



Canonical Truth

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Abstract

We introduce and study some variants of a notion of canonical set theoretical truth. By this, we mean truth in a transitive proper class model M of ZFC that is uniquely characterized by some \in -formula. We show that there are interesting statements that hold in all such models, but do not follow from ZFC, such as the ground model axiom and the nonexistence of measurable cardinals. We also study a related concept in which we only require M to be fixed up to elementary equivalence. We show that this theory-canonicity also goes beyond provability in ZFC, but it does not rule out measurable cardinals and it does not fix the size of the continuum.

Keywords Set theory · Mathematical realism · Philosophy of mathematics · Justification of Axioms · Mathematical Platonism · Ground Axiom · Axiom of Choice Forcing

1 Introduction

It is an old logical dream to devise an effectively describable axiomatic system for mathematics that uniquely describes ‘mathematical reality’; in modern logical language, this should mean at least that it uniquely fixes a model. It is well-known that this dream is unattainable in first-order logic: By the Löwenheim-Skolem theorem, we get models of all infinite cardinalities once there is one infinite model; and by Gödel’s incompleteness theorem, if the theory is strong enough to express elementary arithmetic, it will have different models that are not even elementary equivalent.

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Focusing on ZFC set theory, one of the main foundational frameworks for mathematics, these two effects can in a certain sense be cancelled out by asking not for arbitrary models, but for transitive models that are proper class-sized, i.e. contain all ordinals. When we restrict the allowed models in this way, there are extensions of ZFC that uniquely fix a model. The most prominent example is $V = L$: It is well-known (and provable in ZFC) that $ZFC + V = L$ has exactly one transitive class-model, provided that ZFC is consistent.

This form of canonicity gives the axiom of constructibility a certain attractiveness: It seems to describe, up to the unavoidable weakness of first-order logic, a unique ‘mathematical reality’. However, it is usually seen as too restrictive since many objects of set-theoretical interest are ruled out under this assumption.

But $V = L$ is by far not the only theory that uniquely fixes a transitive class model: Other examples include $V = L[0^\#]$ and $V = L[x]$, where x is an absolute Π_2^1 -singleton (see below). The ‘true mathematical reality’ that the adherents of the logical dream mentioned in the beginning believe in would have to be one of those ‘canonical’ models. Hence, whatever holds in all of these ‘canonical’ models will have to be accepted as true by someone who believes in a uniquely describable mathematical reality. We call such statements ‘canonically necessary’. If there are no such statements that go beyond what is derivable from ZFC, then this kind of mathematical realism would be mathematically neutral: the belief in a uniquely describable mathematical reality would merely be a way of interpreting set theory, without influencing it. On the other hand, if there are statements that hold in all canonical models without following from ZFC, this realistic mindset would be mathematically informative.

In this paper, we investigate statements that hold in all ‘canonical’ models of ZFC, i.e. in all transitive class models that are uniquely fixed by some extension of ZFC by finitely many extra statements.¹ It turns out that the realistic mindset is indeed mathematically informative: Examples of canonically necessary statements that do not follow from ZFC are the ground model axiom of (Reitz 2007) (Theorem 23) and the non-existence of measurable cardinals (Theorem 28).

This approach generalizes in a natural way to the concept of ‘canonical consequence’: Namely, a sentence ψ is a ‘canonical consequence’ of a theory T if and only if, for all ϕ such that $T + \phi$ has (provably in ZFC^2) exactly one transitive class model M , we have $M \models \psi$.

¹ After most of the work in this paper was done, we noticed that in Friedman (1973) and Friedman (1978), H. Friedman defined and investigated a similar concept for countable set-sized models of a fixed height α under the name ‘ β -categoricity’. However, there is otherwise no overlap in the settings, the questions considered and the results; in particular, in Friedman’s setting, one of the main results is that L is the unique ‘categorical’ model (Friedman 1973, p. 543), while in our setting, there are infinitely many under sufficient large cardinal assumptions.

² It would also be natural to replace ZFC by T here as well; but for the time being, we keep ZFC as our base theory.

One can then ask for the canonicity of the ZFC axioms themselves: Are there proper subsystems T of ZFC that canonically imply the ZFC axioms? We give some preliminary results in this respect.³

A natural weakening of the concepts of canonical truth and consequence would be to merely demand that the theory of the transitive class model M , rather than M itself, is uniquely determined by an \in -sentence. Let us say that an \in -theory is canonical if and only if there is an \in -sentence ϕ such that all transitive class models of $\text{ZFC}+\phi$ are elementary equivalent. Then we can say that a sentence ψ is ‘theory-canonical’ if and only if it is contained in all canonical theories. If we replace ZFC by some \in -theory T in this definition, we obtain the notion of a T -canonical theory. We can then say that a sentence ψ is a ‘theory-canonical consequence of T ’ if and only if it is contained in all T -canonical theories. This ‘theory canonicity’ turns out to be strictly weaker than canonicity (for example, it does not exclude measurable cardinals), but is still informative: There are theory-canonical consequences of ZFC that are not first-order consequences of ZFC.

We conclude with various open questions; in particular, we do not know whether the continuum hypothesis is canonically necessary (we conjecture that it is not) or whether there are canonical models of $\text{ZF}+\neg\text{AC}$ (i.e. whether the axiom of choice is canonically necessary over ZF).

2 Basic Definitions

We start by giving a formal counterpart to the intuitive idea that a theory T ‘uniquely fixes a transitive class model’ and ‘uniquely fixes a transitive class model up to elementary equivalence’. This is not straightforward, as quantifying over proper classes is not possible in ZFC. This might be solvable by instead working in NBG, but we prefer to stick to ZFC for the moment, partly because the methods we intend to use (forcing, class forcing and inner models) are commonly developed for ZFC models. Thus, a proper class model of ZFC will always be an inner model of V . Of course, this will immediately trivialize our analysis when one assumes $V = L$, so that L is the only transitive class model. To get a sufficient supply of inner models, we will hence assume sufficient large cardinals in our metatheory.

Still, we need to deal with our inability, due to the lack of a truth predicate, to quantify over all inner models. This will be solved by formulating the uniqueness not as a single statement, but as a scheme. This leaves us with the problem of expressing that the class defined by a formula ϕ is a model of ZFC. Again, this is not trivial, since ZFC is not finitely axiomatizable. Fortunately, for the case we are interested in, there is a workaround:

³ A related question is which subsystems T of ZFC have the property that ZFC holds in all inner models of T . One result in this direction, due to Philip Welch, will be given below.

Lemma 1 (See Jech 2002, Theorem 13.9.) *A transitive class C is a model of ZF if and only if C is closed under Gödel operations and almost universal (i.e. for every subset $x \subseteq C$, there is $y \in C$ with $x \subseteq y$).*

We fix a natural enumeration $(\psi_i : i \in \omega)$ of the \in -formulas in order type ω .

2.1 Canonical Implication

Definition 2 Let ϕ be an \in -formula, $i, j \in \omega$. Let $IM_i^{ZFC}(\phi, y)$ (‘inner model’) abbreviate the statement ‘ $M_{\psi_i, y} := \{x : \psi_i(x, y)\}$ is transitive, almost universal, closed under Gödel operations, contains all ordinals and satisfies AC and ϕ ’. More generally, when T is an \in -theory, we let $IM_i^T(\phi, y)$ denote the claim that $M_{\psi_i, y}$ is a transitive class model of T .

The uniqueness statement $U_{ij}^{\phi, T}$ is the following \in -formula:

$$\forall y, y' [(IM_i^T(\phi, y) \wedge IM_j^T(\phi, y')) \rightarrow \forall x (\psi_i(x, y) \leftrightarrow \psi_j(x, y'))].$$

Now, ϕ is a uniqueness statement over a theory T if and only if all elements of $U_\phi^T := \{U_{ij}^{\phi, T} : i, j \in \omega\}$ are provable in T ⁴. When T is ZFC, we will usually drop ‘over T ’.

Moreover, for T an extension of KP, ϕ is a T -canonical statement if and only if there is some $i \in \omega$ such that $\exists y IM_i(\phi, y)$ and ϕ is a uniqueness statement. When T is ZFC, we simply call ϕ canonical.

Remark 3 Note that we do *not* require $\exists y IM_i(\phi, y)$ to be provable in T ; we only want it to be true (in V). These existence statements will usually be derived from stronger meta-theories, such as ZFC with large cardinals.

Remark 4 Typically, T will just be ZFC. Below, we will also consider cases where T is much weaker; however, some base theory is necessary to exclude unwanted cases such as ‘ \in is a total ordering’ (which has On as its only transitive proper class model) from our consideration.

The paradigmatical example for a uniqueness statement is $V = L$, which is a Π_2 -statement. This is indeed the minimal complexity for a uniqueness statement:

Proposition 5 *No Σ_2 -sentence is a uniqueness statement.*

Proof Suppose otherwise, let $\psi \equiv \exists x \forall y \phi(x, y)$ be a uniqueness statement, where ϕ is Δ_0 , and let M be the unique transitive class model of $ZFC + \psi$. Pick $a \in M$ such that $M \models \forall y \phi(a, y)$, and let $\kappa \in \text{Card}$ be large enough so that $a \in V_\kappa^M$. As a cardinal,

⁴ Alternatively, we could also demand that all elements of U_ϕ^T hold in V . We will take up this idea below as C^1 -canonicity.

κ is Σ_1 -reflecting, so $V_\kappa^M \models \forall y\phi(a, y)$. Let \mathbb{P}_{κ^+} be the forcing for adding a new subset of κ^+ described in (Kunen (1973), section 6) (i.e., the set of partial functions $f : \kappa^+ \rightarrow \{0, 1\}$ of cardinality $< \kappa^+$), and let $M[G]$ be a generic extension for \mathbb{P}_{κ^+} . Since \mathbb{P}_{κ^+} is κ -closed, we have $V_\kappa^{M[G]} = V_\kappa^M$; thus $V_\kappa^{M[G]} \models \forall y\phi(a, y)$, and since κ is also a cardinal in $M[G]$, we have that $M[G] \models \forall y\phi(y, a)$ by Σ_1 -reflection. Thus $M[G] \models \text{ZFC} + \psi$, contradicting the assumption that ψ is a uniqueness statement. \square

Definition 6 A statement ϕ is canonically necessary (c.n.) if and only if ϕ holds in all canonical models of ZFC.

A statement ϕ is canonically possible (c.p.) if and only if there is a canonical model $M \models \phi$ of ZFC, i.e., if and only if its negation is not canonically necessary.

Definition 7 Let T be an extension of KP.

If M is a transitive class model of T , then M is T -canonical if and only if there is a T -canonical statement ϕ such that $M \models \phi$.

If A is any \in -theory and ϕ is an \in -statement, then ϕ canonically follows from A if and only if ϕ holds in all canonical models in which A holds. In this case, we write $A \models_c \phi$.

The preceding notions can be generalized by replacing the single tc statement ϕ with elements from a class \mathcal{T} of theories. We would then, e.g., say that $T \models_c^{\mathcal{T}} \phi$ if and only if ϕ holds in every transitive proper class model of T that is fixed by some element of \mathcal{T} . Particularly interesting cases might be the set of recursive theories, of countable theories, of Σ_n -axiomatizable theories, or even the class of all \in -theories with ordinal parameters. However, in this work, we will focus on the singleton case and only briefly mention when our results easily generalize to other variants.

For a class \mathcal{A} of \in -theories with ordinal parameters, we say that a formula ϕ is \mathcal{A} -canonical, written $\text{ZFC} \models_c^{\mathcal{A}} \phi$, if and only if ϕ holds in all transitive class models M of ZFC such that, for some $T \in \mathcal{A}$, M is the unique transitive class model of $\text{ZFC} + T$.⁵

Let \mathcal{C} denote the class of \in -theories with ordinal parameters. For a cardinal κ , \mathcal{C}^κ denotes the subclass of \mathcal{C} consisting of the elements with cardinality κ .⁶ Finally, let \mathcal{C}^L be the class of constructible countable \in -theories.

2.1.1 Weak Canonicity

In our definition above, we required that ZFC must be capable of proving the uniqueness of a model of $\text{ZFC} + \phi$. A somewhat reasonable weaker requirement would be that $\text{ZFC} + \phi$ proves this.

⁵ Note that we do not require that the uniqueness of the respective models is provable in ZFC, but only that it is true (in V). This relaxation seems unavoidable, as no analogue of the schematic approach taken for single statements is available for arbitrary theories.

⁶ Note that, by the last footnote, \mathcal{C}^L -canonical necessity is not the same as canonical necessity.

Definition 8 ϕ is weakly canonical if and only if $ZFC+\phi$ proves the uniqueness statements in the definition of canonicity. If M is a model of $ZFC+\phi$ for some weakly canonical ϕ , then M is weakly canonical. If ψ holds in all weakly canonical models, then ψ is weakly canonically necessary (weakly c.n.).

2.2 Theory-Canonical Implication

We now give a formal counterpart to the claim that a statement ϕ fixes a transitive class model of a theory T up to elementary equivalence. Intuitively, a sentence ϕ is ‘theory-canonical’ (tc) if and only if it is provable in ZFC that any two inner models M and N of $ZFC+\phi$ are elementary equivalent. Formally, we again need to express this as a scheme: Whenever ϕ_0 and ϕ_1 define transitive proper classes M_0 and M_1 in which ϕ holds, we have that ZFC proves $\psi^{M_0} \leftrightarrow \psi^{M_1}$.

Definition 9 For $i, j, k \in \omega$, ϕ an \in -formula and T an \in -theory, the theory-uniqueness statement $TU_{ijk}^{\phi, T}$ is the following \in -formula:

$$\forall y, y' [(IM_i^T(\phi, y) \wedge IM_j^T(\phi, y')) \rightarrow (\psi_k^{M_{w_i, y}} \leftrightarrow \psi_k^{M_{w_j, y'}})].$$

An \in -sentence ϕ is a theory-uniqueness statement for T if and only if ZFC proves all elements of $TU_{\phi}^T := \{TU_{ijk}^{\phi, T} : i, j, k \in \omega\}$.

We say that a theory-uniqueness sentence ϕ for T is theory-canonical (tc) over a theory T if and only if $\exists y IM_i \phi(\phi, y)$ is true (in V) for some $i \in \omega$.

For theories T and S , we say that S is a canonical extension of T if and only if there is a theory-canonical sentence ϕ over T such that S is the (unique) theory of the transitive class models of $T + \phi$. When T is ZFC , we call S a canonical theory.

A transitive class model M of T is tc over T if and only if some ϕ holds in M which is tc over T .

A sentence ψ is ‘theory-canonically necessary’ (tcn) if and only if ψ belongs to all canonical extensions of ZFC . Likewise, ψ is ‘theory-canonically possible’ (tcp) if and only if it belongs to some canonical extension of ZFC .

We can then extend this as above to obtain a notion of ‘theory-canonical implication’:

Definition 10 If T is an \in -theory and ϕ is an \in -sentence, then we say that ϕ is a ‘theory-canonical consequence (tc-consequence) of T ’, or that T ‘tc-implies ϕ ’ if and only if we have $M \models \phi$ for every transitive class model M that is tc over T . In this case, we write $T \vDash_{tc} \phi$.

As above, the notions of theory-canonicity and tc-implication can be generalized to theories, rather than single statements.

2.3 General Properties of \vDash_c and \vDash_{tc}

In this section, we consider some general logical and proof-theoretic properties of \vDash_c and \vDash_{tc} .

Definition 11 For a theory T , we let $\mathcal{C}(T) := \{\phi : T \vDash_c \phi\}$ and $\mathcal{TC}(T) := \{\phi : T \vDash_{tc} \phi\}$ be the sets of canonical and theory-canonical consequences of T , respectively.

Note that, for any T , we have $\mathcal{C}(T) \subseteq \mathcal{TC}(T)$. It is easy to see that both \vDash_c and \vDash_{tc} are closure operators on the set of \vDash -theories.⁷

Clearly, for any theory T , $\mathcal{C}(T)$ and $\mathcal{TC}(T)$ are closed under first-order inferences. There are, however, additional inference rules with this property, such as the ω -rule (for which, however, the restriction to transitive models suffices). We do not know whether there are inference rules that are sound for \vDash_c or \vDash_{tc} but not already for transitive models.

We also note that, again due to the restriction to transitive models, compactness fails for both notions. Since in the setting of (theory-)canonicity, we want to keep our basic theory even if it is infinite, we define the relevant compactness property as follows:

Definition 12 Let T be an \vDash -theory. We say that \vDash_c has the compactness property over T if and only if, for every set S of \vDash -sentences ϕ , $T \cup S \vDash_c \phi$ implies that there is a finite subset $S' \subseteq S$ such that $T \cup S' \vDash_c \phi$. The definition for \vDash_{tc} is analogous.

Proposition 13 *Neither \vDash_c nor \vDash_{tc} have the compactness property. That is, there are a theory T and an \vDash -sentence ϕ such that $T \vDash_{tc} \phi$ (and thus $T \vDash_c \phi$), but there is no finite subtheory of T' of T such that $T' \vDash_c \phi$ (and thus, neither $T' \vDash_{tc} \phi$).*

Proof Define $0^{\sharp,i}$ as the i -th iterate of the sharp operator applied to 0. For $i \in \omega$, let ϕ_i be the sentence ‘ $0^{\sharp,i}$ exists’, let $S := \{\phi_i : i \in \omega\}$, let T be ZFC \cup S, and let ψ be the sentence ‘For every $i \in \omega$, $0^{\sharp,i}$ exists’. Then $T \vDash_{tc} \psi$ (and hence $T \vDash_c \psi$).

However, if S' is a finite subset of S , then there is a $k \in \omega$ that is maximal with the property that $\phi_k \in S'$. But now, $V = L[0^{\sharp,k}]$ is a uniqueness statement over ZFC and in the unique transitive class model of $V = L[0^{\sharp}]$, ψ clearly fails. Hence $T' \not\vDash_c \psi$ (and thus also $T' \not\vDash_{tc} \psi$).

Thus, T and ψ are as desired. □

Since compactness fails, there cannot be sets of finite inference rules (i.e., rules with finitely many premises) that are both sound and complete for \vDash_c or \vDash_{tc} .

⁷ Note a slight pathology for theories T that do not have (theory-)canonical models: For such theories, the definition of (theory-)canonical implication universally quantifies over the empty set and is thus vacuously fulfilled for every statement ϕ , so that, in this case, we have that $\mathcal{C}(T)$ and $\mathcal{TC}(T)$ both coincide with the set of all \vDash -sentences.

Proposition 14 *There is an infinite \in -theory T such that, for any finite subset $S \subseteq T$, $ZFC+S$ has a canonical model, but $ZFC+T$ does not.*

Proof Let T and ψ be as in the proof of Proposition 13, then $T + \neg\psi$ is as desired. □

By taking ψ to be \perp , Proposition 13 follows from Proposition 14. However, we consider the given presentation to be more illustrative.

3 Examples of Canonical Truth

Obvious examples for uniqueness statements are $V = L$ or $V = L[0^\sharp]$ with corresponding canonical models L and $L[0^\sharp]$. These actually give rise to a larger class of examples:

Definition 15 A real number x is a relative Π_2^1 -singleton if and only if there is a Π_2^1 -statement ϕ such that x is the unique element y of $L[x]$ with $L[x] \models \phi(y)$.

A real number x is an absolute Π_2^1 -singleton⁸ if and only if there is a Π_2^1 -statement ϕ such that x is the unique element y of V with $\phi(y)$.

Corollary 16 *An absolute Π_2^1 -singleton x is the unique element satisfying its defining Π_2^1 -formula ϕ in each transitive inner model that contains x , while all other models will not contain such a witness.*

Proof By Shoenfield absoluteness, if M is a transitive class model of ZF^- , $x \in M$ and $M \models \phi(x)$, then $V \models \phi(x)$. Hence, if some transitive inner model had two distinct elements satisfying ϕ , the same would hold for V , contradicting uniqueness. Similarly, if M was some transitive inner model with $x \notin M$ but $M \models \phi(y)$ for some $y \in M$, then $V \models \phi(x) \wedge \phi(y) \wedge x \neq y$, again contradicting the uniqueness. □

The existence of 0^\sharp has consistency strength. However, no such assumption is needed to obtain canonical models beyond the constructible universe:

Proposition 17 *It is consistent relative to ZFC that there are canonical models besides L .*

Proof (Sketch) Force a Π_2^1 -singleton over L as described in chapter 6 of Friedman (2000). The generic extension satisfies that there is a real number r satisfying the Π_2^1 -statement ψ (which is unique) and $V = L[r]$ and is unique with this property. □

⁸ For the notion of an absolute Π_2^1 -singleton, cf., e.g., David (1982).

Lemma 18 *There is a subtheory T of $ZFC + '0^\sharp \text{ exists}'$ that is consistent with $V = L$ such that there are canonical models M of T with $L \subsetneq M \subseteq L[0^\sharp]$.*

Proof In Friedman (2000), Theorem 6.23, it is shown that there is a Π_2^1 -singleton x and a theory T as in the statement of the lemma such that $x \notin L$, $0^\sharp \notin L[x]$ and $L[x]$ is canonical with respect to T . □

According to the discussion before Theorem 6.23 in Friedman (2000), it is open whether the x in the preceding proof can be proved to be unique in ZFC rather than $ZFC + 0^\sharp \text{ exists}$.

Question 19 Is there a canonical model M with $L \subsetneq M \subsetneq L[0^\sharp]$?

Our first observation concerning canonical implication is that it does not coincide with first-order provability, i.e., there are canonically necessary statements that are not provable in ZFC:

Lemma 20 *There is some \in -formula ϕ such that ϕ does not hold in all transitive class models of ZFC, but ϕ is canonically necessary.*

Proof Let ϕ be the statement: ‘It is not the case that there is a Cohen-generic filter G over L such that $V = L[G]$ ’. (Thus, intuitively, ϕ says: ‘I am not a Cohen-extension of L ’). This is an \in -statement. Clearly, ϕ is false in a Cohen-extension $L[G]$ of L .

On the other hand, let M be canonical and assume that $M \models \phi$. Let ψ be a uniqueness statement for M . Then there is some G Cohen-generic over L with $M = L[G]$. Moreover, as $M \models \psi$, there is some condition p such that $p \Vdash \psi$. Let G' be Cohen-generic over L relative to G such that $p \in G'$. Then $L[G'] \models \psi$ but $L[G'] \neq L[G]$, a contradiction to the assumption that ψ is a uniqueness statement. □

Remark 21 For any \in -sentence $\phi(\alpha)$ with ordinal parameters, the homogeneity of Cohen forcing implies that, if $\phi(\alpha)$ holds in the generic extension, then $\phi(\alpha)$ is forced by $\mathbb{1}$. It follows that the sentence ϕ just constructed is also \mathcal{C} -canonical.

This example can be considerably strengthened: In fact, no set forcing extension can be canonical. It is not obvious that the statement ‘I am not a set forcing extension’ is expressible in the first-order language of set theory at all, but by Reitz (2007), where it is introduced under the name ‘ground model axiom’ or ‘ground axiom’, it turns out to be so.

Definition 22 (See Reitz 2007) The Ground Model Axiom (GMA) is the statement that there is no transitive class model M of ZFC such that, for some forcing $\mathbb{P} \in M$ and some \mathbb{P} -generic filter G over M , we have $V = M[G]$. It is proved in Reitz (2007) that GMA is expressible as an \in -formula.

Theorem 23 *The ground model axiom GMA is canonically necessary.*

Proof Assume that M is canonical, witnessed by ϕ , and M does not satisfy the ground axiom, e.g. $M = N[G]$, where N is an inner model of M and G is a generic filter for a forcing $\mathbb{P} \in N$. As ϕ holds in M , there is some $p \in \mathbb{P}$ such that $p \Vdash \phi$ over N .

We pass from M to a generic extension $M[H]$ in which $\mathfrak{P}^M(\mathbb{P})$ is countable (via the appropriate Levy collapse). By Rasiowa-Sikorski, we find in $M[H]$ two mutually \mathbb{P} -generic filters over N containing p , namely G_1, G_2 . Hence $N[G_1] \vDash \phi$ and $N[G_2] \vDash \phi$, but $N[G_1] \neq N[G_2]$, as e.g. $G_1 \in N[G_1] \setminus N[G_2]$, so $M = N[G]$ cannot be unique with this property, a contradiction. □

Although the preceding argument generalizes to show that forcing extensions by homogenous, as well as countably closed, forcings cannot be \mathcal{C} -canonical, we do not know whether GMA is \mathcal{C}^L -canonical. It is, however, not \mathcal{C} -canonical:

Theorem 24 *GMA is not \mathcal{C} -canonically necessary.*

Proof (Sketch) We use an iterated forcing of length ω . (A similar construction is described in Reitz (2009), proof of Theorem 26.) The first forcing \mathbb{P}_0 adds a Cohen-real, and then, for $i > 1$, the i th forcing encodes the generic objects of the $(i - 1)$ th forcing in the continuum function. In the extension by this forcing, the generic filter can then be read off from the continuum function. Let \mathbb{P} denote this forcing, and, for some \mathbb{P} -generic extension $L[G]$ of L , let T be the theory that tells us sufficiently many values of the continuum function in $L[G]$ to reconstruct G . Then $L[G]$ is the unique transitive class model of $T + \text{‘I am a } \mathbb{P}\text{-generic extension of } L\text{’}$. Hence $L[G]$ is \mathcal{C} -canonical, but $L[G] \notin \text{GMA}$. □

Question 25 Is GMA \mathcal{C}^L -canonically necessary?

By a similar argument, now using the forcing theorem for symmetric extensions (see Hayut and Karagila (2020), p. 453; also see Karagila (2019)), we obtain:

Corollary 26 *No symmetric extension of a ZFC-model is canonical. Thus, the statement that V is not a generic extension of a ZFC-model is canonically necessary.*

We can further exploit this argument in a different direction, yielding an upper bound on the elements of a canonical model. To this end, recall, e.g. from Fuchs et al. (2015) that, in set-theoretical geology, the mantle is the intersection of all transitive class models M of ZFC such that V is a generic extension of M .

Proposition 27 *If M is canonical, then M is a subclass of the mantle.*

Proof Suppose that M is canonical with uniqueness statement ϕ and N is a ground for V . Then $V = N[G]$ for some \mathbb{P} -generic filter G over N and some forcing $\mathbb{P} \in N$. Moreover, for some formula $\psi(x, y)$ and some parameter y , there is a condition

$p \in G$ which forces over N that φ holds in the inner model that is defined by a $\psi(x, y)$. Let H be \mathbb{P} -generic over V with $p \in H$. Then G and H are mutually generic. Since $p \in H$, there is an inner model M' of $N[H]$ in which φ holds. By uniqueness of inner models of φ in $N[G \times H]$, we have $M = M'$. Since G and H are mutually generic over N , we have $N[G] \cap N[H] = N$ and hence $M \subseteq N$, as required. □

Given Theorem 23, one might wonder whether GMA captures the full strength of canonical necessity, i.e. whether there are canonically necessary statements that do not follow from GMA. This also turns out to be true:

Theorem 28

- (1) *The statement that ‘There is no measurable cardinal’ is canonically necessary.*
- (2) *The statement that ‘There are only finitely many measurable cardinals’ is \mathcal{C}^1 -canonically necessary.*

Proof We start with the second statement. Suppose for a contradiction that M is a canonical model with uniqueness statement $\phi(\alpha)$ ($\alpha \in \text{On}$) and that M contains infinitely many measurable cardinals ($\kappa_i : i \in \omega$). By a theorem of Kunen (see, e.g., Theorem 19.17 of Kanamori (2003)), there is an ultrapower $\text{Ult}(M, U)$ by a normal ultrafilter U on some κ_i that fixes α . But then, $\text{Ult}(M, U)$ is a transitive class inner model of $\text{ZFC} + \phi(\alpha\alpha)$ different from M , contradicting the assumption that M was unique with these properties.

The first statement can be proved in the same way, by just taking the ultrapower with any normal ultrafilter on a measurable cardinal in M (since there is no α to preserve). □

Theorem 29 *Suppose that there is a measurable cardinal in V . Then every canonical model has a proper class of order indiscernibles.*

Proof Suppose that M is a canonical model with uniqueness statement ϕ , and suppose that M is defined in V as $\{x : \psi(x, \vec{p})\}$, for some $\vec{p} \in V$. Let $M_0 := M$. Let U be a normal ultrafilter on κ , and let $\pi : V \rightarrow \text{Ult}(V, U)$ be the ultrapower embedding. Define M_1 as $\{x : \psi(x, \pi(\vec{p}))\}^{\text{Ult}(V, U)}$. Since π is elementary, so is $(\pi \upharpoonright M_0) : M_0 \rightarrow M_1$. Thus M_1 is a transitive class model of $\text{ZFC} + \phi$, and thus, $M_1 = M$.

Let $\kappa_1 = \pi(\kappa)$, so $\kappa_1 > \kappa$. Iterating this procedure, and taking direct limits at limit ordinals, we obtain a sequence $(M_\iota : \iota \in \text{On})$ of models, all of which are equal to M , on the one hand and a strictly increasing sequence $(\kappa_\iota : \iota \in \text{On})$ of cardinals in M on the other. But now, it is standard to show that $(\kappa_\iota : \iota \in \text{On})$ is a sequence of order indiscernibles for M . □

Corollary 30 *There are canonically necessary statements that do not follow from GMA.*

Proof By results of J. Reitz (see Reitz 2007), the fine structural models for measurable cardinals satisfy GMA. Hence, the nonexistence of measurable cardinals does not follow from GMA. □

The canonical impossibility of measurable cardinals suggests further considerations about the canonical possibility, or otherwise, of large cardinals. First, a positive observation:

Theorem 31

- (i) *Let ϕ^\sharp be the statement ‘For every $x \subseteq \omega$, x^\sharp exists’. If ϕ^\sharp holds in V , then ϕ^\sharp is canonically possible.*
- (ii) *If ϕ^\dagger is the statement that x^\dagger exists for every $x \subseteq \omega$, and ϕ^\dagger holds in V , then ϕ^\dagger is canonically possible.*

Proof (i) Let us add the unary function symbol \sharp to the language of set theory, with the obvious intended meaning. For a set X , let $\text{Def}^\sharp(X)$ be the set of subsets of X that are definable over X using the language of set theory extended by \sharp . Now define a class $\hat{L} \subseteq V$ in analogy with L as follows:

- $\hat{L}_0 = \emptyset$
- $\hat{L}_{\alpha+1} = \text{Def}^\sharp(\hat{L}_\alpha)$
- $\hat{L}_\lambda = \bigcup_{\iota < \lambda} \hat{L}_\iota$ for $\iota < \lambda$
- Finally, $\hat{L} := \bigcup_{\iota \in \text{On}} \hat{L}_\iota$.

By Π_2^1 -absoluteness of sharps, the sharp function and thus \hat{L} is a definable subclass of V via the recursion just stated. It is clear from the definition that $\hat{L} \models \phi^\sharp$. That $\hat{L} \models \text{ZFC}$ can be checked similarly to the fact that ZFC holds in L . Thus \hat{L} is a canonical model in which ϕ^\sharp holds.

(ii) is now proved similarly, as the \dagger -operation is still Π_2^1 . □

We can do the same for other inner model operators whose iterability is Π_2^1 .

3.1 Canonicity and the Continuum Function

All examples of canonical models that we have constructed so far satisfy the continuum hypothesis, and even the generalized continuum hypothesis. An attractive question is then whether $\text{ZFC} \vDash_c \text{CH}$, or even whether $\text{ZFC} \vDash_c \text{GCH}$. We conjecture that this is false (see the last section). Here, we mention some related results. We denote by c the continuum (class) function that maps each ordinal α to $2^{\text{card}(\alpha)}$.

Theorem 32 $\neg CH$ is weakly c.p.

Proof In Theorem 19 of Groszek (1988), a set forcing extension M of L is constructed such that $M \models \neg CH$, but CH holds in every transitive class N such that $L \subseteq N \subseteq M$ (and this is provable in ZFC). The forcing \mathbb{P} used is definable over M without parameters.

Consider the \in -statement ϕ ‘I am a \mathbb{P} -extension of L ’. Then every proper inner model of a transitive class model M of $ZFC+\phi$ will satisfy CH, so that M is the only inner model of M in which CH fails (and all of this is provable in $ZFC+\phi$). Hence M is a weakly canonical model of $ZFC+\neg CH$. □

3.2 How Canonical is ZFC?

Besides asking which statements are canonically necessary over ZFC, one can also step backwards and ask whether the axioms of ZFC canonically follow from weaker subtheories.

Even forgetting about uniqueness statements, the restriction to transitive class models alone yields certain results.

The following observation was communicated to us by Philip Welch:

Recall that Σ_2 -KP is Kripke-Platek set theory (see, e.g., Mansfield and Weitkamp 1985) supplemented with the Σ_2 -collection and the Σ_2 -separation scheme. Moreover, Pot denotes the power set axiom.

Theorem 33 (Welch) *All axioms of ZF are implied over transitive classes by Σ_2 -KP+Pot (the power set axiom). In particular, Σ_2 -KP+Pot \models ZF.*

Proof Let M be an transitive class model Σ_2 -KP+Pot, defined in V . By Lemma 1, it suffices to show that M is closed under Gödel functions and almost universal.

Since $M \models KP$, M is clearly closed under Gödel functions.

It remains to show that M is almost universal. To this end, we will show how to define inside M the cumulative hierarchy $(V_\alpha^M : \alpha \in \text{On})$ and show that $M = \bigcup_{\iota \in \text{On}} V_\iota^M$.

Once this is done, the argument finishes as follows: If $x \subseteq M$ is a set, we consider the functional class $F : x \rightarrow \text{On}$ (defined in V) that maps each $y \in x$ to the smallest $\iota \in \text{On}$ such that $y \in V_\iota^M$. By Replacement in V , $F[x]$ is a set of ordinals in V , and so $\alpha = \sup F[x]$ is an ordinal. Then, by definition of α , we have $x \subseteq V_\alpha^M$, which suffices.

Inside M , define a class function $F : \text{On} \times M \rightarrow M$ as follows:

For successor ordinals, let $F(\iota + 1, x) = z$ if and only if $z = \mathcal{P}(x(\iota))$, i.e. if and only if there is exactly one set a with $(\iota, a) \in x$ and $\forall y(y \in z \leftrightarrow y \subseteq x)$; if there is no such set, then $F(\iota + 1, x) = \emptyset$. Since the first existential (and implicit, via the uniqueness condition, universal) quantifier is bounded, this is Π_1 .

For a limit ordinal δ , $F(\delta, x) = z$ if and only if $z = \bigcup x$, i.e., if and only if $\forall y(y \in z \leftrightarrow \exists b \in x(y \in b))$. Again, this is Π_1 .

Now the class function $V^M : \text{On} \rightarrow M$ defined by applying the recursion principle to F is Σ_2 over M .

We claim that $V^M := \bigcup_{i \in \text{On}} V_i^M = M$.

It is clear that $V^M \subseteq M$.

To see the reverse direction, suppose otherwise, and let $x \in M$ be \in -minimal (by transitivity of M) such that $x \notin V^M$. Then, by transitivity of M , we have $x \subseteq M$, and by minimality of x , we have $x \subseteq V^M$. Thus (by replacement in V), there must be an ordinal α such that $x \subseteq V_\alpha^M$. But then, we have $x \in V_{\alpha+1}^M$. □

Corollary 34 *Let M be a canonical model of ZF with uniqueness statement ϕ , where ϕ is Σ_n , and let $n = \max\{3, n\}$. Then M has no proper transitive Σ_n -elementary substructure that contains all ordinals.*

Proof Suppose that $N \prec_{\Sigma_n} M$ and $\text{On} \subseteq N$. Then $N \vDash \phi$. Moreover, since $n \geq 3$, $N \vDash \Sigma_2\text{-KP}$ and $N \vDash \text{Pot}$. By Theorem 33, $N \vDash \text{ZF}$. Hence $N \vDash \text{ZF} + \phi$, and since ϕ is a uniqueness statement, we have $N = M$. □

Remark 35 By a similar argument, one can see that a canonical model M with a Σ_n -uniqueness statement coincides with the Σ_n -Skolem hull of the ordinals in M .

Question 36 Is there a transitive proper class inner model of $\text{KP} + \text{Pot}$ in which ZF fails? Do we have $\text{KP} \vDash_{\subset} \text{ZF}$?

3.2.1 The Axiom of Choice

We do not know whether there is a canonical model of ZF in which the axiom of choice fails. One of the most studied theories contradicting choice is $\text{ZF} + \text{AD}$, where AD is the axiom of determinacy. Whether or not AD is canonically possible is open. However, we have the following partial results:

Theorem 37 *Assume that A^\sharp exists for every $A \subseteq \mathbb{R}$. Then there is no canonical model of $T := \text{ZF} + V = L(\mathbb{R}) + \text{AD}$.*

Proof Suppose that M is a canonical (proper class) inner model and that T holds in M ; let ϕ be the canonical sentence for M .

We first claim that \mathbb{R}^M is countable. Otherwise, we can find a countable elementary substructure R^\sharp of $(\mathbb{R}^M)^\sharp$, but then $L(R)$ is an inner model M' that has the same theory as M ; in particular, $M \vDash \phi$. However, we will clearly have $M' \neq M$, contradicting the uniqueness of M .

It now follows from Theorem 0.1 in Schindler and Steel (2006) that we can pass to a generic extension of M and find an extender sequence \vec{E} with extenders below ω_1^M such that the symmetric collapse of $\text{Col}(\omega, < \omega_1^M)$ over M is equal to $L(\mathbb{R})^V$ (in fact, it is shown in Schindler and Steel (2006) that ω_1^M is a limit of Woodin cardinals in $L[\vec{E}]$).

Since \mathbb{R}^M is countable and since $(\mathbb{R}^M)^\#$ exists, the required generic extension exists in V , as it can be chosen as a collapse below the least \mathbb{R}^M -indiscernible. Thus we can choose $L[\vec{E}]$ in V and since ω_1^M is countable and $\vec{E}^\#$ exists, $P(\omega_1^M)^M$ is countable in V . Hence we can choose two different $\text{Col}(\omega, <\omega_1^M)$ -generics over M in V . By homogeneity of the forcing, the extensions have the same theory, but this contradicts the uniqueness of M . □

4 Theory Canonicity

The fact that canonicity rules out measurable cardinals almost trivially may suggest that it is too strong a demand; and indeed, uniquely fixing a transitive class model is rather much to ask. A reasonable weakening would be that the statement ϕ only fixes transitive class models up to elementary equivalence, which is the intuition behind the concept of theory-canonicity, as defined above.

We observe that theory-canonicity still goes beyond ZFC, but allows for measurable cardinals:

Proposition 38 *The existence of measurable cardinals is theory-canonically possible.*

Proof Let ϕ be the statement ‘ $V = L[U]$, where U is a normal ultrafilter’. Now, every two transitive class models M and M' of ϕ can be coiterated to a common target model N (see, e.g., Jech 2002, Theorem 19.14), which will be elementary equivalent to both M and M' ; thus M and M' are elementary equivalent. Hence ϕ is tc. Clearly, any transitive class model M of ϕ contains a measurable cardinal. □

Proposition 39 *There is an \in -sentence ψ that is theory-canonically necessary, but does not follow from ZFC.*

Proof For $x \subseteq \omega$, let T_x be the theory $\{2^{\aleph_i} = \aleph_{i+1} : i \in x\} \cup \{2^{\aleph_i} = \aleph_{i+2} : i \notin x\}$. Thus, in a model of T_x , x is encoded in the continuum function. Denote the Easton forcing that forces T_x to be true over a model of GCH by P_x .

Now let ϕ be the sentence that claims that V is not of the form $(L[x])[P_x]$, where x is Cohen-generic over L . Clearly, ϕ does not follow from ZFC, as it e.g. fails in models of the form $(L[x])[P_x]$ just described.

We claim that ϕ is tcn. Suppose otherwise; then there are a tc statement ψ and a transitive class model M of $\text{ZFC} + \psi + \neg\phi$. Then there is a real number x which is Cohen-generic over L such that $M = (L[x])[P_x]$. By the forcing theorem, ψ is forced by some Easton-condition q in P_x . The fact that q belongs to P_x and that ψ is forced by q in P_x , which holds in $L[x]$, is in turn forced by some Cohen-condition p over L .

Now let r and r' be two incompatible extensions of p , and let x_r and $x_{r'}$ be Cohen-generic reals extending r and r' , respectively.

Then both in $L[x_r]$ and in $L[x_{r'}]$, we have that $q \Vdash \psi$. Pick a P_{x_r} -generic filter G_r and a $P_{x_{r'}}$ -generic filter $G_{r'}$, both containing q ; this is possible as q belong both to P_{x_r} and to $P_{x_{r'}}$, because r and r' extend p , which forces this.

Then ψ holds both in $M_r := (L[x_r])[G_r]$ and in $M_{r'} := (L[x_{r'}])[G_{r'}]$; however, as $x_r \neq x_{r'}$, there is some $i \in \omega$ such that $M_r \models 2^{\aleph_i} = \aleph_{i+1}$ and $M_{r'} \models 2^{\aleph_i} = \aleph_{i+2}$ or vice versa. Thus M_r and $M_{r'}$ are two models of $ZFC+\psi$ that are not elementary equivalent, which contradicts the assumption that ψ is tc. □

Concerning the size of the continuum however, theory canonicity has little information to offer:

Proposition 40 *When ι is definable in L and $\text{cf}(\iota) > \omega$, then there is a tc model of $ZFC+2^{\aleph_0} = \aleph_\iota$. That is, any possible value of the continuum that is definable is also theory-canonically possible.*

Proof Let P_ι be the Cohen forcing notion for achieving $2^{\aleph_0} = \aleph_\iota$ over L . Then P_ι is definable from ι , which is definable by assumption, so P_ι is definable in L . Now the statement $\phi \equiv \forall V$ is a P_ι -generic extension of L in a transitive class model M of ZFC implies that $M \models 2^{\aleph_0} = \aleph_\iota$ and moreover, by homogeneity of P_ι , all sentences true in M are forced by 1 and thus hold in all such extensions. Thus M is a tc model in which $2^{\aleph_0} = \aleph_\iota$ holds. □

Remark 41 : Note that the proof of Proposition 40 also implies that GMA is not theory-canonically necessary, as all Cohen-extensions of L have the same theory, ‘ $\forall V$ is a Cohen-extension of L ’ is expressible as an \in -sentence ϕ and GMA clearly fails in every transitive class model of ϕ .

We can also show that $\neg\text{AC}$ is theory-canonically possible. We know the following lemma from a talk of Karagila; however, it has apparently so far not been made explicit in the literature. We thank Asaf Karagila for a sketch of the proof (personal communication) and the kind permission to include it here.

We say that a transitive class model of ZF is a ‘Cohen-model of the first type’ if and only if it is a symmetric extension of L in the sense of the first Cohen model as described, e.g., in Jech (2008), chapter 5.3.

Lemma 42 *There is an \in -formula ϕ_{cs} such that, for a transitive class $M \models ZF$, we have that $M \models \phi_{\text{cs}}$ if and only if M is a Cohen model of the first type.⁹*

Proof By Theorem 4.2 of [16], we have that, if M is a Cohen model of the first type, then there is an M -generic filter F such that $M[F] = L[F]$, $L[F] \models ZFC$ and M is a

⁹ We thank Asaf Karagila for informing us about this following folklore result and sketching the proof below to us.

symmetric submodel of $L[F]$. By Remark 3.3 of [16], we have $M = L(A)$, where A is the set of Cohen reals added by F over L .

So we can express ϕ_{cs} as ‘There is a generic filter F such that M is the symmetric submodel of $L[F]$ ’. (More precisely, we mean that M is the class of the F -evaluations of the symmetric names.) □

Proposition 43 *There is a canonical theory T that extends $ZF+\neg AC$.*

Proof Consider the statement ϕ_{cs} from Lemma 42, and let M be a transitive class model of $ZFC+\phi_{cs}$.

For such an M , we have $M \models \neg AC$; moreover, if ψ is a sentence such that $M \models \psi$, then, by homogeneity of the Cohen-forcing, we have $\mathbb{1} \Vdash \psi$, and hence, any two transitive class models of ϕ_{cs} are elementary equivalent. □

5 Further Ideas and Questions

Question 44 Is $V = HOD$ (theory-)canonically necessary?

Question 45 Is CH canonically necessary? Is GCH canonically necessary?

We conjecture that the answer to the last question is negative; a natural approach is to iterate Friedman’s forcing for adding a Π_2^1 -singleton (described in Friedman (2000), chapter 6) ω_2 many times to generate a canonical model in which CH fails.

Question 46 (Dominik Klein): Is AC canonical for models of ZF? That is, is there a formula ϕ such that there is a unique transitive class model M of $ZF+\phi$ and such that AC fails in M ?

More boldly, is there a canonical model of $ZF+AD$?

By a theorem of Kunen (1973), AC fails in the model C^{ω_2} , which is constructed in analogy with L , but using definability in the infinitary language L_{ω_1, ω_1} . This motivates the investigation of canonical implication for infinitary languages, which we plan to consider in future work.

Question 47 Can $M_1^\#$ be an element of a canonical model?

Question 48 In general, which ZFC axioms are (theory-)canonical over the others? Or over KP? Are there e.g. canonical models for ZFC^- in which power set fails? Are there canonical models of ZFC without replacement in which replacement is false?

The Ehrenfeucht principles asserts that, when a is definable from b , but $a \neq b$, then a and b have different types; there are several variantes of this principle, discussed in, e.g., (Enayat 2004a; Mycielski 1995; Fuchs et al. 2018). Following

(Enayat 2004b), a model M of ZF is ‘Leibnizean’ if and only if any two elements of M have different types with ordinal parameters.

Question 49 Does every canonical model of ZFC satisfy the Ehrenfeucht principle? Is every canonical model of ZFC Leibnizean?

Question 50 Is there a proof calculus that is sound and complete for \vDash_c and \vDash_{tc} ? (Note that, by failure of compactness, such a calculus cannot consist of a finite set of finite rules.) Are there inference rules \mathcal{R} such that, if \mathcal{R} allows deducing ϕ from T , then $T \vDash_c \phi$ (or $T \vDash_{tc} \phi$), but ϕ can fail in transitive class models of T ?

A concept closely related to, but different from, (theory-)canonicity is that of ‘unique describability’: Namely, we say that $\phi(\alpha)$ has *unique models* if and only if, for every transitive class model N of ZFC, there is at most one inner model M of N such that $M \vDash \phi(\alpha)$. This more liberal notion is closely related to recognizability by Ordinal Turing Machines (see Carl et al. 2018) and is considered in ongoing work with Philip Welch.

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