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Hadamard Product of Monomial Ideals and the Hadamard Package

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Abstract: In this paper, we generalize and study the concept of Hadamard product of ideals of projective varieties to the case of monomial ideals. We have a research direction similar to the one of the join of monomial ideals contained in a paper of Sturmfels and Sullivant. In the second part of the paper, we give a brief tutorial on the `Hadamard.m2` package of `Macaulay2`.

Keywords: Hadamard product; Segre product; Macaulay2; monomial ideals

MSC: 13F20; 13D02; 13C40; 14N20; 14M99

1. Introduction

The Hadamard product is a sort of multiplication of matrices and in spite of the usual product, it is commutative. This product of matrices is a classical topic in linear algebra that is useful in statistics and physics and it is also studied and used in combinatorial and probabilistic problems. Its applications can be found not only in mathematics but also, for instance, in cryptography, information theory, and data compression such as jpeg format.

Around 2010, in [1,2], the Hadamard product of matrices was extended to the Hadamard product of varieties in the study of the geometry of *binary Boltzmann machine*. The Hadamard product of varieties is also related to *tropical geometry* [3–5].

Successively, in the last few years, the Hadamard product of projective varieties has been widely studied from the point of view of projective geometry and tropical geometry.

The paper [3], where Hadamard products of general linear spaces are studied, can be considered the first step in this direction, while other papers, for example, Refs. [6–11], study the Hadamard product of varieties in relation with star configurations, Hilbert functions, apolarity, and so on.

The computation of the Hadamard product $X \star Y$ of two varieties X and Y can be conducted in terms of suitable projections of Segre Embedding (Definitions 1 and 2) or working with the ideals of X and Y and using Elimination Theory (Definition 3). This second approach has two main benefits: first, we can use Computer Algebra software, such as `Macaulay2` [12], to produce explicit computations and examples; secondly, we can generalize the Hadamard product to any kind of ideal, that is, to ideals which are not associated with varieties (for example, monomial ideals, edge ideals, clutter ideals).

This second fact led us to study carefully the behavior of the Hadamard product of monomial ideals. In Proposition 1, we prove that the Hadamard product of square-free monomial ideals is square-free. The class of square-free monomial ideals is a classical object in commutative algebra, which has a strong connection with combinatorial structures and invariants of hypergraphs. We address our research in the spirit of the paper [13], where another operation between ideals, the so-called join (Definition 4), is taken into consideration and the join of monomial ideals is studied. Moving as in [13], but using the Hadamard product instead of the join, we discover that the Hadamard product of two

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monomial ideals is nothing more than their sum (Theorem 1), a much easier behavior with respect to the join.

Coming back to the first benefit of the algebraic approach, that is, the use of computer algebra software, it is clear that writing down all the instructions for the computation of the Hadamard product of two ideals is not so difficult, but requires some time. For this reason, the first author created the `Hadamard.m2` package of `Macaulay2` which has been verified for Macaulay2 development and it can be found at https://github.com/imanbj/Hadamard-m2 (accessed on 1 September 2021). The second part of this paper is intended to be a tutorial that provides a brief introduction to the use of `Hadamard.m2` for the Hadamard product of varieties.

2. Preliminaries

We work over an algebraically closed field $\mathbb{K}$. We start defining the Hadamard product of projective varieties in two different ways.

Definition 1 ([3]). *Given varieties $X, Y \subset \mathbb{P}^n$, we consider the usual Segre product*

$$X \times Y \subset \mathbb{P}^N$$

$$([a_0 : \cdots : a_n], [b_0 : \cdots : b_n]) \mapsto [a_0b_0 : a_0b_1 : \cdots : a_nb_n]$$

and we denote with z_{ij} the coordinates in $\mathbb{P}^N$. Let $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$ be the projection map from the linear space Λ defined by equations $z_{ii} = 0, i = 0, \ldots, n$. The Hadamard product *of X and Y is*

$$X \star Y = \overline{\pi(X \times Y)},$$

where the closure is taken in the Zariski topology.

Definition 2 ([2]). *Let $p, q \in \mathbb{P}^n$ be two points of coordinates $[a_0 : a_1 : \ldots : a_n]$ and $[b_0 : b_1 : \ldots : b_n]$, respectively. If $a_ib_i \neq 0$ for some i, their Hadamard product $p \star q$ of p and q, is defined as*

$$p \star q = [a_0b_0 : a_1b_1 : \cdots : a_nb_n].$$

If $a_ib_i = 0$ for all $i = 0, \ldots, n$ then we say $p \star q$ is not defined.

We have that

$$X \star Y = \overline{\{p \star q : p \in X, q \in Y, p \star q \text{ is defined}\}}.$$

Hence, Definition 1 of the Hadamard product corresponds to Definition 2.

Remark 1. *We have that*

$$X \star Y = \overline{\{x \star y : x \in X, y \in Y, x \star y \textit{ is defined }\}}.$$

Note that the closure of the map π is necessary unless it is defined on all of $X \times Y$, see ([3], Remark 2.4) and Example 5.

We pass on now to defining the Hadamard product using Elimination Theory.

Let $S = \mathbb{K}[\mathbf{x}] = \mathbb{K}[x_0, \ldots, x_N]$ be a polynomial ring over an algebraically closed field.

Let $I_1, I_2, \ldots I_r$ be ideals in S. We introduce $(N+1)r$ new unknowns, grouped in r vectors $\mathbf{y}_j = (y_{j0}, \ldots, y_{jN}), j = 1, 2, \ldots, r$ and we consider the polynomial ring $\mathbb{K}[\mathbf{x}, \mathbf{y}]$ in all $(N+1)r + N + 1$ variables.

Let $I_j(\mathbf{y}_j)$ be the image of the ideal I_j in $\mathbb{K}[\mathbf{x}, \mathbf{y}]$ under the map $\mathbf{x} \mapsto \mathbf{y}_j$.

Definition 3. *Given ideals $I_1, I_2, \dots I_r \subset S$, their Hadamard product $I_1 \star I_2 \star \cdots \star I_r$ is the elimination ideal*

$$(I_1(\mathbf{y}_1) + \cdots + I_r(\mathbf{y}_r) + \langle x_i - y_{1i}y_{2i}\cdots y_{ri} \mid i = 0, \dots, N\rangle) \bigcap \mathbb{K}[\mathbf{x}].$$

Thus, the defining ideal of the Hadamard product $X \star Y$ of two varieties X and Y, that is, the ideal $I(X \star Y)$, equals the Hadamard product of the ideals $I(X) \star I(Y)$ (see ([3], Remark 2.6)).

According to [13], we give the following.

Definition 4. *Given ideals $I_1, I_2, \dots I_r \subset S$, their join $I_1 * I_2 * \cdots * I_r$ is the elimination ideal*

$$(I_1(\mathbf{y}_1) + \cdots + I_r(\mathbf{y}_r) + \langle x_i - y_{1i} - y_{2i} - \cdots - y_{ri} \mid i = 0, \dots, N\rangle) \bigcap \mathbb{K}[\mathbf{x}].$$

We define the r-th secant of an ideal $I \subset \mathbb{K}[\mathbf{x}]$ to be the r-fold join I with itself:

$$I^{*r} := I * I * \cdots * I.$$

Similarly, we define the r-th Hadamard power of an ideal $I \subset \mathbb{K}[\mathbf{x}]$ to be the r-fold Hadamard product of I with itself:

$$I^{\star r} := I \star I \star \cdots \star I.$$

3. Hadamard Products of Monomial Ideals

We start with a preliminary more general result.

Lemma 1. *The Hadamard product distributes over intersections:*

$$\left(\bigcap_{l \in \mathcal{L}} J_l\right) \star K = \bigcap_{l \in \mathcal{L}} (J_l \star K).$$

Proof. The proof is analogous to ([14], Lemma 2.6). A polynomial f belongs to $(\cap J_l) \star K$ if and only if $f(\mathbf{y}_1\mathbf{y}_2) \in (\cap J_l)(\mathbf{y}_1) + K(\mathbf{y}_2)$ if and only if $f(\mathbf{y}_1\mathbf{y}_2) \in J_l(\mathbf{y}_1) + K(\mathbf{y}_2)$ for all $l \in \mathcal{L}$ if and only if $f \in \cap(J_l \star K)$. □

Example 1. *Let $N = 0$ and consider the ideals $I = \langle x^i \rangle$ and $J = \langle x^j \rangle$. If $i \leq j$; then, $I \star J = \langle x^i \rangle$. The proof follows from the definition.*

$$\begin{aligned} I \star J =& (< y^i > + < z^j > + < x - yz >) \cap \mathbb{K}[x] = \\ & (< y^i, z^j, x - yz, x^i - y^i z^i >) \cap \mathbb{K}[x] = < x^i > . \end{aligned}$$

Definition 5. *An ideal $I \subseteq \mathbb{K}[\mathbf{x}]$ is a* monomial ideal *if it is generated by monomials. A monomial ideal I is a* square-free monomial ideal *if it is generated by monomials of the form $x_0^{a_0} \cdots x_N^{a_N}$ with $a_i \in \{0, 1\}$.*

A special family of monomial ideals is the class of *irreducible* monomial ideals. An irreducible monomial ideal is represented by an integer vector $\mathbf{u} = (u_0, \dots, u_N)$ as follows:

$$\mathbf{m}^{\mathbf{u}} = \langle x_i^{u_i} : u_i > 0 \rangle.$$

Example 1 generalizes to irreducible monomial ideals in $N + 1$ variables.

Lemma 2. *The Hadamard product of two irreducible monomial ideals* $\mathbf{m}^{\mathbf{u}}$ *and* $\mathbf{m}^{\mathbf{v}}$ *is an irreducible monomial ideal* $\mathbf{m}^{\mathbf{w}} = \langle x_i^{w_i} : w_i > 0 \rangle$ *where*

$$w_i = \begin{cases} \min\{u_i, v_i\} & \text{if } \quad u_i > 0 \text{ and } v_i > 0, \\ u_i & \text{if } \quad v_i = 0, u_i > 0, \\ v_i & \text{if } \quad u_i = 0, v_i > 0, \\ 0 & \text{if } \quad u_i = v_i = 0. \end{cases}$$

Proof. A polynomial $f(\mathbf{x}) = \sum_{\mathbf{a}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ lies in the monomial ideal $\mathbf{m}^{\mathbf{u}} \star \mathbf{m}^{\mathbf{v}}$ if and only if $f(\mathbf{y}_1\mathbf{y}_2) = \sum_{\mathbf{a}} c_{\mathbf{a}} (\mathbf{y}_1\mathbf{y}_2)^{\mathbf{a}}$ lies in the monomial ideal

$$\mathbf{m}^{\mathbf{u}}(\mathbf{y}_1) + \mathbf{m}^{\mathbf{v}}(\mathbf{y}_2) = \langle y_{1i}^{u_i} : u_i > 0 \rangle + \langle y_{2i}^{v_i} : v_i > 0 \rangle. \tag{1}$$

This happens if and only if every term $(\mathbf{y}_1\mathbf{y}_2)^{\mathbf{a}}$ lies in (1). Now, $\mathbf{x}^{\mathbf{a}}$ lies in $\mathbf{m}^{\mathbf{u}} \star \mathbf{m}^{\mathbf{v}}$ if and only if every term of

$$(\mathbf{y}_1\mathbf{y}_2)^{\mathbf{a}} = \prod_{i=0}^{N} (y_{1i} y_{2i})^{a_i}$$

lies in $\mathbf{m}^{\mathbf{u}}(\mathbf{y}_1) + \mathbf{m}^{\mathbf{v}}(\mathbf{y}_2)$ if and only if $w_i \le a_i$ for some i with $u_i \ne 0$ or $v_i \ne 0$ if and only if $\mathbf{x}^{\mathbf{a}}$ lies in $\mathbf{m}^{\mathbf{w}}$. □

The next result shows that the Hadamard product of ideals maintains some property of the starting ideals.

Proposition 1. *Let* $I_1, \ldots, I_r$ *be monomial ideals in* $\mathbb{K}[\mathbf{x}]$*. Then* $I_1 \star \cdots \star I_r$ *is a monomial ideal. If* $I_1, \ldots, I_r$ *are square-free, then monomial ideal* $I_1 \star \cdots \star I_r$ *is square-free.*

Proof. It is enough to consider the case $r = 2$ since, then, the general statement follows by induction on r. If $I_1 = \mathbf{m}^{\mathbf{u}}$ and $I_2 = \mathbf{m}^{\mathbf{v}}$ are irreducible ideals then both statements follow from Lemma 2. Otherwise, we decompose $I_1 = \bigcap_{\mathbf{u}} \mathbf{m}^{\mathbf{u}}$ and $I_2 = \bigcap_{\mathbf{v}} \mathbf{m}^{\mathbf{v}}$ as intersections of irreducible monomials ideals. Using Lemma 1, we then write

$$I_1 \star I_2 = \bigcap_{\mathbf{u},\mathbf{v}} \mathbf{m}^{\mathbf{u}} \star \mathbf{m}^{\mathbf{v}}. \tag{2}$$

Thus, $I_1 \star I_2$ is a monomial ideal, and its set of standard monomials is the union of the sets of standard monomials of its irreducible components $\mathbf{m}^{\mathbf{u}} \star \mathbf{m}^{\mathbf{v}}$. Assume that I_1 and I_2 are square-free. We can decompose $I_1 = \bigcap_{\mathbf{u}} \mathbf{m}^{\mathbf{u}}$ and $I_2 = \bigcap_{\mathbf{v}} \mathbf{m}^{\mathbf{v}}$ with $0 < u_i, v_i \le 1$, as intersections of irreducible linear monomials ideals. Again by Lemma 1, we then write $I_1 \star I_2$ as in (2). Since each $\mathbf{m}^{\mathbf{u}} \star \mathbf{m}^{\mathbf{v}}$ is square-free, $I_1 \star I_2$ is square-free. □

We can now prove the main result of this section, showing that the Hadamard product of two monomial ideals is equal to their sum.

Theorem 1. *Let* I *and* J *be monomial ideals in* $\mathbb{K}[\mathbf{x}]$*. Then* $I \star J = I + J$*.*

Proof. First, assume that the given ideals are irreducible, say, $I = \mathbf{m}^{\mathbf{u}}$ and $J = \mathbf{m}^{\mathbf{v}}$. Then the result follows by applying Lemma 2. For the general case, we decompose the two given monomial ideals into their irreducible components: $I = \cap_\nu I_\nu$ and $J = \cap_\mu J_\mu$. By Lemma 1 and the result for irreducible ideals, we get

$$I \star J = \bigcap_{\nu,\mu} (I_\nu \star J_\mu) = \bigcap_{\nu,\mu} (I_\nu + J_\mu) = I + J.$$

This completes the proof of Theorem 1. □

As a corollary, we get immediately the following result.

Corollary 1. *Let I be a monomial ideal in $\mathbb{K}[\mathbf{x}]$. Then $I^{\star r} = I$.*

We end this section with a few results about the join and Hadamard product of irreducible monomial ideals, also in connection with the irrelevant ideal $M = \langle x_0, \ldots, x_n \rangle$.

Proposition 2. $M^m * M^n = M^{m+n-1}$.

Proof. We decompose $M^m = \cap_{\mathbf{u}} \mathbf{m}^{\mathbf{u}}$ and $M^n = \cap_{\mathbf{v}} \mathbf{m}^{\mathbf{v}}$. The proof follows from ([13], Lemma 2.3) and ([14], Lemma 2.6). □

Corollary 2. *If $m \leq n$, then $M^m \star M^n = M^m$.*

Proof. The proof follows from Theorem 1. □

Proposition 3. *Given two irreducible monomial ideals $\mathbf{m}^{\mathbf{u}}$ and $\mathbf{m}^{\mathbf{v}}$, one has*

$$\mathbf{m}^{\mathbf{u}} * \mathbf{m}^{\mathbf{v}} \subset \mathbf{m}^{\mathbf{u}} \star \mathbf{m}^{\mathbf{v}}.$$

Proof. By Lemma 2.3 in [13], the minimal generators of $\mathbf{m}^{\mathbf{u}} * \mathbf{m}^{\mathbf{v}}$ are of the form $x_i^{u_i+v_i-1}$ when $u_i > 0$ and $v_i > 0$. For the same values of u_i and v_i, the corresponding generator of $\mathbf{m}^{\mathbf{u}} \star \mathbf{m}^{\mathbf{v}}$ is given by $x_i^{\min\{u_i,v_i\}}$. Since $u_i + v_i - 1 \geq \min\{u_i, v_i\}$ the claim easily follows. □

4. Hadamard.m2

We give now some explanations of the commands in `Hadamard.m2` together with examples. Looking at the documentation page, we can note that the package contains:

Types

- `Point` - a new type for points in projective space.

Functions and commands

- `hadamardPower` - computes the Hadamard powers of varieties;
- `hadamardProduct` - computes the Hadamard product of varieties;
- `idealOfProjectivePoints` - computes the ideal of a set of points;
- `point` - constructs a projective point from the list or array of coordinates.

Methods

- `hadamardPower(Ideal,ZZ)` - computes the r-th Hadamard power of varieties;
- `hadamardPower(List,ZZ)` - computes the r-th Hadamard power of a set of points;
- `hadamardProduct(Ideal,Ideal)` - computes the Hadamard product of two homogeneous ideals;
- `hadamardProduct(List)` - computes the Hadamard product of a list of homogeneous ideals or points;
- `hadamardProduct(List,List)` - computes the Hadamard product of two sets of points;
- ‘‘`idealOfProjectivePoints(List,Ring)`’’ - see idealOfProjectivePoints - computes the ideal of a set of points;
- ‘‘`point(VisibleList)`’’(see point) - constructs a projective point from the list or array of coordinates.
- `Point * Point` - entry-wise product of two projective points;
- `Point == Point` - check equality of two projective points.

In this package, a *point* can be constructed in two different and equivalent ways: both from a list `{}` and both from an array `[]`. Hence both the commands `p = point {1,2,3}` and `p = point [1,2,3]` define the same point $p = [1 : 2 : 3] \in \mathbb{P}^2$. In the following examples, we use both methods to define points.

Once the package is loaded the Hadamard product of two points is defined by `*`.

Example 2. *Consider the points* $[1:2:3]$, $[-1:2:5]$, $[1:1:2]$, $[2:2:4]$.

```
Macaulay2, version 1.21
i1 : loadPackage"Hadamard"
i2 : p = point {1,2,3};
i3 : q = point [-1,2,5];
i4 : p * q
o4 = Point{-1, 4, 15}
i5 : x = point [1,1,2];
i6 : y = point {2,2,4};
i7 : x == y
o7 = true
i8 : p == q
o8 = false
```

The function `hadamardProduct` is one of the main functions in `Hadamard.m2`.

`hadamardProduct` is multi-functional and works in different ways: it computes the Hadamard products of two ideals, two sets of points, or a list of ideals or points. Hence, according to the aim of computation, `hadamardProduct` can have a different number of inputs.

Before showing some examples, we mention the mathematical approach inside `hadamardProduct`.

Let X and Y be two varieties in $\mathbb{P}^n$. Denote by $I := I(X) \subset R = K[x_0, \ldots, x_n]$ and $J := J(Y) \subset T = K[y_0, \ldots, y_n]$ their corresponding vanishing ideals. We consider

$$S = \frac{R}{I} \otimes \frac{T}{J}.$$

Let $W = K[w_0, \ldots, w_n]$. We define the map

$$\begin{aligned} \Phi : W &\longrightarrow S \\ w_0 &\mapsto x_0 y_0 \\ &\vdots \\ w_n &\mapsto x_n y_n. \end{aligned}$$

Hence, we have that $X \star Y = \ker \Phi$.

Example 3. *Consider the ideals I and J.*

```
i9  : S = QQ[x,y,z,w];
i10 : I = ideal(x-y+z,z+y+w);
i11 : J = ideal(2*x-y,w+x+y+z);
i12 : hadamardProduct(I,J)
o12 = ideal(6x^2 - 9x*y + 3y^2 - 2x*z + 2y*z + 2x*w - y*w)
```

Example 4. *Using* `idealOfProjectivePoints`*, one can compute the vanishing ideal of a list of points. Here, we use two different methods to compute the Hadamard product of two points. First, we use* `*` *and then* `idealOfProjectivePoints`.

```
i13 : p = point {1,-1,4,2};
i14 : q = point {1,2,0,-1};
i15 : idealOfProjectivePoints({p*q},S)
o15 = ideal (z, y - w, 2x + w)
```

Now, we first use `idealOfProjectivePoints` *to compute the defining ideals of the points; then, we apply* `hadamardProduct`.

```
i23 : Ip = idealOfProjectivePoints({p},S)
o23 = ideal (z - 2w, 2y + w, 2x - w)
i24 : Iq = idealOfProjectivePoints({q},S)
o24 = ideal (z, y + 2w, x + w)
i25 : hadamardProduct(Ip,Iq)
o25 = ideal (z, y - w, 2x + w)
i26 : o25 == o15
o26 = true
```

Example 5. *Due to Remark 1, consider the curve $X \subset \mathbb{P}^2$ defined by $xy - z^2 = 0$ and the point $p = [1 : 0 : 2]$. The Hadamard product $p \star X$ is the line $y = 0$. However, the point $[0 : 0 : 1]$ cannot be obtained as $p \star q$, with $q \in X$.*

```
i27 : T = QQ[x,y,z];
i28 : IX = ideal(x*y-z^2)
i29 : p = point[1, 0, 2]
i30 : Ip = idealOfProjectivePoints({p},T)
o30 = ideal (y, 2x - z)
i31 : hadamardProduct(Ip,IX)
o31 = ideal y
```

Given two sets of points L and M, `hadamardProduct` returns the list of (well-defined) entry-wise multiplications of pairs of points in the Cartesian product $L \times M$.

Example 6. *Consider the two sets of points L and M.*

```
i32 : L = point\{{0,1}, {1,2}};
i33 : M = point\{{1,0}, {2,2}};
i34 : hadamardProduct(L,M)
o34 = {Point{1, 0}, Point{0, 2}, Point{2, 4}}
```

Note that the map π is not defined on all $L \times M$ since `point{0,1} * point{1,0} = point{0,0}`.

The Hadamard products of a list of ideals or points are constructed by using the binary function `hadamardProduct(Ideal,Ideal)`, or `Point * Point` iteratively. So, in this case, `hadamardProduct` requires in input only one argument given by the list, as shown in the following example.

Example 7. *Consider again the ideals I and J of Example 3 and let K be an ideal different from I and J.*

```
i35 : K = ideal(2*x-y,w+x+y+z, z-w)
o35 = ideal (2x - y, x + y + z + w, z - w)
i36 : L = {I,J,K};
i37 : hadamardProduct(L)
o37 = ideal(72x^2  - 54x*y + 9y^2  + 16x*z - 8y*z - 16x*w + 4y*w)
i38 : P = point\{{1,2,3},{-1,1,1},{1,1/2,-1/3}};
i39 : hadamardProduct(P)
o39 = Point{-1, 1, -1}
```

For any projective variety X, we defined, in Section 2, its Hadamard square $X^{\star 2} = X \star X$ and its higher Hadamard powers $X^{\star r} = X \star X^{\star r-1}$.

Give a homogeneous ideal I or a set of points L, the r-th Hadamard power of I or L is computed via the function `hadamardPower`. The function computes the r-times Hadamard products of I or L to itself.

Example 8. *Consider the ideals I and K of Examples 3 and 7.*

```
i40 : hadamardPower(J,3)
o40 = ideal(8x - y)
i41 : hadamardPower(K,3)
o41 = ideal (z - w, 27y + 64w, 27x + 8w)
i42 : L = {point{1,1,1/2},point{1,0,1},point{1,2,4}};
i43 : hadamardPower(L,2)
o43 = {Point{1, 0, 1}, Point{1, 0, 1/2}, Point{1, 0, 4},
       Point{1, 4, 16}, Point{1, 2, 2}, Point{1, 1, 1/4}}
```

Example 9. *Here is another example of Hadamard power.*

```
i44 : S = QQ[x,y,z];
i45 : X = point\ {{1,1,0},{0,1,1},{1,2,-1}};
i46 : A = idealOfProjectivePoints(X,S)
o46 = ideal(3x*z - y*z + z^2, 3x*y - 3y^2 - y*z + 4z^2,
            3x^2 - 3y^2 - 2y*z + 5z^2, y^2z + y*z^2 - 2z^3)
i47 : A2 = hadamardPower(A,2)
o47 : ideal(y^2*z - 18x*z^2+ y*z^2 - 2z, x*y*z - 4x*z^2,
            x^2*z - x*z^2, 2x^3 - 3x^2*y + x*y^2 - 6x*z^2)
i48 : X2 = hadamardPower(X,2);
i49 : A2 == idealOfProjectivePoints(X2,S)
o49 = true
```

Theorem 2 ([5])**.** *Let L and M be two lines in $\mathbb{P}^3$. Then $L \star M$ is a quadratic form.*

Example 10 (Example 6.1 in [3])**.** *Let L be the line in $\mathbb{P}^3$ through points* $[2:3:5:7]$ *and* $[11:13:17:19]$ *and M the line through* $[23:29:31:37]$ *and* $[41:43:47:53]$*. Let x, y, z, w be the homogeneous coordinates of $\mathbb{P}^3$. We compute $L \star M$ as follows:*

```
i50 : S = QQ[x,y,z,w];
i51 : L = matrix{{2, 3, 5, 7},{11, 13, 17, 19}};
i52 : IL = ideal flatten entries(matrix{gens S} * gens ker L);
i53 : M = matrix {{23, 29, 31, 37},{41, 43, 47, 53}};
i54 : IM = ideal flatten entries(matrix{gens S} * gens ker M);
i55 : hadamardProduct(IL,IM)
o55 = ideal(88128x^2 - 89280x*y - 5299632y^2 - 817938x*z + 8896641y*z -
            1481805z^2 - 321510x*w - 1777545y*w - 54250z*w + 116375w^2)
```

Example 11. *Let M1 and M2 be two monomial ideals. By Theorem 1, we know that the Hadamard product $M1 \star M2$ is equal to $M1 + M2$.*

```
i56 : M1 = ideal(x*y,x*z,w);
i57 : M2 = ideal(y^2,x*w,x^2*y);
i58 : hadamardProduct(M1,M2)
o58 = ideal(w, x*z, y^2 , x*y)
i59 : M1+M2
o59 = ideal(x*y, x*z, w, y^2 , x*w, x^2y)
i60 : o58 == o59
o60 = true
i61 : trim o59
o61 = ideal(w, x*z, y^2 , x*y)
```

Since coordinate hyperplanes are defined by monomial ideals (with only one variable), Theorem 1 gives another proof of part of the following Lemma.

Lemma 3 (Lemma 4.1 in [15])**.** *Let H_i be any coordinate hyperplane, for $i = 0, \dots, n$. Then,*

(i) $H_{i_1} \star \cdots \star H_{i_t}$ *is the linear subspace* $Z(x_{i_j}; j = 1, \dots, t)$*.*

(ii) $H_i \star C = H_i$ *for any hypersurface C different from a coordinate hyperplane.*

Example 12. *Let $L1 = V(x)$ and $L2 = V(y)$ be two lines in $\mathbb{P}^2$. One can see that the Hadamard product $L1 \star L2$ is the point $[0, 0, 1]$ obtained by the intersection of $L1$ and $L2$. We have that $L1 \star L2 = L1 \cap L2$.*

```
i62 : S = QQ[x,y,z];
i63 : IL1 = ideal(x);
i64 : IL2 = ideal(y);
i65 : hadamardProduct(IL1,IL2)
o65 = ideal(y,x)
i66 : IL1+IL2
o66 = ideal(y,x)
```

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References

1. Cueto, M.A.; Morton, J.; Sturmfels, B. Geometry of the restricted Boltzmann machine. In *Algebraic Methods in Statistics and Probability*; Contemporary Mathematics; Viana, M., Wynn, H., Eds.; American Mathematicals Society: Providence, RI, USA, 2010; Volume 516, pp. 135–153.
2. Cueto, M.A.; Tobis, E.A.; Yu, J. An implicitization challenge for binary factor analysis. *J. Symb. Comput.* **2010**, *45*, 1296–1315. [CrossRef]
3. Bocci, C.; Carlini, E.; Kileel, J. Hadamard products of linear spaces. *J. Algebra* **2016**, *448*, 595–617. [CrossRef]
4. Friedenberg, N.; Oneto, A.; Williams, R.L. Minkowski sums and Hadamard product of algebraic vatieties. In *Combinatorial Algebraic Geometry*; Fields Institute Communications; Smith, G., Sturmfels, B., Eds.; Springer: New York, NY, USA, 2017; Volume 80, pp. 133–157.
5. Maclagan, D.; Sturmfels, B. *Introduction to Tropical Geometry*; Graduate Studies in Mathematics; American Mathematicals Society: Providence, RI, USA, 2015; p. 161.
6. Jafarloo, I.B.; Bocci, C.; Guardo, E.; Malara, G. Hadamard products of symbolic powers and Hadamard fat grids. *Mediterr. J. Math.* **2023**, *20*, 162. [CrossRef]
7. Jafarloo, I.B.; Calussi, G. Weak Hadamard star configuration and Apolarity. *Rocky Mt. J. Math.* **2020**, *50*, 851–862.
8. Bocci, C.; Calussi, G.; Fatabbi, G.; Lorenzini, A. On Hadamard products of linear varieties. *J. Algebra Appl.* **2017**, *16*, 1750155. [CrossRef]

9. Bocci, C.; Calussi, G.; Fatabbi, G.; Lorenzini, A. The Hilbert function of some Hadamard products. *Collect. Math.* **2018**, *69*, 205–220. [CrossRef]
10. Calussi, G.; Carlini, E.; Fatabbi, G.; Lorenzini, A. On the Hadamard products of degenerate subvarieties. *Port. Math.* **2019**, *76*, 123–141. [CrossRef]
11. Carlini, E.; Catalisano, M.V.; Guardo, E.; Van Tuyl, A. Hadamard star configurations. *Rocky Mt. J. Math.* **2019**, *49*, 419–432. [CrossRef]
12. Grayson, D.R.; Stillman, M.E. Macaulay2, a Software System for Research in Algebraic Geometry. The Current Stable Version is 1.23, Released 20 March 2024. Available online: https://www.macaulay2.com/ (accessed on 20 March 2024).
13. Sturmfels, B.; Sullivant, S. Combinatorial Secant Varieties. *Pure Appl. Math. Q.* **2006**, *2*, 867–891. [CrossRef]
14. Sullivant, S. Combinatorial symbolic powers. *J. Algebra* **2008**, *319*, 115–142. [CrossRef]
15. Bocci, C.; Carlini, E. Hadamard products of hypersurfaces. *J. Pure Appl. Algebra* **2022**, *226*, 107078. [CrossRef]