Projectivity and unification in substructural logics of generalized rotations

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Abstract

We develop a unifying approach to study projectivity and unification in substructural logics corresponding to varieties of residuated lattices generated by generalized rotation constructions. These include many interesting varieties especially in the realm of mathematical fuzzy logics. Our main results pertain what we shall call radical-determined varieties of rotations, which include all of the most relevant varieties in this framework. We characterize free algebras in a radical-determined variety V of rotations in terms of weak Boolean products of rotations of free algebras in the variety of radicals, the latter being the intersections of maximal filters of the algebras in V. Then we use such description to study projectivity in these varieties of rotations, characterizing finitely generated projective algebras. Moreover, we show that the strong unitary unification type of a variety of radicals implies the strong unitary type for the generated variety of rotations, which can be used to deduce the decidability of the admissibility of rules. As relevant applications of our general results, we obtain that product logic and nilpotent minimum logic have (strong) unitary unification type.

1 Introduction

In this paper we develop a uniform approach to study unification problems in a large class of algebraizable logics, which we shall call *substructural logics of generalized rotations*.

The start of the study of unification problems can be traced back to Herbrand's doctoral thesis, and in the pioneering work of Robinson on resolution [57]. The classical syntactic unification problem consists of a pair of terms (s, t), built from functions symbols and variables, and a solution or *unifier* is a uniform replacement of the variables occurring in s and t by other terms in the same language that makes s and t identical. Here we are interested in unification problems where syntactical identity is replaced by equality modulo logical equivalence, or, from an algebraic point of view, modulo an equational theory. This approach to unification, originating in influential works such as [56], has acquired increasing interest in recent years [7, 8]. Ghilardi in particular shows in [42] that, in an algebraizable logic, the study of a unification problem modulo logical equivalence can be carried out purely in the algebraic framework. Moreover, the *unification type*, which essentially measures the cardinality of the set of "best solutions" to a unification problem, can be studied algebraically as well. In more detail, if a logic has a variety V as equivalent algebraic semantics in the sense of Blok-Pigozzi [11], a unification problem corresponds to a *finitely presented* algebra in V (a finitely generated quotient of a finitely generated free algebra), and a unifier consists in a homomorphism to a *projective* algebra in V (that is, a retract of a free algebra in V). Thus, the study of projective algebras in varieties corresponding to a logic gives relevant information on the logical unification problems.

Our study is grounded in the unifying framework of substructural logics. The latter are a large class of logical systems containing a variety of important non-classical logics, such as intuitionistic logic, mathematical fuzzy logics, relevance logics, linear logic to name a few, and encompass classical logic as well as a limit case. The importance of substructural logics as a framework is given by the semantical approach which allows to study all these logics uniformly, as axiomatic extensions of the Full Lambek Calculus $\mathbb{FL}$ (see [41]). A key fact is indeed that substructural logics are all algebraizable in the sense of Blok-Pigozzi, that is, the logical deducibility relation is characterized by the equational consequence in the corresponding equivalent algebraic semantics, which allows to study logical properties from the algebraic point of view and vice versa. The equivalent algebraic semantics are classes of residuated lattice-ordered monoids, or *residuated lattices* for short. The property they all share is indeed the residuation property, which in algebraic models can be expressed as: $a \cdot b \leq c$ if and only if $b \leq a \to c$.

In this work, we will show new results involving a large class of substructural logics, which on the algebraic side are characterized by the fact that they are generated by particular *rotations* of algebras introduced in [19]. In particular, this class includes many logics that are relevant in the Mathematical Fuzzy Logic framework, which was first introduced by Hájek in the influential 1998 book [45]. Mathematical fuzzy logics, here seen as axiomatic extensions of Esteva and Godo's Monoidal t-norm based Logic $\mathbb{MTL}$ [35], are meant to interpret reasoning with *degrees of truth*. They play a relevant role in the framework of substructural logics, as they correspond to those systems satisfying weakening (i.e., extra assumptions and conclusions can be added to derivations), exchange (the main conjunction connective is commutative), and where the intended truth-values of formulas are *comparable*. Substructural logics of generalized rotations include many relevant mathematical fuzzy logics, such as finite- and infinite-valued Gödel logic, product logic, nilpotent minimum logic, the logic associated to Chang's MV-algebra, the logics corresponding to Stonean BL-algebras and many others. The rotation construction introduced in [19] allows a uniform study of this large class of logics. This framework allows in particular to transfer results from relatively tame logics that are extensions of Hájek Basic Logic [45] to axiomatic extensions of $\mathbb{MTL}$ previously either unknown or hard to study (see [3], [2] for other extension results). We remark that our work does not restrict to the fuzzy logic framework alone: other interesting systems that can be seen as substructural logics of generalized rotation include for instance Nelson Constructive Logic with strong negation [18] and the intermediate De Morgan Logic (whose corresponding equivalent algebraic semantics are, respectively, regular Nelson lattices and Stonean Heyting algebras).

As a first main result, we give a characterization of the free algebras for the varieties of interest, which we call *varieties of generalized n-rotations*, for a fixed $n \in \mathbb{N}, n \geq 2$. This, from a logical point of view, yields a description of the algebras of formulas of the logics of generalized rotations. The description will prove to be particularly transparent and useful in a large class of varieties, that we will call *radical-determined*, which includes all of the relevant examples. In such a variety V, free algebras can be described as particular weak Boolean products of rotations of free algebras in the variety of residuated lattices obtained as the class of radicals (i.e., intersections of maximal filters) of algebras in V. This is inspired by, and generalizes, several previous works, for instance by Busaniche [15], Busaniche and Cignoli [16], Cignoli [23], Cignoli and Torrens [25, 26]. Building on our results, we then characterize finitely generated projective algebras in the relevant varieties of rotations as finite direct products of rotations of projective algebras in the variety of radicals. Moreover, we show how properties related to unification and admissibility are also transferred from the variety of radicals to the variety of rotations. Most notably, what we will define as *strong* unitary type transfers from the variety of radicals to the corresponding variety of rotations. It is relevant to stress that strong unitary unification type implies the decidability of the admissibility of rules for the logics whose equational theory is decidable.

The general results we obtain show a unifying framework to study unification in a large class of substructural logics of rotations. As particularly relevant new results, we obtain that the unification problem of product logic, the main fuzzy logic for which the unification type was still unknown, has (strong) unitary type, and the same holds for the logic of nilpotent minimum.

2 Preliminaries

While we refer the reader interested in axiomatizations of substructural logics to [41], we shall develop our study in the universal algebraic framework. For the unexplained basic notions from Universal Algebra, we refer the reader to [14]. We proceed to introduce the algebras which are objects of our study, that is, (bounded) commutative and integral residuated lattices. A *commutative integral residuated lattice* (or *CIRL*) is an algebra $\langle A, \vee, \wedge, \cdot, \to, 1\rangle$ such that

1. $\langle A, \vee, \wedge, 1\rangle$ is a lattice with largest element 1;
2. $\langle A, \cdot, 1\rangle$ is a commutative monoid;
3. $(\cdot, \to)$ form a residuated pair w.r.t. the lattice ordering, i.e. for all $a, b, c \in A$

$$a \cdot b \leq c \qquad \text{if and only if} \qquad b \leq a \to c.$$

In what follows, we will usually write ab for $a \cdot b$. A *bounded* commutative residuated lattice is a commutative integral residuated lattice with an extra constant 0 in the signature that is the least element in the lattice order. In bounded structures we can also define a negation connective in the usual way: $\neg x := x \to 0$. Since residuation can be expressed equationally [12, 49], (bounded) commutative and integral residuated lattices form varieties called $(\mathsf{FL_{ew}})$ CIRL. The name $\mathsf{FL_{ew}}$ comes from logic, indeed this variety corresponds to the Full Lambek calculus with exchange and weakening $\mathbb{FL}_{ew}$. Mathematical fuzzy logics seen as extensions of Esteva-Godo's *Monoidal t-norm based logic* $\mathbb{MTL}$ [35] are extensions of $\mathbb{FL}_{ew}$. Their equivalent algebraic semantics is the variety of MTL-*algebras*, that is given by $\mathsf{FL_{ew}}$-algebras further satisfying *prelinearity*:

$$(x \to y) \vee (y \to x) \approx 1. \tag{prel}$$

It can be shown ([12, 49]) that a subvariety of $\mathsf{FL_{ew}}$ satisfies the prelinearity equation (prel) if and only if it is generated by totally ordered algebras (also called *chains*). Such varieties are called *representable* or *semilinear*, and MTL-algebras are the largest subvariety of $\mathsf{FL_{ew}}$ that is representable. This property is the algebraic counterpart to the mentioned *comparability* of truth-values characterizing mathematical fuzzy logics. Hájek's Basic Logic $\mathbb{BL}$, the logic of continuous t-norms [45], has as equivalent algebraic semantics the variety of BL-algebras BL, which are MTL-algebras satisfying *divisibility*:

$$x(x \to y) \approx y(y \to x); \tag{div}$$

The most relevant subvarieties of BL are the varieties of MV-algebras, Gödel algebras, and product algebras, which we denote with $\mathsf{MV}, \mathsf{G}, \mathsf{P}$, respectively, and that correspond to infinite-valued Łukasiewicz logic, Gödel logic, and product logic. These three subvarieties of BL are generated by standard algebras over the real-unit interval $[0, 1]$ determined by, respectively, Łukasiewicz, Gödel, and product t-norm (playing the role of the monoidal operation). They play a fundamental role since by Mostert-Shields' Theorem all continuous t-norms can be constructed by these three via the *ordinal sum* construction [50] (the algebraic counterpart of this result is in [1]).

Some subvarieties of MV will play a special role in this paper. The variety MV is the subvariety of BL-algebras satisfying involution:

$$\neg\neg x \approx x. \tag{inv}$$

Finite chains (i.e., totally ordered algebras) in MV can be described as particular subalgebras of the standard MV-algebra

$$[0, 1]_{Ł} = ([0, 1], \cdot_{Ł}, \to_{Ł}, \min, \max, 0, 1),$$

where $x \cdot_{Ł} y = \max(0, x + y - 1), x \to_{Ł} y = \min(1, 1 - x + y)$. For $n \in \mathbb{N}, n \geq 2$, the n-valued MV-chain $\mathbf{Ł}_n$ is the subalgebra of $[0, 1]_{Ł}$ with domain:

$$(0,\ \frac{1}{n-1},\ \ldots,\ \frac{n-2}{n-1},\ 1).$$

For $n \geq 2$, we denote with $\mathsf{MV_n}$ the subvariety of MV generated by $\mathbf{Ł}_n$ ([43]). We remark that a finite MV-chain $\mathbf{Ł}_m$ belongs to $\mathsf{MV_n}$ if and only if $m - 1$ divides $n - 1$ (in symbols $m - 1|n - 1$). $\mathsf{MV_n}$-algebras are *locally finite*, that is, finitely generated subalgebras are finite. $\mathsf{MV_n}$-algebras will act as a *skeleton* of algebras in varieties of rotations, in a sense that we will make precise in the following subsection.

With respect to the structure theory, algebras in $\mathsf{FL_{ew}}$ are very well behaved, indeed, any subvariety of $\mathsf{FL_{ew}}$ is *ideal determined* with respect to 1, in the sense of [44] (but see also [5]). In particular this implies that congruences are totally determined by their 1-blocks (i.e., the set of elements in relation with 1). If $\mathbf{A} \in \mathsf{FL_{ew}}$, the 1-block of a congruence of $\mathbf{A}$ is called a *congruence filter* (or *filter* for short). Filters corresponds to the *deductive filters* induced by the corresponding logic, which, in the algebra on formulas, are exactly deductively closed theories (see [41]). It can be shown that a filter of $\mathbf{A}$ is a lattice filter containing 1 and closed under multiplication [41]. Filters form an algebraic lattice isomorphic with the congruence lattice of $\mathbf{A}$ and if $X \subseteq A$ then the filter generated by X is

$$\mathrm{Fil}_{\mathbf{A}}(X) = \{a \in A : x_1 \cdot \ldots \cdot x_n \leq a, \text{for some } n \in \mathbb{N} \text{ and } x_1, \ldots, x_n \in X\}.$$

The isomorphism between the filter lattice and the congruence lattice is given by the maps:

$$\theta \longmapsto 1/\theta$$
$$F \longmapsto \theta_F = \{(a, b) : a \to b,\ b \to a \in F\},$$

where θ is a congruence and F a filter. In what follows, if $F = 1/\theta$ is a filter of an algebra $\mathbf{A}$, we shall write the corresponding quotient with either $\mathbf{A}/\theta$ or $\mathbf{A}/F$.

Given an $\mathsf{FL_{ew}}$ algebra $\mathbf{A}$, we call *Boolean skeleton of* $\mathbf{A}$ and write $\mathsf{Bool}(\mathbf{A})$, the set of Boolean (i.e., complemented) elements in $\mathbf{A}$: $x \in A$ such that $x \vee \neg x = 1$, $x \wedge \neg x = 0$.

Moreover, given an $\mathsf{FL_{ew}}$-algebra $\mathbf{A}$, we will call *radical* the intersection of its maximal filters $\mathsf{Rad}(\mathbf{A})$, and *coradical* the set of elements whose negation is in the radical: $\mathsf{Corad}(\mathbf{A}) = \{x \in A : \neg x \in \mathsf{Rad}(\mathbf{A})\}$. Since the radical is a filter itself, it is the subuniverse of a CIRL. Indeed, it follows from the definition that filters are closed under all the operations in the signature $\{\cdot, \to, \wedge, \vee, 1\}$. By a slight abuse of notation we will denote by $\mathsf{Rad}(\mathbf{A})$ both the radical of $\mathbf{A}$ seen as a filter and the corresponding CIRL. Given any $\mathbf{A} \in \mathsf{FL_{ew}}$, $\mathsf{Rad}(\mathbf{A}) \cup \mathsf{Corad}(\mathbf{A})$ is a directly indecomposable subalgebra of $\mathbf{A}$ ([26, Theorem 3.3]), since its only Boolean elements are 0 and 1 (and this property characterizes directly indecomposable $\mathsf{FL_{ew}}$-algebras as shown in [51]).

2.1 Varieties of generalized rotations

We introduce here the needed context on the varieties of algebras that are the main focus of our study, which are generated by a particularly generalized notion of rotation construction.

The *generalized* n*-rotation*, for a fixed $n \geq 2$, is defined in [19] and it is inspired by ideas in [25, 6]. It generalizes both (dis)connected rotation constructions (developed by Jenei [47, 48] for ordered semigroups) and the n-liftings of residuated lattices (see [19]), given by an ordinal sum of a n-valued MV-chain and a CIRL (see the

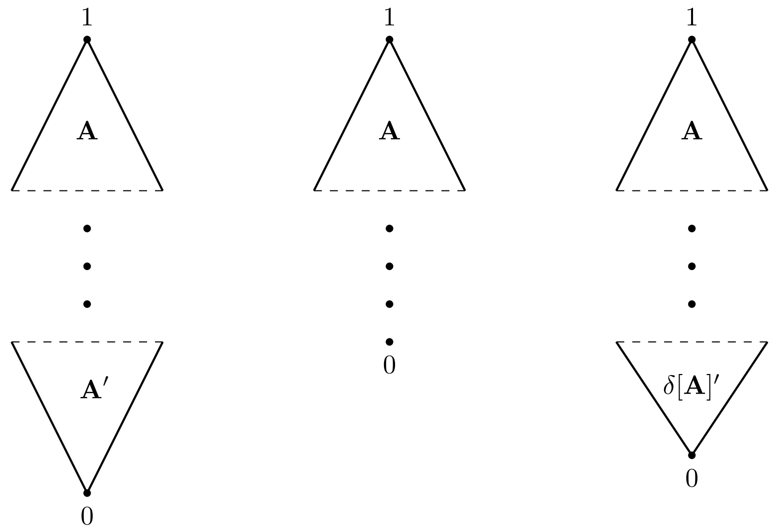


Figure 1: Respectively: a disconnected 5-rotation, a 5-lifting, and a generalized 5-rotation of a CIRL **A**.

center structure in Figure 1 for a picture). Let us first give a pictorial intuition of this construction. In the disconnected rotation, given a CIRL (and also more generally in [39] given a topped residuated lattice, not necessarily commutative or integral), a bounded involutive structure is produced, obtained by attaching below the original CIRL a rotated copy of it. The generalized rotation in [6] takes a CIRL and generates a bounded CIRL, which is not necessarily involutive, by attaching below it a rotated (possibly proper) *nuclear image* of the original. The generalized n-rotation, for $n \geq 2$, further adds a Łukasiewicz chain of n elements, $n-2$ of which are between the original structure and its rotated nuclear image (see the right-most image in Figure 1 for a sketch).

More precisely, let us first define the disconnected rotation of an IRL $\mathbf{A}$ as the $\mathsf{FL_{ew}}$-algebra $\mathbf{A}^*$ whose lattice reduct is given by the union of A and its disjoint copy $A' = \{a' : a \in A\}$ with dualized order, placed below A: for all $a, b \in A$,

$$a' < b, \text{ and } a' \leq b' \text{ iff } b \leq a.$$

In particular, the top element of $\mathbf{A}^*$ is the top 1 of $\mathbf{A}$ and the bottom element of $\mathbf{A}^*$ is the copy $0 := 1'$ of the top 1. $\mathbf{A}$ is a subalgebra, the products in A' are all defined to be the bottom element $0 = 1'$, and furthermore, for all $a, b \in A$,

$$a \cdot b' = (a \to b)', \quad b' \to a = 1;$$

$$a \to b' = (b \cdot a)', \quad a' \to b' = b \to a.$$

A *nucleus* on a residuated lattice $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, 1)$ is a closure operator δ on $\mathbf{A}$ that satisfies $\delta(x)\delta(y) \leq \delta(xy)$, for all $x, y \in A$. It is known that then $\mathbf{A}_\delta = (\delta[A], \wedge, \vee_\delta, \cdot_\delta, \to, \delta(1))$ is a residuated lattice, where $x \vee_\delta y = \delta(x \vee y)$ and $x \cdot_\delta y = \delta(xy)$.

Now, let us call a *rotation operator* for an algebra $\mathbf{A}$ a map $\delta : A \to A$ that is a nucleus and a lattice homomorphism. Given such a rotation operator, the *generalized δ-rotation* $\mathbf{A}^\delta$ (defined in [6]) differs from the disconnected rotation above in that it replaces A' with $\delta[A]' = \{\delta(a)' : a \in A\}$, where $\delta(a)'$ is short for $(\delta(a))'$. Moreover, for all $a \in A$, $b \in \delta[A]$, the only changed operation is:

$$a \to b' = (\delta(ba))'.$$

The *generalized n-rotation via δ*, $\mathrm{Rot}_n^\delta(\mathbf{A})$ or $\mathbf{A}_n^\delta$, of an IRL $\mathbf{A}$ with respect to a rotation operator δ and a natural number $n \geq 2$ is defined on the disjoint union of A^δ and $\{\ell_i : 0 < i < n-1\}$. We also set $\ell_0 = 0$ and $\ell_{n-1} = 1$, the bounds of $\mathbf{A}^\delta$. The order extends the order of $\mathbf{A}^\delta$ by

$$b < \ell_1 < \ldots < \ell_{n-2} < a,$$

for all $a \in A$ and $b \in \delta[A]'$. The operations extend those of $\mathbf{A}^\delta$, of the n-element Łukasiewicz chain $\mathbf{Ł}_n$, where $0 = \ell_0 < \ell_1 < \ldots < \ell_{n-2} < \ell_{n-1} = 1$, and for $0 < i < n-1$, $a \in A$ and $b \in \delta[A]'$:

$$a\ell_i = \ell_i, \quad b\ell_i = 0.$$

The resulting structure is an $\mathsf{FL_{ew}}$-algebra [19]. In particular, if $\delta = id$, we call the construction a *disconnected n-rotation*, and if $\delta(x) = 1$ we call the construction an *n-lifting* (see Figure 1).

Recall that the radical of an FL_{ew}-algebra $\mathbf{A}$ is the intersection of its maximal filters, and its *coradical* the set of elements whose negation is in the radical. In a generalized n-rotation, the starting CIRL is the radical, and its rotated nuclear image is the coradical, which actually coincides with the negations of elements in the radical: $\mathsf{Corad}(\mathbf{A}) = \neg\mathsf{Rad}(\mathbf{A}) = \{\neg x : x \in \mathsf{Rad}(\mathbf{A})\}$. All generalized n-rotations are directly indecomposable FL_{ew}-algebras. Moreover, δ coincides with the restriction to the radical of the double negation $\neg\neg$ of the generalized rotation.

For any $n \in \mathbb{N}, n \geq 2$, we call $\mathsf{MVR_n}$ the subvariety of $\mathsf{FL_{ew}}$ generated by all the generalized n-rotations of CIRLs.

$\mathsf{MVR_n}$, for each $n \geq 2$, can be axiomatized and described using the following unary terms:

$$\begin{aligned}
\nabla_n(x) &= \neg(\neg x^n)^2 && (1)\\
\Delta_n(x) &= (\neg(\neg x)^n)^2 && (2)\\
\beta_n(x) &= (x \vee \neg x) \vee \neg\nabla_n(x \vee \neg x) && (3)\\
\gamma_n(x) &= \Delta_n(x) \wedge (x \vee \nabla_n(x \vee \neg x)). && (4)
\end{aligned}$$

The idea is that ∇_n "separates" the radical of the algebra, Δ_n "separates" the coradical, the image of β_n is the radical, and γ_n is the term that gives the retraction onto an $\mathsf{MV_n}$-algebra.

Now, $\mathsf{MVR_n}$ is the subvariety of $\mathsf{FL_{ew}}$-algebras such that:

- the term functions ∇_n, Δ_n result in Boolean elements;
- γ_n is a retraction (i.e., a homomorphism that is idempotent: $\gamma_n \circ \gamma_n = \gamma_n$) into an $\mathsf{MV_n}$-algebra;
- De Morgan laws hold;
- the coradical is involutive.

Since all these conditions can be expressed equationally (see [19, Definition 4.1]), $\mathsf{MVR_n}$ is indeed a variety. A particularly relevant case is given by $\mathsf{MVR_2}$, where γ_2 is a Boolean retraction term in the sense of [26] (but see also [6, 59]), that is a term that gives a retraction onto the Boolean skeleton of the algebras.

Given any algebra $\mathbf{A} \in \mathsf{MVR_n}$, let us define its *MV-skeleton* to be the $\mathsf{MV_n}$-algebra $\gamma_n(\mathbf{A})$.

Proposition 2.1 ([19]). *Let $\mathbf{A} \in \mathsf{MVR_n}$, $n \geq 2$. Then:*

$$\mathsf{Rad}(\mathbf{A}) = \gamma_n^{-1}(\{1\}) = \beta_n(\mathbf{A}), \mathsf{Corad}(\mathbf{A}) = \gamma_n^{-1}(\{0\}),\ \gamma_n(x) = x \textit{ iff } x \in \gamma_n(\mathbf{A}).$$

Moreover, if $\mathbf{A}$ is directly indecomposable, $A = \mathsf{Rad}(\mathbf{A}) \cup \mathsf{Corad}(\mathbf{A}) \cup \gamma_n(\mathbf{A})$ and

$$\beta_n(x) = \begin{cases} x & \textit{if } x \in \mathsf{Rad}(\mathbf{A}),\\ 1 & \textit{if } x \in \gamma_n(\mathbf{A}),\\ \neg x & \textit{if } x \in \mathsf{Corad}(\mathbf{A}).\end{cases}$$

Given a class K of FL_{ew}-algebras, let the *radical class of* K, in symbols $\mathsf{R_K}$, be:

$$\mathsf{R_K} = \{\mathbf{R} \in \mathsf{CIRL} : \mathbf{R} \text{ is isomorphic to the radical of some } \mathbf{A} \in \mathsf{K}\}$$

Definition 2.2. Given a class K of CIRLs, we call *term-defined rotation* for K a rotation operator for every algebra in K defined by a term t in the language of residuated lattices. We call *term-defined* a subvariety W of $\mathsf{MVR_n}$ if there is a fixed term-defined rotation t_{W} for $\mathsf{R_W}$ which coincides with the double negation operator for every $\mathbf{A}$ in W.

Notice that, for instance, the identity map $\delta(x) = x$ and the map constantly equal to 1, $\delta(x) = 1$, are examples of term-defined rotations for all varieties of CIRLs (and actually, the only examples we know so far). The papers [2, 3] extensively study term-defined subvarieties of $\mathsf{MVR_n}$. The two examples, $\delta(x) = x$ and $\delta(x) = 1$, identify two relevant subvarieties of $\mathsf{MVR_n}$. The variety term-defined by $\delta(x) = x$ is the variety of *involutive* $\mathsf{MVR_n}$-algebras, that is, $\mathsf{MVR_n}$-algebras satisfying the double negation law:

$$\neg\neg x = x. \tag{dn}$$

We shall denote the variety by $\mathsf{IMVR_n}$. The variety term-defined by $\delta(x) = 1$ is instead the variety of $\mathsf{MVR_n}$-algebras that satisfy:

$$\neg\neg(\beta_n(x)) = 1. \tag{l}$$

We call this the *lifting* subvariety of $\mathsf{MVR_n}$, and denote it with $\ell\mathsf{MVR_n}$, since it is generated by n-liftings. While such varieties do not constitute all possible subvarieties of $\mathsf{MVR_n}$-algebras, they do contain all the relevant known examples. As particular subvarieties of $\ell\mathsf{MVR_2}$ we have, among others, the varieties of: Gödel algebras, product algebras, Stonean Heyting algebras, and more generally Stonean residuated lattices. As subvarieties of $\mathsf{IMVR_2}$: DLMV-algebras (that is, the variety generated by perfect MV-algebras), nilpotent minimum algebras without negation fixpoint $\mathsf{NM^-}$, $\mathsf{IBP_0}$-algebras studied in [54]. Varieties of generalized n-rotation where $n = 3$ includes in particular regular Nelson lattices and nilpotent minimum algebras, both subvarieties of $\mathsf{IMVR_3}$. See [6, 19, 59] for more examples and details.

In what follows, our aim will be to describe properties of $\mathsf{MVR_n}$-algebras deriving them from the corresponding properties of their radicals. In general, if we consider a subvariety V of $\mathsf{MVR_n}$, it is easy to see that the radical class of V is closed under homomorphic images and direct products.

We first show a lemma that will be useful in the rest of the paper.

Lemma 2.3. *Let* $\mathbf{R} \in \mathsf{CIRL}$, $n \in \mathbb{N}$ *and* $n \geq 2$, δ *a rotation operator for* $\mathbf{R}$. *Then given* F *a filter of* $\mathbf{R}$, $\mathrm{Rot}_n^{\delta}(\mathbf{R})/F \cong \mathrm{Rot}_n^{\bar{\delta}}(\mathbf{R}/F)$, *where* $\bar{\delta}(x/F) = \delta(x)/F$.

Proof. Notice that $\mathrm{Rot}_n^{\bar{\delta}}(\mathbf{R}/F)$ is well-defined. Indeed $\bar{\delta}$ is a rotation operator for $\mathbf{R}/F$ since δ is the double negation $\neg\neg$ of $\mathrm{Rot}_n^{\delta}(\mathbf{R})$ restricted to $\mathbf{A}$.

We now show that $\mathrm{Rot}_n^{\delta}(\mathbf{R})/F$ is isomorphic to $\mathrm{Rot}_n^{\bar{\delta}}(\mathbf{R}/F)$. First, it follows by the definition of a generalized rotation that the universe of $\mathrm{Rot}_n^{\delta}(\mathbf{R})/F$ is given by

$$\mathsf{Rad}(\mathrm{Rot}_n^{\delta}(\mathbf{R}))/F \ \cup\ \text{Ł}_n/F \ \cup\ \mathsf{Corad}(\mathrm{Rot}_n^{\delta}(\mathbf{R}))/F,$$

and the universe of $\mathrm{Rot}_n^{\bar{\delta}}(\mathbf{R}/F)$ is

$$R/F \ \cup\ \text{Ł}_n \ \cup\ \bar{\delta}[R/F]'.$$

Now, $R/F = \mathsf{Rad}(\mathrm{Rot}_n^{\delta}(\mathbf{R}))/F$, and it is easily seen that $\text{Ł}_n/F \cong \text{Ł}_n$. Consider now $y \in \mathsf{Corad}(\mathrm{Rot}_n^{\delta}(\mathbf{R}))$, then $y = \neg x = \neg\delta(x)$ for some $x \in R$. Thus $y/F = \neg x/F = \neg\delta(x)/F = \neg\bar{\delta}(x/F)$. The claim follows. □

The following is in [26, Theorem 3.3, Corollary 3.4].

Lemma 2.4 ([26])**.** *Let* $\mathbf{A} \in \mathsf{FL_{ew}}$*, then* $\mathsf{Rad}(\mathbf{A}) \cup \mathsf{Corad}(\mathbf{A})$ *is the domain of a directly indecomposable subalgebra of* $\mathbf{A}$*. Therefore, given any variety* V *of* $\mathsf{FL_{ew}}$*-algebras,* $\mathsf{R_V} = \mathsf{R}_{\mathsf{V}_{\mathrm{di}}}$*, where* V_{di} *is the class of directly indecomposable algebras in* V*.*

Proposition 2.5. *Let* V *be a variety of* $\mathsf{MVR_n}$*-algebras. The radical class of* V*,* $\mathsf{R_V}$*, is closed under homomorphic images and direct products.*

Proof. By Lemma 2.4, $\mathsf{R_V} = \mathsf{R}_{\mathsf{V}_{\mathrm{di}}}$, where V_{di} is the class of directly indecomposable algebras in V. By [19, Theorem 4.8], every directly indecomposable algebra in V is a generalized m-rotation of its radical, for some m such that $m-1$ divides $n-1$.

From Lemma 2.3 it follows that $\mathsf{R}_{\mathsf{V}_{\mathrm{di}}} (= \mathsf{R_V})$ is closed under quotients, thus, equivalently, it is closed under homomorphic images.

We now check closure under direct products. By Proposition 2.1, an element $x \in \mathbf{A} \in \mathsf{V}$ is in the radical of $\mathbf{A}$ if and only if $\gamma_n(x) = 1$. Since in a direct product the operations are defined componentwise, it follows that the radical of a direct product of algebras in V is the direct product of their radicals. Therefore, $\mathsf{R_V}$ is closed under direct products. □

We have shown that given an algebra $\mathbf{R} \in \mathsf{R_V}$, its homomorphic images are in $\mathsf{R_V}$, and if we take a family of algebras $\{\mathbf{R}_i\}_{i\in I}$ in $\mathsf{R_V}$, their direct product is in $\mathsf{R_V}$ as well. Let us now show that in general, given a subvariety V of $\mathsf{MVR_n}$, the class $\mathsf{R_V}$ is not a variety, that is, it is not closed under subalgebras. The reason intuitively is the following. Consider a rotation of a CIRL $\mathbf{R}$, $\mathrm{Rot}_n^{\delta}(\mathbf{R})$, and consider a subalgebra $\mathbf{S}$ of the radical $\mathbf{R}$. This in general generates a subalgebra of $\mathrm{Rot}_n^{\delta}(\mathbf{R})$ whose intersection with $\mathbf{R}$ properly contains $\mathbf{S}$. More precisely, it will contain all of the double-negations

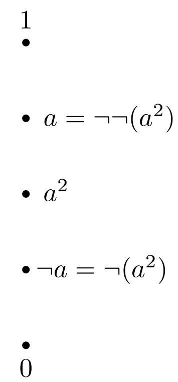


Figure 2: $\mathrm{Rot}_2^\delta(\mathbf{W}_3)$ from Example 2.6.

of elements of $\mathbf{S}$. Thus, unless the double-negation (or, equivalently, the rotation operator δ) is term-definable in the language of CIRL, closure under subalgebras is not granted. We give an example of this fact.

Example 2.6. Consider the 0-free reduct of the three-element MV-chain (or the three-element Wajsberg hoop) $\mathbf{W}_3 = \{1, a, a^2\}$, and the following operator δ: $\delta(a) = \delta(a^2) = a$, $\delta(1) = 1$. It is easy to check that δ is a rotation operator on $\mathbf{W}_3$. Thus we can consider the generalized rotation $\mathrm{Rot}_2^\delta(\mathbf{W}_3)$, as in Figure 2. Since $a = \neg\neg(a^2)$, it can be directly checked that a^2 generates $\mathrm{Rot}_2^\delta(\mathbf{W}_3)$, while it does not generate $\mathbf{W}_3$ as a CIRL.

Let us now consider the subvariety W of MVR_2 generated by $\mathrm{Rot}_2^\delta(\mathbf{W}_3)$. We show that the radical class R_W is not a variety, that is, it is not closed under subalgebras. In particular, notice that $\{1, a^2\}$ is the domain of a subalgebra $\mathbf{W}_2$ of $\mathbf{W}_3$ that is isomorphic to the 0-free reduct of the two-element Boolean algebra. Suppose by way of contradiction that $\mathbf{W}_2$ is the radical of an algebra in W. Then by Lemma 2.4, $\mathbf{W}_2$ is the radical of a directly indecomposable algebra $\mathbf{A}$ in W. Moreover, $\mathbf{A}$ is a generalized rotation of $\mathbf{W}_2$. Since $\mathbf{W}_2$ is simple, $\mathbf{A}$ is subdirectly irreducible. By Jònsson's Lemma (see for instance [14, Theorem 6.8]), $\mathbf{A} \in \mathbf{HSP_u}(\mathrm{Rot}_2^\delta(\mathbf{W}_3))$ (where $\mathbf{H}, \mathbf{S}, \mathbf{P_u}$ are, respectively, the operators of homomorphic images, subalgebras, and ultraproducts). Since $\mathrm{Rot}_2^\delta(\mathbf{W}_3)$ is finite, its ultrapowers are isomorphic to itself ([14, Lemma 6.5]), thus $\mathbf{HSP_u}(\mathrm{Rot}_2^\delta(\mathbf{W}_3)) = \mathbf{HS}(\mathrm{Rot}_2^\delta(\mathbf{W}_3))$. Moreover, it can be directly checked that $\mathbf{HS}(\mathrm{Rot}_2^\delta(\mathbf{W}_3)) = \{\mathrm{Rot}_2^\delta(\mathbf{W}_3), \mathbf{2}, \mathbf{T}\}$ (where $\mathbf{T}$ is the trivial algebra), since every non-constant element generates the whole algebra, and the only non-trivial congruence filter is the radical. Clearly, $\mathbf{W}_2$ is not the radical of either $\mathrm{Rot}_2^\delta(\mathbf{W}_3)$ or $\mathbf{2}$, a contradiction. Thus, $\mathbf{W}_2$ is not in R_W, which is therefore not closed under subalgebras.

Thus, in general, the radical class of a variety of $\mathsf{MVR_n}$-algebras is not a variety. In fact, the radical class of a variety of $\mathsf{MVR_n}$-algebras is better described as a class of algebras with also the rotation operator in the signature.

Definition 2.7. We call a commutative integral residuated lattice with a rotation operator on it $(\mathbf{R}, \delta)$ a *commutative integral rotation algebra*, or *rotation algebra* for short.

Rotation algebras form a variety that we shall denote by CIRot. Since every algebra in the radical class of a variety arises as the kernel of the MV-retraction term, rotation algebras have been called *kernel DL*-algebras in [25, 19]. Here we choose a different naming since we find it more evocative in our framework.

Given a class K of $\mathsf{MVR_n}$-algebras, let the *rotation-radical class of* K, in symbols $\mathsf{RR_K}$, be:

$$\mathsf{RR_K} = \{(\mathbf{R}, \delta) \in \mathsf{CIRot} : (\mathbf{R}, \delta) \text{ is isomorphic to } (\mathsf{Rad}(\mathbf{A}), \neg\neg) \text{ for some } \mathbf{A} \in \mathsf{K}\}.$$

Proposition 2.8. *Let* V *be a variety of* $\mathsf{MVR_n}$*-algebras. Then* $\mathsf{RR_V}$ *is a variety of rotation algebras.*

Proof. The closure under homomorphic images and direct products is shown as in the proof of Proposition 2.5. We show closure under subalgebras. Let $(\mathbf{A}, \delta) \in \mathsf{RR_V}$, so that $\mathrm{Rot}^{\delta}_{m}(\mathbf{A}) \in \mathsf{V}$. Consider a subalgebra $(\mathbf{B}, \delta)$ of $(\mathbf{A}, \delta)$, given by a subalgebra $\mathbf{B}$ of $\mathbf{A}$ that is also closed under δ. Thus, δ is a rotation operator on $\mathbf{B}$, and we can consider $\mathrm{Rot}^{\delta}_{m}(\mathbf{B})$. It follows directly from the construction of generalized rotation that $\mathrm{Rot}^{\delta}_{m}(\mathbf{B})$ is a subalgebra of $\mathrm{Rot}^{\delta}_{m}(\mathbf{A})$, and thus $(\mathbf{B}, \delta) \in \mathsf{RR_V}$. Therefore $\mathsf{RR_V}$ is closed under subalgebras, homomorphic images, and direct products. We conclude that $\mathsf{RR_V}$ is a variety of rotation algebras. ☐

The following is an easy observation.

Lemma 2.9. *Let* V *be a variety of rotation algebras where the rotation operator is term-defined in the language of* CIRL*. Then* V *is term-equivalent to the subvariety* V_δ *of* CIRL *given by the* δ*-free reducts of the algebras in* V*.*

Proposition 2.10. *Let* V *be a variety of term-defined* $\mathsf{MVR_n}$*-algebras. The radical class of* V *is a variety.*

Notice that, in particular, the previous proposition applies to $\mathsf{IMVR_n}$, $\ell\mathsf{MVR_n}$, and all of their subvarieties. E.g., Gödel algebras, product algebras, DLMV-algebras, nilpotent minimum algebras, Stonean residuated lattices, Stonean Heyting algebras, for which this result is already known, but we derive it here uniformly from a more general perspective.

3 Free algebras in varieties of rotations

Free algebras play an important role in the study of algebraic varieties, indeed the equations valid in a variety V are exactly the equations valid in all free algebras (equivalently, in the ω-generated one), and every algebra in V is a homomorphic image of some free algebra. Moreover, free algebras are also relevant in the varieties that are the equivalent algebraic semantics of a logic; indeed, for any $n \in \mathbb{N}$, the free n-generated algebra of the variety is (isomorphic to) the Lindenbaum algebra of formulas with n variables of the corresponding logic.

In order to understand the results of this section, we remind the reader some basic facts about free algebras (for details see [14]). Given a class of algebras K, and an algebra $\mathbf{U}$ generated by a set X, we say that $\mathbf{U}$ has the *universal mapping property* for K over X if every map $\alpha : X \to \mathbf{A} \in \mathsf{K}$ extends to a (unique) homomorphism $\beta : \mathbf{U} \to \mathbf{A}$. An algebra $\mathbf{F} \in \mathsf{K}$ generated by a set X is *free in* K *over* X if it has the universal mapping property for K over X. All classes of algebras closed under isomorphic images, subalgebras, and direct products (thus, in particular, varieties and quasivarieties) contain all of their free algebras. Free algebras over sets with the same cardinality are isomorphic, therefore when we write $\mathbf{F}_{\mathsf{V}}(\kappa)$, for any cardinal κ, we mean (unambiguously, up to isomorphism) the free algebra in V over κ generators.

In this section we show how one can characterize free algebras in varieties generated by generalized n-rotations. In particular, we will identify a large class of varieties including all of the most relevant examples; given a variety V in that class, the κ-generated free algebra in V (κ any cardinal), can be obtained as a weak Boolean product indexed by the Stone space of the Boolean skeleton of the free $\mathsf{MV_n}$-algebra over the same number of generators. Importantly, the factors are generalized rotations of free algebras in $\mathsf{R_V}$. This will follow from a general characterization result for any variety V generated by generalized n-rotations. In the general case, we show that the factors in the representation are rotations of algebras in $\mathsf{RR_V}$, free with respect to the rotation operator (in a sense that will be made precise). The results we find are in analogy with the cases studied in several papers by Cignoli [23], Cignoli and Torrens [25, 26], Busaniche and Cignoli [16], Busaniche [15], which are all particular cases of our more general results.

Let us explain how one can obtain the weak Boolean product representation. Since subvarieties of $\mathsf{FL_{ew}}$ are congruence distributive, they have the *Boolean Factor Congruence property* or BFC: the set of factor congruences of any algebra is a distributive sublattice of its congruence lattice. This notion has been introduced by Chang, Jónsson and Tarski [22] who proved it equivalent to a strict version of the refinement property. The BFC implies that one can, to some extent, use the Stone representation Theorem for Boolean algebras to characterize algebras in less manageable varieties. Indeed algebras in varieties with the BFC are representable as *weak Boolean products* of directly indecomposable algebras [29]. Precisely, a weak Boolean product of a family $\{\mathbf{A}_i\}_{i\in I}$ of algebras is a subdirect product $A \leq \prod_{i\in I} \mathbf{A}_i$, where I can be endowed with a Boolean space topology such that: the set $\{i \in I : a_i = b_i\}$ is open for all $a, b \in A$; if $a, b \in A$ and $N \subseteq I$ is clopen, then the element c, defined by $c_i = a_i$ for $i \in N$ and $c_i = b_i$ for $i \in I \setminus N$, belongs to A.

The relevant fact is that each algebra in a subvariety of $\mathsf{FL_{ew}}$ can be represented as a weak Boolean product of directly indecomposable algebras over the Stone space of its Boolean skeleton. More precisely, let $\mathbf{A} \in \mathsf{FL_{ew}}$, and let $\mathsf{Bool}(\mathbf{A})$ be its Boolean skeleton.

Theorem 3.1 ([26]). *Let $\mathbf{A} \in \mathsf{FL_{ew}}$. Then $\mathbf{A}$ is representable as the weak Boolean product of the family*

$$\{\mathbf{A}/Fil_{\mathbf{A}}(U) : U \text{ is an ultrafilter of } \mathsf{Bool}(\mathbf{A})\},$$

over the Boolean space given by the Stone topology on the ultrafilters of $\mathsf{Bool}(\mathbf{A})$.

We will use this description to characterize free algebras in varieties of generalized n-rotations. Thus we now proceed to characterize their Boolean skeleton and the factors of the decomposition. We first observe the following.

Proposition 3.2. *Let V be a variety of $\mathsf{MVR_n}$-algebras, $n \geq 2$. Given any set X, the MV-skeleton of the free algebra in V over X, $\gamma_n(\mathbf{F}_{\mathsf{V}}(X))$, is the free algebra in $\mathsf{MV_n}$ generated by $\gamma_n(X) = \{\gamma_n(x) : x \in X\}$.*

Proof. We show that $\gamma_n(\mathbf{F}_{\mathsf{V}}(X))$ has the universal mapping property for $\mathsf{MV_n}$ over $\gamma_n(X)$. Indeed, let $\mathbf{A} \in \mathsf{MV_n}$, and α be any map from $\gamma_n(X)$ to A. Consider $\bar{\alpha} = \alpha \circ \gamma_n : X \to A$. Since $\mathsf{MV_n} \subseteq \mathsf{MVR_n}$ and γ_n is a homomorphism, $\bar{\alpha}$ extends uniquely to a homomorphism $\bar{\beta} : \mathbf{F}_{\mathsf{V}}(X) \to \mathbf{A}$, whose restriction β to $\gamma_n(\mathbf{F}_{\mathsf{V}}(X))$ is a homomorphism to $\mathbf{A}$ that extends α. □

In the case $n = 2$, the MV-skeleton coincides with the Boolean skeleton. In general, given an algebra $\mathbf{A} \in \mathsf{MVR_n}$, its Boolean skeleton is the Boolean skeleton of its MV-skeleton, $\mathsf{Bool}(\mathbf{A}) = \mathsf{Bool}(\gamma_n(\mathbf{A}))$ ([19, Lemma 4.3]). Thus, in order to understand the Boolean skeleton of free $\mathsf{MVR_n}$-algebras we shall use a characterization of the Boolean skeleton of free $\mathsf{MV_n}$-algebras in [17], to which we refer the reader. This makes use of the so-called *Moisil operators*. Let $\mathbf{A} \in \mathsf{MV_n}$, the Moisil operators in

$\mathbf{A}$ are one-variable terms $\sigma_1^n(x), \ldots, \sigma_{n-1}^n(x)$ in the language $(\neg, \to, 1)$ that map $\mathbf{A}$ to $\mathsf{Bool}(\mathbf{A})$, such that:

$$x \in \mathsf{Bool}(\mathbf{A}) \text{ if and only if } \sigma_{n-1}^n(x) = x \tag{5}$$

$$\sigma_{n-1}^n(x) = \min\{b \in \mathsf{Bool}(\mathbf{A}) : x \le b\} \tag{6}$$

$$\sigma_1^n(x) = \max\{a \in \mathsf{Bool}(\mathbf{A}) : a \le x\} \tag{7}$$

In particular, if they are evaluated in the MV-chain of n elements $\text{Ł}_n = \{\frac{0}{n-1}, \ldots, \frac{n-1}{n-1}\}$ they are such that

$$\sigma_i^n(\frac{j}{n-1}) = 1 \text{ if } i+j \ge n, \text{ 0 otherwise.} \tag{8}$$

Moisil operators applied on a set X of generators of a free $\mathsf{MV_n}$-algebra result in a poset $(X_\sigma, \le)$, where for each $x \in X$, $\sigma_1^n(x) < \ldots < \sigma_{n-1}^n(x)$, and elements from different such chains are incomparable. We say that an algebra $\mathbf{F}$ generated by a set X in a variety V is *free over a poset* $(X, \le)$ if for each $\mathbf{A} \in \mathsf{V}$ and for each non-decreasing function $f : X \to \mathbf{A}$, f can be uniquely extended to a homomorphism from $\mathbf{F}$ to $\mathbf{A}$.

Theorem 3.3 ([17]). *Let $n \in \mathbb{N}$, $n \ge 2$. Then:*

1. $\mathsf{Bool}(\mathbf{F}_{\mathsf{MV_n}}(X))$ *is the free Boolean algebra over the poset $(X_\sigma, \le)$ where $X_\sigma = \{\sigma_i^n(x) : x \in X, i = 1, \ldots, n-1\}$.*

2. *The correspondence that assigns each upwards closed subset $S \subseteq X_\sigma$ to the Boolean filter U_S generated by the set $S \cup \neg(X_\sigma - S)$, where $\neg(X_\sigma - S) = \{\neg y : y \in X_\sigma - S\}$, defines a bijection from the set of upwards closed subsets of X_σ onto the ultrafilters of* $\mathsf{Bool}(\mathbf{F}_{\mathsf{MV_n}}(X))$.

Thus, given any variety V of $\mathsf{MVR_n}$-algebras, the free algebra $\mathbf{F}_\mathsf{V}(X)$ can be represented as a weak Boolean product indexed by the upwards closed subset $S \subseteq (\gamma_n(X))_\sigma$, or equivalently, by the ultrafilters of the kind:

$$U_S := \text{ultrafilter of } \mathsf{Bool}(\mathbf{F}_\mathsf{V}(X)) \text{ generated by } S \cup \neg((\gamma_n(X))_\sigma - S). \tag{9}$$

Each factor is then going to be a quotient of $\mathbf{F}_\mathsf{V}(X)$ with respect to the filter generated in $\mathbf{F}_\mathsf{V}(X)$ by the Boolean ultrafilter U_S, $Fil_{\mathbf{F}_\mathsf{V}(X)}(U_S)$. To ease the notation, we shall write:

$$\mathbf{F}_\mathsf{V}(X)/U_S := \mathbf{F}_\mathsf{V}(X)/Fil_{\mathbf{F}_\mathsf{V}(X)}(U_S). \tag{10}$$

Since U_S induces a prime filter on $\gamma_n(\mathbf{F}_\mathsf{V}(X))$, the algebra $\mathbf{F}_\mathsf{V}(X)/U_S$ is directly indecomposable, indeed the resulting Boolean skeleton is the 2-element Boolean algebra. The next technical lemma will prove useful in what follows.

Lemma 3.4. *Let X be any set of generators, and $\mathbf{F}_{\mathsf{V}}(X)/U_S$ defined as above. Given any $x \in X$, the following hold.*

1. $\gamma_n(x/U_S) = 1$ *iff* $\sigma_1^n(\gamma_n(x)) \in S$ *iff* $\sigma_i^n(\gamma_n(x)) \in S$ *for all* $i = 1 \ldots n-1$.

2. $\gamma_n(x/U_S) = 0$ *iff* $\sigma_{n-1}^n(\gamma_n(x)) \notin S$ *iff* $\sigma_i^n(\gamma_n(x)) \notin S$ *for all* $i = 1 \ldots n-1$.

Proof. We prove (1) first. By (7), $\sigma_1^n(\gamma_n(x)) = \max\{a \in \mathsf{Bool}(\mathbf{F}_{\mathsf{V}}(X)) : a \leq \gamma_n(x)\}$. Thus if $\sigma_1^n(\gamma_n(x)) \in S$, $\gamma_n(x) \in Fil_{\mathbf{F}_{\mathsf{V}}(X)}(U_S)$ and thus $\gamma_n(x/U_S) = 1$. Vice versa, if $\gamma_n(x/U_S) = 1$, equivalently $\gamma_n(x) \in Fil_{\mathbf{F}_{\mathsf{V}}(X)}(U_S)$, which means there is a Boolean element $a \in U_S, a \leq \gamma_n(x)$. Again by (7), $a \leq \sigma_1^n(\gamma_n(x))$, which is then in S. Moreover, $\sigma_1^n(\gamma_n(x)) \in S$ iff $\sigma_i^n(\gamma_n(x)) \in S$ for all $i = 1 \ldots n-1$ since Moisil operators on a particular element form a chain.

Let us now prove (2). By (6), $\gamma_n(x) \leq \sigma_{n-1}^n(\gamma_n(x))$, thus $\neg\sigma_{n-1}^n(\gamma_n(x)) \leq \neg\gamma_n(x)$. Thus if $\sigma_{n-1}^n(\gamma_n(x)) \notin S$, $\neg\sigma_{n-1}^n(\gamma_n(x)) \in U_S$, and then $\neg\gamma_n(x) \in Fil_{\mathbf{F}_{\mathsf{V}}(X)}(U_S)$. Equivalently, $\gamma_n(\neg x/U_S) = 1$, thus $\gamma_n(x/U_S) = 0$. Vice versa, if $\gamma_n(x/U_S) = 0$, $\gamma_n(\neg x/U_S) = 1$, thus $\neg\gamma_n(x) \in Fil_{\mathbf{F}_{\mathsf{V}}(X)}(U_S)$. That is, there is a Boolean element $a \in U_S$ such that $a \leq \neg\gamma_n(x)$, thus $\gamma_n(x) \leq \neg a$. By (6), $\sigma_{n-1}^n(\gamma_n(x)) \leq \neg a$, thus $a \leq \neg\sigma_{n-1}^n(\gamma_n(x)) \in U_S$. Thus, $\sigma_{n-1}^n(\gamma_n(x)) \notin S$. The second equivalence follows again from the fact that Moisil operators on a particular element form a chain. □

Since $\mathbf{F}_{\mathsf{V}}(X)/U_S$ is directly indecomposable, it is a generalized rotation of its radical. Thus, we are now going to characterize $\mathsf{Rad}(\mathbf{F}_{\mathsf{V}}(X)/U_S)$. We will follow a line of reasoning similar to [16, 15], abstracting the ideas therein to our more general context. Let

$$X_S = \{x/U_S : x \in X\},$$

and let us show that the following set generates the radical as a rotation algebra:

$$\beta_n(X_S) = \{\beta_n(x/U_S) : x \in X\} \setminus \{1\}. \tag{11}$$

It follows by Proposition 2.1 that

$$\beta_n(X_S) = \{x/U_S : x \in X, \gamma_n(x/U_S) = 1\} \cup \{\neg x/U_S : x \in X, \gamma_n(x/U_S) = 0\}.$$

Lemma 3.5. *Let V be a subvariety of $\mathsf{MVR_n}$, $n \geq 2$, and let X be any set. Then $\beta_n(X_S)$ defined above generates $(\mathsf{Rad}(\mathbf{F}_{\mathsf{V}}(X)/U_S), \neg\neg)$ as a rotation algebra.*

Proof. Since the algebra $\mathbf{F}_{\mathsf{V}}(X)/U_S$ is directly indecomposable, by Proposition 2.1 $\mathbf{F}_{\mathsf{V}}(X)/U_S$ is the union of its radical, its coradical, and its MV-skeleton (which is an MV-chain of m elements, with $m-1$ divisor of $n-1$). In this proof we will use the definition of the operations in a generalized rotation as described in the previous section.

Notice first that the set of elements in X_S generates $\mathbf{F}_{\mathsf{V}}(X)/U_S$. We show the following by induction on the complexity of the term y constructed in the signature $\{\cdot, \to, \wedge, \vee, 1, \neg\}$ from X_S:

(i) if $y \in \mathsf{Rad}(\mathbf{F}_{\mathsf{V}}(X)/U_S)$, then y is in the subalgebra $\mathbf{B}_S$ generated by $\beta_n(X_S)$ in the language $\{\cdot, \to, \wedge, \vee, 1, \neg\neg\}$;

(ii) if $y \in \mathsf{Corad}(\mathbf{F}_{\mathsf{V}}(X)/U_S)$, then $y = \neg z$, with $z \in \mathbf{B}_S$.

More precisely, we consider the complexity of a term t, $c(t)$, to be defined as follows, for $* \in \{\cdot, \to, \wedge, \vee\}$:

$$c(1) = c(x/U_S) = 1,\ c(\neg t) = 1 + c(t),\ c(t * u) = 1 + c(t) + c(u).$$

Let y be of complexity 1. Then if y is in the radical, either $y = 1 \in \mathbf{B}_S$, or $y = x/U_S$ and $\gamma_n(x/U_S) = 1$ (Proposition 2.1), thus $x/U_S \in \beta_n(X_S) \subseteq \mathbf{B}_S$ and (i) holds. If y is in the coradical, then $y = x/U_S$ with $\gamma_n(x/U_S) = 0$. Thus $\neg x/U_S \in \beta_n(X_S) \subseteq \mathbf{B}_S$, and since the coradical is involutive in $\mathsf{MVR_n}$, $x/U_S = \neg(\neg x/U_S)$, thus (ii) holds.

Suppose now that y has complexity m, and that the claim holds for all terms of minor complexity. We need to check the cases where $y = a \star b$, for $\star \in \{\cdot, \to, \wedge, \vee\}$, and $y = \neg a$. Suppose first that $y \in \mathsf{Rad}(\mathbf{F}_{\mathsf{V}}(X)/U_S)$, we show (i).

- If $y = a * b$ with $* \in \{\cdot, \wedge\}$, since $a * b \leq a, b$, both a and b belong to the radical, thus the conclusion follows from inductive hypothesis.

- Suppose now $y = a \to b$. If $a \leq b$, then $y = 1$. Let us then consider the case $a \not\leq b$. There are only two possibilities, that is, both a, b are in the radical, or both in the coradical. In the former case, the result follows from inductive hypothesis. If instead a, b are in the coradical, it means that their negation is in the radical, and, given that the coradical is involutive in algebras in $\mathsf{MVR_n}$,

$$y = a \to b = \neg\neg a \to \neg\neg b = \neg b \to \neg a$$

  Since $\neg a, \neg b$ have complexity smaller than m, and they belong to the radical, by inductive hypothesis $\neg a, \neg b \in \mathbf{B}_S$, thus $y \in \mathbf{B}_S$.

- If $y = a \vee b$, at least one of a, b is in the radical, say a. If also b is in the radical, then the result follows from the inductive hypothesis. If b is not in the radical, then $b < a$, thus $y = a$ and the claim follows.

- It $y = \neg a$, then necessarily $a \in \mathsf{Corad}(\mathbf{F}_\mathsf{V}(X)/U_S)$, thus by inductive hypothesis $a = \neg b$, $b \in \mathbf{B}_S$. Thus $y = \neg a = \neg\neg b$, which is in $\mathbf{B}_S$ (since the latter is generated in a language which includes $\neg\neg$).

Let us now suppose that $y \in \mathsf{Corad}(\mathbf{F}_\mathsf{V}(X)/U_S)$, and we show (ii).

- Let $y = a \wedge b$. Then at least one of a and b is in the coradical, say a. If b is not in the coradical, then $y = a$ and the conclusion follows by inductive hypothesis. If both a and b are in the coradical, then by inductive hypothesis $a = \neg c, b = \neg d$ and $c, d \in \mathbf{B}_S$. Thus by De Morgan laws, which hold in $\mathsf{MVR_n}$:

$$y = a \wedge b = \neg c \wedge \neg d = \neg(c \vee d),$$

and clearly $c \vee d$ is in $\mathbf{B}_S$.

- The case $y = a \vee b$ follows again by the inductive hypothesis and the De Morgan laws.

- Suppose now $y = a \cdot b$. By the definition of the operations in a rotation, either $y = 0 = \neg 1$, or either a or b, say a, belongs to the radical and the other, hence b, to the coradical. Thus by inductive hypothesis $a \in \mathbf{B}_S$, and $b = \neg c, c \in \mathbf{B}_S$. Thus $y = a \cdot b = a \cdot \neg c = \neg(a \to c)$, and $a \to c \in \mathbf{B}_S$.

- Let now $y = a \to b$. Thus necessarily a is in the radical, and b is in the coradical. Therefore, $a \in \mathbf{B}_S$, and $b = \neg c, c \in \mathbf{B}_S$. We get $y = a \to b = a \to \neg c = \neg(a \cdot c)$, and $a \cdot c \in \mathbf{B}_S$.

- Lastly, say $y = \neg a$. Necessarily, $a \in \mathsf{Rad}(\mathbf{F}_\mathsf{V}(X)/U_S)$, and by inductive hypothesis $a \in \mathbf{B}_S$.

By (i) it follows that the elements in $\beta_n(X_S)$ generate $(\mathsf{Rad}(\mathbf{F}_\mathsf{V}(X)/U_S, \neg\neg)$ as a rotation algebra, thus the proof is complete. □

If V is a term-defined variety of $\mathsf{MVR_n}$, Lemma 3.5 reads as follows.

Lemma 3.6. *Let* V *be a term-defined subvariety of* $\mathsf{MVR_n}$*,* $n \geq 2$*, and let* X *be any set. Then* $\beta_n(X_S)$ *defined above generates* $\mathsf{Rad}(\mathbf{F}_\mathsf{V}(X)/U_S)$ *as a CIRL.*

Notice that, up to this point, the results we have shown hold for any variety V of $\mathsf{MVR_n}$-algebras. For the next results, we are going to need that the variety V is a variety of generalized n-rotations, in the following sense. We remind the reader that we assume $n \in \mathbb{N}, n \geq 2$.

Definition 3.7. A *variety of generalized n-rotations* is a subvariety V of $\mathsf{MVR_n}$, that is generated by generalized n-rotations of CIRLs, in the sense that: if the class of directly indecomposable algebras V_{di} contains a generalized m-rotation $\text{Rot}^{\delta}_{m}(\mathbf{A})$, for some rotation δ and m such that $m-1$ divides $n-1$, then V_{di} also contains $\text{Rot}^{\delta}_{n}(\mathbf{A})$.

We show that in this case, the radical $\mathsf{Rad}(\mathbf{F}_{\mathsf{V}}(X)/U_S)$, seen as a rotation algebra with the double-negation operator, satisfies a particular kind of universal mapping property.

Definition 3.8. Let $(\mathbf{A}, \delta)$ and $(\mathbf{B}, \delta')$ be algebras in a class K of rotation algebras, and let $(\mathbf{A}, \delta)$ be generated by a set X. We say that a map $f : X \to B$ *respects the rotation operator* if, whenever $x \in X$ is a fixpoint of δ, i.e. $x = \delta(x)$, $f(x)$ is a fixpoint of δ', that is, $\delta'(f(x)) = f(x)$.

Let again $(\mathbf{A}, \delta) \in \mathsf{K}$ be generated by a set X. We say that $(\mathbf{A}, \delta)$ is *free in* K *over* (X, δ) if given any $(\mathbf{B}, \delta') \in \mathsf{K}$, and any map $f : X \to B$ which respects the rotation operator, f can be uniquely extended to a homomorphism from $(\mathbf{A}, \delta)$ to $(\mathbf{B}, \delta')$.

Theorem 3.9. *Let V be a variety of generalized n-rotations, and let X be any set. Then $(\mathsf{Rad}(\mathbf{F}_{\mathsf{V}}(X)/U_S), \neg\neg)$ is free in the rotation-radical variety of V, RR_{V}, over $(\beta_n(X_S), \neg\neg)$.*

Proof. Consider any $(\mathbf{R}, \delta) \in \mathsf{RR}_{\mathsf{V}}$, and any rotation-preserving map $\alpha : \beta_n(X_S) \to R$. That is, α maps each $\neg x/U_S \in \beta_n(X_S)$ to fixpoints of δ, i.e., $\delta(\alpha(\neg x/U_S)) = \alpha((\neg x/U_S))$. We show that α can be extended to a homomorphism β from the rotation algebra $(\mathsf{Rad}(\mathbf{F}_{\mathsf{V}}(X)/U_S), \neg\neg)$ to $(\mathbf{R}, \delta)$.

First, since $(\mathbf{R}, \delta) \in \mathsf{RR}_{\mathsf{V}}$, by definition $(\mathbf{R}, \delta) \cong (\mathsf{Rad}(\mathbf{A}), \neg\neg)$ for some $\mathbf{A} \in \mathsf{V}$. By Lemma 2.4, $\text{Rot}^{\delta}_{2}(\mathbf{R}) \cong (\mathsf{Rad}(\mathbf{A}) \cup \mathsf{Corad}(\mathbf{R})) \in \mathsf{V}$. Since by hypothesis V is a variety of generalized n-rotations in the sense of Definition 3.7, also $\text{Rot}^{\delta}_{n}(\mathbf{R}) \in \mathsf{V}$. In order to be able to use the universal mapping property of $\mathbf{F}_{\mathsf{V}}(X)$, we will now suitably define a map $\hat{\alpha} : X \to \text{Rot}^{\delta}_{n}(\mathbf{R})$. In order to do so, recall that by Lemma 3.4, $\gamma_n(x/U_S) = 1$ if and only if $\sigma^n_1(\gamma_n(x)) \in S$, if and only if $\sigma^n_i(\gamma_n(x)) \in S$ for all $i = 1 \dots n-1$. This means that if $\gamma_n(x) \notin Fil_{\mathbf{F}_{\mathsf{V}}(X)}(U_S)$, there is a maximum index i such that $\sigma^n_i(\gamma_n(x)) \notin S$. Let us then define j_x (in analogy with [16]) as:

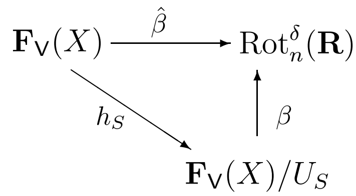
$$j_x = \max\{i : \sigma^n_i(\gamma_n(x)) \notin S\} \quad \text{if } \gamma_n(x/U_s) \neq 1,$$

and let us set $j_x = 0$ otherwise. Let then $\hat{\alpha} : X \to \text{Rot}^{\delta}_{n}(\mathbf{R})$ be such that for $x \in X$:

$$\hat{\alpha}(x) = \begin{cases} \alpha(x/U_S) & \text{if } \gamma_n(x/U_s) = 1, \\ \neg(\alpha(\neg x/U_S)) & \text{if } \gamma_n(x/U_s) = 0, \\ (n-1-j_x)/(n-1) & \text{otherwise.} \end{cases}$$

By the universal mapping property of the free algebra $\mathbf{F}_{\mathsf{V}}(X)$, there is a unique $\hat{\beta} : \mathbf{F}_{\mathsf{V}}(X) \to \mathrm{Rot}_n^{\delta}(\mathbf{R})$ such that for all $x \in X$, $\hat{\beta}(x) = \hat{\alpha}(x)$. Suppose now the following claim holds.

Claim 3.10. *There is a (unique) homomorphism β such that, for all $z \in \mathbf{F}_{\mathsf{V}}(X)$, $\beta(z/U_S) = \hat{\beta}(z)$, closing the following diagram:*



Now, $\beta_n(X_S)$ generates the radical as a rotation algebra by Lemma 3.5. Moreover, any homomorphism of $\mathsf{MVR_n}$-algebras maps the radical to the radical ([19, Corollary 4.10]). Thus, the image of the restriction of β to the radical of $\mathbf{F}_{\mathsf{V}}(X)/U_S$ is contained in $\mathbf{R}$. Moreover, β respects the double negation, and so it is a homomorphism of rotation algebras. Let us show that β extends α. Let us consider first $x/U_S \in \beta_n(X_S)$, with $\gamma_n(x/U_S) = 1$, then:

$$\beta(x/U_S) = \hat{\beta}(x) = \hat{\alpha}(x) = \alpha(x/U_S).$$

Now, let $\neg x/U_S \in \beta_n(X_S)$ with $\gamma_n(x/U_S) = 0$. Since α preserves the rotation operator we get:

$$\beta(\neg x/U_S) = \neg\hat{\beta}(x) = \neg\hat{\alpha}(x) = \neg\neg\alpha(\neg x/U_S) = \delta(\alpha(\neg x/U_S)) = \alpha(\neg x/U_S).$$

Thus $(\mathsf{Rad}(\mathbf{F}_{\mathsf{V}}(X)/U_S), \neg\neg)$ is free in $\mathsf{RR_V}$ over $(\beta_n(X_S), \neg\neg)$. We complete the proof by proving Claim 3.10.

In particular, we show that $Fil_{\mathbf{F}_{\mathsf{V}}(X)}(U_S) \subseteq \hat{\beta}^{-1}\{1\}$, then the claim follows from the Second Isomorphism Theorem (see [14, Theorem 6.15]) and the fact that congruences are determined by (the congruence filter given by) their 1-block. More precisely, we prove that all generators of $Fil_{\mathbf{F}_{\mathsf{V}}(X)}(U_S)$ are in $\hat{\beta}^{-1}\{1\}$ (recall the definition in (9)), which will complete the proof.

Notice first that if $\gamma_n(x/U_S) = 1$, $\hat{\alpha}(x) = \alpha(x/U_S)$ and then for each i:

$$\hat{\beta}(\sigma_i^n(\gamma_n(x)) = \sigma_i^n(\gamma_n(\alpha(x/U_S))) = \sigma_i^n(1) = 1.$$

If instead $\gamma_n(x/U_S) = 0$, then $\hat{\alpha}(x) = \neg(\alpha(\neg x/U_S))$ which is in the coradical of $\mathrm{Rot}_n^{\delta}(\mathbf{R})$, thus:

$$\hat{\beta}(\sigma_i^n(\gamma_n(x)) = \sigma_i^n(\gamma_n(\neg(\alpha(\neg x/U_S)))) = \sigma_i^n(0) = 0.$$

Finally, if $\gamma_n(x/U_S) \notin \{0,1\}$, then:

$$\hat{\beta}(\sigma_i^n(\gamma_n(x))) = \sigma_i^n\left(\gamma_n\left(\frac{n-j_x-1}{n-1}\right)\right) = \sigma_i^n\left(\frac{n-j_x-1}{n-1}\right).$$

We first then consider a generator in S, of the kind $\sigma_i^n(\gamma_n(x))$ for some $x \in X$. If $\gamma_n(x/U_S) = 1$, we have already seen that $\hat{\beta}(\sigma_i^n(\gamma_n(x))) = 1$. Suppose now $\gamma_n(x/U_S) \neq 1$. Notice that $\gamma_n(x/U_S) \neq 0$ since otherwise $\sigma_i^n(\gamma_n(x)) \notin S$ for each $i = 1 \ldots n-1$ by Lemma 3.4. Now, by the definition of j_x, $j_x < i$, thus by (8) we get $\hat{\beta}(\sigma_i^n(\gamma_n(x))) = \sigma_i^n\left(\frac{n-j_x-1}{n-1}\right) = 1$.

Let us now consider a generator in $\neg((\gamma_n(X))_\sigma - S)$, that is, $\neg\sigma_i^n(\gamma_n(x))$ for some $x \in X$, $\sigma_i^n(\gamma_n(x)) \notin S$. Thus by Lemma 3.4, $\gamma_n(x/U_S) \neq 1$. If $\gamma_n(x/U_S) = 0$, we have seen that $\hat{\beta}(\sigma_i^n(\gamma_n(x)) = 0$, thus $\hat{\beta}(\neg(\sigma_i^n(\gamma_n(x))) = 1$. Suppose now $\gamma_n(x/U_S) \neq 0$. Then $i \leq j_x$, thus by (8) we get $\hat{\beta}(\neg\sigma_i^n(\gamma_n(x))) = \neg\sigma_i^n\left(\frac{n-j_x-1}{n-1}\right) = \neg 0 = 1$.

Since we have shown that all generators of $Fil_{\mathbf{F}_\mathsf{V}(X)}(U_S)$ are in $\hat{\beta}^{-1}\{1\}$, we obtain that there is a (unique) homomorphism β such that, for all $z \in \mathbf{F}_\mathsf{V}(X)$, $\beta(z/U_S) = \hat{\beta}(z)$, which proves Claim 3.10 and completes the proof of the theorem. □

We observe that even in the case of a term-defined variety, where the radical $\mathsf{Rad}(\mathbf{F}_\mathsf{V}(X)/U_S)$ is generated by the set $\beta_n(X_S)$ as a CIRL by Lemma 3.6, the previous theorem does not reduce to saying that the radical $\mathsf{Rad}(\mathbf{F}_\mathsf{V}(X)/U_S)$ is free in the radical variety R_V. However, the latter holds if we consider the two most relevant subvarieties of $\mathsf{MVR_n}$, $\mathsf{IMVR_n}$ and $\ell\mathsf{MVR_n}$. This is due to the fact that the condition of preserving the rotation operator is trivially satisfied in such cases. Indeed, consider V to be a subvariety of $\mathsf{IMVR_n}$, thus the rotation operator is given by $\delta = id$, and let $(\mathbf{A}, id)$ and $(\mathbf{B}, id)$ in V, with $(\mathbf{A}, id)$ generated by a set X. Any map $f : X \to B$ respects the rotation operator, since given $x \in X$, x is a fixpoint of id, and $f(x)$ is also a fixpoint of id, that is, $id(f(x)) = f(x)$. If V is instead a subvariety of $\ell\mathsf{MVR_n}$, the rotation operator is the map $\delta(x) = 1$. Thus an element x is a fixpoint of δ if and only if $x = 1$. Thus if one considers the set $\beta_n(X_S)$ of generators of $\mathsf{Rad}(\mathbf{F}_\mathsf{V}(X)/U_S)$, $1 \notin \beta_n(X_S)$, and the condition is trivially satisfied. Notice also that respecting the rotation operator for a map f would just mean that $f(1) = 1$. This means that for both subvarieties of $\mathsf{IMVR_n}$ and $\ell\mathsf{MVR_n}$, Theorem 3.9 shows that $\mathsf{Rad}(\mathbf{F}_\mathsf{V}(X)/U_S)$ is free in R_V in the usual sense. That is, *any* map from the set $\beta_n(X_S)$ of generators of $\mathsf{Rad}(\mathbf{F}_\mathsf{V}(X)/U_S)$ to another algebra in R_V can be extended to a homomorphism.

Following this intuition, we now define a large class of algebras where Theorem 3.9 can be simplified. The idea is to consider a variety V where the radicals $\mathsf{Rad}(\mathbf{F}_\mathsf{V}(X)/U_S)$ are free in the usual sense in the radical variety R_V. This can be achieved by considering algebras in a term-defined variety where any map respects the term-defined rotation operator (as it is the case for $\mathsf{IMVR_n}$ and $\ell\mathsf{MVR_n}$).

Definition 3.11. Let V be a term-defined variety of generalized n-rotations. We call V a *radical-determined variety of n-rotations* if for all $\mathbf{A}, \mathbf{B} \in \mathsf{R}_{\mathsf{V}}$, and X generating set for $\mathbf{A}$, any map $\alpha : X \to \mathbf{B}$ respects the rotation operator of V.

We then obtain the following result.

Corollary 3.12. *Let V be a radical-determined variety of n-rotations, and let X be any set. Then $\mathsf{Rad}(\mathbf{F}_{\mathsf{V}}(X)/U_S)$ is free in the radical variety of V, R_{V}, over $\beta_n(X_S)$.*

Remark 3.13. Radical-determined varieties of n-rotations are a general framework in which the description of free algebras is particularly effective and transparent. Notice that this is a very broad class of algebras, containing all of the well-known and most relevant examples of $\mathsf{MVR_n}$-algebras (in particular, all subvarieties of $\mathsf{IMVR_n}$ and $\ell\mathsf{MVR_n}$). In the following sections we will therefore focus on this class of algebras.

Let us now summarize our findings about the weak Boolean product representation of free algebras in radical-determined varieties of rotations in the following theorem. Given any variety of $\mathsf{MVR_n}$-algebras V, and any set X, in order to describe $\mathbf{F}_{\mathsf{V}}(X)$ recall that: given any upwards closed subset $S \subseteq (\gamma_n(X))_\sigma$, U_S is the ultrafilter of $\mathsf{Bool}(\mathbf{F}_{\mathsf{V}}(X))$ generated by $S \cup \neg((\gamma_n(X))_\sigma - S)$, and $\mathbf{F}_{\mathsf{V}}(X)/U_S := \mathbf{F}_{\mathsf{V}}(X)/Fil_{\mathbf{F}_{\mathsf{V}}(X)}(U_S)$.

Theorem 3.14. *Given a radical-determined variety of n-rotations V, and any set X, the free algebra in V over X is isomorphic to the weak Boolean product of the family*

$$\{\mathbf{F}_{\mathsf{V}}(X)/U_S : U_S \textit{ ultrafilter of } \mathsf{Bool}(\mathbf{F}_{\mathsf{V}}(X))\}.$$

For each U_S ultrafilter of $\mathsf{Bool}(\mathbf{F}_{\mathsf{V}}(X))$, there is $m \geq 2$ such that $m-1$ divides $n-1$ and

$$\mathbf{F}_{\mathsf{V}}(X)/U_S \cong \mathrm{Rot}_m^\delta[\mathbf{F}_{\mathsf{R}_{\mathsf{V}}}(\beta_n(X_S))],$$

where δ is the term-defined rotation operator coinciding with $\neg\neg$ in V.

Since weak Boolean products over finite discrete spaces coincide with direct products, whenever X is finite we can refine our result. In particular, if the cardinality of X is some $k \in \mathbb{N}$, it follows from Theorem 3.3 that the cardinality of the Stone space of $\mathsf{Bool}(\mathbf{F}_{\mathsf{V}}(X))$ is n^k (see [17]).

Corollary 3.15. *Given a radical-determined variety of n-rotations V and $k \in \mathbb{N}$, the free algebra in V with k generators is isomorphic to a direct product of generalized*

m_i-rotations of free algebras in $\mathsf{R_V}$ over an appropriate number of generators, for $m_i - 1$ dividing $n-1$ and $i = 1 \dots n^k$:

$$\mathbf{F}_{\mathsf{V}}(k) = \prod_{i=1}^{n^k} \mathrm{Rot}^{\delta}_{m_i}[\mathbf{F}_{\mathsf{R_V}}(k_i)] \tag{12}$$

where m_i, k_i depend on i, m_i divides $n-1$, δ is the term-defined rotation of V.

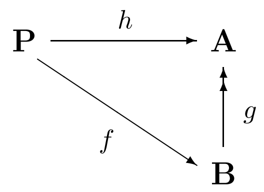
We remark that, while in the previous statement we left unspecified the parameters m_i, k_i since they are not needed for the results we are going to obtain in the following sections, the interested reader can obtain them following Section 4 in [17]. The authors indeed specify the cardinalities of the MV-chains in the representation of the free MV-skeleton $\gamma_n(\mathbf{F}_{\mathsf{V}}(k))$, and considering the associated Boolean ultrafilter U_S, one can calculate k_i since it is the cardinality of $\beta_n(X_S)$.

4 Projectivity and unification

The classical syntactic unification problem, given two term s, t (built from function symbols and variables), finds a *unifier* for them, that is, a uniform replacement of the variables occurring in s and t by other terms that makes s and t identical. When the latter syntactical identity is replaced by equality modulo a given equational theory E, one speaks of E-unification, which is what we study here. Thus, the unification problems we consider are up to logical equivalence. Ghilardi [42] shows that there is an equivalent algebraic way of studying unification problems, which we follow here, which makes use of finitely presented and projective algebras.

An algebra is *finitely presented* if it can be defined by a finite number of generators and finitely many identities. For any set X and any variety V we will denote by $\mathbf{F}_{\mathsf{V}}(X)$ the free algebra in V over X. Thus, an algebra $\mathbf{A} \in \mathsf{V}$ is *finitely presented* in V if there is a finite set X and a finitely generated congruence $\theta \in \mathrm{Con}(\mathbf{F}_{\mathsf{V}}(X))$ such that $\mathbf{F}_{\mathsf{V}}(X)/\theta \cong \mathbf{A}$. We remark that we here blur the distinction between a finitely presented and a finitely presentable algebra, as it does not affect our study. Notice moreover that if V has finite type then any finite algebra in V is finitely presented, and if V is locally finite any finitely presented algebra in V is finite. We also remark that the notion of finitely presented algebra is a categorical notion [38] and thus it is preserved under categorical equivalence.

Given a class K of algebras, an algebra $\mathbf{P} \in \mathsf{K}$ is *projective* in K if for all $\mathbf{A}, \mathbf{B} \in \mathsf{K}$, for any homomorphism $h : \mathbf{P} \longmapsto \mathbf{A}$, and any surjective homomorphism $g : \mathbf{B} \longmapsto \mathbf{A}$ there is a homomorphism $f : \mathbf{P} \longmapsto \mathbf{B}$ such that $h = gf$.



A unification problem for a variety V is a finitely presented algebra $\mathbf{A}$ in V; a *solution* is a homomorphism $u : \mathbf{A} \longrightarrow \mathbf{P}$, where $\mathbf{P}$ is a projective algebra in V. In this case u is called a *unifier* for $\mathbf{A}$ and we say that $\mathbf{A}$ is *unifiable*. If u_1, u_2 are two different unifiers for an algebra $\mathbf{A}$ (with projective targets $\mathbf{P}_1$ and $\mathbf{P}_2$) we say that u_1 is *more general* than u_2 if there exists a homomorphism $m : \mathbf{P}_1 \longmapsto \mathbf{P}_2$ such that $mu_1 = u_2$. The relation "being less general of" is a preordering on the set of unifiers of $\mathbf{A}$, thus we can consider the associated equivalence relation; then the equivalence classes (i.e., the unifiers that are "equally general") form a partially ordered set $U_{\mathbf{A}}$. It is customary to assign a type to the finitely presented algebra according to how many maximal elements $U_{\mathbf{A}}$ has: unitary type if $U_{\mathbf{A}}$ has a maximum; finitary, if there are finitely many maximal elements; infinitary, if there are infinitely many; nullary, otherwise. The type of a variety V, and of its corresponding logic by Ghilardi's results, is the worst unitary type of its finitely presented algebras.

In the case where the type of $\mathbf{A}$ is unitary then the maximum in $U_{\mathbf{A}}$ is called the *most general unifier* (*mgu* for short) of $\mathbf{A}$, which can be thought of as a "best solution" for the unification problem, since all the other ones can be obtained by the mgu by further substitution. We separate the case in which the *mgu* is of a special kind: we say that $\mathbf{A}$ has *strong unitary type* if its *mgu* is the identity. A variety V has strong unitary type if every unifiable finitely presented algebra in V has strong unitary type. This kind of unification has been first studied in [63], and then later for instance in [33, 58]. From the results in [13], it follows that discriminator varieties have strong unitary type, which include Boolean algebras, monadic algebras and MV_n-algebras, while [33] shows the same for k-potent extensions of Basic Logic and basic hoops.

Let us now give more details on finitely presented and projective algebras that will be useful in the rest of the paper. In particular, we mention that if K contains all the free algebras on K (in particular, if K is a (quasi)variety of algebras), then projectivity admits a simpler formulation. An algebra $\mathbf{B}$ is a *retract* of an algebra $\mathbf{A}$ if there is a homomorphism $g : \mathbf{A} \longmapsto \mathbf{B}$ and a homomorphism $f : \mathbf{B} \longmapsto \mathbf{A}$ with $gf = \mathrm{id}_{\mathbf{B}}$ (and thus, necessarily, f is injective and g is surjective). The following theorem was proved first by Whitman for lattices [62] but it is well-known to hold for any class of algebras.

Theorem 4.1. *Let* K *be a class of algebras containing all the free algebras in* K *and let* $\mathbf{A} \in \mathsf{K}$*. Then the following are equivalent:*

1. $\mathbf{A}$ *is projective in* K*;*
2. $\mathbf{A}$ *is a retract of a free algebra in* K*.*
3. $\mathbf{A}$ *is a retract of a projective algebra in* K*.*

In particular every free algebra in K *is projective in* K*.*

For finitely presented algebras, we get the following refinement. The proof is standard, but see [42].

Theorem 4.2. *For a finitely presented algebra* $\mathbf{A} \in \mathsf{V}$ *the following are equivalent:*

1. $\mathbf{A}$ *is projective in* V*;*
2. $\mathbf{A}$ *is projective in the class of all finitely presented algebras in* V*;*
3. $\mathbf{A}$ *is a retract of a finitely generated free algebra in* V*.*

Let us observe that the algebraic definition of projectivity we have given corresponds to a categorical notion. Indeed, one can consider the algebraic category associated to the class of algebras K, where the objects are the algebras in K and the morphisms are the homomorphisms between algebras in K. Then we can rephrase the definition in a completely equivalent way in categorical terms, where the surjective homomorphism is a regular epimorphism (see for instance [42]). Since the definition only involves properties of morphisms, it follows that projectivity is preserved by categorical equivalence. The following is an easy observation.

Proposition 4.3. *Let* V *be any variety; then the following are equivalent.*

1. V *has strong unitary type;*
2. *for any finitely presented algebra* $\mathbf{A} \in \mathsf{V}$*,* $\mathbf{A}$ *is unifiable if and only if it is projective.*

Notice that for every non-trivial subvariety V of $\mathsf{FL_{ew}}$ the 2-element Boolean algebra is the free algebra over the empty set of generators, thus it is projective. Moreover, every free algebra in V has $\mathbf{2}$ as a homomorphic image. Thus, as it is noticed in [4], having a homomorphism to a projective algebra in V is equivalent to having a homomorphism to the two-element Boolean algebra.

Proposition 4.4 ([4]). *Let* V *be a variety of* $\mathsf{FL_{ew}}$*-algebras. If* $\mathbf{A}$ *is projective in* V*, then* $\mathbf{A}$ *has* $\mathbf{2}$ *as a homomorphic image. Moreover, a finitely presented algebra* $\mathbf{A}$ *in a variety* V *of* $\mathsf{FL_{ew}}$*-algebras is unifiable iff it has* $\mathbf{2}$ *as a homomorphic image.*

Therefore we get the following.

Proposition 4.5. *For a subvariety* V *of* $\mathsf{FL_{ew}}$ *the following are equivalent:*

1. V *has strong unitary type.*
2. *for any finitely presented* $\mathbf{A} \in \mathsf{V}$*,* $\mathbf{A}$ *has* $\mathbf{2}$ *as a homomorphic image if and only if* $\mathbf{A}$ *is projective;*

Notice that instead, in a subvariety of CIRL every finitely presented algebra is unifiable: since the 1-element algebra is free, and therefore projective, one can always find the unifier mapping everything to 1.

In our framework, we first get the following immediate consequence of Proposition 3.2.

Theorem 4.6. *Let* V *be any subvariety of* $\mathsf{MVR_n}$*, for any* $n \in \mathbb{N}, n \geq 2$*; then any projective* $\mathsf{MV_n}$*-algebra is projective in* V*.*

Proof. By Proposition 3.2, the free $\mathsf{MV_n}$-algebra over any set X is a retract of $\mathbf{F}_{\mathsf{V}}(X)$, and thus is a projective algebra in V. Hence, so are all of its retracts, i.e., projective $\mathsf{MV_n}$-algebras. □

In particular, for the case $n = 2$, every projective Boolean algebra (e.g., for instance, all finite Boolean algebras) is projective in every variety with a Boolean retraction term in $\mathsf{MVR_2}$, e.g., Gödel algebras, product algebras, nilpotent minimum algebras without negation fixpoint, the variety generated by perfect MV-algebras.

Using the description of free algebras, we can characterize all finitely generated projective algebras in varieties of rotations.

Theorem 4.7. *Let* V *be a radical-determined variety of* n*-rotations. Then finitely generated projective algebras in* V *are (isomorphic to) finite direct products of generalized* m_i*-rotations of projective algebras in* $\mathsf{R_V}$*, for* $m_i - 1$ *dividing* $n-1$*,* $i = 1 \dots l \leq n^k$*, for some* $k \in \mathbb{N}$*.*

Proof. Let $\mathbf{A}$ be a projective and finitely generated algebra in V. Thus $\mathbf{A}$ is a retract of a finitely generated free algebra $\mathbf{F}_{\mathsf{V}}(k)$, for some $k \in \mathbb{N}$. That is, $\mathbf{A}$ is isomorphic to a quotient: $\mathbf{A} \cong \mathbf{F}_{\mathsf{V}}(k)/\theta$ that is a retract of $\mathbf{F}_{\mathsf{V}}(k)$.

From identity (12), $\mathbf{F}_{\mathsf{V}}(k) = \prod_{i=1}^{n^k} \mathrm{Rot}^{\delta}_{m_i}[\mathbf{F}_{\mathsf{R_V}}(k_i)]$, where δ is the term-defined rotation in V. Now, in congruence distributive varieties (as it is the case for varieties of residuated lattices) all congruences of a finite direct product are product congruences, thus we can see $\theta = \prod_{i=1}^{n^k} \theta_i$ where each θ_i is exactly the restriction of the congruence θ on the i-th component of the direct product. Then

$$\mathbf{A} \cong \mathbf{F}_{\mathsf{V}}(k)/\theta \cong \left(\prod_{i=1}^{n^k} \mathrm{Rot}^{\delta}_{m_i}[\mathbf{F}_{\mathsf{R_V}}(k_i)] \right) /\theta \cong \prod_{i=1}^{n^k} \mathrm{Rot}^{\delta}_{m_i}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i. \tag{13}$$

Notice that some of the factors could be trivial (i.e., $0_i/\theta_i = 1_i/\theta_i$). We assume (without loss of generality) that the non-trivial factors are the first l: $i \in \{1, \dots, l\}$ for $1 \leq l \leq n^k\}$. Since $\mathbf{A}$ is projective it is not trivial, thus at least one factor is not trivial. Let $g : \prod_{i=1}^{n^k} \mathrm{Rot}^{\delta}_{m_i}[\mathbf{F}_{\mathsf{R_V}}(k_i)] \to \prod_{i=1}^{n^k} \mathrm{Rot}^{\delta}_{m_i}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i$ be the onto homomorphism defined as:

$$g(x_1, \dots, x_{n^k}) = (x_1/\theta_1, \dots, x_{n^k}/\theta_{n^k}).$$

Since $\mathbf{A}$ is projective, there is an embedding

$$f : \prod_{i=1}^{n^k} \mathrm{Rot}^{\delta}_{m_i}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i \to \prod_{i=1}^{n^k} \mathrm{Rot}^{\delta}_{m_i}[\mathbf{F}_{\mathsf{R_V}}(k_i)]$$

such that $g \circ f = id$. Thus,

$$f(x_1/\theta_1, \dots, x_{n^k}/\theta_{n^k}) = (y_1, \dots, y_\kappa) \text{ where } x_j \theta_j y_j \text{ for } j = 1 \dots n^k. \tag{14}$$

Now, for each $i = 1, \dots, l$ (not associated to a trivial factor), let us call F_i the congruence filter associated to θ_i. Since the radical in a generalized rotation is the only maximal filter, F_i is either a filter of the radical $\mathbf{F}_{\mathsf{R_V}}(k_i)$, or it coincides with the radical itself. In the latter case, $\mathrm{Rot}^{\delta}_{m_i}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i$ coincides with the $\mathsf{MV_n}$-chain $\mathbf{Ł}_{m_i}$, which is the generalized m_i-rotation of the 1-element algebra, which is projective since it is the free algebra in $\mathsf{R_V}$ over the empty set of generators. Suppose now that F_i is a proper filter of $\mathbf{F}_{\mathsf{R_V}}(k_i)$, then:

$$\mathrm{Rot}^{\delta}_{m_i}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i = \mathrm{Rot}^{\delta}_{m_i}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/F_i \cong \mathrm{Rot}^{\bar{\delta}}_{m_i}[\mathbf{F}_{\mathsf{R_V}}(k_i)/F_i], \tag{15}$$

where in the last equality we used Lemma 2.3, and thus $\bar{\delta}(x/F_i) = \delta(x)/F_i$. We now show that f and g induce homomorphisms f_i, g_i that testify that $\mathbf{F}_{\mathsf{R_V}}(k_i)/F_i$ is a retract of $\mathbf{F}_{\mathsf{R_V}}(k_i)$.

Call π_i the projection on the i-th component of $\prod_{i=1}^{n^k} \mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]$. Let f_i be the function mapping every element of the radical

$$x_i/\theta_i \in \mathsf{Rad}(\mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i) \cong \mathbf{F}_{\mathsf{R_V}}(k_i)/F_i$$

to the following:

$$f_i(x_i/\theta_i) = \pi_i \circ f(1/\theta_1, \ldots, 1/\theta_{i-1},\ x_i/\theta_i,\ 1/\theta_{i+1}, \ldots, 1/\theta_{n^k}). \tag{16}$$

Now, f_i is well-defined because f is. Moreover, it can be shown by direct computation that f_i is a homomorphism of residuated lattices, mapping the radical of $\mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i$ to $\mathbf{F}_{\mathsf{R_V}}(k_i)$. Let now $g_i : \mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)] \to \mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i$ be the onto homomorphism such that:

$$g_i(x_i) = x_i/\theta_i. \tag{17}$$

Clearly g_i restricts to a homomorphism on the radical $\mathbf{F}_{\mathsf{R_V}}(k_i)$. Let us compute $g_i \circ f_i$ and show that it is the identity on the radical of $\mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i$. Consider any $x_i/\theta_i \in \mathsf{Rad}(\mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i)$. Then, combining (14), (16), and (17), we get:

$$g_i \circ f_i(x_i/\theta_i) = g_i(y_i) = y_i/\theta_i = x_i/\theta_i.$$

Thus, it follows that $\mathbf{F}_{\mathsf{R_V}}(k_i)/F_i$ is a retract of $\mathbf{F}_{\mathsf{R_V}}(k_i)$, which means that $\mathbf{F}_{\mathsf{R_V}}(k_i)/F_i$ is a finitely generated projective algebra in $\mathsf{R_V}$.

Via (15) and (13) this implies the thesis. □

Given the previous characterization, we can also obtain the following consequence.

Theorem 4.8. *Let* V *be a radical-determined variety of* n*-rotations, and* $\mathsf{R_V}$ *be such that every finitely generated projective algebra is finitely presented. Then every finitely generated projective algebra in* V *is finitely presented.*

Proof. Let $\mathbf{A}$ be a projective and finitely generated in V. Thus, given the description in Corollary 3.15, as in the proof of Theorem 4.7, $\mathbf{A}$ is isomorphic to the following:

$$\left(\prod_{i=1}^{n^k} \mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]\right)/\theta \cong \prod_{i=1}^{n^k} \mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i.$$

For the trivial factors, the congruence θ_i is finitely generated (by the pair $(0_i, 1_i)$). For the other factors, let us recall that

$$\mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i \cong \mathrm{Rot}_{m_i}^{\bar{\delta}}[\mathbf{F}_{\mathsf{R_V}}(k_i)/F_i],$$

and each $\mathbf{F}_{\mathsf{R}_\mathsf{V}}(k_i)/F_i$ (possibly trivial) is projective in R_V, thus by hypothesis finitely presented. Hence, each congruence θ_i is finitely generated. Since this holds for each $i \in \{1 \dots n^k\}$, we can conclude that $\theta = \prod_{i=1}^{n^k} \theta_i$ is finitely generated. Therefore, $\mathbf{A}$ is finitely presented. □

Moreover, we can also show a converse to the previous theorem, which is particularly relevant with respect to the study of unification problems. We first show a lemma that describes finitely presented algebras in terms of rotations.

Lemma 4.9. *Let V be a radical-determined variety of n-rotations. Then every non-trivial finitely presented algebra in V is (isomorphic to) a finite direct product of generalized m_i-rotations of finitely presented algebras in R_V, for $m_i - 1$ dividing $n-1, i \in \{1, \dots, l\}, 1 \leq l \leq n^k$, for some $k \in \mathbb{N}$.*

Proof. Let $\mathbf{A}$ be a finitely presented algebra in V, that is, there is $k \in \mathbb{N}$ and a finitely generated congruence $\theta \in \mathrm{Con}(\mathbf{F}_\mathsf{V}(k))$ such that $\mathbf{F}_\mathsf{V}(k)/\theta \cong \mathbf{A}$. From identity (12), $\mathbf{F}_\mathsf{V}(k) = \prod_{i=1}^{n^k} \mathrm{Rot}^\delta_{m_i}[\mathbf{F}_{\mathsf{R}_\mathsf{V}}(k_i)]$. Now, as in the proof of Theorem 4.7, we notice that θ is a product congruence and thus we can write it as $\theta = \prod_{i=1}^{n^k} \theta_i$ where each θ_i is exactly the restriction of the congruence θ on the i-th component of the direct product. Then

$$\mathbf{A} \cong \mathbf{F}_\mathsf{V}(k)/\theta \cong \left(\prod_{i=1}^{n^k} \mathrm{Rot}^\delta_{m_i}[\mathbf{F}_{\mathsf{R}_\mathsf{V}}(k_i)] \right) /\theta \cong \prod_{i=1}^{n^k} \mathrm{Rot}^\delta_{m_i}[\mathbf{F}_{\mathsf{R}_\mathsf{V}}(k_i)]/\theta_i.$$

Notice once again that either a factor is trivial (say for $i \in \{l+1, \dots, n^k\}$), or the quotient $\mathrm{Rot}^\delta_{m_i}[\mathbf{F}_{\mathsf{R}_\mathsf{V}}(k_i)]/\theta_i$ is determined by a congruence filter F_i, which is either the radical $\mathbf{F}_{\mathsf{R}_\mathsf{V}}(k_i)$ or one of its congruence filters. Since θ is finitely generated so is each θ_i, thus for each non-trivial factor with index i the algebra $\mathbf{F}_{\mathsf{R}_\mathsf{V}}(k_i)/F_i$ is finitely presented in R_V, and the proof is complete. □

Now, recall that a finitely presented algebra that does not have a homomorphic image onto $\mathbf{2}$ cannot be projective. However, we can show the following.

Theorem 4.10. *Let V be a radical-determined variety of n-rotations, and R_V be such that every finitely presented algebra is projective. Then every unifiable finitely presented algebra in V is projective.*

Proof. Let $\mathbf{A}$ be a unifiable finitely presented algebra in V. As shown in the proof of Lemma 4.9, we can write:

$$\mathbf{A} \cong \mathbf{F}_\mathsf{V}(k)/\theta \cong \left(\prod_{i=1}^{n^k} \mathrm{Rot}^\delta_{m_i}[\mathbf{F}_{\mathsf{R}_\mathsf{V}}(k_i)] \right) /\theta \cong \prod_{i=1}^{n^k} \mathrm{Rot}^\delta_{m_i}[\mathbf{F}_{\mathsf{R}_\mathsf{V}}(k_i)]/\theta_i.$$

Since $\mathbf{A}$ is unifiable, there is a homomorphism from $\mathbf{A}$ onto the 2-element Boolean algebra. Equivalently, there is a quotient of $\mathbf{A}$ that is isomorphic to $\mathbf{2}$. Seeing $\mathbf{A}$ as its isomorphic copy $\prod_{i=1}^{n^k} \mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i$, the congruence associated to such quotient is a factor congruence. Since $\mathbf{2}$ is directly indecomposable, one of the resulting factors is $\mathbf{2}$. Without loss of generality, let us assume that the index of such factor is $i = 1$. Thus, $\mathrm{Rot}_{m_1}^{\delta_1}[\mathbf{F}_{\mathsf{R_V}}(k_1)]/\theta_1$ is unifiable, that is, there is a homomorphism h_0 onto $\mathbf{2}$.

Going back to the initial quotient, let us assume (again without loss of generality) that the trivial factors are in the last positions, say for $i \in \{l+1, \ldots, n^k\}$, for some l: $1 \leq l \leq n^k - 1$. Thus

$$\mathbf{A} \cong \prod_{i=1}^{l} \mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i.$$

For each $i \in \{1, \ldots, l\}$, we call again F_i the congruence filter associated to θ_i. Then $\mathbf{F}_{\mathsf{R_V}}(k_i)/F_i$ is a finitely presented algebra in $\mathsf{R_V}$, thus by hypothesis it is projective. Equivalently, $\mathbf{F}_{\mathsf{R_V}}(k_i)/F_i$ is a retract of $\mathbf{F}_{\mathsf{R_V}}(k_i)$ and then it is easily seen that $\mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i$ is a retract of $\mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]$. Calling g_i the natural epimorphism $g_i : \mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)] \to \mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i$, we call the embedding testifying the retraction h_i, thus $g_i \circ h_i = id$. Let us consider the following map $h : \prod_{i=1}^{l} \mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i \to \prod_{i=1}^{n^k} \mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]$

$$h(a_1/\theta_1, \ldots, a_l/\theta_l) = (h_1(a_1/\theta_1), \ldots, h_l(a_l)/\theta_l, h_0(a_1/\theta_1), \ldots, h_0(a_1/\theta_1)).$$

It can be directly checked that h is a homomorphism since the maps $h_i : i = 0, \ldots, l$ are, and it is injective since the maps $h_i : i = 1, \ldots, l$ are. Moreover, consider the homomorphism $g : \prod_{i=1}^{n^k} \mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)] \to \prod_{i=1}^{l} \mathrm{Rot}_{m_i}^{\delta}[\mathbf{F}_{\mathsf{R_V}}(k_i)]/\theta_i$ that projects onto the first l factors:

$$g(a_1, \ldots, a_{n^k}) = (g_1(a_1), \ldots, g_l(a_l)).$$

It follows from their definition that $g \circ h = id$, and thus $\mathbf{A}$ is a retract of $\mathbf{F}_{\mathsf{V}}(k)$ and the proof is complete. □

Therefore, we get the following consequence on unification.

Corollary 4.11. *Let* V *be a radical-determined variety of* n*-rotations. If* $\mathsf{R_V}$ *has strong unitary unification type, so does* V.

Interesting case studies of the general results obtained in this section will be explored in the final section of the paper. In particular, we characterize finitely generated projective product algebras and nilpotent minimum algebras, besides obtaining a different proof of known results, as it is the case for Gödel algebras.

4.1 Admissibility of clauses

A relevant problem closely connected to unification is the admissibility of clauses; a *clause* in a logic is an ordered pair (Σ, Γ) where Σ, Γ are finite sets of formulas. We usually write a clause as $\Sigma \Rightarrow \Gamma$ and a clause is *admissible* in a logic if, when added to its calculus, it does not produce new theorems. A *rule* is a clause $\Sigma \Rightarrow \Gamma$ where Γ contains only one formula, $\Gamma = \{\gamma\}$. We call *negative* a clause where Γ is empty. An admissible clause is not necessarily provable in the logic; a famous example is Harrop's clause for intuitionistic logic (also known as Kreisel-Putnam rule):

$$\{\neg p \to (q \vee r)\} \;\Rightarrow\; \{(\neg p \to q) \vee (\neg p \to r)\}$$

which is admissible but not derivable. A logic is said to be *universally complete* if every admissible clause is derivable and *structurally complete* if every admissible rule is derivable. Since Harrop's rule is admissible but not derivable, intuitionistic logic is not structurally complete. Admissibility can be studied algebraically in varieties of algebras that are the equivalent algebraic semantics of a logic. Now, via algebraizability, formulas of a logic are translated to equations in the algebraic semantics. In particular, a formula φ of $\mathbb{FL}_{ew}$ is translated to the equation $\varphi \approx 1$ in FL_{ew}, and φ is valid in a logic if and only if $\varphi \approx 1$ is valid in the corresponding variety (see [41] for more details). In a variety V, a *clause* $\Sigma \Rightarrow \Gamma$ is then meant to consider Σ, Γ as finite sets of equations in the language of V. A *quasiequation* is a clause in which $\Gamma = \{\gamma\}$. A clause $\Sigma \Rightarrow \Gamma$ is V-*admissible* if every substitution that unifies all equations of Σ in V unifies at least one equation of Γ in V. Notice that a negative clause is admissible if and only if the premises in Σ are not unifiable. A clause $\Sigma \Rightarrow \Gamma$ is instead *valid* in an algebra $\mathbf{A}$, and we write $\mathbf{A} \vDash \Sigma \Rightarrow \Gamma$, if whenever $\mathbf{A} \vDash \Sigma$ there is a $p \approx q \in \Gamma$ with $\mathbf{A} \vDash p \approx q$.

Notice that the varieties where the trivial algebra is not unifiable (as it is the case for all subvarieties of FL_{ew}), cannot be universally complete given the presence of admissible negative clauses that are not derivable. But they can be *non-negative universally complete*, that is, every non-negative clause that is admissible is valid, as it is the case for instance for classical logic (see [20]). Clearly, non-negative universal completeness implies structural completeness. Moreover, note that if a variety has strong unitary type, whenever the premises of a clause are unifiable, admissibility reduces to derivability. The latter property applied to rules has been referred to as *almost structural completeness*, and is studied in [32, 34]. In the same way, we shall call *almost universally complete* a variety (or a logic) such that admissible clauses with unifiable premises are derivable. We get the following observation.

Proposition 4.12. *If a variety has strong unitary unification type, then it is almost universally complete.*

Note that the converse of this theorem fails spectacularly; the variety D of distributive lattices is universally complete ([20], Theorem 16) but has nullary unification type [42]. Nevertheless, combining Proposition 4.12 and Corollary 4.11, we obtain the following.

Corollary 4.13. *Let V be a radical-determined variety of n-rotations such that R_V has strong unitary unification type. Then V is almost universally complete.*

The previous result can be refined if all (nontrivial) finitely presented algebras are unifiable, which, as previously mentioned, corresponds to having the 2-element Boolean algebra as homomorphic image. Notice that, since finite MV-chains are simple, the only unifiable finite MV-chain is the 2-element Boolean algebra. Therefore, we obtain the following.

Proposition 4.14. *Let V be a subvariety of $\mathsf{MVR_n}$ that is not a subvariety of $\mathsf{MVR_m}$ with $m < n$. Then every nontrivial finitely presented algebra in V is unifiable if and only if $n = 2$.*

Proof. Suppose V is a subvariety of $\mathsf{MVR_2}$. Then every algebra has its Boolean skeleton as a homomorphic image, via the retraction term γ_2. Thus, since every nontrivial Boolean algebra has $\mathbf{2}$ as a homomorphic image, every nontrivial finitely presented algebra in V is unifiable.

Vice versa, if V is not a subvariety of $\mathsf{MVR_2}$, there is at least an MV-chain $\mathbf{Ł}_m$ in V with $m > 2$. Since $\mathbf{Ł}_m$ is finite, it is finitely presented, and since it is simple, it does not have $\mathbf{2}$ as homomorphic image, and is therefore not unifiable. ☐

Theorem 4.15. *Let V be a radical-determined variety of 2-rotations such that R_V has strong unitary unification type. Then V is non-negative universally complete, and thus also structurally complete.*

Moreover, in a variety with decidable equational theory and (at most) finitary unification type, admissibility of rules is decidable if there is an algorithm which, for any unification problem $\mathbf{A}$, produces a complete finite set M of maximal unifiers. That is, given any unifier u for $\mathbf{A}$, there is a unifier v from M that is more general than u. The latter condition is clearly satisfied in the particular case of strong unitary type, thus we obtain the following.

Corollary 4.16. *Let V be a radical-determined variety of n-rotations with decidable equational theory, such that R_V has strong unitary unification type. Then admissibility of clauses is decidable in V.*

5 Case studies

In the previous sections we have presented a general approach to the study of projectivity, unification, and admissibility in varieties corresponding to substructural logics of rotations. Moreover, we have seen several transfer-results from varieties of CIRLs to the varieties they generate via generalized n-rotations, for $n \geq 2$. In this section we will see some relevant applications of our findings in the realm of mathematical fuzzy logics. Among the most relevant results in this respect, it is well-known that Łukasiewicz and Gödel logic have, respectively, nullary [52] and strong unitary type (in fact, Gödel algebras are the largest variety of Heyting algebras with this property [58]). Moreover, Dzik shows [31] that k-potent BL-algebras and k-potent basic hoops (prelinear divisible CIRLs) have unitary unification type. Structural completeness in fuzzy logics has been deeply studied in [28, 40].

5.1 Product logic

As previously mentioned, product logic is, together with Łukasiewicz and Gödel logic, one of the main propositional fuzzy logics arising from a continuous t-norm. Product logic has been introduced by Hájek, Godo, Esteva in [46], and has been deeply studied in recent years. In particular, relevant results have been obtained about: the functional representation of its free finitely generated algebras [27]; structural completeness [28]; categorical representation [53] and duality [37]; SMT-solvers [60]; modal extensions [61]; probability theory [36]. However, to the best of our knowledge, the unification type of product logic remained an open problem so far.

The corresponding equivalent algebraic semantics of product logic is the variety of product algebras, which is generated by the standard product algebra $[0,1]_{\mathsf{P}} = ([0,1], \cdot, \to, \min, \max, 0, 1)$ where the t-norm is the product between real numbers and its residuum is such that $x \to y = 1$ if $x \leq y$ and $x \to y = \frac{y}{x}$ otherwise. Since the variety of product algebras P can also be seen as the variety generated by 2-liftings of cancellative hoops [24], it is a subvariety of $\ell\mathsf{MVR}_2$. Thus, it is a radical-determined variety of generalized 2-rotations. Therefore our results apply and R_{P} is the variety of cancellative hoops CH, which is the subvariety of prelinear divisible CIRLs where the monoidal operation satisfies the usual cancellativity law.

Cancellative hoops are (term-equivalent to) negative cones of lattice-ordered abelian groups [9], and are therefore categorically equivalent to ℓ-groups. Finitely generated projective ℓ-groups have been characterized in [10] and coincide with finitely presented ℓ-groups. Since the properties of being projective, being finitely presented, and being finitely generated are categorical, i.e. they can be described in the abstract categorical setting as properties of morphisms, they are preserved by categorical equivalences. Therefore we obtain the following.

Proposition 5.1. *Finitely generated projective cancellative hoops are exactly the finitely presented algebras in their variety.*

Therefore, combining Theorems 4.8 and 4.10 with Proposition 5.1 we obtain the following result.

Corollary 5.2. *Finitely generated projective product algebras are exactly the non-trivial finitely presented algebras in their variety.*

Rephrased in terms of unification, the previous result directly implies the following, given that the equational theory of product logic is decidable.

Corollary 5.3. *Product algebras, and product logic, have strong unitary unification type. Therefore, they are non-negative universally complete and admissibility of clauses is decidable.*

The same holds for the variety DLMV generated by perfect MV-algebras (i.e., disconnected rotations of cancellative hoops), the varieties that have been called *nilpotent product* in [3, 2], and more in general:

Corollary 5.4. *Let* V *be a radical-determined variety of* n*-rotations such that* R_{V} *is the variety of cancellative hoops. Then* V *has strong unitary unification type. If* $n = 2$*,* V *is non-negative universally complete. In particular it holds for: product algebras and the variety generated by perfect MV-algebras.*

5.2 Nilpotent minimum

The variety of Gödel hoops has strong unitary unification type, as it follows from [42, Theorem 5.3]. Indeed, the author shows that any time a finite (equivalently, finitely presented) Brouwerian semilattice ($\{\wedge, \to, 1\}$-reducts of Heyting algebras) is a homomorphic image of another finite Brouwerian semilattice, it is also its retract. Gödel hoops are the subvariety of representable (equivalently, prelinear) Brouwerian semilattices, where the join is definable from the other operations, and thus Ghilardi's result holds for Gödel hoops as well.

Well-known varieties whose radical class is given by the variety of Gödel hoops are the varieties of Gödel algebras G, nilpotent minimum algebras NM, and nilpotent minimum algebras without negation fixpoint NM^-. They are generated, respectively, by 2-liftings, connected rotations, and disconnected rotations of Gödel hoops, thus our results apply.

The variety of Gödel algebras is known to indeed have strong unitary unification type, and the same result follows for nilpotent minimum algebras without negation fixpoint, since they are categorically equivalent to Gödel algebras ([36]). Our approach allows to obtain these results uniformly, together with a new result about the whole variety of nilpotent minimum algebras.

Corollary 5.5. *Let* V *be a radical-determined variety of* n*-rotations such that* $\mathsf{R_V}$ *is the variety of Gödel hoops. Then* V *has strong unitary unification type. In particular: Gödel algebras and* NM^-*, which are both non-negative universally complete, and* NM*.*

We notice that Gispert shows that NM is not even structurally complete in [40]. In [4], it is actually shown that more generally, all locally finite varieties of hoops have strong unitary unification type. Thus we actually get the following more general result.

Corollary 5.6. *Let* V *be a radical-determined variety of* n*-rotations such that* $\mathsf{R_V}$ *is a locally finite variety of hoops. Then* V *has strong unitary unification type. If* $n = 2$*,* V *is non-negative universally complete.*

6 Conclusions

In this work we have developed general results involving a large class of $\mathsf{FL_{ew}}$-algebras. In particular, it is relevant to stress that given any variety of CIRLs, one can generate infinitely many different radical-determined varieties of n-rotations, considering for instance any $n \geq 2$ and as rotation either the identity map or the map constantly equal to 1. For all such varieties our findings hold. Thus, whenever new results about free algebras, projective algebras, unification type, are obtained in a variety of CIRLs, our theorems can be applied. This has particular value in varieties of fuzzy logics, since divisible commutative integral residuated lattices (BL-algebras and their 0-free reducts) are quite well understood, while the structure theory of commutative integral chains (MTL-algebras and their 0-free reducts) is still not understood.

As future work, we believe that a deeper investigation in the algebraic approach to unification could help to refine our methods and allow to transfer also different unification types among varieties of rotations and their radicals.

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References

[1] P. Aglianò and F. Montagna, Varieties of BL-algebras I: general properties, Journal of Pure and Applied Algebras 181 (2003), 105–129.

[2] P. Aglianò and S. Ugolini, *Rotation logics*, Fuzzy Sets and Systems **388** (2020), 1–25.

[3] ______, *MTL-algebras as rotations of basic hoops*, J. Logic Comput. (2021), 763–784.

[4] ______, *Projectivity in (bounded) commutative integral residuated lattices*, submitted, 2022.

[5] P. Aglianò and A. Ursini, *On subtractive varieties II: General properties*, Algebra Universalis **36** (1996), 222–259.

[6] S. Aguzzoli, T. Flaminio, and S. Ugolini, *Equivalences between subcategories of MTL-algebras via Boolean algebras and prelinear semihoops*, J. Log. Comput. **27** (2017), 2525–2549.

[7] F. Baader, S. Ghilardi, Unification in modal and description logics, Log. J. IGPL 19 (6) (2011) 705–730.

[8] F. Baader, W. Snyder, Unification theory, in: J.A. Robinson, A. Voronkov (Eds.), Handbook of Automated Reasoning, Elsevier/MIT Press, 2001, pp. 445–532.

[9] P. Bahls, J. Cole, N. Galatos, P. Jipsen, and C. Tsinakis, Cancellative residuated lattices, Algebra universalis 50, 83–106, 2003.

[10] W.M. Beynon, *Applications of duality in the theory of finitely generated lattice-ordered abelian groups*, Can. J. Math. **29** (1977), 243–254.

[11] W.J. Blok and D. Pigozzi, *Algebraizable Logics*, Mem. Amer. Math. Soc., no. 396, American Mathematical Society, Providence, Rhode Island, 1989.

[12] K. Blount and C. Tsinakis, *The structure of residuated lattices*, Internat. J. Algebra Comput. **13** (2003), no. 4, 437–461.

[13] Burris, S., Discriminator Varieties and Symbolic Computation, J. of Symbolic Computation, 13 (1992), 175–207.

[14] S. Burris and H.P. Sankappanavar, *A course in universal algebra*, Graduate Texts in Mathematics, Springer, Berlin, 1981.

[15] M. Busaniche, Free Nilpotent Minimum Algebras, Math. Log. Quart. 52, No. 3, 219-236, 2006.

[16] M. Busaniche and R. Cignoli, *Free algebras in varieties of BL-algebras generated by a* BL_n*-chain*, J. Aust. Math. Soc. **80** (2006), 419–439.

[17] ———, *Free* MV_n*-algebras*, Algebra Universalis, 58, 335-340, 2008.

[18] ———, *Constructive logic with strong negation as a substructural logic*, Journal of Logic and Computation 20: 761–793, 2010.

[19] M. Busaniche, M. Marcos, and S. Ugolini, *Representation by triples of algebras with an MV-retract*, Fuzzy Sets and Systems **369** (2019), 82–102, DOI: 10.1016/j.fss.2018.10.024.

[20] L. Cabrer and G. Metcalfe, *Admissibility via natural dualities*, J. Pure Appl. Algebra **219** (2015), 4229–4253.

[21] ———, *Exact unification and admissibility*, Logical Methods in Computer Science **11** (2015), 1–15.

[22] C. C. Chang, B. Jónsson, and A. Tarski, *Refinement properties for relational structures*, Fund. Math. 54, 249–281, (1964).

[23] R. Cignoli, *Free algebras in stonean residuated lattices*, Soft Computing **12** (2008), 315–320.

[24] Cignoli, R., and A. Torrens, An algebraic analysis of product logic, Multiple-Valued Logic 5:45–65, 2000.

[25] R. Cignoli and A. Torrens, *Free Algebras in varieties of Glivenko MTL-algebras satisfying the equation* $2(x^2) = (2x)^2$, Studia Logica **83** (2006), 157–181.

[26] ______, *Varieties of commutative integral bounded residuated lattices admitting a Boolean retraction term*, Studia Logica **100** (2012), 1107–1136.

[27] P. Cintula, B. Gerla, *Semi-normal forms and functional representation of product fuzzy logic*, Fuzzy Sets and Systems, 143 (1), (2004), 89–110.

[28] P. Cintula, G. Metcalfe, *Structural completeness in fuzzy logic*, Notre Dame J. Formal Logic 50(2), 153–182 (2009).

[29] S. Comer, *Representations by algebras of sections over Boolean spaces*, Pacific Journal of Mathematics 38, 29–38 (1971).

[30] O. D'Antona and V. Marra. Computing coproducts of finitely presented Gödel algebras. Ann. Pure Appl. Logic, 142:202–211, 2006.

[31] W. Dzik. Unification in some substructural logics of BL-algebras and hoops. Rep. Math. Log., 43:73–83, 2008.

[32] W. Dzik, *Transparent unifiers in modal logics with self-conjugate operators*, Bull. Sect. Logic Univ. Lodz 35 (2,3) (2006), 73–83.

[33] W. Dzik, *Unification in some substructural logics of BL-algebras and hoops*, Reports on Mathematical Logic, 43 (2008), 73–83.

[34] Wo. Dzik, M. Stronkowski, *Almost structural completeness; an algebraic approach*, Annals of Pure and Applied Logic, 167 (7), 2016, 525–556.

[35] F. Esteva and L. Godo, *Monoidal t-norm based logic: towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems **124** (2001), 271–288.

[36] T. Flaminio, L. Godo, S. Ugolini, *Towards a probability theory for product logic: states, integral representation and reasoning*, International Journal of Approximate Reasoning, 93: 199–218, 2018.

[37] W. Fussner, S. Ugolini, *A topological approach to MTL-algebras*, Algebra Universalis, 80(38), 2019.

[38] P. Gabriel and F. Ullmer, *Lokar präsentiertbare Kategorien*, Lecture Notes in Math., no. 221, Springer Verlag, 1971.

[39] N. Galatos J. G. Raftery, *Adding involution to residuated structures*, Studia Logica **77** (2004), 181–207.

[40] J. Gispert, *Finitary Extensions of the Nilpotent Minimum Logic and (Almost) Structural Completeness*, Studia Logica volume 106, pages 789–808 (2018).

[41] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono, *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, Studies in Logics and the Foundations of Mathematics, vol. 151, Elsevier, Amsterdam, The Netherlands, 2007.

[42] S. Ghilardi, *Unification through projectivity*, J. Logic Comput. **7** (1997), 733–752.

[43] R.S. Grigolia, An algebraic analysis of Łukasiewicz-Tarski n-valued logical systems. In: R. Wojcicki, G. Malinowski (Eds.), Selected papers on Łukasiewicz sentential calculus, Ossolineum, Wroclaw (1977), 81–92.

[44] H.P. Gumm and A. Ursini, *Ideals in universal algebra*, Algebra Universalis **19** (1984), 45–54.

[45] P. Hájek, *Metamathematics of fuzzy logic*, Trends in Logic—Studia Logica Library, no. 4, Kluwer Academic Publ., Dordrecht/ Boston/ London, 1998.

[46] P. Hájek, L. Godo, F. Esteva, A complete many-valued logic with product conjunction, Arch. Math. Logic 35 (1996), 191–208.

[47] S. Jenei, *Structure of left-continuous triangular norms with strong induced negations (i) Rotation construction*, J. Appl. Non-Class. Log. **10** (2000), 83–92.

[48] ______, *On the structure of rotation-invariant semigroups*, Arch. Math. Logic **42** (2003), 489–514.

[49] P. Jipsen and C. Tsinakis, *A Survey of Residuated Lattices*, Ordered Algebraic Strucures (J. Martinez, ed.), Kluwer Academic Publisher, 1982, pp. 19–56.

[50] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.

[51] T. Kowalski, H. Ono, Residuated lattices: an algebraic glimpse at logic without contraction. Monograph, Japan Advanced Institute of Science and Technology, 2001.

[52] V. Marra, L. Spada, Duality, projectivity, and unification in Łukasiewicz logic and MV- algebras, Annals of Pure and Applied Logic, 164, 192–210, 2013.

[53] F. Montagna, S. Ugolini, *A categorical equivalence for product algebras*, Studia Logica, Volume 103 (2), 345–373, 2015.

[54] C. Noguera, Algebraic Study of Axiomatic Extensions of Triangular Norm Based Fuzzy Logics, PhD Thesis, Monografies de l'Institut d'Investigacio en Intelligencia Artificial, vol. 27, 2007, Barcelona.

[55] J.G.Raftery J.S. Olson and C.J. Van Alten, *Structural completeness in substructural logics*, Logic Journal of the IGPL **16** (2008), no. 5, 453–495.

[56] G. Plotkin, Building-in equational theories, Mach. Intell. 7 (73–90) (1972) 30.

[57] J.A. Robinson, A machine-oriented logic based on the resolution principle, J. ACM 12 (1) (1965) 23–41.

[58] K. Slomczynska, *Unification and projectivity in Fregean Varieties*, Logic Journal of the IGPL 20 (2012) 73–93.

[59] S. Ugolini, *Varieties of Residuated Lattices with an MV-Retract and an Investigation into State Theory*, PhD Thesis, University of Pisa, 2018.

[60] A. Vidal, *MNiBLoS: A SMT-based solver for continuous t-norm based logics and some of their modal expansions*, Information Sciences, 372 (2016), 709–730.

[61] A. Vidal, F. Esteva, L. Godo, *On modal extensions of Product fuzzy logic*, J. Log. Comput, 27(1), 299–336 (2017).

[62] P.M. Whitman, *Free lattices*, Ann. of Math. **42** (1941), 325–330.

[63] A. Wronski, *Transparent Unification Problem*, Reports Math. Log. 29: 105–107 (1995).