that of the $\{\rightarrow, \wedge, \otimes, \mathbf{1}\}$-fragment is just the variety of BCI-monoids. This was really the motivating fact for our investigation.

The EAS of the *exponential free* fragment of Linear Logic is the variety of *arabesques* [2]. An arabesque **A** is an algebra of type $\{\vee, \wedge, \rightarrow, \urcorner, 1, \top\}$ such that

1. $\langle A, \vee, \wedge, \urcorner, \urcorner\top, \top\rangle$ is a bounded lattice with involution (a *De Morgan lattice*) i.e. for $a, b \in A$

$$\urcorner\urcorner a = a$$
$$\urcorner(a \vee b) = \urcorner a \wedge \urcorner b$$

2. For any $a, b, c \in A$

$$\begin{aligned} 1 \rightarrow a &= a \\ a \rightarrow a &\geq 1 \\ a \rightarrow b &\leq (b \rightarrow c) \rightarrow (a \rightarrow c) \\ a \rightarrow (b \rightarrow c) &\leq b \rightarrow (a \rightarrow c) \\ (a \rightarrow b) \wedge (a \rightarrow c) &= a \rightarrow (b \wedge c) \\ a \rightarrow \urcorner b &\leq b \rightarrow \urcorner a \\ a &\leq ((a \rightarrow b) \wedge 1) \rightarrow b \end{aligned}$$

The variety arabesques is (termwise equivalent to) a variety of BCI-monoids with compatible operations. A *bounded*[1)] BCI-monoid is an algebra $\langle A, \rightarrow, \wedge, \cdot, 1, 0\rangle$ where $\langle A, \rightarrow, \wedge, \cdot, 1\rangle$ is a BCI-monoid and 0 is a constant satisfying

$$(a \rightarrow 0) \rightarrow 0 = a.$$

We claim that the variety of arabesques is termwise equivalent to the variety of algebras $\langle A, \rightarrow, \wedge, \cdot, 1, 0, \top, \rangle$ where

1. $\langle A, \rightarrow, \wedge, \cdot, 1, 0\rangle$ is a bounded BCI-monoid;
2. $\top$ is nullary and for each $a \in A$, $a \wedge \top = a$;
3. for any $a, b, c \in A$

$$(a \rightarrow b) \wedge (a \rightarrow c) = (((a \rightarrow 0) \wedge (b \rightarrow 0)) \rightarrow 0) \rightarrow c.$$

The proof is easy once one observes that in a bounded BCI-monoid satisfying 3. above one can define $\urcorner a = a \rightarrow 0$ and $a \vee b = \urcorner(\urcorner a \wedge \urcorner b)$ in such a way that $\langle A, \vee, \wedge, \urcorner\rangle$ is a De Morgan lattice. We will often make use of this fact in the sequel.

The EASes of *bounded linear logics* [18] satisfy Theorem 3.1 and so they have a commutative, regular deduction term. By Corollary 3.3 the EAS of any fragment containing $\wedge, \rightarrow$ and 1 has the same property.

1) It is clear that 0 in a bounded BCI-monoid is *not* necessarily the smallest element in the semilattice order.

The EAS for *full* Linear Logic is the variety of *girales* [2]. A *girale* is an algebra $\langle A, \to \vee, \wedge, \urcorner, 1, \top, !\rangle$, where $\langle A, \to \vee, \wedge, \urcorner, 1, \top\rangle$ is an arabesque, ! is unary and for any $a, b \in A$

$$!a \leq a \wedge 1 \tag{4.1}$$

$$!a \otimes !b = !(a \wedge b) \tag{4.2}$$

$$!!a = !a \tag{4.3}$$

It is easy to see that the variety of girales is a variety of BCI-monoids with just one normal operator. In [2] Theorem 2.2 it is proved that the variety of girales satisfies Corollary 3.4 (in fact $t(x) = !x$) so it has a commutative, regular TD-term. Since the only operator is unary, then the variety of girales satisfies also Theorem 3.8. By the same token any fragment of full Linear Logic containing $\to, \wedge, 1$ and ! has an EAS with a commutative, regular TD-term.

4.2. Comparative logics. Comparative logics have been introduced by E. Casari in [11], as an attempt to axiomatize the notion of "*p is less A than q*" (*A* any property). The primitive connective in the forefront is denoted by $\leq$: $p \leq q$ expresses that the degree of truth of p is less than the degree of truth of q and behaves like an implication satisfying (B), (C) and (I). Any Comparative logic, regarded as a deductive system, is algebraizable, since it is an axiomatic extension of the $\{\to, \wedge, 1\}$ fragment of Linear Logic. In his paper Casari gives algebraic semantics that are based on the notion of *abelian ℓ-pseudogroup.* With some computations one sees that abelian ℓ-pseudogroups are termwise equivalent to BCI-monoids with compatible operations.

More precisely the EAS of the "minimal theory" whose axioms and rules are in [11] p. 163 is (termwise equivalent) to the variety of arabesques satisfying

$$x \to x = 1$$
$$\urcorner 1 \leq 1.$$

4.3. Logics without exchange. Usually the lack of exchange in a logic makes things hopelessly complicated. On the algebraic side this corresponds to dealing with BI-semilattices and monoids. There are at lest two logics in the literature that try to explore this situation. The first is a modification of Linear Logic, introduced by Abrusci [1], usually called *noncommutative Linear Logic.* In that logic the tensor product is no longer commutative and there are two negations and two implications. On the algebraic side this corresponds to studying *double BI-monoids.* A *double BI-monoid* is an algebra $\langle A, \to, \leftarrow, \wedge, \cdot, 1\rangle$ such that $\langle A, \to, \wedge, \cdot, 1\rangle$ is a semilattice ordered left-residuated monoid and $\langle A, \leftarrow, \wedge, \cdot, 1\rangle$ is a semilattice ordered right-residuated monoid. It is not hard to check that $\rightarrow$ and $\leftarrow$ must be related. For instance

$$a \rightarrow (b \leftarrow c) = b \leftarrow (a \rightarrow c)$$

must hold for any $a, b, c \in A$. This is sufficient for proving that the variety of such objects is congruence permutable (just look at Proposition 2.6 and guess the right term) and we get a description of congruence filters similar to the one for BI-monoids. Unfortunately noncommutativity prevents us from getting a good interplay between semilattice filters and congruence filters.

The second logic has been introduced by Ono and Komori [23]. The logic $\mathrm{L}_{\mathrm{BCC}}$ is obtained by removing contraction and exchange from the Genzten formulation of intuitionistic logic LJ and adding a new binary connective. The fact that the rule of weakening holds for such a logic, forces its algebraic counterpart to be integral, i.e. 1 is the top element of the semilattice structure. It is clear that $\mathrm{L}_{\mathrm{BCC}}$ is algebraizable and its EAS is a variety of BI-monoids with compatible operations. In particular the EAS of its $\{\rightarrow, \wedge, \cdot, 1\}$-fragment is the variety of integral BI-monoids, i.e. those satisfying $x \leq y \rightarrow x$. A subvariety of integral BI-monoids has been extensively studied by Bosbach [10]. A *left-complemented monoid* is an integral BI-monoid, satisfying also

$$(x \rightarrow y)x = (y \rightarrow x)y.$$

Under this hypotheses it can be shown that the semilattice ordering coincides with the *inverse right-divisibility ordering* i.e.

$$x \leq y \qquad \text{iff} \qquad \exists\, z(x = zy)$$

and $\wedge$ is definable in terms of $\rightarrow$ and $\cdot$: $x \wedge y = (x \rightarrow y)x$.

4.4. Relevance logic. In the system R of relevance logic to the usual connectives $\vee$, $\wedge$ and $\neg$ is joined an implication satisfying (B), (C) and (W) and a binary connective $\circ$ (*fusion* or *cotenability*). In the original system there is no "truth" symbol, but it is customary to extend conservatively R to a system R_{t} with truth.

Both systems R and R_{t} are algebraizable [15]. The EAS for R_{t} is the variety of De Morgan monoids, that is a variety of BCI-monoids with compatible operations. More precisely a De Morgan monoid is termwise equivalent to an algebra $\langle A, \rightarrow, \vee, \wedge, \cdot, 0, 1\rangle$ where

1. $\langle A, \rightarrow, \wedge, \cdot, 0, 1\rangle$ is a bounded BCI-monoid;
2. for any $a, b, c \in A$

$$a \leq a \cdot a$$

$$a \rightarrow (a \rightarrow b) \leq a \rightarrow b$$

$$(a \rightarrow c) \wedge (b \rightarrow c) = (a \vee b) \rightarrow c;$$

3. $\langle A, \vee, \wedge \rangle$ is a distributive lattice ($\vee$ is defined in the usual way).

It is clear that the variety of De Morgan monoids satisfies Theorem 3.1 so it has a commutative, regular TD-term

$$p(x, y, z) = (x \leftrightarrow y \wedge 1)z.$$

4.5. BCK-algebras and related structures. A BCK-algebra is an algebra $\langle A, \to, 1 \rangle$ such that, for any $a, b, c \in A$

$$\begin{aligned} a \to a &= 1 \\ a \to 1 &= 1 \\ 1 \to a &= a \\ (a \to b) \to ((c \to a) \to (c \to b)) &= 1 \\ a \to b = b \to a = 1 \quad &\text{implies} \quad a = b. \end{aligned}$$

BCK-algebras were introduced by Iseki [20] who proposed them as a semantics for a certain logic that arises in connection with combinatory logic. They form a quasivariety that is not a variety [26]. It is clear that the relation

$$a \leq b \qquad \text{iff} \qquad a \to b = 1$$

is a partial order and the connection with the theory presented in this paper lies in the fact that BCK-algebras coincide with $\{\to, 1\}$-subreducts of *integral commutative residuated partially ordered monoids*. This suggests that the possibility of characterizing TD-terms for varieties of BCK-algebras is strong and M. Palasiński [24] did exactly that in a preprint that circulated about 1990[2]. Let $\mathcal{E}_n$ be the class of BCK-algebras satisfying the identity

$$x \to^{n+1} y \approx x \to^2 y$$

where $x \to^n y$ is defined in the obvious way. $\mathcal{E}_n$ turns out to be a variety for all n [12]. Palasiński showed that

- For each n any subvariety of $\mathcal{E}_n$ has a commutative TD-term

$$p_n(x, y, z) = ((x \to^n y) \to ((y \to^n x)) \to z.$$

- A variety $\mathcal{V}$ of BCK-algebras has a commutative TD-term if and only if it is contained in $\mathcal{E}_n$ for some n.

[2] We thank the referee for making us aware of this fact.

If we add to BCK-algebra a meet operation in such a way that the underlying partial order becomes a semilattice order, then we obtain a so called BCK-semilattice. In our framework a BCK-semilattice is simply an *integral* BCI-semilattice. Filters and congruences on BCK-semilattices have been investigated by P. Idziak in [19], where he developed a theory of filters very similar to the one in the present paper.

4.6. Hoop logics. A *hoop* is a commutative left-complemented monoid. Therefore the varieties of hoops is a particular variety of (integral) BCI-monoids. Moreover our definition of normal operator for BCI-monoids coincides with the one of *dual normal operator* for hoops as given in [7]. It follows that our theory of filters is a proper extension of the one given there. Examples of logics whose EAS is a class of hoops with dual normal operators are:

- The logic L_{BCK} introduced in [23]. L_{BCK} is obtained by adding exchange to L_{BCC} and its EAS is the variety of *residuated lattices.*
- The logic LJ^* (see again [23]), obtained by adding contraction to L_{BCK}. This logic satisfies all structural rules, but it is different from LJ. Its EAS is the variety of residuated lattices satisfying

$$x \to (x \to y) \leq x \to y.$$

  It takes a second to see that $x \leq x^2$ holds in such variety and hence (by Theorem 3.1) the variety has a commutative, regular TD-term.
- The many-valued logic of Łukasiewicz. Its EAS is the variety of *Wajsberg algebras* [16]. It turns out that Wajsberg algebras are termwise equivalent to residuated lattices satisfying "Łukasiewicz's law"

$$(x \to y) \to y \leq (y \to x) \to x$$

- Classical and intuitionistic logic and all their normal modal extensions.

5. Final remarks

The ultimate purpose of this paper was to show that BCI-monoids are a more natural class then hoops for developing a theory of filters. However they are still an arbitrary choice. In particular a paper by Blok and Raftery [9], that appeared while this paper was going through the usual refereeing process, seems to point in a more general direction. The authors investigate partially ordered integral commutative residuated monoids. They do not form a variety but seem to admit a nice filter theory. It is plausible that the theory of TD-terms developed in this paper would go through in this context, possibly even in the nonintegral case. We plan to investigate closely this matter in the near future.

Another question is the following: how close can one come to a complete characterization of algebras with a commutative, regular TD-term, by continuing to generalize the notion of residuated monoid? In view of the characterization of such algebras given in [8] the answer seems to be: not very close at all. So a more workable question is: what are the special features of TD-terms coming from residuated monoids? What seems to be needed is some additional condition on the TD-term that brings it in convergence with residuated monoids.

Finally we wish to thank the anonymous Referee of this paper for his careful work and his many useful suggestions.

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P. Agliano, Dipartimento di Matematica, Via del Capitano 15, 53100, Siena, Italy; *e-mail*: agliano@unisi.it