Information types in intuitionistic predicate logic with constant domains

Information Types in Intuitionistic Predicate Logic with Constant Domains*

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Abstract

In this paper, we study two operators, inquisitive disjunction and inquisitive existential quantifier, in the context of first-order intuitionistic logic with constant domains. We explain that these operators allow us to express types of intuitionistic propositions. We first provide this language with a relational semantics and formulate a sound axiomatic system. Completeness of the system for the full language is presented as an open problem but we prove completeness for a rich fragment of the language adapting the methods developed by Gianluca Grilleti in the context of classical inquisitive logic. We also develop a general algebraic framework in which we characterize the class of "Kripkean algebras" generated by the relational semantics. In this way, we obtain our first algebraic characterization of the logic of intuitionistic types. We further characterize the class of all homomorphic images of "Kripkean algebras" that we call "inquisitive algebras", thus obtaining a second, more general algebraic semantics for this logic.

Keywords: intuitionistic predicate logic, inquisitive logic, information types, constant domains, algebraic semantics

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1 Introduction

Following (Ciardelli, 2018) we can distinguish between a concrete piece of information and a generic information type. While the sentence *the white queen is on b5* (when uttered in a suitable context of a chess game) provides a piece of information, the expression *the position of the white queen* represents an information type under which various concrete pieces of information fall, namely the pieces expressed by the sentences *the white queen is on b5*, *the white queen is on a3*, and so on.

Even though expressions representing information types are not declarative sentences, one can observe that it makes sense to combine them by logical operators. The type *the position of the white queen* when combined via conjunction with the type *the position of the black king* forms a new type, namely *the position of the white queen and the position of the black king*. For instance, the piece of information *the white queen is on b5 and the black king is on a1* falls under this complex type. In a similar sense, one can form complex information types by means of universal quantifier, obtaining, for example, the type *the positions of all chess pieces*. A piece of information falls under this type if it specifies for all the chessmen their position.

One can generalize the notion of entailment to be applicable not only to pieces of information but also to information types. For example, the type *the positions of all chess pieces* "entails" in this generalized sense the type *the position of the white queen*, meaning that every piece of information falling under the former type entails (in the usual sense) some piece of information falling under the latter type.

One can codify a logic of information types. This is done by first-order inquisitive logic (Ciardelli, 2009, 2010, 2016, 2023; Grilletti, 2020) where information types are identified with questions. The standard inquisitive logic is based on classical logic of declarative sentences. In other words, the standard framework allows us to express information types only in the context of classical predicate logic.

The goal of this paper is to generalize the framework so that we can express information types also in the context of non-classical predicate logics. We will focus on the case of first-order intuitionistic logic with constant domains. In particular, we will formulate a non-standard semantic framework for intuitionistic predicate logic with constant domains and prove completeness with respect to this semantics. Then, we extend the first-order language with new expressive means that allow us to formulate information types. The non-standard framework enables us to equip these new expressions with a suitable semantics. We define the notion of entailment and straightforwardly generalize the ideas from (Grilletti, 2020) to obtain an axiomatization for a rich fragment of the language. Finally, we will formulate also an equivalent algebraic semantics. We will characterize the class of "Kripkean algebras" generated by the relational semantics by which we obtain our first algebraic characterization for the first-order intuitionistic logic of information types. Finally, we introduce a more general framework within which we characterize the class of all homomorphic images of Kripkean algebras, which we call "inquisitive algebras", thus obtaining a second, more general algebraic semantics for this logic.

2 Algebraic and informational semantics for intuitionistic logic with constant domains

We start with the following first-order language $\mathcal{L}$. For the sake of simplicity, terms are just individual variables and individual constants. Formulas are defined in the following way:

$$\alpha ::= \bot \mid Pt_1 \dots t_n \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \to \alpha \mid \forall x\alpha \mid \exists x\alpha$$

In this language, negation $\neg\alpha$ can be defined, as usual, as $\alpha \to \bot$.

This language is interpreted with the help of the following algebraic structures. A *complete Heyting algebra* is a structure of the form $\mathcal{H} = \langle H, \bigsqcup, \bigsqcap, \Rightarrow, 0\rangle$ where $\langle H, \bigsqcup, \bigsqcap\rangle$ is a complete lattice, 0 is the least element of the lattice and $\Rightarrow$ is the relative pseudocomplement, i.e. a binary operation on H such that for all $s, t, u \in H$, $s \sqcap t \leq u$ iff $s \leq t \Rightarrow u$. It can be shown that the presence of the relative pseudocomplement forces the lattice to be distributive, it must even satisfy the following infinite ditributive law:

$$s \sqcap \bigsqcup_{i\in I} t_i = \bigsqcup_{i\in I} (s \sqcap t_i).$$

If the structure satisfies also the dual law:

$$s \sqcup \bigsqcap_{i\in I} t_i = \bigsqcap_{i\in I} (s \sqcup t_i),$$

we call it *complete infinitely distributive Heyting algebra*, or *CD*-algebra, for short, because of the tight connection between this additional constraint and the constant domain axiom.

The top element can be defined as $1 = 0 \Rightarrow 0$. We regard $\Rightarrow$ and 0 as explicit components of these structures, even though they might be defined in terms of join and meet:

$$x \Rightarrow y = \bigsqcup\{z \in H \mid z \sqcap x \leq y\} \text{ and } 0 = \bigsqcap H.$$

A *signature* is just a set of predicates and individual constants. A $\tau\mathcal{L}$-*formula* is a formula of $\mathcal{L}$ in the signature τ. A τ-model is a tuple $\mathcal{M} = \langle \mathcal{H}, U, V\rangle$, where $\mathcal{H}$ is a *CD*-algebra (the elements of which can be viewed as propositions or information states), U is a non-empty set (representing the domain of quantification, also called the universe of discourse), and V is a τ-valuation, i.e. a function that assigns to each individual constant from τ an element of U, and to each n-ary predicate from τ a function that assigns to each n-tuple of elements from U an element of $\mathcal{H}$.

An *assignment* in U is a function that assigns to each variable of the language an element of U. If e is an assignment, x a variable, and $m \in U$, then $e(m/x)$ is the assignment that assigns m to x and $e(y)$ to any other variable y. For any term t, $V^e(t)$ is identical with $V(t)$ if t is an individual constant, and with $e(t)$ if t is a variable.

Let $\mathcal{M} = \langle \mathcal{H}, U, V \rangle$ be a τ-model and e an assignment in $\mathcal{M}$. The algebraic value of an *$\tau\mathcal{L}$-formula* in $\mathcal{M}$ and relative to e is defined recursively in the usual way:

- $|\bot|_e^{\mathcal{M}} = 0$,
- $|Pt_1 \dots t_n|_e^{\mathcal{M}} = V(P)(V^e(t_1), \dots, V^e(t_n))$,
- $|\alpha \wedge \beta|_e^{\mathcal{M}} = |\alpha|_e^{\mathcal{M}} \sqcap |\beta|_e^{\mathcal{M}}$,
- $|\alpha \vee \beta|_e^{\mathcal{M}} = |\alpha|_e^{\mathcal{M}} \sqcup |\beta|_e^{\mathcal{M}}$,
- $|\alpha \rightarrow \beta|_e^{\mathcal{M}} = |\alpha|_e^{\mathcal{M}} \Rightarrow |\beta|_e^{\mathcal{M}}$,
- $|\exists x \alpha|_e^{\mathcal{M}} = \bigsqcup_{m \in U} |\alpha|_{e(m/x)}^{\mathcal{M}}$,
- $|\forall x \alpha|_e^{\mathcal{M}} = \bigsqcap_{m \in U} |\alpha|_{e(m/x)}^{\mathcal{M}}$.

An algebraic consequence relation $\vDash_{alg}$ is defined as follows: if $\Delta \cup \{\alpha\}$ is a set of $\tau\mathcal{L}$-formulas, then $\Delta \vDash_{alg} \alpha$ iff $\bigsqcap_{\beta \in \Delta} |\beta|_e^{\mathcal{M}} \leq |\alpha|_e^{\mathcal{M}}$ (or, equivalently, $\bigsqcap_{\beta \in \Delta} |\beta|_e^{\mathcal{M}} \Rightarrow |\alpha|_e^{\mathcal{M}} = 1$), for every τ-model $\mathcal{M}$ and assignment e in $\mathcal{M}$.

We also define an alternative, Kripke-style "informational" semantics based on a relation of support $\Vdash_e$ between the states of any τ-model $\mathcal{M}$ (i.e. elements of H), viewed as information states, and $\tau\mathcal{L}$-formulas.[1] The support relation is defined in the following way:

- $s \Vdash_e \bot$ iff $s = 0$,
- $s \Vdash_e Pt_1 \dots t_n$ iff $s \leq V(P)(V^e(t_1), \dots, V^e(t_n))$,
- $s \Vdash_e \alpha \wedge \beta$ iff $s \Vdash_e \alpha$ and $s \Vdash_e \beta$,
- $s \Vdash_e \alpha \vee \beta$ iff there are $t, u \in H$ such that $s \leq t \sqcup u$ and $t \Vdash_e \alpha$, $u \Vdash_e \beta$,
- $s \Vdash_e \alpha \rightarrow \beta$ iff for every $t \leq s$, if $t \Vdash_e \alpha$, then $t \Vdash_e \beta$,

[1] A related semantics for propositional intuitionistic logic was introduced for example in (Punčochář, 2017).

- $s \Vdash_e \forall x\alpha$ iff for every $m \in U$, $s \Vdash_{e(m/x)} \alpha$,
- $s \Vdash_e \exists x\alpha$ iff there is a function $g\colon U \to H$ such that $s \leq \bigsqcup_{m\in U} g(m)$ and for all $m \in U$, $g(m) \Vdash_{e(m/x)} \alpha$.

If $s \Vdash_e \alpha$, we also say that s *e-supports* α. The support relation is downward persistent: if $s \Vdash_e \varphi$ and $t \leq s$, then $t \Vdash_e \varphi$. In this respect, the semantics contrasts with the usual Kripke semantics for intuitionistic logic in which propositions are upward closed. This peculiar orientation is motivated by our informational interpretation and a close connection to the algebraic semantics (see Lemma 1 below) where $s \leq t$ also means that s is informationally stronger than t.

A relational consequence relation $\models_{rel}$ is defined as preservation of support, i.e. if $\Delta\cup\{\alpha\}$ is a set of $\tau\mathcal{L}$-formulas, then $\Delta \models_{rel} \alpha$ iff for every τ-model $\mathcal{M}$, every state s of $\mathcal{M}$, and every assignment e in $\mathcal{M}$, if s e-supports all formulas from Δ, then s e-supports α.

Let $||\alpha||_e^{\mathcal{M}}$ be the set of states in $\mathcal{M}$ that e-support α. The following lemmas show that the relational semantics based on the support relation is closely related to the algebraic semantics and that the two consequence relations coincide. For a fixed τ-model $\mathcal{M}$ and a state s in $\mathcal{M}$, let $\downarrow s$ denote the *downset generated by s in $\mathcal{M}$*, i.e. $\downarrow s = \{t \in H \mid t \leq s\}$.

Lemma 1. *For every $\tau\mathcal{L}$-formula α, $||\alpha||_e^{\mathcal{M}} = \downarrow|\alpha|_e^{\mathcal{M}}$.*

Proof. By induction on α. The base cases are immediate. We will go through the inductive steps.

The step for $\wedge$: $s \Vdash_e \alpha \wedge \beta$ iff $s \Vdash_e \alpha$ and $s \Vdash_e \beta$ iff $s \leq |\alpha|_e^{\mathcal{M}}$ and $s \leq |\beta|_e^{\mathcal{M}}$ iff $s \leq |\alpha|_e^{\mathcal{M}} \sqcap |\beta|_e^{\mathcal{M}}$ iff $s \leq |\alpha \wedge \beta|_e^{\mathcal{M}}$.

The step for $\vee$: $s \Vdash_e \alpha \vee \beta$ iff there are $t, u \in H$ such that $s \leq t \sqcup u$ and $t \Vdash_e \alpha$, $u \Vdash_e \beta$ iff there are $t, u \in H$ such that $s \leq t \sqcup u$ and $t \leq |\alpha|_e^{\mathcal{M}}$, $u \leq |\beta|_e^{\mathcal{M}}$ iff $s \leq |\alpha|_e^{\mathcal{M}} \sqcup |\beta|_e^{\mathcal{M}}$ iff $s \leq |\alpha \vee \beta|_e^{\mathcal{M}}$.

The step for $\to$: $s \Vdash_e \alpha \to \beta$ iff for every $t \leq s$, if $t \Vdash_e \alpha$, then $t \Vdash_e \beta$ iff for every $t \leq s$, if $t \leq |\alpha|_e^{\mathcal{M}}$, then $t \leq |\beta|_e^{\mathcal{M}}$ iff for every $t \leq s \sqcap |\alpha|_e^{\mathcal{M}}$, $t \leq |\beta|_e^{\mathcal{M}}$ iff $s \sqcap |\alpha|_e^{\mathcal{M}} \leq |\beta|_e^{\mathcal{M}}$ iff $s \leq |\alpha|_e^{\mathcal{M}} \Rightarrow |\beta|_e^{\mathcal{M}}$ iff $s \leq |\alpha \to \beta|_e^{\mathcal{M}}$.

The step for $\forall$: $s \Vdash_e \forall x\alpha$ iff for every $m \in U$, $s \Vdash_{e(m/x)} \alpha$ iff for every $m \in U$, $s \leq |\alpha|_{e(m/x)}^{\mathcal{M}}$ iff $s \leq \bigsqcap_{m\in U} |\alpha|_{e(m/x)}^{\mathcal{M}}$ iff $s \leq |\forall x\alpha|_e^{\mathcal{M}}$.

The step for $\exists$: $s \Vdash_e \exists x\alpha$ iff there is a function $g\colon U \to S$ such that $s \leq \bigsqcup_{m\in U} g(m)$ and for all $m \in U$, $g(m) \Vdash_{e(m/x)} \alpha$ iff there is a function $g\colon U \to S$ such that $s \leq \bigsqcup_{m\in U} g(m)$ and for all $m \in U$, $g(m) \leq |\alpha|_{e(m/x)}^{\mathcal{M}}$ iff $s \leq \bigsqcup_{m\in U} |\alpha|_{e(m/x)}^{\mathcal{M}}$ iff $s \leq |\exists x\alpha|_e^{\mathcal{M}}$. □

Lemma 2. *If $\Delta \cup \{\alpha\}$ is a set of $\tau\mathcal{L}$-formulas, then $\Delta \models_{alg} \alpha$ iff $\Delta \models_{rel} \alpha$.*

Proof. $\Delta \not\models_{alg} \alpha$ iff $\bigsqcap_{\beta\in\Delta} |\beta|_e^{\mathcal{M}} \not\leq |\alpha|_e^{\mathcal{M}}$, for some $\mathcal{M}, e$, iff $\bigcap_{\beta\in\Delta} ||\beta||_e^{\mathcal{M}} \not\subseteq ||\alpha||_e^{\mathcal{M}}$, for some $\mathcal{M}, e$, iff there are $\mathcal{M}, e$ and there is a state in $\mathcal{M}$ that e-supports all formulas from Δ but not α, iff $\Delta \not\models_{rel} \alpha$. □

$$\frac{\alpha \qquad \beta}{\alpha \wedge \beta}\ (\wedge\text{-intro}) \qquad\qquad \frac{\alpha \wedge \beta}{\alpha}\ (\wedge\text{-elim}) \quad \frac{\alpha \wedge \beta}{\beta}\ (\wedge\text{-elim})$$

$$\frac{\alpha}{\alpha \vee \beta}\ (\vee\text{-intro}) \quad \frac{\beta}{\alpha \vee \beta}\ (\vee\text{-intro}) \qquad\qquad \frac{\alpha \vee \beta \qquad \begin{array}{c}[\alpha]\\ \gamma\end{array} \qquad \begin{array}{c}[\beta]\\ \gamma\end{array}}{\gamma}\ (\vee\text{-elim})$$

$$\frac{\begin{array}{c}[\alpha]\\ \beta\end{array}}{\alpha \to \beta}\ (\to\text{-intro}) \qquad\qquad \frac{\alpha \qquad \alpha \to \beta}{\beta}\ (\to\text{-elim})$$

$$\frac{\alpha(y/x)}{\forall x \alpha}\ (\forall\text{-intro}) \qquad\qquad \frac{\forall x \alpha}{\alpha(t/x)}\ (\forall\text{-elim})$$

$$\frac{\alpha(t/x)}{\exists x \alpha}\ (\exists\text{-intro}) \qquad\qquad \frac{\exists x \alpha \qquad \begin{array}{c}[\alpha(y/x)]\\ \beta\end{array}}{\beta}\ (\exists\text{-elim})$$

$$\frac{\bot}{\alpha}\ (\text{EFQ}) \qquad\qquad \frac{\forall x(\alpha \vee \beta)}{\alpha \vee \forall x \beta}\ (\text{CD})$$

Figure 1: A system of natural deduction for intuitionistic logic with constant domains, where the usual restrictions are assumed. In the introduction rule for universal quantifier it is assumed that y is not free in α and any undischarged assumption. In the elimination rule for existential quantifier y does not occur free in α, β and in any undischarged assumption. In the last rule related to constant domains, it is assumed that x is not free in α. In universal elimination and existential introduction, t stands for an arbitrary term.

Let $\vdash_{ilcd}$ be the derivability relation determined by the system of natural deduction for intuitionistic predicate logic with constant domains that is presented in Fig. 1. It is well-known that the system is sound and complete w.r.t. the algebraic semantics that we just defined. For our purposes, it will be useful to sketch the completeness proof with the help of the following canonical model as a suitable completion of the Lindenbaum–Tarski algebra (for more details, see Section 3.3 of (Gabbay, 1981)). This construction and the related auxiliary results will be needed in the next section.

The canonical model that we will use coincides with the complex algebra of the canonical model obtained by the usual Henkin construction for Kripke semantics. Fix a signature τ. We denote by τ^+ the set $\tau \cup C$ where $C = \{c_1, c_2, \ldots\}$ is a countably infinite set of individual constants not occurring in τ. A set Γ of $\tau^+\mathcal{L}$-formulas will be called a *saturated $\tau\mathcal{L}$-theory* if the following conditions are satisfied:

(a) for each $\tau^+\mathcal{L}$-formula α, if $\Gamma \vdash_{ilcd} \alpha$, then $\alpha \in \Gamma$,

(b) for each $\tau^+\mathcal{L}$-formula $\forall x\alpha \notin \Gamma$, there is a τ^+-term t s.t. $\alpha(t/x) \notin \Gamma$,

(c) for each $\tau^+\mathcal{L}$-formula $\exists x\alpha \in \Gamma$, there is a τ^+-term t s.t. $\alpha(t/x) \in \Gamma$,

(d) $\alpha \vee \beta \in \Gamma$ only if $\alpha \in \Gamma$ or $\beta \in \Gamma$.

We denote by $Th(\tau)$ the set of saturated $\tau\mathcal{L}$-theories. Notice that the set of all $\tau\mathcal{L}$-formulas, which can be denoted as $Fle(\tau\mathcal{L})$, is also a $\tau\mathcal{L}$-theory. Let us denote by $UpTh(\tau)$ the set of all non-empty upward closed sets of saturated $\tau\mathcal{L}$-theories, i.e. $X \in UpTh(\tau)$ iff $\emptyset \neq X \subseteq Th(\tau)$ and for all $\Delta, \Gamma \in Th(\tau)$, if $\Delta \in X$ and $\Delta \subseteq \Gamma$, then $\Gamma \in X$. Observe that the intersection of any subset of $UpTh(\tau)$ is a member of $UpTh(\tau)$. In particular, the intersection of any such subset is non-empty because every member of $UpTh(\tau)$ contains $Fle(\tau\mathcal{L})$. The *canonical τ-model* is defined as the structure $\mathcal{M}_\tau = \langle \mathcal{H}_\tau, U_\tau, V_\tau \rangle$, where

- $\mathcal{H}_\tau = \langle UpTh(\tau), \bigcup, \bigcap, \Rrightarrow, \{Fle(\tau\mathcal{L})\} \rangle$,[2]
- U_τ is the set of τ^+-terms (i.e. variables and constants from $\tau \cup C$),
- $V_\tau(c) = c$, $V_\tau(P)(t_1, \ldots, t_n) = \{\Gamma \in Th(\tau) \mid Pt_1, \ldots, t_n \in \Gamma\}$.

We also need to define a canonical assignment by $e(x) = x$. It is obvious that $\mathcal{H}_\tau$ is a *CD*-algebra, and thus $\mathcal{M}_\tau$ is indeed a τ-model. The order relation in this structure is just inclusion. We obtain the standard duality result.

[2]The operation $\Rrightarrow$, as a relative pseudocomplement, is determined in the usual way:

$$X \Rrightarrow Y = \bigcup\{Z \in UpTh(\tau) \mid Z \cap X \subseteq Y\} = \{\Gamma \in Th(\tau) \mid \forall\Delta \supseteq \Gamma, \text{ if } \Delta \in X, \Delta \in Y\}.$$

Lemma 3. $|\alpha|_e^{\mathcal{M}_\tau} = \{\Gamma \in Th(\tau) \mid \alpha \in \Gamma\}$*, for any* $\tau^+\mathcal{L}$*-formula* α.

From this, we obtain the following connection with the informational semantics.

Lemma 4. $X \Vdash_e \alpha$ *in* $\mathcal{M}_\tau$ *iff* $\alpha \in \bigcap X$*, for any* $\tau^+\mathcal{L}$*-formula* α *and any* $X \in UpTh(\tau)$.

Proof. $X \Vdash_e \alpha$ in $\mathcal{M}_\tau$ iff (by Lemma 1) $X \subseteq |\alpha|_e^{\mathcal{M}_\tau}$ iff (by Lemma 3) for all $\Gamma \in X$, $\alpha \in \Gamma$ iff $\alpha \in \bigcap X$. □

Lemma 5. *Let* $\Delta \cup \{\alpha\}$ *be a set of* $\tau\mathcal{L}$*-formulas such that* $\Delta \nvdash_{ilcd} \alpha$*. Then, there is a saturated* $\tau\mathcal{L}$*-theory* Γ *such that* $\Delta \subseteq \Gamma$ *and* $\alpha \notin \Gamma$.

Proof. This is the usual extension lemma. Note that the constant domain rule is needed for its proof. □

Theorem 1. $\Delta \vDash_{rel} \alpha$ *iff* $\Delta \vdash_{ilcd} \alpha$*, for any set of* $\tau\mathcal{L}$*-formulas* $\Delta \cup \{\alpha\}$.

Proof. Completeness can now be proved as follows: Assume $\Delta \nvdash_{ilcd} \alpha$. We will show that $\Delta \nvDash_{rel} \alpha$. By Lemma 5, there is a saturated $\tau\mathcal{L}$-theory Γ such that $\Delta \subseteq \Gamma$ and $\alpha \notin \Gamma$. Take $\uparrow\Gamma = \{\Omega \in Th(\tau) \mid \Gamma \subseteq \Omega\}$ as an element of the model $\mathcal{M}_\tau$. Note that $\bigcap \uparrow\Gamma = \Gamma$. By Lemma 4, it holds in $\mathcal{M}_\tau$ that $\uparrow\Gamma \Vdash_e \beta$, for every $\beta \in \Delta$, but $\uparrow\Gamma \nVdash_e \alpha$. □

3 Expressing information types

Now we extend the language $\mathcal{L}$ with inquisitive disjunction and inquisitive existential quantifier in the following way, where α stands for $\mathcal{L}$-formulas:

$$\varphi ::= \alpha \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x \varphi \mid \varphi \vee\!\!\!\vee\, \varphi \mid \exists x \varphi$$

The resulting language will be called $\mathcal{L}_{inq}$. Non-inquisitive existential quantifier and disjunction are applicable only to formulas from $\mathcal{L}$. The reason for this syntactic restriction is explained below. We will systematically use α, β, γ to refer to $\mathcal{L}$-formulas, and φ, ψ, χ for $\mathcal{L}_{inq}$-formulas. Moreover, we will be using the letters Δ, Γ for sets of $\mathcal{L}$-formulas and Σ, Φ for sets of $\mathcal{L}_{inq}$-formulas.

In inquisitive semantics $\vee\!\!\!\vee$ and $\exists$ are usually interpreted as question-generating operators, but their meaning can also be specified in terms of the notion of an information type (Ciardelli, 2018).

In this interpretation, formulas of the language $\mathcal{L}_{inq}$ can be viewed as representing types of information. Different information tokens may fall under the same type. For example:

- A piece of information falls under the type $\forall x \exists y Rxy$ if it specifies for every x some y such that Rxy.

- A piece of information falls under the type $\exists y \forall x Rxy$ if it specifies some y such that, for all x, Rxy.

Within this language, the relational semantic clauses for the operators from $\mathcal{L}$ are as before and for the new operators they are defined as follows:

- $s \Vdash_e \varphi \mathbin{\vee\!\!\!\vee} \psi$ iff $s \Vdash_e \varphi$ or $s \Vdash_e \psi$,
- $s \Vdash_e \exists x\varphi$ iff for some $m \in U$, $s \Vdash_{e(m/x)} \varphi$.

We extend the consequence relation $\vDash_{rel}$ to the language $\mathcal{L}_{inq}$. So, in general, if $\Sigma \cup \{\varphi\}$ is a set of $\tau\mathcal{L}_{inq}$-formulas, then $\Sigma \vDash_{rel} \varphi$ iff for every τ-model $\mathcal{M}$, every state s of $\mathcal{M}$ and every assignment e in $\mathcal{M}$, if s e-supports all formulas from Σ, then s e-supports φ.

The following lemma explains in which sense the inquisitive operators introduce a new type of proposition.

Lemma 6. *For every $\varphi \in \mathcal{L}_{inq}$, $||\varphi||_e^{\mathcal{M}}$ is a non-empty downset, but not necessarily a principal downset in $\mathcal{M}$.*

Proof. This claim reflects the fact that non-empty downsets are closed under all the semantic operations. However, the union of principal non-empty downsets is not always principal. □

Now we can explain why the non-inquisitive existential quantifier and disjunction are applicable only to formulas from $\mathcal{L}$. The reason for this syntactic restriction is partly conceptual and partly technical. On the conceptual side, in contrast to $\wedge$, $\rightarrow$, and $\forall$, the application of $\vee$ and $\exists$ to information types does not lead to constructions that have an intuitively clear meaning. On the technical side, we observe that, if unrestricted application of $\vee$ and $\exists$ were allowed, these operations would lose some basic properties. For example, take $\varphi = Pa \mathbin{\vee\!\!\!\vee} Qa$. If we were allowed to form the formulas $\exists x\varphi$ and $\varphi \vee \varphi$, then we would obtain that they would not be equivalent to φ because such equivalence would require the set of states supporting φ to be closed under join, which is not generally the case.[3]

Note that models of standard inquisitive semantics, in which information states are modeled as sets of first-order structures on a common domain of quantification, can be viewed as particular instances of our more general notion of a model. More precisely, in these particular cases, the algebra of states can be identified with the Boolean algebra of all subsets of a set X of classical first-order structures based on a common domain of quantification U. The valuation V is then determined in the expected way. It assigns to each n-ary predicate P the function assigning to $m_1, \ldots, m_n \in U$ the set

[3]For versions of propositional intuitionistic inquisitive logic, where non-inquisitive disjunction is applicable without any restriction, see (Punčochář, 2017; Ciardelli, Iemhoff & Yang, 2020; Punčochář & Pezlar, 2024).

of all structures from X in which the formula $Px_1 \dots x_n$ is true under the assignment respectively assigning the elements $m_1, \dots, m_n$ to the variables $x_1, \dots, x_n$ (see Grilletti, 2020). Let us call such models *classical*.

The logic of classical models differs from the logic determined by our semantics. As shown in the previous section, for the language $\mathcal{L}$, the logic of our semantics coincides with the intuitionistic predicate logic with constant domains. In contrast, the $\{\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}, \exists\}$-free fragment of standard inquisitive logic is just classical predicate logic.

Moreover, the structure of complete atomic Boolean algebras makes universally valid (i.e. valid not only for $\mathcal{L}$-formulas) also some schemata that are invalid in our framework. For example, for every $\mathcal{L}_{inq}$-formula φ, the formula $\forall x \neg\neg\varphi \to \neg\neg\forall x\varphi$ is supported by every state of any classical model.[4] In contrast, one can construct the following counterexample to $\forall x\neg\neg Px \to \neg\neg\forall xPx$ in our more general setting. Let $\mathcal{H}$ be the closed real interval $[0, 1]$, viewed as a CD-algebra, let U be the set of positive natural numbers and V a valuation assigning to the predicate P a function that assigns to each n the state $1/n$. Then, for any e, $1 \Vdash_e \forall x\neg\neg Px$ in the resulting model, because for every natural number n and every $u \in (0, 1]$ there is $0 < v \leq u$ such that $v \Vdash_{e(n/x)} Px$ (for example, one can put $v = min\{u, 1/n\}$). On the other hand, $1 \nVdash_e \neg\neg\forall xPx$. The support of this formula would imply that there is some $v \in (0, 1]$ such that for all n, $v \Vdash_{e(n/x)} Px$. This would mean that v is greater than 0 but below each $1/n$.

This observation points to a significant contrast between propositional and predicate inquisitive logic. On the propositional level, there is no principle that is schematically valid in classical but not in intuitionistic inquisitive logic. In other words, classical and intuitionistic inquisitive logics have the same propositional schematic fragment, which coincides with the Medvedev logic of finite problems. This follows from the main result of (Grilletti, 2022). In contrast, in the first-order setting, $\forall x\neg\neg\varphi \to \neg\neg\forall x\varphi$ is schematically valid in classical but not in intuitionistic inquisitive logic.

A sound axiomatic system for our semantics is obtained by taking a presentation of intuitionistic logic with constant domains (extrapolated to the language $\mathcal{L}_{inq}$) and enriching it with the usual introduction and elimination rules for inquisitive disjunction and inquisitive existential quantifier and the following two "split" rules and the constant domain rule for inquisitive disjunction:

$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-split $\alpha \to (\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi) \;/\; (\alpha \to \varphi) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (\alpha \to \psi)$,

$\exists$-split $\alpha \to \exists x\varphi \;/\; \exists x(\alpha \to \varphi)$, if x is not free in α,

CD2 $\forall x(\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi) \;/\; \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \forall x\psi$, if x is not free in φ.

[4] But note that, because of the presence of the inquisitive operators, $\neg\neg\varphi$ is not generally equivalent to φ in classical models.

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\ (\wedge\text{-intro}) \qquad \frac{\varphi \wedge \psi}{\varphi}\ (\wedge\text{-elim}) \quad \frac{\varphi \wedge \psi}{\psi}\ (\wedge\text{-elim})$$

$$\frac{\alpha}{\alpha \vee \beta}\ (\vee\text{-intro}) \quad \frac{\beta}{\alpha \vee \beta}\ (\vee\text{-intro}) \qquad \frac{\alpha \vee \beta \quad \begin{array}{c}[\alpha]\\ \gamma\end{array} \quad \begin{array}{c}[\beta]\\ \gamma\end{array}}{\gamma}\ (\vee\text{-elim})$$

$$\frac{\varphi}{\varphi \mathbin{\vee\!\!\!\vee} \psi}\ (\mathbin{\vee\!\!\!\vee}\text{-intro}) \quad \frac{\psi}{\varphi \mathbin{\vee\!\!\!\vee} \psi}\ (\mathbin{\vee\!\!\!\vee}\text{-intro}) \qquad \frac{\varphi \mathbin{\vee\!\!\!\vee} \psi \quad \begin{array}{c}[\varphi]\\ \chi\end{array} \quad \begin{array}{c}[\psi]\\ \chi\end{array}}{\chi}\ (\mathbin{\vee\!\!\!\vee}\text{-elim})$$

$$\frac{\begin{array}{c}[\varphi]\\ \psi\end{array}}{\varphi \to \psi}\ (\to\text{-intro}) \qquad \frac{\varphi \qquad \varphi \to \psi}{\psi}\ (\to\text{-elim})$$

$$\frac{\varphi(y/x)}{\forall x \varphi}\ (\forall\text{-intro}) \qquad \frac{\forall x \varphi}{\varphi(t/x)}\ (\forall\text{-elim})$$

$$\frac{\alpha(t/x)}{\exists x \alpha}\ (\exists\text{-intro}) \qquad \frac{\exists x \alpha \qquad \begin{array}{c}[\alpha(y/x)]\\ \beta\end{array}}{\beta}\ (\exists\text{-elim})$$

$$\frac{\varphi(t/x)}{\exists\!\!\exists x \varphi}\ (\exists\!\!\exists\text{-intro}) \qquad \frac{\exists\!\!\exists x \varphi \qquad \begin{array}{c}[\varphi(y/x)]\\ \psi\end{array}}{\psi}\ (\exists\!\!\exists\text{-elim})$$

$$\frac{\bot}{\varphi}\ (\text{EFQ})$$

$$\frac{\forall x(\alpha \vee \beta)}{\alpha \vee \forall x \beta}\ (\text{CD1}) \qquad \frac{\forall x(\varphi \mathbin{\vee\!\!\!\vee} \psi)}{\varphi \mathbin{\vee\!\!\!\vee} \forall x \psi}\ (\text{CD2})$$

$$\frac{\alpha \to (\varphi \mathbin{\vee\!\!\!\vee} \psi)}{(\alpha \to \varphi) \mathbin{\vee\!\!\!\vee} (\alpha \to \psi)}\ (\mathbin{\vee\!\!\!\vee}\text{-split}) \qquad \frac{\alpha \to \exists\!\!\exists x \varphi}{\exists\!\!\exists x(\alpha \to \varphi)}\ (\exists\!\!\exists\text{-split})$$

Figure 2: A sound system of natural deduction for inquisitive intuitionistic predicate logic with constant domains, where α, β, γ range over $\mathcal{L}$-formulas and φ, ψ, χ range over $\mathcal{L}_{inq}$-formulas. Moreover, the usual restrictions are assumed: In the introduction rule for universal quantifier, it is assumed that y is not free in any undischarged assumption and in φ. In the elimination rules for the existential quantifiers, y does not occur free in $\alpha, \beta, \varphi, \psi$ and in any undischarged assumption. In the constant domain rules, it is assumed that x is not free in α and φ. In the split rule for $\exists\!\!\exists$, it is assumed that x is not free in α. In universal elimination and the two kinds of existential introduction, t stands for an arbitrary term.

The whole system is presented in Fig. 2. Note for example that the elimination rule for $\vee\!\!\!\vee$ allows us to infer $\alpha \vee \beta$ from $\alpha \vee\!\!\!\vee \beta$, but the elimination rule for $\vee$ does not allow us to infer $\alpha \vee\!\!\!\vee \beta$ from $\alpha \vee \beta$ because it requires that the conclusion of the rule be $\{\vee\!\!\!\vee, \exists\!\!\!\exists\}$-free. This suggests that $\alpha \vee\!\!\!\vee \beta$ is strictly stronger than $\alpha \vee \beta$ in this system.

One can easily establish soundness of this system with respect to our semantics. Whether this system is also complete is an open question. But by applying the methods developed by Grilletti (2020, 2021), we can establish completeness of the system for a strong fragment of $\mathcal{L}_{inq}$ denoted as $\mathcal{L}^{-}_{inq}$:

$$\varphi ::= \alpha \mid \varphi \wedge \varphi \mid \alpha \to \varphi \mid \forall x \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid \exists\!\!\!\exists x \varphi$$

In this fragment, antecedents of implications are required to be from $\mathcal{L}$. The completeness for this fragment can be proved by an adjustment of the argument from (Grilletti, 2020, 2021) which is restricted to classical models. Let us now use the symbol $\vdash_{inq}$ for the derivability relation in the system and fix a signature τ. A set Σ of $\tau^{+}\mathcal{L}^{-}_{inq}$-formulas will be called a *saturated* $\tau\mathcal{L}^{-}_{inq}$*-theory* if the following conditions are satisfied:

(a) for every $\tau^{+}\mathcal{L}^{-}_{inq}$-formula φ, if $\Sigma \vdash_{inq} \varphi$, then $\varphi \in \Sigma$,

(b) for every $\tau^{+}\mathcal{L}^{-}_{inq}$-formula $\forall x\varphi \notin \Sigma$, there is a τ^{+}-term t s.t. $\varphi(t/x) \notin \Sigma$,

(c) for every $\tau^{+}\mathcal{L}^{-}_{inq}$-formula $\exists\!\!\!\exists x\varphi \in \Sigma$, there is a τ^{+}-term t s.t. $\varphi(t/x) \in \Sigma$,

(d) $\varphi \vee\!\!\!\vee \psi \in \Sigma$ only if $\varphi \in \Sigma$ or $\psi \in \Sigma$.

Notice the difference between the notion of a saturated $\tau\mathcal{L}$-theory, which is defined in terms of $\exists$ and $\vee$, and the notion of a saturated $\tau\mathcal{L}^{-}_{inq}$-theory defined in terms of $\exists\!\!\!\exists$ and $\vee\!\!\!\vee$.

For any state s of any τ^{+}-model $\mathcal{M}$, and any assignment e in $\mathcal{M}$, let $[s]^{\mathcal{M}}_{e}$ denote the set of $\tau^{+}\mathcal{L}^{-}_{inq}$-formulas e-supported by s in $\mathcal{M}$.

Lemma 7. *For any state X in the canonical model $\mathcal{M}_{\tau}$, the set $[X]^{\mathcal{M}_{\tau}}_{e}$ is a saturated $\tau\mathcal{L}^{-}_{inq}$-theory.*

Proof. This is a straightforward consequence of soundness of the system, the support conditions for $\forall, \exists\!\!\!\exists, \vee\!\!\!\vee$, and the fact that the terms cover the entire domain of the canonical model. □

Let φ be a $\tau^{+}\mathcal{L}^{-}_{inq}$-formula and Σ a deductively closed set of $\tau^{+}\mathcal{L}^{-}_{inq}$-formulas. Then, $\Sigma \oplus \varphi$ will denote the set of $\tau^{+}\mathcal{L}^{-}_{inq}$-formulas defined as follows:

$$\psi \in \Sigma \oplus \varphi \text{ iff } \varphi \to \psi \in \Sigma.$$

Since in the current setting the deduction theorem holds, the set $\Sigma \oplus \varphi$ could be equivalently defined as the set of $\tau^+\mathcal{L}^-_{inq}$-formulas derivable from $\Sigma \cup \{\varphi\}$.[5]

The following lemma can be seen as a natural intuitionistic counterpart of Lemma 26 from (Grilletti, 2021).

Lemma 8. *If Σ is a saturated $\tau\mathcal{L}^-_{inq}$-theory and α a $\tau^+\mathcal{L}$-formula, then $\Sigma \oplus \alpha$ is a saturated $\tau\mathcal{L}^-_{inq}$-theory.*[6]

Proof. (a) Assume $\Sigma \oplus \alpha \vdash_{inq} \varphi$. Then, there are $\psi_1, \ldots, \psi_n \in \Sigma \oplus \alpha$ such that $\psi_1, \ldots, \psi_n \vdash_{inq} \varphi$. It follows that $\alpha \to (\psi_1 \wedge \ldots \wedge \psi_n) \in \Sigma$ and so $\alpha \to \varphi \in \Sigma$. Hence, $\varphi \in \Sigma \oplus \alpha$.

(b) Assume $\forall x\varphi \notin \Sigma \oplus \alpha$, i.e. $\alpha \to \forall x\varphi \notin \Sigma$. Without loss of generality, we can assume that x is not free in α. Hence, $\forall x(\alpha \to \varphi) \notin \Sigma$. Since Σ is saturated, there is a τ^+-term t such that $\alpha \to \varphi(t/x) \notin \Sigma$, i.e. $\varphi(t/x) \notin \Sigma\oplus\alpha$.

(c) Assume $\exists x\varphi \in \Sigma \oplus \alpha$, i.e. $\alpha \to \exists x\varphi \in \Sigma$. Without loss of generality, we can assume that x is not free in α. Hence, $\exists x(\alpha \to \varphi) \in \Sigma$ (by $\exists$-split). Since Σ is saturated, there is a τ^+-term t such that $\alpha \to \varphi(t/x) \in \Sigma$, i.e. $\varphi(t/x) \in \Sigma \oplus \alpha$.

(d) Assume $\varphi \vee\!\!\!\vee \psi \in \Sigma \oplus \alpha$. Thus, $\alpha \to (\varphi \vee\!\!\!\vee \psi) \in \Sigma$ and hence, by $\vee\!\!\!\vee$-split, $(\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi) \in \Sigma$. Since Σ is saturated, $\alpha \to \varphi \in \Sigma$ or $\alpha \to \psi \in \Sigma$, i.e. $\varphi \in \Sigma \oplus \alpha$ or $\psi \in \Sigma \oplus \alpha$. □

Let Σ^* denote the set of $\mathcal{L}$-formulas contained in Σ.

Lemma 9. *If Σ and Φ are saturated $\tau\mathcal{L}^-_{inq}$-theories such that $\Sigma^* = \Phi^*$, then $(\Sigma \oplus \alpha)^* = (\Phi \oplus \alpha)^*$ for each $\tau^+\mathcal{L}$-formula α.*

Proof. Assume $\Sigma^* = \Phi^*$. Then, $\beta \in \Sigma \oplus \alpha$ iff $\alpha \to \beta \in \Sigma$ iff $\alpha \to \beta \in \Phi$ iff $\beta \in \Phi \oplus \alpha$. □

Likewise, the following lemma can be seen as an intuitionistic counterpart of Lemma 27 in (Grilletti, 2021).

Lemma 10. *For all saturated $\tau\mathcal{L}^-_{inq}$-theories Σ, Φ, if $\Sigma^* = \Phi^*$, then $\Sigma = \Phi$.*

Proof. We will prove by induction that, for every $\tau^+\mathcal{L}^-_{inq}$-sentence φ, the following holds: For all saturated $\tau\mathcal{L}^-_{inq}$-theories Σ and Φ, if $\Sigma^* = \Phi^*$, then $\varphi \in \Sigma$ iff $\varphi \in \Phi$. The base cases are immediate. We need to show the inductive steps for $\wedge, \to, \vee\!\!\!\vee, \forall, \exists$. The inductive steps are straightforward:

The step for $\wedge$: $\varphi \wedge \psi \in \Sigma$ iff $\varphi \in \Sigma$ and $\psi \in \Sigma$ iff $\varphi \in \Phi$ and $\psi \in \Phi$ iff $\varphi \wedge \psi \in \Phi$. The step for $\to$: $\alpha \to \varphi \in \Sigma$ iff $\varphi \in \Sigma \oplus \alpha$ iff (using Lemma 9) $\varphi \in \Phi \oplus \alpha$ iff $\alpha \to \varphi \in \Phi$. The step for $\vee\!\!\!\vee$: $\varphi \vee\!\!\!\vee \psi \in \Sigma$ iff $\varphi \in \Sigma$ or $\psi \in \Sigma$

[5] However, the two notions would not be equivalent in a substructural setting where deduction theorem fails. In such a setting the former formulation would be needed.

[6] The rules $\vee\!\!\!\vee$-split and $\exists$-split are needed for this proposition.

iff $\varphi \in \Phi$ or $\psi \in \Phi$ iff $\varphi \vee\!\!\!\vee\, \psi \in \Phi$. The step for $\forall$: $\forall x\varphi \in \Sigma$ iff for every τ^+-term t, $\varphi(t/x) \in \Sigma$ iff for every τ^+-term t, $\varphi(t/x) \in \Phi$ iff $\forall x\varphi \in \Phi$. The step for $\exists$: $\exists x\varphi \in \Sigma$ iff for some τ^+-term t, $\varphi(t/x) \in \Sigma$ iff for some τ^+-term t, $\varphi(t/x) \in \Phi$ iff $\exists x\varphi \in \Phi$. □

Now we are ready to show that failure of entailment in $\vdash_{inq}$ can always be witnessed by a saturated $\tau\mathcal{L}^-_{inq}$-theory.

Lemma 11. *Let $\Sigma \cup \{\varphi\}$ be a set of $\tau\mathcal{L}^-_{inq}$-formulas such that $\Sigma \nvdash_{inq} \varphi$. Then, there is a saturated $\tau\mathcal{L}^-_{inq}$-theory Φ such that $\Sigma \subseteq \Phi$ and $\varphi \notin \Phi$.*

Proof. The usual argument, using the constant domain rule CD2. See the proof of the corresponding Lemma 33 in (Grilletti, 2021). □

For any saturated $\tau\mathcal{L}^-_{inq}$-theory Σ, let $X_\Sigma = \{\Delta \in Th(\tau) \mid \Sigma^* \subseteq \Delta\}$. Note that $X_\Sigma \in UpTh(\tau)$, so it is a state in the canonical model $\mathcal{M}_\tau$.

Lemma 12. $\Sigma^* = \bigcap X_\Sigma$*, for any saturated $\tau\mathcal{L}^-_{inq}$-theory Σ.*

Proof. It is obvious that any $\alpha \in \Sigma^*$ is contained in every member of X_Σ, and thus $\Sigma^* \subseteq \bigcap X_\Sigma$. Moreover, if $\alpha \notin \Sigma^*$ then, by Lemma 5, there is a $\Delta \in X_\Sigma$ such that $\alpha \notin \Delta$. Hence, $\bigcap X_\Sigma \subseteq \Sigma^*$. □

The following lemma is an intuitionistic counterpart of Theorem 31 in (Grilletti, 2021).

Lemma 13. *For any $\tau^+\mathcal{L}^-_{inq}$-formula φ and any saturated $\tau\mathcal{L}^-_{inq}$-theory Σ, $X_\Sigma \Vdash_e \varphi$ in $\mathcal{M}_\tau$ iff $\varphi \in \Sigma$.*

Proof. By Lemma 7, $[X_\Sigma]^{\mathcal{M}_\tau}$ is a saturated $\tau\mathcal{L}^-_{inq}$-theory. By Lemma 10, this theory is uniquely determined by its declarative fragment. By Lemmas 4 and 12, this fragment coincides with Σ^*, so $[X_\Sigma]^{\mathcal{M}_\tau} = \Sigma$. □

Theorem 2. *Let $\Sigma \cup \{\varphi\}$ be a set of $\mathcal{L}^-_{inq}$-formulas. Then, $\Sigma \vDash_{rel} \varphi$ iff $\Sigma \vdash_{inq} \varphi$.*

Proof. Assume $\Sigma \nvdash_{inq} \varphi$. Then, by Lemma 11, there is a saturated $\tau\mathcal{L}^-_{inq}$-theory Φ such that $\Sigma \subseteq \Phi$ and $\varphi \notin \Phi$. So, by Lemma 13, X_Φ supports everything in Σ but not φ. □

We have proved completeness for the inquisitive intuitionistic logic with constant domains for the restricted language $\mathcal{L}^-_{inq}$. It remains an open question whether this result could be extended to the full language $\mathcal{L}_{inq}$. There are further open questions concerning some modifications of our semantic setting. Note that the system of natural deduction contains two versions of the constant domain rule, CD1 for $\vee$, and CD2 for $\vee\!\!\!\vee$. These two rules are mutually independent and both of them were needed, the former in the proof of Lemma 5 and the latter in the proof of Lemma 11. It is not clear how the validity of CD2 could be reasonably avoided without changing too much the whole framework but it seems that CD1 should be optional and it would become invalid if the class of τ-models was based on arbitrary complete Heyting algebras rather than just on *CD*-algebras. In other words, one can expect that dropping CD1 should correspond to dropping the constraint that any τ-model satisfies the distributive law: $s \sqcup \bigsqcap_{i\in I} t_i = \bigsqcap_{i\in I}(s \sqcup t_i)$. All the other rules of the system would be preserved under this expansion of the class of admissible models (which shows that CD1 is independent of CD2). A natural question of completeness arises for this modified setting. Nevertheless, even for the restricted language $\mathcal{L}^-_{inq}$, the strategy of a potential completeness proof would have to be adapted to this modification and we leave this task for future research.

The semantic setting can be extended in a straightforward way to encompass also functional symbols and identity. Functional symbols can be interpreted by valuations as functions on the universe with their corresponding arity. Identity can be interpreted as a binary predicate for which we impose constraints forcing it to behave as a congruence relation with respect to a given language. That is, a valuation assigns to the identity predicate any function $Eq\colon U \times U \to U$ satisfying the following conditions for any $m, n, o, m_1, \ldots, m_k, n_1, \ldots, n_k \in U$, any k-ary function f on U that is a value of some functional symbol in the language, and any k-ary predicate P of the language:

- $Eq(m, m)$ is the top element,
- $Eq(m, n) = Eq(n, m)$,
- $Eq(m, n) \sqcap Eq(n, o) \leq Eq(m, o)$,
- $Eq(m_1, n_1) \sqcap \ldots \sqcap Eq(m_k, n_k) \leq Eq(f(m_1, \ldots, m_k), f(n_1, \ldots, n_k))$,
- $Eq(m_1, n_1) \sqcap \ldots \sqcap Eq(m_k, n_k) \sqcap V(P)(m_1, \ldots, m_k) = \\ = Eq(m_1, n_1) \sqcap \ldots \sqcap Eq(m_k, n_k) \sqcap V(P)(n_1, \ldots, n_k)$.

The information state $Eq(m, n)$ can be viewed as representing the information that m and n are equal. The completeness proof can be easily adapted to prove that this extension can be axiomatized by the system in Fig. 2 enriched with the following two standard rules of identity:

$$\frac{}{t = t} \qquad \frac{t = u \qquad \varphi[t/x]}{\varphi[u/x]}$$

4 An algebraic perspective

In this section, we provide an algebraic definition of the consequence relation $\vDash_{rel}$ in the language $\mathcal{L}_{inq}$. Our approach is inspired by the frameworks from (Punčochář, 2021; Punčochář & Pezlar, 2024) and (Quadrellaro, 2022) that provide an algebraic semantics for propositional intermediate inquisitive logics.

Let us introduce the main notions of our algebraic setting. Take any complete Heyting algebra $\mathcal{H}$. An operation $j\colon H \to H$ is called a *nucleus* on $\mathcal{H}$, if it satisfies, for every $x, y \in H$, the following conditions:

(a) $x \leq j(x)$,

(b) $j(j(x)) = j(x)$,

(c) $j(x \sqcap y) = j(x) \sqcap j(y)$.

A nucleus is *dense* if $j(0) = 0$. Note that every nucleus is a closure operator, but not the other way around. A complete Heyting algebra equipped with a nucleus will be called a *nuclear algebra*. If its nucleus is dense, the nuclear algebra is also called *dense*. Take a nuclear algebra $\mathcal{H} = \langle H, \bigsqcup, \bigsqcap, \Rightarrow, 0, j\rangle$. The fixed points of j represent *information tokens*. The set of all information tokens of $\mathcal{H}$ will be called *the core of* $\mathcal{H}$ and it will be denoted as jH, so that we have

$$jH = \{x \in H \mid x = j(x)\} = \{j(x) \mid x \in H\}.$$

The core is closed under arbitrary meets and under $\Rightarrow$. A crucial feature of nuclear algebras is that their cores also form complete Heyting algebras. For any nuclear algebra $\mathcal{H} = \langle H, \bigsqcup, \bigsqcap, \Rightarrow, 0, j\rangle$, one can define the structure

$$j\mathcal{H} = \langle jH, \bigsqcup{}^j, \bigsqcap{}^j, \Rightarrow^j, j(0)\rangle,$$

where $\bigsqcap^j$ and $\Rightarrow^j$ are $\bigsqcap$ and $\Rightarrow$ restricted to jH and $\bigsqcup^j X = j(\bigsqcup X)$. Then, one can observe that $j\mathcal{H}$ is a complete Heyting algebra.[7] If $\mathcal{H}$ is a nuclear algebra such that $j\mathcal{H}$ is a *CD*-algebra, then we say that $\mathcal{H}$ is *core-balanced.*

For a given signature τ, we define a *nuclear frame* as a pair $\mathcal{F} = \langle \mathcal{H}, U\rangle$, where $\mathcal{H} = \langle H, \bigsqcup, \bigsqcap, \Rightarrow, 0, j\rangle$ is a nuclear algebra and U is a non-empty set. A *nuclear τ-model* is a tuple $\mathcal{N} = \langle \mathcal{H}, U, V\rangle$, where $\langle \mathcal{H}, U\rangle$ is a nuclear frame and V is a τ-valuation, now defined as a function that assigns an element of U to every individual constant from τ, and to every n-ary predicate from τ a function that assigns an element *of the core* to every n-tuple of elements from U. Assignments are defined as before. Then, the semantic clauses for the language $\mathcal{L}_{inq}$, given an assignment e, are defined as follows:

- $|\bot|^{\mathcal{N}}_e = 0$,
- $|Pt_1 \dots t_n|^{\mathcal{N}}_e = V(P)(V^e(t_1), \dots, V^e(t_n))$,
- $|\varphi \wedge \psi|^{\mathcal{N}}_e = |\varphi|^{\mathcal{N}}_e \sqcap |\psi|^{\mathcal{N}}_e$,

[7] For more details, see (Bezhanishvili & Holliday, 2016, 2019).

- $|\alpha \vee \beta|_e^{\mathcal{N}} = j(|\alpha|_e^{\mathcal{N}} \sqcup |\beta|_e^{\mathcal{N}})$,
- $|\varphi \to \psi|_e^{\mathcal{N}} = |\varphi|_e^{\mathcal{N}} \Rightarrow |\psi|_e^{\mathcal{N}}$,
- $|\varphi \mathbin{\vee\!\!\!\vee} \psi|_e^{\mathcal{N}} = |\varphi|_e^{\mathcal{N}} \sqcup |\psi|_e^{\mathcal{N}}$,
- $|\exists x \alpha|_e^{\mathcal{N}} = j(\bigsqcup_{m \in U} |\alpha|_{e(m/x)}^{\mathcal{N}})$,
- $|\forall x \varphi|_e^{\mathcal{N}} = \bigsqcap_{m \in U} |\varphi|_{e(m/x)}^{\mathcal{N}}$,
- $|\exists\!\!\exists x \varphi|_e^{\mathcal{N}} = \bigsqcup_{m \in U} |\varphi|_{e(m/x)}^{\mathcal{N}}$.

A $\tau\mathcal{L}_{inq}$-formula φ is e-valid in $\mathcal{N}$, if $|\varphi|_e^{\mathcal{N}} = 1$. φ is valid in $\mathcal{N}$ if for every assignment e in $\mathcal{N}$, φ is e-valid in $\mathcal{N}$. φ is valid in a nuclear frame if it is valid in every nuclear model based on that frame.

Given a class $\mathcal{C}$ of nuclear τ-models, the algebraic consequence relation of $\mathcal{C}$ is defined in the following way: $\Phi \vDash_{alg}^{\mathcal{C}} \varphi$ iff $\bigsqcap_{\psi \in \Phi} |\psi|_e^{\mathcal{N}} \leq |\varphi|_e^{\mathcal{N}}$, for every $\mathcal{N} \in \mathcal{C}$ and every e in $\mathcal{N}$.

Consider two nuclear algebras $\mathcal{A} = \langle A, \bigsqcup^{\mathcal{A}}, \bigsqcap^{\mathcal{A}}, \Rightarrow^{\mathcal{A}}, 0^{\mathcal{A}}, j^{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle B, \bigsqcup^{\mathcal{B}}, \bigsqcap^{\mathcal{B}}, \Rightarrow^{\mathcal{B}}, 0^{\mathcal{B}}, j^{\mathcal{B}} \rangle$, and a function h from A to B. Then, h is called a *(complete) homomorphism from $\mathcal{A}$ to $\mathcal{B}$* if it satisfies the following conditions for every $x, y, x_i \in A$ (for all $i \in I$ of any index set I):

- $h(\bigsqcup_{i \in I}^{\mathcal{A}} x_i) = \bigsqcup_{i \in I}^{\mathcal{B}} h(x_i)$,
- $h(\bigsqcap_{i \in I}^{\mathcal{A}} x_i) = \bigsqcap_{i \in I}^{\mathcal{B}} h(x_i)$,
- $h(x \Rightarrow^{\mathcal{A}} y) = h(x) \Rightarrow^{\mathcal{B}} h(y)$,
- $h(0^{\mathcal{A}}) = 0^{\mathcal{B}}$,
- $h(j^{\mathcal{A}}(x)) = j^{\mathcal{B}}(h(x))$.

If h is moreover a bijection, it is called an *isomorphism.* If there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are called *isomorphic.* Dropping the condition for nucleus, we will talk also about *isomorphic complete Heyting algebras.*

Note that, for every homomorphism h, we have $h(jA) \subseteq jB$ and, if $jB \subseteq h(A)$, then $h(jA) = jB$. Note also that a bijection h is an isomorphism iff (a) h preserves the nucleus: $h(j^{\mathcal{A}}(x)) = j^{\mathcal{B}}(h(x))$, and (b) h preserves and reflects the order: $x \leq^{\mathcal{A}} y$ iff $h(x) \leq^{\mathcal{B}} h(y)$.

Lemma 14. *Let $\mathcal{A}$, $\mathcal{B}$ be nuclear algebras, h a homomorphism from $\mathcal{A}$ to $\mathcal{B}$, U a non-empty set, and φ an $\mathcal{L}_{inq}$-formula. Then, the following hold:*

(a) φ is valid in $\langle \mathcal{B}, U \rangle$ only if φ is valid in $\langle \mathcal{A}, U \rangle$,

(b) if, moreover, $jB \subseteq h(A)$, then φ is valid in $\langle \mathcal{A}, U \rangle$ only if φ is valid in $\langle \mathcal{B}, U \rangle$.

Proof. We will prove only (b). Assume that $jB \subseteq h(A)$ and thus $jB = h(jA)$. Assume further that φ is not valid in $\langle \mathcal{B}, U \rangle$. Then, there is a nuclear model $\mathcal{N} = \langle \mathcal{B}, U, V \rangle$ such that φ is not valid in $\mathcal{N}$, i.e. for some e, $|\varphi|_e^{\mathcal{N}} \neq 1^{\mathcal{B}}$. Since $jB \subseteq h(jA)$, we can take, for every x from the core of $\mathcal{B}$, an element x^* from the core of $\mathcal{A}$ such that $h(x^*) = x$. Then, the valuation V in $\mathcal{B}$ determines a valuation V' in $\mathcal{A}$ such that $V'(a) = V(a)$, for every individual constant a, and $V'(P)(m_1, \ldots, m_n) = V(P)(m_1, \ldots, m_n)^*$, for every n-ary predicate P and $m_1, \ldots, m_n \in U$. Let $\mathcal{N}' = \langle \mathcal{A}, U, V' \rangle$. By a simple induction, one can show that $h(|\psi|_e^{\mathcal{N}'}) = |\psi|_e^{\mathcal{N}}$, for every $\mathcal{L}_{inq}$-formula ψ. It follows that $|\varphi|_e^{\mathcal{N}'} \neq 1^{\mathcal{A}}$, and thus φ is not valid in $\langle \mathcal{A}, U \rangle$. □

We say that a homomorphism h from $\mathcal{A}$ to $\mathcal{B}$ is a *c-homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if $jB \subseteq h(A)$ (and thus $jB = h(jA)$). $\mathcal{B}$ is called a *homomorphic c-image* of $\mathcal{A}$ if there is a c-homomorphism from $\mathcal{A}$ to $\mathcal{B}$. Note that it follows from Lemma 14 that, if $\mathcal{B}$ is a homomorphic c-image of $\mathcal{A}$ and a universe U is fixed, then $\langle \mathcal{A}, U \rangle$ and $\langle \mathcal{B}, U \rangle$ validate exactly the same formulas.

The prototypical examples of nuclear algebras that will be especially important for our purposes are the algebras of non-empty downsets of complete Heyting algebras. Let $\mathcal{H} = \langle H, \bigsqcup, \bigsqcap, \Rightarrow, 0 \rangle$ be a complete Heyting algebra. It determines the structure $Dw(\mathcal{H}) = \langle Dw(H), \bigcup, \bigcap, \Rrightarrow, \{0\}, dj \rangle$, where $Dw(H)$ is the set of all non-empty downsets of $\mathcal{H}$; the operations $\bigcup$ and $\bigcap$ are (infinitary) union and intersection, $dj \colon Dw(H) \to Dw(H)$ is defined as follows: $dj(X) = {\downarrow} \bigsqcup X$; and $\Rrightarrow$ is defined as follows:

$$X \Rrightarrow Y = \{u \in H \mid \text{for all } v \leq u, \text{ if } v \in X \text{ then } v \in Y\}.$$

The following lemma can be easily verified.

Lemma 15. *If $\mathcal{H}$ is a complete Heyting algebra, $Dw(\mathcal{H})$ is a nuclear algebra such that $djDw(\mathcal{H})$ is isomorphic to $\mathcal{H}$. Hence, if $\mathcal{H}$ is a CD-algebra then $Dw(\mathcal{H})$ is core-balanced.*

Note that core-balanced downset algebras correspond to the algebras of propositions in our relational Kripke-style semantics. We will introduce the notion of a Kripkean algebra in order to characterize directly downset algebras. A *Kripkean algebra* is defined as a dense nuclear algebra which is join-generated by its core and the core is formed by the set of all completely join-irreducible elements (an element $y \in H$ is *completely join-irreducible* if for any non-empty $X \subseteq H$, $y \leq \bigsqcup X$ only if $y \leq x$, for some $x \in X$). The set of completely join-irreducible elements of $\mathcal{H}$ will be denoted as $CJI(\mathcal{H})$. Note that according to our definition $0 \in CJI(\mathcal{H})$.

So, a Kripkean algebra is a dense nuclear algebra that satisfies the following two conditions:

(K1) for every $x \in H$, there is some $Y \subseteq jH$ such that $x = \bigsqcup Y$,

(K2) $jH = CJI(\mathcal{H})$.

Lemma 16. *For every complete Heyting (CD-)algebra $\mathcal{H}$, the nuclear algebra $Dw(\mathcal{H})$ is Kripkean. Moreover, every (core-balanced) Kripkean algebra is isomorphic to $Dw(\mathcal{H})$ for some complete Heyting (CD-)algebra $\mathcal{H}$.*

Proof. In Lemma 15, we showed that, for any complete Heyting (CD-)algebra $\mathcal{H}$, $Dw(\mathcal{H})$ is a (core-balanced) nuclear algebra. Note that $dj(\{0\}) = \{0\}$ and so dj is a dense nucleus. Moreover, it is clear that every non-empty downset is the union of a set of principal downsets and that the core elements, i.e. the principal downsets, are exactly the completely join-irreducible elements of $Dw(\mathcal{H})$. Hence, (K1) and (K2) from the definition of a Kripkean algebra hold for $Dw(\mathcal{H})$.

Now we have to prove that every Kripkean algebra arises in this way. Take any Kripkean algebra $\mathcal{H} = \langle H, \bigsqcup, \bigsqcap, \Rightarrow, 0, j\rangle$. We will prove that $\mathcal{H}$ is isomorphic to $Dw(j\mathcal{H}) = \langle Dw(jH), \bigcup, \bigcap, \Rightarrow, \{0\}, dj\rangle$. An isomorphism h from $Dw(j\mathcal{H})$ to $\mathcal{H}$ is defined as follows:

$$h(X) = \bigsqcup X, \text{ for every } X \in Dw(jH).$$

It is clear that h is a function from $Dw(jH)$ to H, i.e. from $Dw(j\mathcal{H})$ to $\mathcal{H}$. First, we will show that h preserves the nucleus. Since $\bigsqcup^j$ is the join in $j\mathcal{H}$, we have $djX = \downarrow \bigsqcup^j X$. So, the following holds:

$$h(dj(X)) = \bigsqcup \downarrow \bigsqcup{}^j X = \bigsqcup{}^j X = j(\bigsqcup X) = j(h(X)).$$

Second, we will show that h preserves and reflects the order:

$$X \subseteq Y \text{ iff } h(X) \leq h(Y).$$

If $X \subseteq Y$, then clearly $\bigsqcup X \leq \bigsqcup Y$. For the other direction assume that $\bigsqcup X \leq \bigsqcup Y$ and $a \in X$. Then, $a \leq \bigsqcup Y$ and, since a is completely join-irreducible and Y is a downset, it follows that $a \in Y$.

Since h reflects the order, it is injective. Moreover, since every element from $\mathcal{A}$ is the least upper bound of a set of core elements, the function h is also surjective. We have shown that h is a bijection which preserves the nucleus and preserves and reflects the order, which implies that h is an isomorphism. □

Informally speaking, the previous result shows that core-balanced Kripkean algebras correspond exactly to the algebras of propositions generated by our relational Kripke-style semantics discussed in the previous sections, where the set of states supporting an $\mathcal{L}$-formula was always a principal downset (Lemma 1) and the set of states supporting an $\mathcal{L}_{inq}$-formula was always a downset, though not necessarily a principal one (Lemma 6). The operations of the algebraic semantics also match the operations of the relational semantics. We obtain a correspondence between the two frameworks.

Theorem 3. *Let $\mathcal{K}$ be the class of all nuclear τ-models based on core-balanced Kripkean algebras and $\Phi \cup \{\varphi\}$ be a set of $\tau\mathcal{L}_{inq}$-formulas. Then, $\Phi \vDash^{\mathcal{K}}_{alg} \varphi$ iff $\Phi \vDash_{rel} \varphi$.*

Proof. For the right-to-left implication, assume that $\Phi \nvDash^{\mathcal{K}}_{alg} \varphi$. This means that $\bigsqcap_{\psi\in\Phi} |\psi|^{\mathcal{N}}_e \nleq |\varphi|^{\mathcal{N}}_e$, for some nuclear τ-model $\mathcal{N} = \langle \mathcal{A}, U, V\rangle$, where $\mathcal{A}$ is a core-balanced Kripkean algebra, and e is an assignment in $\mathcal{N}$. Thanks to Lemma 16, we can identify $\mathcal{A}$ with $Dw(\mathcal{H})$ for some CD-algebra $\mathcal{H}$. We can construct a τ-model $\mathcal{M} = \langle \mathcal{H}, U, V^{\uparrow}\rangle$, where $V^{\uparrow}(c) = V(c)$, for each individual constant c and, if $V(P)(m_1, \dots, m_n) = {\downarrow}s$, then $V^{\uparrow}(P)(m_1, \dots, m_n) = s$. Then, it can be shown that $||\psi||^{\mathcal{M}}_e = |\psi|^{\mathcal{N}}_e$, for every $\tau\mathcal{L}_{inq}$-formula ψ. It follows that $\bigcap_{\psi\in\Phi} ||\psi||^{\mathcal{M}}_e \nsubseteq ||\varphi||^{\mathcal{M}}_e$, i.e. there is a state which e-supports all formulas from Φ in $\mathcal{M}$ but does not e-support φ. Hence, $\Phi \nvDash_{rel} \varphi$.

For the left-to-right implication, assume that $\Phi \nvDash_{rel} \varphi$. So, there is a τ-model $\mathcal{M} = \langle \mathcal{H}, U, V\rangle$, an assignment e and a state s in $\mathcal{M}$ such that s e-supports all formulas from Φ but not φ. Then, we take $\mathcal{N} = \langle Dw(\mathcal{H}), U, V^{\downarrow}\rangle$, where $V^{\downarrow}$ is defined so that $V^{\downarrow}(c) = V(c)$, for each individual constant c, and for each n-ary predicate P, $V^{\downarrow}(P)(m_1, \dots, m_n) = {\downarrow}V(P)(m_1, \dots, m_n)$. Again, we obtain $||\psi||^{\mathcal{M}}_e = |\psi|^{\mathcal{N}}_e$, for every $\tau\mathcal{L}_{inq}$-formula ψ, and so $\bigcap_{\psi\in\Phi} |\psi|^{\mathcal{N}}_e \nsubseteq |\varphi|^{\mathcal{N}}_e$. Hence, $\Phi \nvDash^{\mathcal{K}}_{alg} \varphi$. □

Note that this algebraic semantics for inquisitive intuitionistic predicate logic with constant domains extended with information types can be used also as an algebraic semantics for standard first-order inquisitive logic as presented for example in (Grilletti, 2021; Ciardelli, 2023). To adjust it to this case, we just need to require that the core forms a complete atomic Boolean algebra. If $\mathcal{N} = \langle \mathcal{H}, U, V\rangle$ is a nuclear τ-model where $\mathcal{H}$ is a Kripkean algebra in which the core forms a complete atomic Boolean algebra, then the atoms from the core can be viewed as first-order structures on the common domain U, other elements of the core correspond to sets of such first-order structures, i.e. information states in the sense of the standard setting, and the elements outside the core coincide with non-empty downsets of sets of first-order structures, i.e. propositions of the standard semantics. Moreover, the usual support conditions correspond to the algebraic operations in $\mathcal{H}$.

However, the algebraic semantics based on Kripkean algebras is not completely natural from an algebraic point of view. Kripkean algebras do not form a well-behaved class of algebraic structures. In particular, this class is not closed under c-homomorphic images which preserve validity of formulas. Consider the example of two nuclear algebras in Fig. 3. The black dots represent the core elements, the white dot in the algebra on the left is an element outside the core. The algebra on the left is a Kripkean algebra but the algebra on the right is not. For instance, we see that the top element is in the core but it is not join-irreducible.

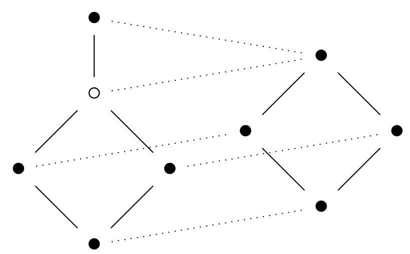

Figure 3: A non-Kripkean c-homomorphic image of a Kripkean algebra

In order to obtain a better behaving class of algebraic structures for inquisitive logic, we introduce the following notion of an inquisitive algebra. An *inquisitive algebra* is a dense nuclear algebra $\mathcal{H} = \langle H, \bigsqcup, \bigsqcap, \Rightarrow, 0, j\rangle$ satisfying the following two conditions for any index sets I, J and every $a_{ij}, a, b_i \in jH$, where $i \in I$ and $j \in J$:

(I1) $\bigsqcap_{i\in I} \bigsqcup_{j\in J} a_{ij} = \bigsqcup_{f\colon I\to J} \bigsqcap_{i\in I} a_{if(i)}$,

(I2) $a \Rightarrow \bigsqcup_{i\in I} b_i = \bigsqcup_{i\in I}(a \Rightarrow b_i)$.

Lemma 17. *Every Kripkean algebra is inquisitive.*

Proof. We will show something slightly stronger, namely that the following holds for every Kripkean algebra $\mathcal{H} = \langle H, \bigsqcup, \bigsqcap, \Rightarrow, 0, j\rangle$ and all $x_{ij}, x, y_i \in H$, where $i \in I$ and $j \in J$:

(a) $\bigsqcap_{i\in I} \bigsqcup_{j\in J} x_{ij} = \bigsqcup_{f\colon I\to J} \bigsqcap_{i\in I} x_{if(i)}$,

(b) $x \Rightarrow \bigsqcup_{i\in I} y_i = \bigsqcup_{i\in I}(x \Rightarrow y_i)$, if $x \in jH$.

(a) Note that $\bigsqcup_{f\colon I\to J} \bigsqcap_{i\in I} x_{if(i)} \leq \bigsqcap_{i\in I} \bigsqcup_{j\in J} x_{ij}$ holds in every nuclear algebra. We prove the other direction. Since $\mathcal{H}$ is Kripkean, we can express $\bigsqcap_{i\in I} \bigsqcup_{j\in J} x_{ij}$ as the join of some set of completely join-irreducible elements. So, we can write $\bigsqcap_{i\in I} \bigsqcup_{j\in J} x_{ij} = \bigsqcup_{k\in K} a_k$. Take any $k \in K$. Then, for every $i \in I$, $a_k \leq \bigsqcup_{j\in J} x_{ij}$. Since a_k is completely join-irreducible, it follows that, for every $i \in I$, there is some $j \in J$ such that $a_k \leq x_{ij}$. In other words, there is a mapping $f\colon I \to J$ such that, for every $i \in I$, $a_k \leq x_{if(i)}$ and hence $a_k \leq \bigsqcap_{i\in I} x_{if(i)}$. Since there is such a mapping for every $k \in K$, it follows that $\bigsqcup_{k\in K} a_k \leq \bigsqcup_{f\colon I\to J} \bigsqcap_{i\in I} x_{if(i)}$, i.e. $\bigsqcap_{i\in I} \bigsqcup_{j\in J} x_{ij} \leq \bigsqcup_{f\colon I\to J} \bigsqcap_{i\in I} x_{if(i)}$, as desired.

(b) Assume $x \in jH$. Note that $\bigsqcup_{i\in I}(x \Rightarrow y_i) \leq x \Rightarrow \bigsqcup_{i\in I} y_i$ holds in every nuclear algebra. We prove the other direction. Since $\mathcal{H}$ is inquisitive, we can express $x \Rightarrow \bigsqcup_{i\in I} y_i$ as the join of some set of completely join-irreducible elements. Hence, we can write $x \Rightarrow \bigsqcup_{i\in I} y_i = \bigsqcup_{k\in K} a_k$. So, for every a_k, we have $a_k \leq x \Rightarrow \bigsqcup_{i\in I} y_i$, and hence $a_k \sqcap x \leq \bigsqcup_{i\in I} y_i$. We have

$a_k, x \in jH$ and, since jH is closed under meet, we obtain that $a_k \sqcap x$ is completely join-irreducible. So, for some $i \in I$, $a_k \sqcap x \leq y_i$ and thus $a_k \leq x \Rightarrow y_i$. Since this holds for every a_k, we obtain $\bigsqcup_{k \in K} a_k \leq \bigsqcup_{i \in I}(x \Rightarrow y_i)$, i.e. $x \Rightarrow \bigsqcup_{i \in I} y_i \leq \bigsqcup_{i \in I}(x \Rightarrow y_i)$. □

The next lemma provides a characterization of the class of homomorphic c-images of Kripkean algebras.

Lemma 18. *A (core-balanced) nuclear algebra is inquisitive if and only if it is a homorphic c-image of a (core-balanced) Kripkean algebra.*

Proof. First, we prove the right-to-left implication. Take any nuclear algebra $\mathcal{A}$ such that there is a c-homomorphism h from some Kripkean algebra $\mathcal{B}$ to $\mathcal{A}$. So, for every element a from the core of $\mathcal{A}$, there is an element a^* from the core of $\mathcal{B}$ such that $h(a^*) = a$. By Lemma 17, $\mathcal{B}$ satisfies the properties (I1) and (I2). One can easily observe that these properties are preserved by the homomorphism h. For example, the preservation of (I1) can be shown as follows: It holds that $\bigsqcap^{\mathcal{B}}_{i \in I} \bigsqcup^{\mathcal{B}}_{j \in J} a^*_{ij} = \bigsqcup^{\mathcal{B}}_{f\colon I \to J} \bigsqcap^{\mathcal{B}}_{i \in I} a^*_{if(i)}$. Moreover, $h(\bigsqcap^{\mathcal{B}}_{i \in I} \bigsqcup^{\mathcal{B}}_{j \in J} a^*_{ij}) = \bigsqcap^{\mathcal{A}}_{i \in I} \bigsqcup^{\mathcal{A}}_{j \in J} a_{ij}$, and $h(\bigsqcup^{\mathcal{B}}_{f\colon I \to J} \bigsqcap^{\mathcal{B}}_{i \in I} a^*_{if(i)}) = \bigsqcup^{\mathcal{A}}_{f\colon I \to J} \bigsqcap^{\mathcal{A}}_{i \in I} a_{if(i)}$. Thus (I1) holds in $\mathcal{A}$. In the case of (I2), one can reason in a similar way.

Now, in order to prove the left-to-right implication, take any nuclear algebra $\mathcal{A} = \langle A, \bigsqcup, \bigsqcap, \Rightarrow, 0, j \rangle$ that satisfies (I1) and (I2). Take the Kripkean algebra $Dw(j\mathcal{A}) = \langle Dw(jA), \bigcup, \bigcap, \Rrightarrow, \{0\}, dj \rangle$ and define a function $h\colon Dw(jA) \to A$ by $h(X) = \bigsqcup X$. We will show that h is a c-homomorphism from $Dw(j\mathcal{A})$ to $\mathcal{A}$.

First, observe that for every $a \in jA$, we obtain $a = h(\downarrow a)$ and thus $jA \subseteq h(Dw(jA))$. We have to further show that h preserves all the operations. In particular, we will show the following for every non-empty downsets X, Y, X_i $(i \in I)$ in $Dw(jA)$:

(a) $h(\bigcup_{i \in I} X_i) = \bigsqcup_{i \in I} h(X_i)$,

(b) $h(\bigcap_{i \in I} X_i) = \bigsqcap_{i \in I} h(X_i)$,

(c) $h(X \Rrightarrow Y) = h(X) \Rightarrow h(Y)$,

(d) $h(\{0\}) = 0$.

(e) $h(dj(X)) = j(h(X))$.

The equation (a) is obvious, it expresses that $\bigsqcup \bigcup_{i \in I} X_i = \bigsqcup_{i \in I} \bigsqcup X_i$.

The equation (b) says that $\bigsqcup \bigcap_{i \in I} X_i = \bigsqcap_{i \in I} \bigsqcup X_i$. First, assume that $a \in \bigcap_{i \in I} X_i$. Then, for each $i \in I$, $a \leq \bigsqcup X_i$. It follows that $a \leq \bigsqcap_{i \in I} \bigsqcup X_i$.

Since this holds for each $a \in \bigcap_{i \in I} X_i$, we obtain $\bigsqcup \bigcap_{i \in I} X_i \leq \bigsqcap_{i \in I} \bigsqcup X_i$. Now in order to prove the other direction, we define:

$$z = \bigsqcup_{f \in \prod_{i \in I} X_i} \bigsqcap_{i \in I} f(i).$$

By (I1), we obtain $\bigsqcap_{i \in I} \bigsqcup X_i = z$. Moreover, $\bigsqcap_{i \in I} f(i) \in \bigcap_{i \in I} X_i$, and hence $\bigsqcap_{i \in I} f(i) \leq \bigsqcup \bigcap_{i \in I} X_i$, for each $f \in \prod_{i \in I} X_i$. So, $z \leq \bigsqcup \bigcap_{i \in I} X_i$. Therefore, $\bigsqcap_{i \in I} \bigsqcup X_i \leq \bigsqcup \bigcap_{i \in I} X_i$.

The equation (c) says that $\bigsqcup X \Rightarrow \bigsqcup Y = \bigsqcup (X \Rrightarrow Y)$. This can be proved in the following steps:

$$\begin{aligned} \bigsqcup X \Rightarrow \bigsqcup Y &= \bigsqcap_{a \in X} (a \Rightarrow \bigsqcup_{b \in Y} b) \\ &= \bigsqcap_{a \in X} \bigsqcup_{b \in Y} (a \Rightarrow b) \\ &= \bigsqcup_{f \colon X \to Y} \bigsqcap_{a \in X} (a \Rightarrow f(a)) \\ &= \bigsqcup (X \Rrightarrow Y). \end{aligned}$$

The step from the first line to the second line is an application of the property (I2). The step from the second line to the third line is an application of the property (I1). To see this, just identify I with X and J with Y and set $a_{ij} = i \Rightarrow j$. Then, (I1) gives us exactly the equation between the second and the third line. It remains to prove the equation between the third and the last line, i.e. we need to show that

$$\bigsqcup_{f \colon X \to Y} \bigsqcap_{a \in X} (a \Rightarrow f(a)) = \bigsqcup \{a \in jA \mid {\downarrow} a \cap X \subseteq Y\}.$$

First, take any $f \colon X \to Y$. Fix $b = \bigsqcap_{a \in X} (a \Rightarrow f(a))$. Since X and Y are non-empty downsets of jA and jA is closed under $\Rightarrow$ and $\bigsqcap$, it holds that $b \in jA$. If $d \in {\downarrow} b \cap X$, we obtain $d \leq f(d)$ and so $d \in Y$, for $f(d) \in Y$ and Y is a downset. We have shown that, for every $f \colon X \to Y$, there is a $b \in jA$, namely $b = \bigsqcap_{a \in X} (a \Rightarrow f(a))$, such that ${\downarrow} b \cap X \subseteq Y$. It follows that $\bigsqcup_{f \colon X \to Y} \bigsqcap_{a \in X} (a \Rightarrow f(a)) \leq \bigsqcup \{a \in jA \mid {\downarrow} a \cap X \subseteq Y\}$.

Second, take any $b \in jA$ such that ${\downarrow} b \cap X \subseteq Y$. We define a function $f_b \colon X \to Y$ by $f_b(a) = b \sqcap a$. It follows that, for every $a \in X$, $b \leq a \Rightarrow f_b(a)$ and thus $b \leq \bigsqcap_{a \in X} (a \Rightarrow f_b(a))$. It follows that $\bigsqcup \{a \in jA \mid {\downarrow} a \cap X \subseteq Y\} \leq \bigsqcup_{f \colon X \to Y} \bigsqcap_{a \in X} (a \Rightarrow f(a))$, which finishes the proof of (c).

We also have $\bigsqcup \{0\} = 0$, which shows that (d) holds as well. (e) can be shown in the same way as in the proof of Lemma 16:

$$h(dj(X)) = \bigsqcup {\downarrow} \bigsqcup^j X = \bigsqcup^j X = j(\bigsqcup X) = j(h(X)).$$

□

As an application of the previous result, we obtain a more general algebraic semantics for the consequence relation $\vDash_{rel}$ in the full language $\mathcal{L}_{inq}$.

Theorem 4. *Let $\mathcal{I}$ be the class of all nuclear τ-models based on inquisitive algebras. Let $\Phi \cup \{\varphi\}$ be a set of $\tau\mathcal{L}_{inq}$-formulas. Then, $\Phi \vDash^{\mathcal{I}}_{alg} \varphi$ iff $\Phi \vDash_{rel} \varphi$.*

Proof. Since every Kripkean algebra is in $\mathcal{I}$, the left-to-right implication is immediate. For the other direction, assume $\Phi \nvDash^{\mathcal{I}}_{alg} \varphi$. So, $\bigsqcap_{\psi\in\Phi} |\psi|^{\mathcal{N}}_e \not\leq |\varphi|^{\mathcal{N}}_e$, for some $\mathcal{N} = \langle \mathcal{A}, U, V\rangle$, where $\mathcal{A}$ is an inquisitive algebra and e is an assignment in $\mathcal{N}$. This means that $\bigsqcap_{\psi\in\Phi} |\psi|^{\mathcal{N}}_e \Rightarrow |\varphi|^{\mathcal{N}}_e \neq 1^{\mathcal{A}}$. By Lemma 18, $\mathcal{A}$ is a c-homomorphic image of some inquisitive algebra $\mathcal{B}$. Let h be the c-homomorphism from $\mathcal{B}$ to $\mathcal{A}$. Since $jB \subseteq h(jA)$, there is an $\mathcal{N}' = \langle \mathcal{B}, U, V'\rangle$ such that $|\psi|^{\mathcal{N}}_e = h(|\psi|^{\mathcal{N}'}_e)$, for every $\tau\mathcal{L}_{inq}$-formula ψ. Hence, $\bigsqcap_{\psi\in\Phi} |\psi|^{\mathcal{N}'}_e \Rightarrow |\varphi|^{\mathcal{N}'}_e \neq 1^{\mathcal{B}}$. It follows that $\bigsqcap_{\psi\in\Phi} |\psi|^{\mathcal{N}'}_e \not\leq |\varphi|^{\mathcal{N}'}_e$ and so $\Phi \nvDash^{\mathcal{K}}_{alg} \varphi$. We obtain $\Phi \nvDash_{rel} \varphi$ by Theorem 3. □

To the best of our knowledge, algebraic semantics for first-order inquisitive logic has not yet been explored in the literature. The approach that we have developed in this section can be viewed as a first-order version of the algebraic semantics for propositional intuitionistic inquisitive logic described in (Punčochář & Pezlar, 2024). Our approach is applicable to various relational semantic frameworks for propositional intuitionistic inquisitive logic, such as those introduced in (Punčochář, 2016) and (Ciardelli, Iemhoff & Yang, 2020), where the algebras of propositions naturally form nuclear algebras. The approach is significantly different from the previous version of the propositional algebraic semantics from (Punčochář, 2021) where the nucleus operation was not considered and the declarative propositions, i.e. our information tokens, were characterized as prime elements in the algebra of propositions.

The relevance of the abstract notion of nucleus for inquisitive semantics was first revealed in (Holliday, 2020), which however works in a different setting of Beth semantics and considers a notion of nucleus that does not naturally validate the split principle which plays a key role in our framework in the form of the condition (I2). Other algebraic treatments of inquisitive logic include (Bezhanishvili, Grilletti & Holliday, 2019; Bezhanishvili, Grilletti & Quadrellaro, 2019; Frittella et al., 2016; Roelofsen, 2013; Quadrellaro, 2022). They are all focused on propositional level and except (Quadrellaro, 2022) they are concerned with the classical version of inquisitive logic. For this reason, they employ the double negation operation rather than the more general notion of nucleus.

5 Conclusion

In this paper we have studied the operations of inquisitive disjunction and inquisitive existential quantifier in the context of intuitionistic predicate logic with constant domains. We have explained how these operators allow us to express types of intuitionistic propositions. We have formulated a relational semantics for these operators and proved completeness with respect to a fragment of the language. The question whether the logic is recursively axiomatizable for the whole language remains open, but we have also formulated an algebraic semantics for this logic and we hope that the algebraic characterization that we provided might be a useful intermediate step in the solution of this open problem.

The completeness results that we have presented (together with the open problem), for each of the three considered languages, are summarized in the following table:

	$\mathcal{L}$	$\mathcal{L}^-_{inq}$	$\mathcal{L}_{inq}$
Algebraic consequences	$\vDash_{alg}$	$\vDash^{\mathcal{K}}_{alg} = \vDash^{\mathcal{I}}_{alg}$	$\vDash^{\mathcal{K}}_{alg} = \vDash^{\mathcal{I}}_{alg}$
	$\Vert$	$\Vert$	$\Vert$
Informational semantics	$\vDash_{rel}$	$\vDash_{rel}$	$\vDash_{rel}$
	$\Vert$	$\Vert$	$\Vert$?
Proof systems	$\vdash_{ilcd}$	$\vdash_{inq}$	$\vdash_{inq}$

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