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Thirlwall's law: Binding constraint or 'centre-of-gravity'?

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Abstract

Thirlwall's law is one of the most powerful empirical regularities in demand-led growth theories. In recent years, the challenges imposed by globalisation have led to a new wave of studies incorporating into this framework topics such as ecological sustainability, the complexity of innovation processes, the role of institutions, the composition of external imbalances, and gender issues. We notice some overlapping between two alternative interpretations: one that sees the law as a *binding constraint* and another that adopts a kind of '*centre-of-gravity*' perspective. It is argued that they may be rather complementary. By means of a simple Keynesian multiplier model compatible with Harrodian instability, we show that assuming a balance-of-payments ceiling to growth gives rise to persistent and bounded fluctuations such that the external constraint works as an asymmetric 'centre-of-gravity'. There is no need to impose a floor to output. Moreover, the model allows for different sources of autonomous demand. Numerical simulations show the robustness of our results with respect to alternative scenarios.

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KEYWORDS
aggregate demand, growth, mutliplier-accelerator, open economies, Thirlwall's law

JEL CLASSIFICATION
F43, O11, O40

1 | INTRODUCTION

Frequently referred to as the dynamic Harrod trade-multiplier, the idea of a balance-of-payments constraint on growth stands as one of the most powerful empirical regularities among alternative theories of growth and distribution. Under the assumption that countries trading in a foreign currency cannot sustain persistent and increasing current account imbalances, Thirlwall's (1979) law states that the long run performance of a given economy is well-approximated by the ratio between the income elasticity of exports over imports multiplied by the rate of growth of the rest of the world. This model offers a unique framework that combines demand and supply constraints to explain international growth rate differences and structural change.

Over the past 4 decades, the model has been generalised in a number of directions, incorporating capital flows, multisectoral issues, and adjustment dynamics. Textbook presentations that provide accessible, though less technical, treatments of the strengths and weaknesses of its structure include Lavoie (2014, pp. 513–540) and Blecker and Setterfield (2019, pp. 425–512). It must be noted, however, that the challenges imposed by globalisation have led to a new wave of studies that incorporate into this framework topics such as ecological sustainability, the role of institutions and innovation, as well as gender issues. Initial debates such as whether the balance-of-payments adjusts through prices, income or trade-elasticities (e.g. Alonso & Garcimart ín, 1998; Krugman, 1989), or concerns that the law was merely expressing a tautology (see McCombie, 2011; Ros, 2013, pp. 223–245; Blecker, 2016) are giving way to a refreshed research agenda. Among these recent developments, we identify some overlapping between two alternative interpretations somehow confronting the formal theories of economic fluctuations vis-à-vis macroeconomic theories of growth.

A more traditional view differentiates between cyclical and long-term output dynamics. It is argued that the rate of growth compatible with equilibrium in the balance-of-payments belongs to the second type and is as a *binding constraint* that results in a stable attractor. In the short-term, a country might grow faster or slower than this rate. In the long-term, nonetheless, the economy will converge towards it. This approach can be seen in theoretical contributions that explicitly or implicitly separate between time frames (Cimoli & Porcile, 2014; Ribeiro et al., 2017) as well as in empirical studies that either apply cointegration or Generalised Method of Moments (GMM) estimators (Gouvêa & Lima, 2010; Romero & McCombie, 2016). On the other hand, we have those who argue that fluctuations are intimately related to growth and the latter is a specimen of the former that should receive a unified mathematical treatment. In that case, Thirlwall's law is seen as a kind of '*centre-of-gravity*' of the economy (e.g. Dávila-Fernández and Sordi, 2019; Garcimartin et al., 2016). The system is never in a state of rest, continuously fluctuating around the demand-led trend.

While the difference might sound subtle, we believe it has important policy implications. As long as international conditions are relatively favourable, the first approach implies that, in the short-term, policy-makers could treat the external constraint as a secondary issue. This is especially the case when the economy is confronted with more 'urgent' needs such as, for example, managing high unemployment rates. However, the conclusion might be misleading, wrongly downplaying the importance of equilibrium in the current account. Alternatively, to conceive the rate of growth compatible with equilibrium in the balance-of-payments as always being satisfied on average might induce an underestimation of the role of other aggregate demand components.

This paper argues that the two views may be rather complementary. The proposed solution goes back to distinguishing elements in the thinking and life contribution to economics of John Hicks and Richard Goodwin. From Hicks (1950), we take the multiplier-accelerator interaction with floor-ceiling limits to investment as a piece-wise linear representation of the economy that allows the persistence of endogenous oscillations (for a recent formalisation along similar lines, see Gallegati et al., 2003; Sushko et al., 2010). From Goodwin (1967), the idea that the growth trend and fluctuations cannot be regarded as separate phenomena because the factors responsible for them are not independent of one another. Moreover, aggregate demand matters over the growth-cycle and time-lags play a fundamental role in investment decisions (see, for example, Goodwin, 1951; Sordi, 2006). The period that elapses from the moment that firms decide to expand their capital until the corresponding additional plant or equipment is ready for production cannot be overlooked (for a recent presentation referring to Kalecki's work, see Franke, 2018).

Our simple Keynesian dynamic model shows that assuming a balance-of-payments ceiling to growth can produce persistent demand-side-driven economic fluctuations. They remain present even when the model is augmented with a basic supply-side mechanism. We implicitly endogenise foreign capital inflows, allowing us to explain why the 'strong' version of the law might deviate from actual long run growth averages. Additionally, it provides an explanation of the empirically relevant asymmetry during the different phases of the cycle. Persistent fluctuations have the following rationale. Facing increasing demand, firms expand productive capacity, raising import requirements. As output accelerates, so do imports relative to foreign demand. Eventually, the country runs a trade deficit. Initially, the economy can finance it through global credit markets. However, international liquidity imposes a ceiling on capital accumulation that is a multiple of the balance-of-payments equilibrium growth rate. The interaction between investment lags and capital stock adjustment costs initiates the contractionary phase of the cycle. Firms are forced to reduce production and cancel investment plans, bringing imports down and correcting the trade deficit. Their balance sheets improve, leading to a recovery of economic activity, thus, reinitiating the cycle.

Our proposal conciliates both views on Thirlwall's law because (i) we assume *ex-ante* that it is a binding constraint that does not necessarily have to be satisfied, but (ii) its mere existence results in the emergence of growth-cycle dynamics in which the law works as an asymmetric 'centre-of-gravity'. This means that the balance-of-payments equilibrium growth rate can exist as a theoretical construct to be satisfied only in an abstract long run, as proposed by the first strand of the literature. Still, given that capital flows are directly or indirectly related to it, Thirlwall's law emerges endogenously as a reference point around which the economy fluctuates, in line with the second group's view. Moreover, we show there is no need to impose a floor to output nor any additional assumption on the behaviour of fiscal policy. The model is compatible with different sources of autonomous demand that may or may not grow at the same rate. Numerical simulations are performed, showing our results' robustness and sensitivity to different scenarios.

The remainder of the paper is organised as follows. In the next section, we revisit the fundamentals of the dynamic Harrod trade-multiplier. We provide a brief overview of recent developments in the field, differentiating between binding constraint and 'centre-of-gravity' perspectives. Section 3 presents a simple stylised demand-led growth model that reconciles both views as well as a numerical example showing the emergence of periodic orbits. Section 4 shows how a rudimentary supply-side technology can be incorporated into the system. Some final considerations follow.

2 | INTERNATIONAL TRADE AND ECONOMIC GROWTH

As pointed out not long ago by Razmi (2016), what distinguishes Thirlwall's law from other models is the role of the demand side in defining the nature of growth. Going back at least to Prebisch (1959), the fundamental assumption is that, in the long run, a country that trades in foreign currency cannot sustain increasing balance-of-payments imbalances. McCombie and Thirlwall (1999, p. 49) argue that the term 'balance-of-payments constraint' indicates that the performance of a country in international markets—ranging from goods to services including the financial sphere—limits growth to a rate lower than the one internal conditions would warrant.

Abstracting for simplicity from any price considerations, equilibrium in the balance-of-payments is approximated by equilibrium in trade, implying:

$$X = M \tag{1}$$

where X are exports and M stands for imports. In reality, we know that the real exchange rate plays a non negligible role in the dynamics of trade. We will briefly come back to this point later by referring to some of the main contributions on the relationship between growth and relative prices in this framework.

Suppose that the following traditional functions for exports and imports hold:

$$\begin{aligned} X &= X\left(Y^*\right), \quad X_{Y^*} > 0 \\ M &= M(Y), \quad M_Y > 0 \end{aligned} \tag{2}$$

where Y^* and Y correspond to foreign and domestic output, respectively.

Substituting Equation (2) into Equation (1) and log-differentiating, we have that the rate of growth compatible with equilibrium in the balance-of-payments, g_{BP}, is given by[1]:

$$g_{BP} = \rho \frac{\dot{Y}^*}{Y^*} \tag{3}$$

where

$$\rho = \frac{\varphi}{\pi} \gtreqless 1$$

corresponds to the ratio between the income elasticities of exports, $\varphi = \frac{\partial X}{\partial Y^*}\frac{Y^*}{X}$, and of imports, $\pi = \frac{\partial M}{\partial Y}\frac{Y}{M}$. When $\rho > 1$, the economy is growing faster than its trade partners. In the case of a developing country, this means it is catching-up with the rest of the world. On the contrary, $\rho < 1$ suggests a process of falling-behind. Hence, ρ can be understood as a measure of non-price competitiveness, something that has proved to explain international growth rate differences effectively.

In the past 40 years, this set-up has been generalised in a number of directions, for example, by incorporating capital flows (see Moreno-Brid, 2003; Thirlwall & Hussain, 1982) or by introducing some initial assessments of technology and institutions (e.g. Cimoli, 1988; Fagerberg, 1988) and multisectoral issues (Araújo & Lima, 2007). The latter became the starting point for a fruitful empirical literature on the law. Building on a Pasinettian framework to explain uneven development, Araújo

[1]For any generic variable x, the time derivative is indicated by $\dot{x}$, while $\dot{x}/x$ corresponds to its rate of growth. On the other hand, the derivative of any generic function $x(\tau)$ will be indicated as $dx/d\tau = x_\tau$.

and Teixeira (2004) disaggregate trade into *n* sectors such that non-price competitiveness responds to both the dynamics *between* and *within* different production activities. Higher growth rates can be achieved by moving the production structure towards sectors with higher income elasticity of exports relatively to imports as well as by changing the respective elasticities of each sector. In the first case, there is a progressive structural change in the composition of trade. In the second, the economy benefits from a broad process of technological upgrading which results from the assimilation and adaptation of new technologies.

2.1 | A note on prices and the real exchange rate

By allowing exchange rates to influence trade performance, Thirlwall (1979) shows g_{BP} depends on the rate of change of relative prices. Under purchasing power parity, however, prices cannot continuously appreciate or depreciate, thus providing an initial justification for our choice to abstract from this element here. Still, one should recognise that the existing empirical evidence suggests a positive association between the real exchange rate level, RER, and economic growth, especially in developing countries. This correspondence appears to be such that an undervalued currency favours growth while overvaluation hurts it (e.g. Rodrik, 2008; for a literature review see Rapetti, 2020).

In fact, a number of scholars have formalised the hypothesis that ρ directly responds to the level of relative prices (see Oreiro, 2016). Alternatively, authors such as Ribeiro et al. (2016), suppose non-price competitiveness mainly depends on technological and distributive variables, which in turn are a non-linear function of RER. Oreiro et al. (2020) adopt a similar route assuming that ρ is crucially determined by the share of modern manufacturing industries in trade, the latter being a function of the exchange rate. The story is basically the same in Missio et al. (2017) who, in a disaggregated framework, propose that the sectorial income-elasticities of exports and imports respond to relative prices. A differentiation between tradable and non-tradable sectors that formally addresses the problem of hidden unemployment is presented in Razmi et al. (2012).

Though we recognise the importance of relative prices to the field of development macroeconomics, the model presented in the next section abstracts from this element. Our motivation is twofold. First, we are interested in a possible 'conciliation' between the two interpretations of Thirlwall's law and the fundamental element that differentiates them does not depend on the behaviour of the RER. Adding this element would significantly complicate the modelling strategy without adding much to our main purpose. Second, multiplier-accelerator models of the type developed in the present paper do not usually give central attention to prices. If we convince the reader of the importance of the idea at hand, future extensions should provide a comprehensive treatment of the RER.

2.2 | External constraint or 'centre-of-gravity'?

It goes beyond the scope of this paper to provide a reconstruction of the history of growth and cycle theories (for this purpose, see Punzo, 2009). For us, it is enough to spend a few words on a basic stylised fact of modern economies: they grow at varying rates over time. Figure 1A plots, in black, the output growth rate of the global economy since the 1960s, g. We can see that most of the time, $g > 0$, though it alternates periods of acceleration and of economic recession. The problem of how to model such dynamics has motivated several scholars in the past decades. A traditional approach assumes it is possible to separate cyclical or short-term dynamics from what happens in the long run, that is, output trends. That would correspond to diagrams (B) and (C) obtained by applying the

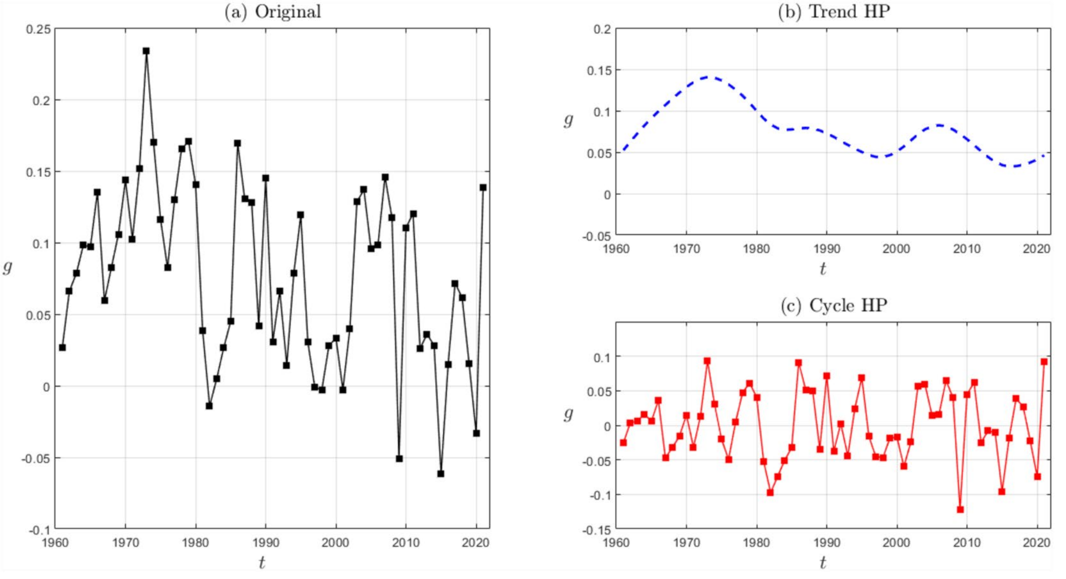

FIGURE 1 Global GDP growth rates. The reader is referred to the web version of this article for interpretation of colour references of this figure. [Colour figure can be viewed at wileyonlinelibrary.com]

Hodrick–Prescott (HP) filter on g. Such a disaggregation is part of undergraduate textbooks that use the IS-LM framework for the short-term and a Solowian setup for growth paths. We can also think of a generic Dynamic Stochastic General Equilibrium model for short run analysis. In contrast, an endogenous or semi-endogenous growth model is used to explain the long-term. Analogous distinctions are made among alternative theories of growth and distribution. In that context, Thirlwall's law is supposed to refer to the blue dotted line in panel (B), with no formalised connection to the business cycle in panel (C).

The challenges created by globalisation have led to a new and ongoing wave of studies that incorporate into this framework topics such as ecological sustainability (see Althouse et al., 2020; Guarini & Porcile, 2016), a deeper assessment of the role of technical change and innovation (e.g. Cimoli et al., 2019; Cimoli & Porcile, 2014; Setterfield, 2011), micro-foundations from an Agent-Based Modelling perspective (Dosi et al., 2019, 2020), endogenous aspects of institutional change (Dávila-Fernández and Sordi, 2020; Porcile & Sanchez-Ancochea, 2021), as well as gender issues (as in Seguino, 2010; Seguino & Setterfield, 2010). Either the trade equations are modified to take into account a particular element of interest, or non-price competitiveness is directly endogenised with respect to a vector of explanatory variables.[2] Frequently, scholars have adopted the traditional view previously mentioned in which the rate of growth compatible with equilibrium in the balance-of-payments is a *binding constraint* that results in a stable attractor.

For example, authors such as Ribeiro et al. (2017) explicitly separate time frames. A certain model is assumed for the short-term, another for the medium-term, and finally, in the long run, growth adjusts to the external constraint, which may or may not be endogenous to other macroeconomic variables (see also Cimoli & Porcile, 2014; Porcile & Lima, 2010). Some models allow for a dynamic

[2]Thirlwall (2011) describes the historical antecedents of the model and revisits some of the main theoretical and empirical contributions. The rapid progress of computational tools and statistical packages has recently led to a large increase in the number of studies in the field that need to be properly assessed and systematised. Providing a comprehensive review of these developments, however, goes beyond the scope of this paper. For a recent survey, see Blecker (2022).

adjustment towards equilibrium in the balance-of-payments while others simply take it for granted as a long run condition. This can also be seen in empirical contributions that either apply cointegration or GMM estimators, both designed to capture the existence of a relationship of that kind (e.g. Gouvêa & Lima, 2010; Romero & McCombie, 2016, 2018).

On the other hand, it has been argued that a comprehensive macroeconomic theory should contemplate the possibility of endogenous fluctuations as part of balance-of-payments constrained growth framework. In the case of Spinola (2020, 2021), cycles have a decreasing amplitude. Nishi (2019) demonstrated the emergence of a stable orbit while Dávila-Fernández (2020) showed that fluctuations might be persistent and irregular. The main underlying assumption is that panels (B) and (C), in Figure 1, are artificial constructs; one should aim at modelling diagram (A) directly. Besides the mechanisms underlying the interaction between labour market, the goods market, and international trade—which always contemplate an endogenous natural rate of growth—the crucial element in these studies is a subtle reinterpretation of Thirlwall's law as a kind of '*centre-of-gravity*'. Thus, growth is not properly constrained by the balance-of-payments, but it actually oscillates around g_{BP} without ever reaching a state of rest because trend and cycle are indissolubly fused. This idea has also been explored from an empirical point of view by Garcimartin et al. (2016), who explicitly propose the development of a balance-of-payments constraint theory for business cycles (see Kvedaras et al., 2020).

Figure 2 provides an additional graphical example of our previous discussion with $g_{BP} = 0.025$ indicated by the dotted blue line. Panel (A) depicts the more traditional scenario in which the rate of growth compatible with equilibrium in the balance-of-payments works as a stable attractor. In the very short-term, the economy might grow faster or slower than that rate. In the long-term, however, a country cannot sustain increasing current account imbalances, hence, growth will converge towards the rate determined by the external constraint. Panel (B) corresponds to the alternative scenario in which the economy fluctuates around g_{BP}, but only satisfies it on average. This difference has important theoretical and policy implications.

There is a certain convergence in terms of policy insights indicating the need to articulate a set of industrial policies targeting R&D investments, the development of firms' innovative and adaptive capabilities, and the design of pro-development institutions. While scholars in both positions accept that international specialisation reflects the economic complexity of countries' productive structures, those who adopt a 'centre-of-gravity' approach might be overestimating the importance of trade performance. In this way, they run the risk of inducing policymakers to underestimate the role of other components of aggregate demand. On the other hand, the separation between time frames might be also misleading. As long as international conditions are relatively favourable—or the country has enough foreign reserves—a pure binding constraint approach implies that policymakers do not have to worry about g_{BP} when facing more 'urgent' needs. This is because such a rate is associated with slow-motion processes of structural change. However, this conclusion may wrongly downplay the relevance of equilibrium in the current account for macroeconomic stability.

3 | A POSSIBLE HICKS-GOODWIN SOLUTION

Our brief assessment of the recent literature on the balance-of-payments constrained growth model suggests some overlapping between the two interpretations. While the difference could sound subtle, it has important implications for the way we understand the dynamics of the economy. In this section, we argue that the two views may actually be complementary rather than substitute. The solution we propose goes back to what we consider to be the deep roots of Hicks's and Goodwin's thinking as well as their lifetime contribution to economics: (i) Hicks' multiplier-accelerator as a piece-wise linear

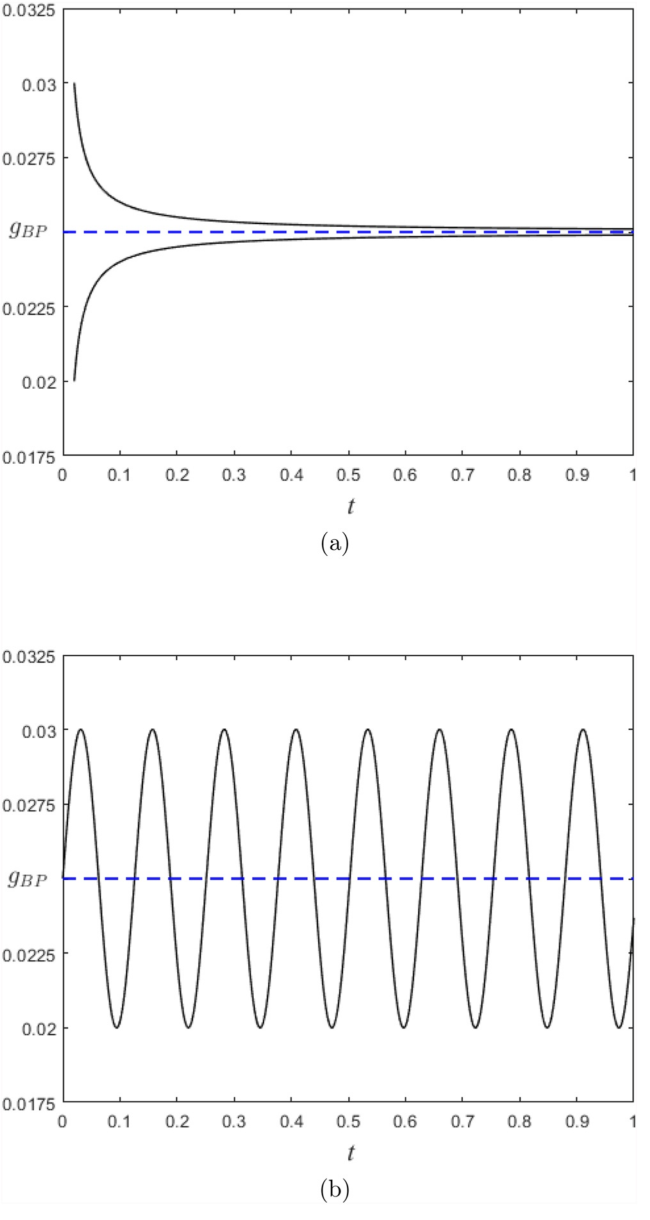

FIGURE 2 Binding constraint versus 'centre-of-gravity'. A generic illustrative example for $g_{BP} = 0.025$. Panel (A) stands for the first case, while panel (B) corresponds to the second. [Colour figure can be viewed at wileyonlinelibrary.com]

representation of the economy that allows growth processes to be coupled with the business cycle (see Hicks, 1950), and (ii) Goodwin's understanding that growth and cycles are indissolubly fused, including his recognition of the fundamental role of time-lags in investment decisions (e.g. Goodwin, 1951, 1967).[3] Our model introduces Thirlwall's law into this setup by implicitly endogenising foreign capital

[3]The presence of these elements in much of Kalecki's work stands as an interesting connection between the three scholars. Besides his reference to the need to pursue a growth-cycle modelling approach (Kalecki, 1968), in many aspects, Kalecki (1935) anticipated the idea of the multiplier-accelerator that later became famous with the names of Samuelson and

inflows. International liquidity is assumed to impose a ceiling on accumulation that is a multiple of the balance-of-payments equilibrium growth rate.

3.1 | The model

In an open economy, the expenditure identity is given by:

$$Y = C + I + G + X - M$$

where C stands for consumption, I is investment, and G corresponds to government expenditures. For our purposes, Y can be divided into two main components:

$$Y = Q(Y) + Z, \;\; 0 < Q_Y < 1 \tag{4}$$

that is, induced demand, $Q(\cdot)$, and non-induced expenditures, Z. Scholars more aligned with Kaldor's ideas may argue that exports are the only truly autonomous component of demand. The so-called 'neo-Kaleckians', on the other hand, would perhaps assume that entrepreneurs' 'animal spirits' make investment partially non-induced. Others have made the case that government consumption is somehow semi-autonomous. For our purposes, the model is compatible with different sources of induced and non-induced demand that may or may not grow at the same rate.

Applying the implicit function theorem, Equation (4) can be rewritten as:

$$Y = H(Z), \;\; H_Z > 1 \tag{5}$$

where $H(\cdot)$ incorporates the Keynesian multiplier effect. This is a well-known textbook result that does not require a lengthy explanation. The level of demand is an increasing function of its non-induced component.

Log-differentiating Equation (5), we obtain:

$$\frac{\dot{Y}}{Y} = \varepsilon \frac{\dot{Z}}{Z}$$

where $\varepsilon = \frac{\partial H}{\partial Z}\frac{Z}{H(\cdot)} > 0$ is the elasticity of aggregate demand with respect to its autonomous component, standing for the dynamic Keynesian multiplier. Notice the similarities with the trade-multiplier in Equation (3). If none of the domestic absorption components is autonomous, the economy will be export-led. Of course, as pointed out by one of the reviewers, such similarity is not a surprise as both share directly or indirectly the aggregate demand identity as starting point. To simplify notation, redefine $\dot{Y}/Y = g$ and $\dot{Z}/Z = z$ so that the expression above becomes:

$$g = \varepsilon z \tag{6}$$

Using the properties of the inverse function, we can also express the dynamic multiplier as $\varepsilon = \frac{Z/Y}{1-Q_Y}$. In the special case in which $Q(\cdot)$ is a linear function, we have that $\varepsilon = 1$. However, in the real world, $Q(\cdot)$ is likely to be highly nonlinear. Empirical evidence suggests multipliers are

Hicks, including the empirical relevance of an implementation lag in investment decisions. Still, as one of the reviewers requested, we address our proposal only as a Hicks-Goodwin possible solution. Her/his request seems to be motivated by our model being closer to Franke's (2018) recent revisitation of Kalecki's business cycle model in a growth context than to the 'neo-Kaleckian' model for open economies (as in Blecker, 1989, and subsequent contributions).

counter-cyclical, being lower than one in the expansive phase of the cycle and higher than the unity during recessions (e.g. Fazzari et al., 2015). Hence, let us rewrite this elasticity as:

$$\varepsilon = 1 + \phi \tag{7}$$

where $\phi \gtreqless 0$ captures the magnitude of the multiplier effect.[4]

We understand ϕ as to be determined by two main forces. On the one hand, we recognise that firms see their prospects improve when demand is accelerating, and thus plan new investment projects, A. In modelling growing economies, the determination of investment refers to relative rather than absolute changes in the capital stock. On the other hand, one should take into account that, ceteris paribus, growth implies a cost of adjusting the current stock of machinery and equipment, B. Accordingly:

$$\phi = A - B \tag{8}$$

While firms respond to an increase in the rate of growth of demand by augmenting investment in fixed capital, this process does not continue indefinitely. As the rate of growth accelerates, there is an increase in import requirements that, ceteris paribus, results in a trade-deficit. Given that, in the long run, a country that trades in foreign currency cannot sustain increasing balance-of-payments imbalances, g_{BP} defines the relevant ceiling. When international conditions are favourable, the external constraint is less stringent and the economy can handle greater trade deficits. In formal terms:

$$A = \min\{v\dot{g}, \gamma g_{BP}\} \tag{9}$$

where $v > 1$ is the response of the piecewise-linear function $A(\cdot)$ to variations in the rate of growth of demand before reaching the ceiling, that is, the accelerator effect. The stringency of international credit markets is captured by $\gamma > 0$, with a high value of this parameter standing for periods of high liquidity.

One of the immediate concerns regarding the original formulation of Thirlwall's law was by how much growth could deviate from g_{BP} as defined in Equation (3). Thirlwall and Hussain (1982) address the issue in a rather elegant way, but it was then Moreno-Brid (2003) who imposed a long-term constraint defined as a constant ratio of the current account deficit and income. Most studies interested in the behaviour of current account deficits have followed a similar route. Minor variants can be found in Bhering et al. (2019), who redefine the external constraint in terms of the capacity to export. Our approach is related to this literature to the extent that we take the constraint as a multiple of g_{BP}.[5]

We could also think of γ as related to foreign exchange reserves and the level of confidence in financial markets. High reserves limit external vulnerability to shocks during times of crisis or when access

[4]It is interesting to notice that Equation (6) is compatible with Freitas and Serrano (2015) and the so-called 'super-multiplier'. After imposing a number of linear functional forms, they basically suppose that ε is equal to one plus the variations in the investment-output ratio (for a recent presentation referring to Thirlwall's law, see Morlin, 2022). Of course, the Hicksian multiplier-accelerator model with a 'floor' and/or a 'ceiling' continues to be the most successful mechanism generating persistent fluctuations (e.g. Gallegati et al., 2003; Sordi, 2006; Sushko et al., 2010). As it will become clear later, our approach has several similarities with this last set of contributions, even though it is conceived for an open economy.

[5]The recent European debt crisis has demonstrated that when internal imbalances are out of hand, they may constrain output and have implications for economic growth in a severe way. In this respect, Soukiazis et al. (2012, 2014), building on Thirlwall's law, have presented a growth model that takes into account both internal and external imbalances. The crucial step consists in disaggregating imports into the various components of aggregate demand. For an assessment of the interplay between capital flows, terms-of-trade and the exchange rate, see Pé rez-Caldentey and Moreno-Brid (2019). In addition, a comparison between wage-led and debt-led regimes in a balance-of-payments constrained economy is investigated by Pé rez-Caldentey and Vernengo (2017).

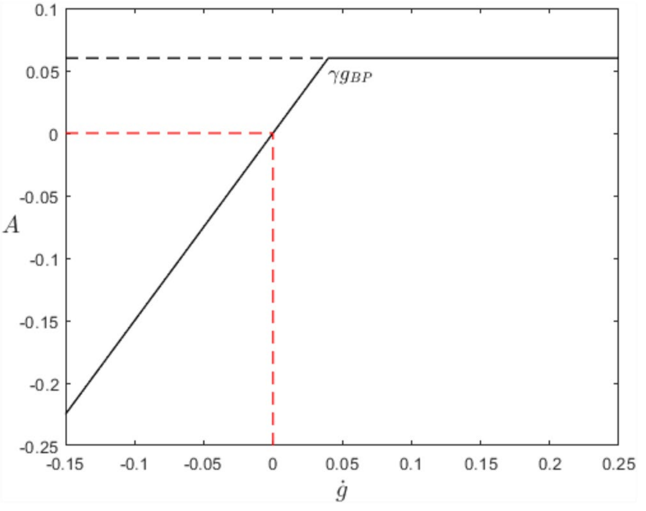

FIGURE 3 A piece-wise linear function for the accelerator effect with only an external constraint ceiling. In this example, we assume parameter values $v = 1.5$, $\gamma = 2$, and $g_{BP} = 0.03$. [Colour figure can be viewed at wileyonlinelibrary.com]

to borrowing is curtailed. In this way, they make it possible to have a less-binding ceiling on accumulation. Figure 3 provides a graphical representation of Equation (9). For values of $\dot{g} \leq \gamma g_{BP}/v$, $A(\cdot)$ is a linear increasing function of $\dot{g}$ with a slope equal to v. After the ceiling is reached, $\dot{g} > \gamma g_{BP}/v$, external conditions impose a constraint on capital accumulation. For example, $\gamma = 2$ implies international markets allow growth to accelerate in principle up to twice the size of the rate compatible with equilibrium in the balance-of-payments.

On the other hand, the cost of adjusting the capital stock to growth is simply:

$$B = \beta \dot{g} \tag{10}$$

where $0 < \beta < 1$ guarantees that there is actually an accelerator effect after all. Equation (10) can be motivated under two different scenarios. First, suppose the economy consists of a representative monopolistic firm. In that case, production is growing at a certain pace, as previously planned by the its managers. Hence, only variations in the rate of capital accumulation require a somehow unexpected restructuring of the production line. The firm is ready to handle a specific rate of expansion in the scale of production. However, an acceleration of growth implies an additional 'fee' either because there are unplanned changes in the composition of tangible and intangible assets or as a consequence of unintended labour and legal problems, among other issues. Alternatively, we could assume oligopoly markets that may be more or less competitive. In this case, a certain growth rate is compatible with a given number of firms entering and exiting the market every period. Thus, the initial result of a faster growth rate is a temporary mismatch between them. This fact implies inefficiencies at the economic system level, with a related adjustment cost for the economy as a whole.

The idea of a lag in investment decisions goes back to Kalecki (1935) and was later taken up by Goodwin (1951) in his nonlinear accelerator model (for a modern revisitation, see Sordi, 2006; Franke, 2018). Indeed, productive capacities are not created instantaneously. The period that elapses from the moment that firms make a decision to expand their capital until the respective additional plant or equipment is in place and ready for production corresponds to the implementation lag, $\theta > 0$. To use Kalecki's terminology, investment goes through a 'gestation period'. This means that $A(\cdot)$ depends on decisions taken at time $t - \theta$ while $B(\cdot)$ happens in t. Substituting Equations (9) and (10) into Equation (8), and taking account of the previous discussion, the magnitude of the multiplier effect can be rewritten as:

$$\phi(t;\theta) = \min\{v\dot{g}(t-\theta), \gamma g_{BP}\} - \beta\dot{g}(t) \tag{11}$$

In order to show that Harrodian instability is not being artificially tamed by the inclusion of lags, we impose:

$$v > \beta + \theta/z$$

so that firms strongly respond to demand signals.

Substituting Equation (11) into Equation (7), we obtain:

$$\varepsilon(t;\theta) = 1 + \min\{v\dot{g}(t-\theta), \gamma g_{BP}\} - \beta\dot{g}(t) \tag{12}$$

such that the dynamic Keynesian multiplier responds with a time-lag. Such a result is in line with evidence indicating ε is counter-cyclical. Increases in the growth rate at t reduce ε in the same period because B costs are paid upfront. A similar argument can be made the other way around. A reduction in output growth is related to an immediate increase in the multiplier, given that firms do not have to pay a series of adjustment costs any longer. The accelerator effect is felt only afterwards as a result of the gestational period in induced investment. An acceleration of growth eventually magnifies the multiplier effect up the point at which the external constraint binds. The respective adjustment costs are always paid after the decision to add a new unit of capital is made.

Introducing Equation (12) into Equation (6) and rearranging, it follows that:

$$z\beta\dot{g}(t+\theta) + g(t+\theta) - z\min\{v\dot{g}(t), \gamma g_{BP}\} = z \tag{13}$$

At this point, we can either work with a delayed differential equation or use a Taylor approximation. We choose the second alternative because it was the strategy followed by Goodwin (1951, p. 12). The procedure consists in expanding in a Taylor series the leading elements $\dot{g}(t+\theta)$ and $g(t+\theta)$, and dropping all but the initial two terms in each:

$$\begin{aligned} \dot{g}(t+\theta) &\approx \dot{g}(t) + \theta\ddot{g}(t) \\ g(t+\theta) &\approx g(t) + \theta\dot{g}(t) \end{aligned} \tag{14}$$

Substituting Equation (14) into Equation (13) and dividing the resulting expression by $z\beta\theta$, we obtain a non-homogeneous piece-wise second-order differential equation:

$$\ddot{g} + F(\dot{g}) + G(g) = \frac{1}{\beta\theta} \tag{15}$$

where

$$F(\dot{g}) = \left(\frac{1}{\theta} + \frac{1}{z\beta}\right)\dot{g} - \frac{\min\{v\dot{g}, \gamma g_{BP}\}}{\beta\theta}$$

and

$$G(g) = \frac{g}{\beta\theta z}$$

For simplicity, given that Equation (15) only depends on variables in t, we omit it from the expression.

In steady-state, $\dot{g} = \ddot{g} = 0$. Thus, the equilibrium rate of growth of the economy, $\bar{g}$, is determined by the rate of growth of autonomous demand:

$$\bar{g} = z$$

If we accept that exports are the only truly non-induced component of aggregate demand and that there is equilibrium in the current account, then $\bar{g} = g_{BP}$. The model is, nonetheless, perfectly compatible with other non-capacity generating autonomous sources, that do not need to grow at the same rate, so that $\bar{g} \gtreqless g_{BP}$. We show, in the Mathematical Appendix, that Equation (15) can be reduced to a dimensionless form and transformed into a so-called *Lord Rayleigh type equation*, which supports persistent and bounded fluctuations. It consists of two distinct phases per cycle—one slow and one fast—and will be referred to in the rest of the paper as a two-phase oscillator. The equilibrium point is unstable, in line with the Harrodian principle of instability, but an oscillator emerges due to the ceiling in the piece-wise function for capital accumulation.

The main structure of the model is very close to Goodwin (1951) flexible-accelerator, with the main modification being that we are now in the context of a growing open economy. From a mathematical point of view, the mechanism generating endogenous and persistent fluctuations is fundamentally the same. Still, we bring some new economic insights into the growth cycle dynamics and the possible conciliation between the *external constrained* and *'centre-of-gravity'* views of Thirlwall's law. During the expansive phase, firms produce more in response to increasing demand. They also need to expand productive capacity, thus raising the rate of capital accumulation, inevitably increasing import requirements because single economies do not operate in autarchy. As output accelerates, so do imports relative to foreign demand. Eventually, the country runs a trade deficit, which only means it grows faster than the rate compatible with equilibrium in the balance-of-payments. Of course, that is not a problem because the economy can finance the deficit through international credit markets.

Nonetheless, international liquidity imposes a ceiling on capital accumulation that is a multiple of g_{BP}. While investment projects take time to be executed, the cost of adjusting the capital stock to growth will initiate the contractionary phase of the cycle. Firms are forced to reduce production and cancel investment plans as aggregate demand falls. The reduction in imports that follows allows for a correction of the trade deficit. Moreover, the time lag in investment decisions works in both ways, meaning that this time there is no adjustment cost to be paid, and, on the contrary, we observe an improvement in firms' balance sheets. Such a mechanism will allow a recovery of economic activity, reinitiating the cycle. The equilibrium point is unstable because the multiplier-accelerator implies that the economy strongly responds to demand signals. Still, Thirlwall's law works as an external constraint sufficient to generate the growth cycle around it. We will return to this description in the next section.

3.2 | A numerical example

We rely on numerical simulations to confirm our model generates periodic persistent oscillations of the variable and better appreciate its economic properties, as well as the robustness of our results with respect to different scenarios. As a note of caution, it is important to highlight that we are not calibrating a real economy. Using a sample of European countries, Smets and Wouters (2003) estimated $\beta \in [0.15; 0.25]$. Here, we adopt their more conservative coefficient. For the accelerator, we follow the magnitudes indicated in Gallegati et al. (2003) and Sushko et al. (2010). We assume that autonomous demand in this representative economy grows annually at 3%. Finally, θ was chosen using a trial-and-error approach, making adjustments so as to obtain trajectories that are economically meaningful:

$$\beta = 0.15, \quad \theta = 0.03, \quad z = 0.03, \quad v = 1.5$$

while the rate of growth compatible with equilibrium in the balance-of-payments was supposed to be lower than the rate of growth of demand:

$$g_{BP} = 0.0275, \quad \gamma = 2$$

such that by all means, there is a binding external constraint. Parameter $\gamma = 2$ indicates that international markets allow the growth rate of output to accelerate up to 2 times the size of g_{BP}. Results are fundamentally the same if we assume higher international liquidity.

Figure 4 shows the emergence of a clockwise periodic oscillator in the $(g, \dot{g})$ phase space. The dotted blue line corresponds to Thirlwall's law while the red one is the rate of growth of the autonomous components of aggregate demand. Panel (A) brings a surprising result. Fluctuations are asymmetric with respect to the equilibrium solution. In fact, g_{BP} works *as if* it were the 'centre-of-gravity' of the economy with a clear orbit around it. We know that by construction this is not the case. The economy never comes to a state of rest, and neither g_{BP} nor z can be considered stable attractors, though the emerging orbit is. Plotting the time series, as in panel (B), reveals the role of the ceiling in 'constraining growth'.[6]

Suppose instead:

$$g_{BP} = 0.03$$

that is, the rate of growth compatible with equilibrium in the balance-of-payments is equal to the actual growth rate of non-induced demand. As previously discussed, this is equivalent to assuming that there is equilibrium in the current account, domestic absorption is fully induced and only exports are autonomous. Panels (C) and (D) illustrate such case. Given that $z = g_{BP}$, red and blue dotted lines overlap. Still, the oscillator is there, giving rise to asymmetric persistent fluctuations. An external observer would have the impression that growth is limited by the external constraint, though in fact it is also the 'centre-of-gravity' of the economy.

A final example is the case in which, for instance:

$$g_{BP} = 0.0325$$

We are acknowledging again that other components of demand such as government expenditures, for example, might be non-induced. However, $z < g_{BP}$ implies that the country in question is accumulating reserves in foreign currency. In panel (E), we have that Thirlwall's law is completely outside the orbit. In fact, time-series reported in panel (F) show g_{BP} is significantly above the peak of the cycle. Notice that the dotted blue line appears now on the right of the red one. While a reader might argue that, in this last scenario, the external constrained does not work as a 'centre-of-gravity', the model indicates trade-cycles emerge only because of the existence of the trade-multiplier. Recall that we are assuming a strong accelerator effect such that Harrodian instability is not artificially tamed. The equilibrium point $\bar{g}$ is unstable in

[6]A careful analysis of the underlying second order differential equation reveals that the emerging periodic attractor corresponds to the so-called two-phase oscillator. Providing a complete mathematical assessment of its properties goes beyond the scope of this paper. In an ongoing research project (Sordi & Dávila-Fernández, 2022), we present a graphical and analytical treatment of this issue. Here, it is enough to highlight that it is sufficient to have a ceiling in the piece-wise accelerator function to obtain the asymmetric cycle. For a formal presentation of this family of oscillators, the reader is invited to see Sordi (2006).

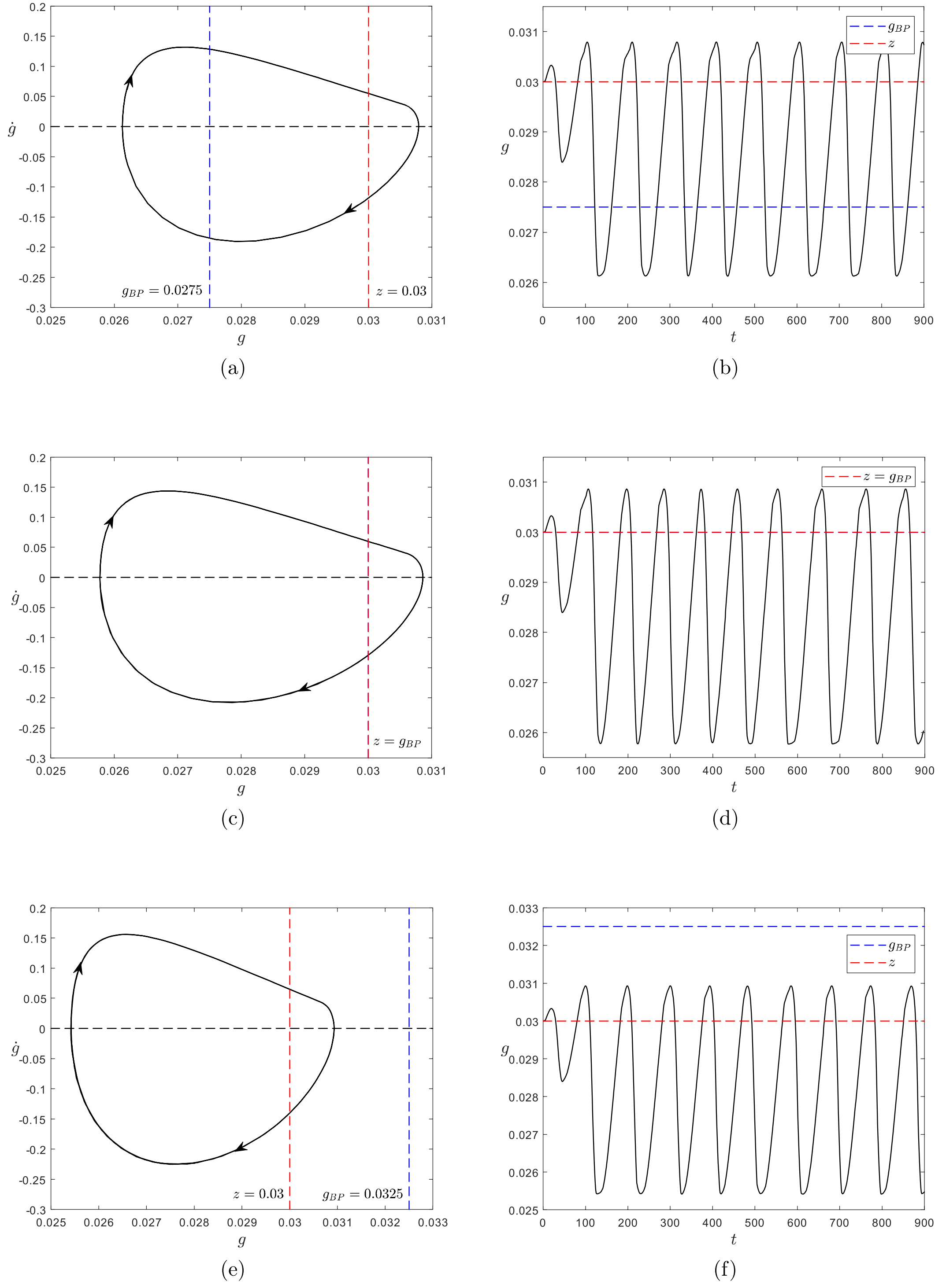

FIGURE 4 Two-phase clockwise oscillations when $z > g_{BP}$, panels (A–B), $z = g_{BP}$, panels (C–D), and $z < g_{BP}$, panels (E–F). The reader is referred to the web version of this article for interpretation of colour references of this figure. [Colour figure can be viewed at wileyonlinelibrary.com]

the absence of the ceiling imposed by international credit conditions on capital accumulation, γg_{BP}. Oscillations result from g_{BP} constraining growth, with no need of a floor to control for Harrodian instability.

The structure above allows us to differentiate between two primary forces. The first consists in Harrodian instability arising from the multiplier-accelerator. An increase in the growth rate of output leads firms to project higher sales and respond by increasing capital accumulation. New investment projects need some time to become concrete and, therefore, the dynamic Keynesian multiplier rises with a time-lag giving a further impulse to output growth. On the other hand, adjustments to the capital stock are costly. The higher the accelerator, the higher the cost for the individual firm and the economic system. This cost is paid without a lag and brings down the Keynesian multiplier, reducing output growth. Persistent fluctuations emerge from the interaction of these two mechanisms intermediated by Thirlwall's law. The external constraint is responsible for establishing a ceiling for the acceleration of capital accumulation:

$$\uparrow \quad g \Rightarrow \uparrow A_{t-\theta} \Rightarrow \uparrow \varepsilon \Rightarrow \uparrow g$$

$$\uparrow \quad g \Rightarrow \uparrow B_t \Rightarrow \downarrow \varepsilon \Rightarrow \downarrow g$$

Given the importance of θ, v, and γ to the emergence of the oscillator, we proceed by accessing its robustness. We are also interested in grasping the economic intuition of changes in the size of the orbit that might follow from variations in those parameters.

3.3 | Robustness checks

One may wonder what is the impact of an increase in the investment implementation time-lags. We know θ has no effect over the equilibrium rate of growth. Nonetheless, a first intuition from Equation (11) is that higher gestation periods are related to business-cycles of lower amplitude. This basically comes from the fact that accumulation costs are paid in t while the decision to invest is made at time $t - \theta$. Figure 5 compares the size of the orbits when

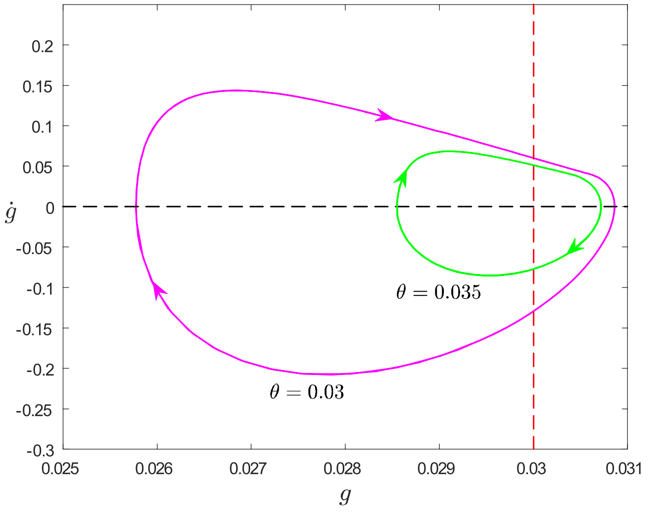

FIGURE 5 Business cycle amplitude under different time-lags in capital accumulation. The reader is referred to the web version of this article for interpretation of colour references of this figure. [Colour figure can be viewed at wileyonlinelibrary.com]

$$\theta = 0.035$$

in green, and

$$\theta = 0.03$$

in magenta, confirming our previous insights. For convenience, we only present the case in which $z = g_{BP}$ but the result is fundamentally the same for $z \gtrless g_{BP}$. The intersection between the dotted black and red lines corresponds to the equilibrium point.

From an economic point of view, it makes sense that more complex projects require additional care in preparation and execution, which in their turn are related to a higher θ. Such a characteristic brings resilience in the form of higher inertia in the system. It is exactly the role of time-lags that makes the multiplier to be counter-cyclical. As shown in Figures 4 and 5, $\dot{g}$ increases (falls) hand in hand with a falling (increasing) g. Altogether, ρ and the investment time-lags stand for a fundamental difference between developed and developing economies and help to explain why growth is more volatile in the latter—or, to use the *Financial Times* terminology, to understand the occurrence of 'chicken flight' growth episodes. The combination of $\rho < 1$ and a relatively low θ results in an economy that is highly volatile and falls behind, even when the level of confidence of international financial markets is high.

Figure 6 shows the model's response to variations in the accelerator effect and international liquidity. Not surprisingly, a stronger multiplier-accelerator is associated with fluctuations of higher amplitude. The more substantial firms respond to demand impulses, output, and capital accumulation will reach the limits imposed by the external constraint more violently. Thus, the recession necessary to bring equilibrium in the balance-of-payments will also be stronger, as seen in panel (A). On the other hand, more favourable international credit conditions paradoxically are associated with higher fluctuations. While this result might not be as intuitive as the previous one, it depends on the nature of this ceiling to investment. Less stringent credit markets mean that a country or region is allowed for a while to raise import requirements and get indebted on foreign currency. The higher γ, the greater the difference $g - g_{BP}$. Hence, aggregate demand will need to fall more strongly to correct the initial deviation, thus justifying the greater amplitude of the fluctuations, depicted in pink on panel (B). That may be what happened in the downward phase of the cycle in Brazil and Turkey after 2014. The first decade of the 2000s was marked by a period in which γ improved as financial markets were more liquid, these countries accumulated significant international reserves, and there was some optimism among foreign creditors. Under those conditions, our model predicts both stronger expansive and contractionary phases of the cycle.

4 | INTRODUCING THE SUPPLY-SIDE

The issue of the adjustment of z towards g_{BP} remains open and goes beyond the scope of this article. If the former consists only in the rate of growth of exports, there is no problem in the first place. Still, it is not difficult to present a basic structure for the supply-side of the economy, showing a simple way in which labour markets and labour productivity might guarantee that actual and natural rates of growth do not fall apart. To avoid creating somehow excessive expectation, it is worth stressing some reservation on the purpose of this section. We aim just to illustrate that similar properties of the cycles are retained whenever the model is augmented with some basic supply-side mechanism.

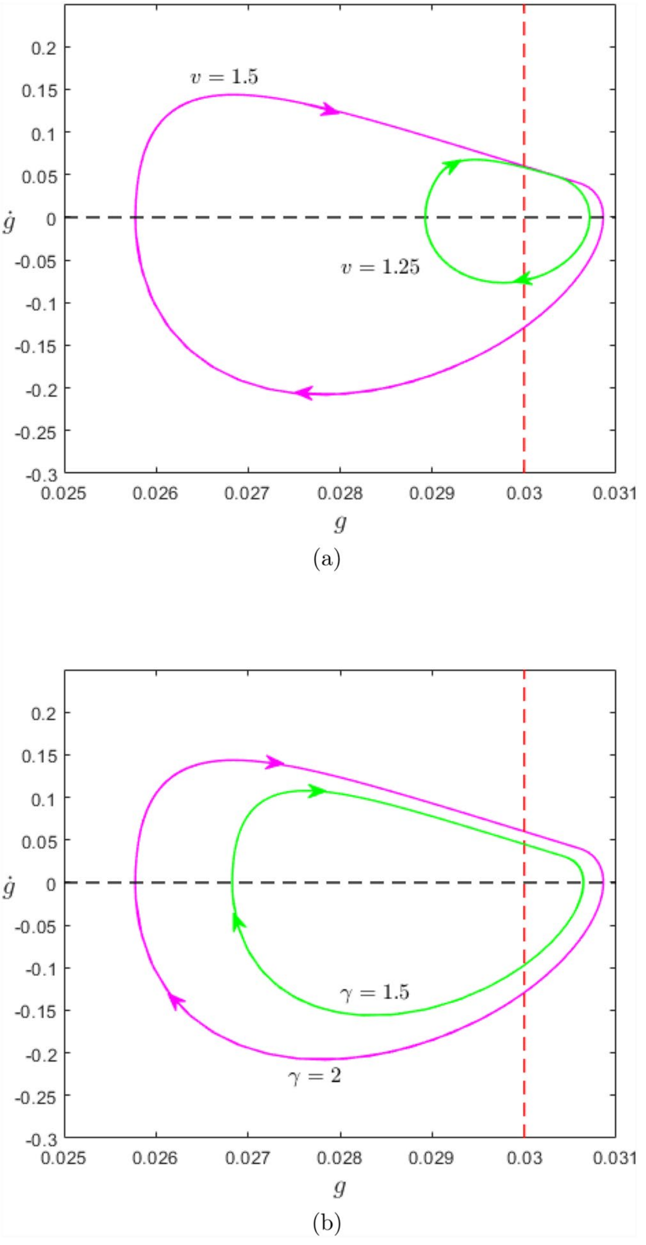

FIGURE 6 Robustness of growth-cycle to variations in the accelerator effect (v), panel (A), and international liquidity (γ), panel (B), when $z = g_{BP} = 0.03$ and $\theta = 0.03$. The reader is referred to the web version of this article for interpretation of colour references of this figure. [Colour figure can be viewed at wileyonlinelibrary.com]

4.1 | Labour productivity and employment rates

Let us consider the following Leontief production technology:

$$Y = \min\left\{ \frac{K}{\vartheta}, qNe \right\} \tag{16}$$

where K stands for the capital stock, ϑ is the capital-output ratio, $q = Y/L$ corresponds to labour productivity, with L indicating the number of workers employed, N is the labour force, and $e = L/N$ is the participation rate.

Dynamic efficiency requires:

$$\frac{\dot{\vartheta}}{\vartheta} = \frac{\dot{K}}{K} - g$$
$$\frac{\dot{e}}{e} = g - \frac{\dot{q}}{q} - \frac{\dot{N}}{N}$$

so that equilibrium in the goods and labour markets, $\dot{\vartheta}/\vartheta = \dot{e}/e = 0$, result in:

$$\frac{\dot{K}}{K} = g \qquad \frac{\dot{q}}{q} + \frac{\dot{N}}{N} = g \tag{17}$$

Capital accumulation directly follows the rate of growth of aggregate demand, as described throughout the paper. Persistent fluctuations in accumulation result from the interaction between the external constraint and the dynamic Keynesian multiplier. Moreover, we have that the so-called natural rate of growth, $\dot{q}/q + \dot{N}/N$, is equal to the actual rate of growth.

The profession has long recognised that, to a large extent, new technologies are capital embodied either in the form of tangible or intangible assets. We could think of such a mechanism in terms of the evolution of hardware and software components in computers. This means that, as the economy expands, the presence of dynamic economies of scale allows a faster increase in labour productivity through a process of learning-by-doing. Such a mechanism is frequently referred to as Kaldor-Verdoorn's law. We shall also acknowledge that a tight labour market has a positive effect on $\dot{q}/q$. An increase in the bargaining power of workers leads to increases in real wages relative to labour productivity, reducing profitability. Firms respond accordingly by increasing their search for labour saving techniques. Hence, the vector of explanatory variables includes, but is not limited to:

$$\frac{\dot{q}}{q} = P(g, e), \quad 0 < P_g < 1, \quad P_e > 0 \tag{18}$$

where P_g stands for Kaldor-Verdoorn's coefficient.

Substituting Equation (18) into Equation (17), and assuming the labour force grows at an exogenous rate, n, we obtain:

$$P(g, e) + n = g \tag{19}$$

Applying the implicit function theorem, it is easy to see that:

$$\frac{de}{dg} = \frac{1 - P_g}{P_e} > 0$$

In this way, we recover the positive correspondence between economic activity and participation rates well documented in the literature.

We suppose a linear specification for the rate of growth of labour productivity[7]:

$$P(g, e) = -\alpha_0 + \alpha_1 g + \alpha_2 e \tag{20}$$

Substituting Equation (20) into Equation (19) and solving for the employment rate:

$$e = \frac{(1 - \alpha_1)g + \alpha_0 - n}{\alpha_2} \tag{21}$$

we find that the employment rate in equilibrium is a function of the output growth rate. Furthermore, from Equation (20), we know that the labour productivity growth rate in steady-state is also a function of the dynamics of g. Notice, however, that output growth is governed by the oscillator Equation (15), which was shown to be compatible with persistent fluctuations. Thus, as g varies, so will e and $\dot{q}/q$. We report, in Figure 7, a graphical representation of the growth cycle in e and $\dot{q}/q$ when Equation (19) is satisfied. This very stylised introduction of the supply-side does not allow for feedback effects from the labour to the goods market but is enough to grasp how the latter impacts the former.

In our simulations, we used the following parameter values:

$$\alpha_0 = 0.05, \quad \alpha_1 = 0.5, \quad \alpha_2 = 0.0\bar{6}, \quad n = 0.015$$

which are in line with studies in the field (e.g. Fazzari & Gonzalez, 2023; McCombie & Spreafico, 2016; Tavani & Zamparelli, 2017). A stagnant economy that does not employ a significant share of its population will actually experience a reduction in q, that is, $P(0, 0) < 0$. This is in accordance with the idea of a deterioration in human capital and in the capacity for learning-by-doing. Oscillations in the $(\dot{g}, e)$ and $(\dot{g}, \dot{q}/q)$ phase space are anti-clockwise, as shown in Figure 7. During the initial part of the recovery, output accelerates leading to higher labour market participation. However, such a process does not go on forever. On the one hand, the external constraint imposes a limit to capital accumulation. On the other hand, strong workers continue pushing q upwards. At a certain point, growth slows down eventually reducing e. Depending on the strength of the Kaldor-Verdoorn effect, the rate of growth of labour productivity will fall accordingly, as depicted in both panels (A) and (B).

We should emphasise that the description just provided depends on Equation (19) being always satisfied. Since we are not taking into account the dynamic adjustment of employment to the difference between actual and natural rates of growth, the picture remains somehow incomplete. At this stage, there are no feedback effects from the labour market or income distribution to aggregate demand, which is also an important limitation. While recognising the importance of these issues, we leave the task of building a full model to future research (for attempts in that direction, see Perrotini-Hernández & Vázquez-Muñoz, 2019; or Dávila-Fernández, 2020). For the time being, it is sufficient to stress the importance of the message that the external binding constraint might be a kind of 'centre-of-gravity' of the economy. This is particularly clear when $g_{BP} \leq z$, that is, the rate of growth compatible with equilibrium in the balance-of-payments is lower or equal to the rate of growth of autonomous demand. The relevance of such a case, especially for developing economies, is quite evident.

[7]McCombie and Spreafico (2016) demonstrated that if a linear form of Eq. (18) is adopted, 'the intercept *cannot* and *should not* be interpreted as the separate contribution to economic growth of the rate of exogenous technical change' while 'the Verdoorn coefficient also *should not* be interpreted as a measure of increasing returns to scale per se' (p. 1131, emphasis added).

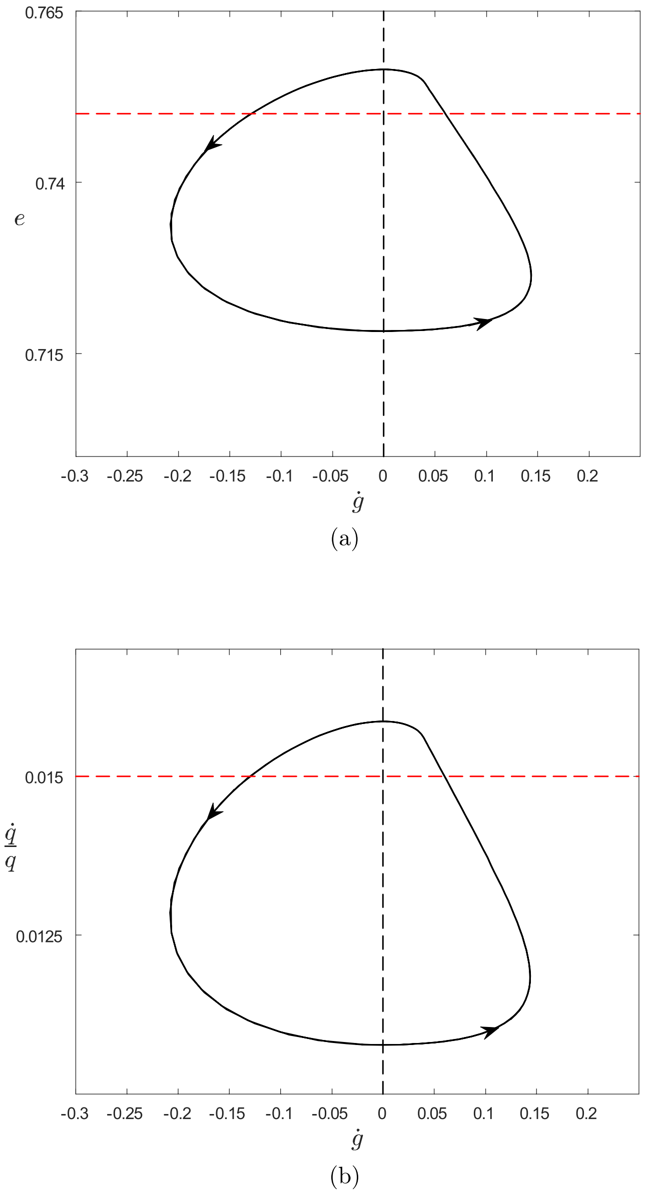

FIGURE 7 Participation rates, panel (A), and the rate of growth of labour productivity, panel (B), when $z = g_{BP} = 0.03$ and $\theta = 0.03$. [Colour figure can be viewed at wileyonlinelibrary.com]

4.2 | 'Endogenising' Thirlwall's law

This framework leaves plenty of space for endogenising g_{BP}. We left this step to the end because it does not influence the main motivation behind our proposed Hicks-Goodwin solution. Modelling the dynamics of non-price competitiveness is perfectly compatible with both the *binding constraint* and '*centre of gravity*' approaches. In our case, a time-varying balance-of-payments equilibrium growth rate is an interesting avenue of research because it comes with the recognition that oscillators of different frequencies coexist. The literature review in the previous sections already indicated a large number of contributions have explored how non-price competitiveness relates to innovation waves, institutional variables, and climate change, among others (for a survey of recent developments, the reader is once more referred to Blecker, 2022).

Given that our aim here is only to illustrate that growth-cycle dynamics are preserved whenever the model is augmented with basic supply-side elements, we allow Thirlwall's law to respond to the growth rate of labour productivity. Such a relationship can be motivated as in Setterfield (2011), appealing to the importance of non-price competition in international trade. The presence of dynamic economies of scale implies that firms use productivity improvements induced by growth via Kaldor-Verdoorn's law to raise the quality of output rather than cut costs or prices. For simplicity, we assume:

$$\rho = \rho_0 + \rho_1 \frac{\dot{q}}{q} \tag{22}$$

where ρ_0 captures all elements relevant to non-price competitiveness that are not explicitly taken into account, and ρ_1 is the part induced by labour productivity growth. As long as consumers value quality, Setterfield argues that the income elasticity of exports is sensitive to efficiency in the use of labour inputs. Empirical evidence supporting this intuition can be found in Romero and McCombie (2016, 2018), who show trade elasticities are higher for medium- and high-tech goods, sectors that usually concentrate productivity gains.

Substituting Equation (22) into Equation (3), we obtain g_{BP} as a function of $\dot{q}/q$. Plugging the resulting expression into Equation (9), the external constraint becomes:

$$A = \min\left\{ \upsilon \dot{g}, \gamma \left(\rho_0 + \rho_1 \frac{\dot{q}}{q} \right) \frac{\dot{Y}^*}{Y^*} \right\} \tag{23}$$

where the rate of growth of labour productivity follows the discussion of Equations (18)–(21). This way, we allow feedback mechanisms from the growth cycle to non-price competitiveness through labour productivity. A proper representation of such an interaction would require the distinction of oscillations with different frequencies, given that pure business cycle oscillations are different from innovation waves, for example, While we refrain from going in that direction to keep this presentation as simple as possible, Equation (23) comes with an important economic intuition. It includes an additional pro-cyclical element in the external constraint. As demand expands, firms respond by raising production. Because of dynamic economies of scale, there is an increase in the labour productivity growth rate through learning by doing. More productive workers deliver products of higher quality, improving non-price competitiveness. This result means that the balance of payments constraint growth rate increases, relaxing the external constraint on capital accumulation. A similar mechanism works in the opposite direction.

Using the new specification of function $A(\cdot)$, the resulting piece-wise second-order differential equation becomes:

$$\ddot{g} + F(\dot{g}, g) + G(g) = \frac{1}{\beta\theta} \tag{24}$$

where

$$F(\dot{g}, g) = \left(\frac{1}{\theta} + \frac{1}{z\beta} \right) \dot{g} - \frac{\min\left\{ \upsilon \dot{g}, \gamma [\rho_0 + \rho_1 (g - n)] \frac{\dot{Y}^*}{Y^*} \right\}}{\beta\theta}$$

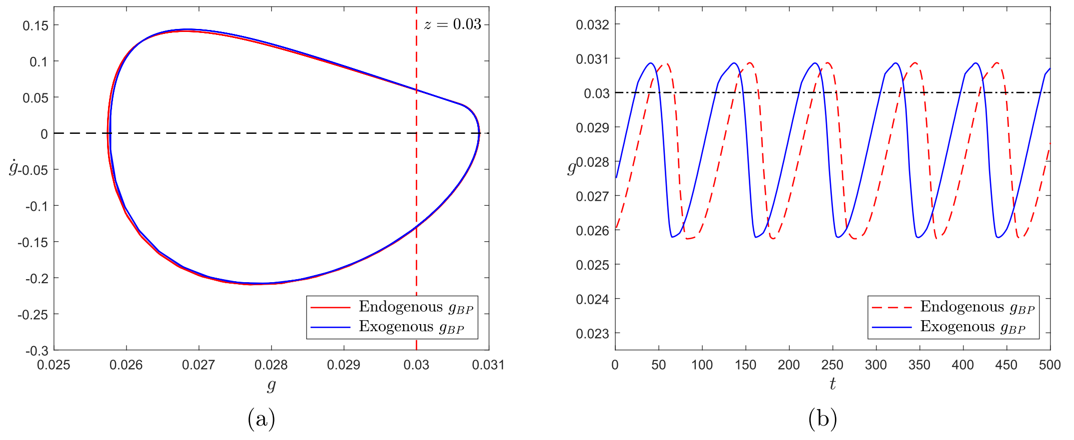

FIGURE 8 Numerical simulations comparing endogenous and exogenous versions of g_{BP}. Exports are assumed to be the only autonomous component of aggregate demand and there is equilibrium in the current account. Values of the additional parameters are $\rho_0 \frac{\dot{Y}^*}{Y^*} = 0.02$ and $\rho_1 \frac{\dot{Y}^*}{Y^*} = 0.66$. The black dotted line in panel (B) marks $z = 0.03$. The reader is referred to the web version of this article for interpretation of colour references of this figure. [Colour figure can be viewed at wileyonlinelibrary.com]

and

$$G(g) = \frac{g}{\beta\theta z}$$

We do not provide a local stability analysis of the modified dynamic equation. Instead, we rely on numerical simulations to highlight the main differences with respect to the initial setup. Figure 8 shows that the main growth-cycle features are preserved. Panel (A) indicates differences between the two scenarios are minimal in terms of the amplitude of the orbit. They can be appreciated by comparing the red and blue lines in the left diagram. Still, an important qualitative distinction appears in panel (B). The continuous blue line represents the case of an exogenous external constraint while the red dotted curve stands for an endogenous g_{BP}. The latter appears to be lagged with respect to the baseline scenario.

5 | FINAL CONSIDERATIONS

Thirlwall's law has proven to be a useful tool not only to explain uneven development but also to search for answers to the new challenges facing modern economies. Initial debates are giving way to a new research agenda in the field and the balance-of-payments constrained growth model has been updated accordingly. This includes considerations on climate change, the complexity of innovation processes, the role of institutions, the composition of external imbalances, and gender issues. There is a certain convergence in terms of policy insights indicating the need to articulate a set of industrial policies targeting R&D investments, the development of firms' innovative and adaptive capabilities, and a positive attitude towards change. Additional elements include the role of green-innovation as a window of opportunity for a new technological paradigm. Price concerns matter and the benefits of a competitive exchange rate have been properly documented. Still, the main message remains that non-price competitiveness is the crucial element. In this respect, the recent use of state-space models to estimate the law creates a unique opportunity to investigate its determinants between and within productive sectors.

To conclude, we notice some overlapping of two alternative interpretations: one that sees the law as a *binding constraint* and another that adopts a '*centre-of-gravity*' perspective. Such a separation reflects a broader confrontation between formal theories of economic fluctuations vis-a-vis the macroeconomic theories of growth. In this paper, we have argued that they might be rather complementary. By means of a simple Keynesian multiplier model compatible with Harrodian instability, we showed that assuming a balance-of-payments ceiling to growth gives rise to persistent and bounded fluctuations such that the external constraint works as a reference point to business cycle oscillations. There is no need to impose a floor on output. Furthermore, the model is compatible with different sources of autonomous demand. Numerical simulations showed the robustness of our results to alternative scenarios.

Our proposal conciliates both views on Thirlwall's law because (i) we assume *ex-ante* that it is a binding constraint that does not necessarily have to be satisfied, but (ii) its mere existence allows for the emergence of growth-cycle dynamics in which the law works as an asymmetric 'centre-of-gravity'. This means that the rate of growth compatible with equilibrium in the balance-of-payments can exist as a theoretical construct to be satisfied only in an abstract long run. Such a starting point is in line with the interpretation proposed by the first strand of the literature. On the other hand, given that capital flows are directly or indirectly related to the law, it emerges endogenously as an element around which the economy fluctuates. This result is in line with the second group's view. We would like to stress that, in the model, there is no need to impose a floor to output nor any additional assumption on the behaviour of fiscal policy. Our system is compatible with different sources of autonomous demand that may or may not grow at the same rate. Numerical simulations are performed, showing our results' robustness and sensitivity to different scenarios.

In terms of policy implications, our analysis indicates that adopting a pure binding constraint approach is problematic because induces policy-makers to wrongly downplay the relevance for macroeconomic stability of equilibrium in the current account. Given that is not clear when the economy actually reaches the 'long run', separating between time-frames hampers the understanding of the role of g_{BP} over the growth-cycle. On the other hand, a pure 'centre-of-gravity' perspective reduces the policy window to changes in the trade income-elasticities. If we recognise that at least part of government expenditures is also autonomous, austerity might result in a scenario with $g_{BP} > z$, which implies unjustified lower economic performance. The proposed solution combines elements of both viewpoints while highlighting that a longer gestational period for investment, associated with more mature and complex productive structures, reduces the amplitude of the cycle. Short-termism in investment decisions only leads to higher volatility.

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APPENDIX: MATHEMATICAL APPENDIX

The present Mathematical Appendix has two main purposes. First, it aims at providing the local stability analysis of the equilibrium of Equation (15). Second, we show that the model can be rewritten in the same form as a Lord Rayleigh's type of equation. This step is crucial for applying the two theorems in de Figueiredo (1958) that fulfil the conditions for the emergence of the limit cycle and confirm we are dealing with a two-phase oscillator. Still, given that de Figueireido's theorems do not have a straightforward economic interpretation, we refrain from providing their full proof. Instead, after obtaining a Lord Rayleigh's form, we move on to numerical simulations to confirm the existence of persistent endogenous fluctuations.

The final second-order differential equation of the model is given by:

$$\ddot{g} + F(\dot{g}) + G(g) = \frac{1}{\beta\theta} \tag{A.1}$$

Define the auxiliary variable w such that we may study deviations from the equilibrium by substituting:

$$w = g - \bar{g}$$

which give us

$$\ddot{w} + F(\dot{w}) + \frac{w}{\beta\theta z} = 0 \tag{A.2}$$

Let us make use of another set of auxiliary variables, defined and given by:

$$\begin{aligned} y &= \sqrt{\frac{1}{\beta\theta z}}\frac{w}{\dot{w}_0} \\ \dot{y} &= \frac{dy}{dt_1} = \frac{\dot{w}}{\dot{w}_0} \\ \ddot{y} &= \frac{d^2y}{dt_1^2} \\ t_1 &= \sqrt{\frac{1}{\beta\theta z}}t \end{aligned} \tag{A.3}$$

where $\dot{w}_0$ is any convenient unit in which to measure velocity.

Noting that:

$$\ddot{w} = \dot{w}_0\frac{d^2y}{dt_1 dt} = \dot{w}_0\sqrt{\frac{1}{\beta\theta z}}\frac{d^2y}{dt_1^2} \tag{A.4}$$

we may substitute Equations (A.3) and (A.4) into Equation (A.2), which becomes, after simplification:

$$\ddot{y} + \frac{F(\dot{w}_0\dot{y})}{\dot{w}_0\sqrt{\beta\theta z}} + y = 0$$

Letting

$$C(\dot{y}) = \frac{F(\dot{w}_0\dot{y})}{\dot{w}_0\sqrt{\beta\theta z}}$$

we get

$$\ddot{y} + C(\dot{y}) + y = 0 \tag{A.5}$$

which is a non-linear ordinary differential equation of order two in the well-known format of a *Lord Rayleigh type*. These algebraic manipulations are basically similar to Goodwin (1951).

Define two new auxiliary variables, x_1 and x_2, such that:

$$x_1 = y$$
$$x_2 = \dot{y}$$

which implies:

$$x_2 = \dot{x}_1$$

Hence, Equation (A.5) can be rewritten as a two-dimensional first-order dynamic system:

$$\dot{x}_1 = x_2 = h_1(x_2)$$
$$\dot{x}_2 = -x_1 - C(x_2) = h_2(x_1, x_2)$$

In equilibrium, $\dot{x}_1 = \dot{x}_2 = 0$. We thus have:

$$\bar{x}_1 + C(\bar{x}_2) = 0$$
$$\bar{x}_2 = 0$$

where the bar above a variable indicates the respective equilibrium value.

Recall that $A(\cdot)$ is a piece-wise function with two regimes. Therefore, we must study the dynamics of the two regimes separately.

- In the **first regime**, where $\dot{g} \leq g_{BP}/v$, the system admits a unique equilibrium solution, $P = (\bar{x}_1, \bar{x}_2)$, defined and given by:

$$\bar{x}_1 = 0$$
$$\bar{x}_2 = 0$$

The respective coefficient matrix is given by:

$$\Gamma = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} \tag{A.6}$$

where

$$\tau_{11} = \frac{\partial h_1(x_2)}{\partial x_1} = 0$$
$$\tau_{12} = \frac{\partial h_1(x_2)}{\partial x_2} = 1$$

$$\tau_{21} = \frac{\partial h_2(x_1, x_2)}{\partial x_1} = -1 < 0$$
$$\tau_{22} = \frac{\partial h_2(x_1, x_2)}{\partial x_2} = -C_{x_2} \gtreqless 0$$

so that

$$|\Gamma - \lambda \mathbf{I}| = \begin{vmatrix} -\lambda & 1 \\ -1 & -C_{x_2} - \lambda \end{vmatrix}$$
$$= \lambda\left(C_{x_2} + \lambda\right) + 1$$
$$= \lambda^2 + C_{x_2}\lambda + 1$$

The characteristic equation becomes

$$\lambda^2 + \frac{\left(\frac{1}{\theta} + \frac{1}{\beta z} - \frac{v}{\beta\theta}\right)}{\sqrt{\beta\theta z}}\lambda + 1 = 0$$

with eigenvalues

$$\lambda_{1,2} = \frac{\frac{-\frac{1}{\theta} - \frac{1}{\beta z} + \frac{v}{\beta\theta}}{\sqrt{\beta\theta z}} \pm \sqrt{\frac{\left(\frac{1}{\theta} + \frac{1}{\beta z} - \frac{v}{\beta\theta}\right)^2}{\beta\theta z} - 4}}{2}$$
$$= \frac{1}{2\sqrt{\beta\theta z}}\left[-\frac{1}{\theta} - \frac{1}{\beta z} + \frac{v}{\beta\theta} \pm \sqrt{\left(\frac{1}{\theta} + \frac{1}{\beta z} - \frac{v}{\beta\theta}\right)^2 - 4\beta\theta z}\right]$$

Given that

$$v > \beta + \frac{\theta}{z}$$

the real part of the eigenvalues is always positive. Hence the unique equilibrium point is either an *unstable node* or an *unstable focus*.

- In the **second regime**, where $\dot{g} > g_{BP}/v$, the coefficient matrix is similar to Equation (A.6) while the characteristic equation is equal to:

$$\lambda^2 + \frac{\left(\frac{1}{\theta} + \frac{1}{\beta z}\right)}{\sqrt{\beta\theta z}}\lambda + 1 = 0$$

with eigenvalues

$$\lambda_{1,2} = \frac{\frac{-\frac{1}{\theta}-\frac{1}{z\beta}}{\sqrt{\beta\theta z}} \pm \sqrt{\frac{\left(\frac{1}{\theta}+\frac{1}{\beta z}\right)^2}{\beta\theta z} - 4}}{2}$$
$$= \frac{1}{2\sqrt{\beta\theta z}}\left[-\frac{1}{\theta} - \frac{1}{z\beta} \pm \sqrt{\left(\frac{1}{\theta}+\frac{1}{\beta z}\right)^2 - 4\beta\theta z}\right]$$
$$= \frac{1}{2\sqrt{\beta\theta z}}\left[-\left(\frac{z\beta+\theta}{\beta\theta z}\right) \pm \left(\frac{z\beta-\theta}{\beta\theta z}\right)\right]$$

so that

$$\lambda_1 = -\frac{1}{\beta z\sqrt{\beta\theta z}}$$
$$\lambda_2 = -\frac{1}{\theta\sqrt{\beta\theta z}}$$

Since both eigenvalues are real and negative, the equilibrium for this regime is a *stable node*.

It is possible to show that Equation (A.5) and consequently Equation (A.1) fulfil the conditions for the emergence of a limit cycle given by de Figueiredo in two theorems (see de Figueiredo, 1958, Theorem 4.1 and 5.1). As confirmed by our numerical simulations, the underlying second order differential equation reveals that the emerging periodic attractor corresponds to the so-called two-stroke or two-pahse oscillator. Providing a complete mathematical assessment of its properties goes beyond the scope of this paper. In an ongoing research project (Sordi & Dávila-Fernández, 2022), we present a graphical and analytical treatment of this issue. Here, we limit ourselves to highlight that it is sufficient to have a ceiling in the piece-wise accelerator function to obtain the asymmetric cycle.