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Controllability and stabilization of a degenerate/singular Schrödinger equation

Genni Fragnelli [a,*], Alhabib Moumni [b], Jawad Salhi [b]

[a] Department of Ecology and Biology, Tuscia University, Largo dell'Università, 01100 Viterbo, Italy
[b] Moulay Ismail University of Meknes, FST Errachidia, MAIS Laboratory, MAMCS Group, 52000 Errachidia, Morocco

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ABSTRACT

The aim of this paper is to prove controllability and stabilization properties for a degenerate and singular Schrödinger equation with degeneracy and singularity occurring at the boundary of the spatial domain. We first address the boundary control problem. In particular, by combining multiplier techniques and compactness-uniqueness argument, we prove direct and inverse inequalities for the associated adjoint system. Consequently, via the Hilbert Uniqueness Method, we deduce exact boundary controllability for the control system under consideration in any time $T > 0$. Moreover, we investigate the stabilization problem for this class of equations in the range of subcritical coefficients of the singular potential. By introducing a suitable linear boundary feedback, we prove that the solution decays exponentially in an appropriate energy space.

1. Introduction

The controllability and stabilization of Schrödinger equations without degeneracies and singularities have received a lot of attention during the past years. Under the so-called geometric control condition, it is shown by G. Lebeau [26] that the Schrödinger equation is exactly controllable for arbitrary short time. This is due to the fact that the Schrödinger equation can be viewed as a wave equation with an infinite speed of propagation. We also quote the article by E. Machtyngier [29] where observability inequalities for the Schrödinger equation are established by means of the multiplier method developed in [28]. The corresponding exponential decay is obtained by E. Machtyngier and E. Zuazua [30] when the boundary dissipation is linear (see also [23]).

* Corresponding author.
E-mail addresses: genni.fragnelli@unitus.it (G. Fragnelli), alhabibmoumni@gmail.com (A. Moumni), j.salhi@umi.ac.ma (J. Salhi).

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The literature on well posedness, controllability, stabilization and inverse problems for the Schrödinger equation is abundant. We refer to [1,2,6,7,9–11,13,19,24,25,31,32,34,35,41–43] and references therein.

In this paper, we are mainly interested in extending the known results on observability inequalities together with exponential stabilization in [29,30] to the Schrödinger equation with degeneracy and singularity at the boundary.

It is interesting to note that observability inequalities for the Schrödinger equation can be obtained from the corresponding ones for the wave equation and vice versa by an abstract framework (see Remark 1 for more details).

Here, we establish observability inequality and exponential stabilization of degenerate/singular Schrödinger equation based on a direct application of the usual multiplier method developed recently by F. Alabau-Boussouira et al. [4] in the context of the controllability and stabilization of purely degenerate wave equations. Although the approach is classical, these results are new for the degenerate/singular Schrödinger equation.

The first objective of this paper is to study the exact boundary controllability for Schrödinger equations of the form

$$\begin{cases} iy_t + (x^\alpha y_x)_x + \frac{\mu}{x^{2-\alpha}} y = 0, & (t,x) \in Q := (0,T) \times (0,1), \\ \begin{cases} y(t,0) = 0, & \text{if } 0 \le \alpha < 1, \\ (x^\alpha y_x)(t,0) = 0, & \text{if } 1 \le \alpha < 2, \end{cases} & t \in (0,T), \\ y(t,1) = f(t), & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{1}$$

where $y = y(t,x)$ is the state and $f = f(t)$ is a control function to be determined which acts on the system by means of the Dirichlet boundary condition at the point $x = 1$. Both are complex valued functions. Here $i \in \mathbb{C}$ is the imaginary unit, while $\alpha \in [0,2)$ and μ are two real parameters, y_0 is regarded as being the initial value and $T > 0$ stands for the length of the time-horizon. In particular, if $\alpha \in (0,1)$ we say that the problem is *weakly degenerate* (WD), if $\alpha \in [1,2)$ then it is *strongly degenerate* (SD).

The control problem we shall address can be formulated, roughly, as follows: *given $T > 0$ and y_0, y_d belonging to a suitable Hilbert space we look for a control function f such that the solution y of* (1) *satisfies* $y(T) = y_d$. This is called an exact controllability problem.

In order to study system (1), we assume that the parameters α and μ satisfy the following assumption:

$$\alpha \in [0,2) \setminus \{1\} \text{ and } \mu \le \mu(\alpha), \tag{2}$$

where

$$\mu(\alpha) := \frac{(1-\alpha)^2}{4} \tag{3}$$

is the constant appearing in the following generalized Hardy inequality: for all $\alpha \in [0,2)$,

$$\frac{(1-\alpha)^2}{4} \int_0^1 \frac{|u|^2}{x^{2-\alpha}}\, dx \le \int_0^1 x^\alpha |u_x|^2\, dx, \tag{4}$$

for all $u \in C_c^\infty(0,1)$ (the space of infinitely smooth functions compactly supported in $(0,1)$). We refer for example to [18, chap 5.3].

We emphasise that (4) ensures that, if $\alpha \in [0,2)\setminus\{1\}$ and if $u \in H^1_{\text{loc}}((0,1])$ is such that $x^{\alpha/2} u_x \in L^2(0,1)$, then $\dfrac{u}{x^{(2-\alpha)/2}}$ belongs to $L^2(0,1)$. On the contrary, in the case $\alpha = 1$, (4) (which reduces to a trivial inequality) does not provide this information anymore. Hence, it is not surprizing if with our techniques we cannot handle this latter special case and we refer to [20] and [39] where this issue is attacked in a different way for the heat equation.

Now, observe that when $\mu = 0$, the problem above is purely degenerate. In this case, controllability properties by means of a locally distributed control have been investigated in [17] using a Carleman approach.

On the other hand, when $\alpha = 0$, system (1) becomes purely singular with a singularity that takes the form of an inverse-square potential. To the best of our knowledge, [14] and [40] are the unique published works on this subject; they are concerned with the problem of exact controllability for the linear multidimensional Schrödinger equation with singular potentials.

As far as we know, there are currently no controllability results for the Schrödinger equation that couples a degenerate variable coefficient in the principal part with a singular potential.

In this work, we are interested in studying precisely this issue, extending the results obtained in [5], where the authors discuss the same issue in the case of wave equations.

Thanks to the linearity and the time reversibility of the Schrödinger system (1) (see [43]), exact controllability is equivalent to null controllability. Henceforth we shall assume that the target $y_d \equiv 0$. Thus, we look for a suitable control f such that the solution of (1) satisfies $y(T) = 0$.

By the now classical HUM (Hilbert Uniqueness Method), this result is actually equivalent to the so-called observability inequality for the solution of the adjoint system (see [43])

$$\begin{cases} iu_t + (x^\alpha u_x)_x + \frac{\mu}{x^{2-\alpha}} u = 0, & (t,x) \in Q, \\ \begin{cases} u(t,0) = 0, & \text{if } 0 \le \alpha < 1, \\ (x^\alpha u_x)(t,0) = 0, & \text{if } 1 < \alpha < 2, \end{cases} & t \in (0,T), \\ u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{5}$$

which formally states that, for any $\mu \le \mu(\alpha)$ and $T > 0$, there exists $C > 0$ such that

$$\int_0^1 \left\{ x^\alpha |u_x(0,x)|^2 - \frac{\mu}{x^{2-\alpha}} |u(0,x)|^2 \right\} dx \le C \int_0^T |u_x(t,1)|^2 \, dt, \tag{6}$$

where u solves (5). The proof of (6) relies on both multiplier method and compactness-uniqueness argument. As a consequence of this inequality, it follows that system (1) is null controllable for arbitrarily small time T by a control acting at $x = 1$ (that is, away from the degenerate and singular point).

Remark 1.

1. In the observability estimate (6), we only prove the existence of some positive constant without explicit constants. This is due to our method which is based on a compactness-uniqueness argument.
2. Note that the proof of (6) can be deduced applying the general theory in [38, Chapter 6], from the result proved for the wave equation in [5]. Indeed, it is well-known that exact observability for an (autonomous) wave equation implies observability for the associated Schrödinger equation. However, as far as we know, this general theory does not work for nonautonomous evolution equation (see [21]). Thus, we believe that our approach, that consists in deriving the observability estimate directly for the Schrödinger equation, is a first step and can be adapted to address the observability of a one-dimensional Schrödinger equation on certain time dependent domain. This equation can be transformed into a non-autonomous equation on a fixed domain, via a change of variable (see [8,21]).

3. Besides being of interest in itself, the Schrödinger equation may serve also as a preliminary step to study an Euler-Bernoulli (plate) equation. We refer to [23], where the connection between these two problems is discussed in details.

In the last part of this paper, we study the energy decay rate of the degenerate and singular Schrödinger equation with a boundary damping. More precisely, we shall consider the following Schrödinger equation

$$iu_t + (x^\alpha u_x)_x + \frac{\mu}{x^{2-\alpha}} u = 0 \quad \text{in } (0,T)\times(0,1), \tag{7}$$

with dissipative boundary condition:

$$u_t(t,1) + u_x(t,1) + \beta u(t,1) = 0, \tag{8}$$

where $\beta \geq 0$.

The main purpose of this part is to show that (8) stabilizes exponentially the corresponding solution of (7) under suitable assumptions on the parameters α, μ and β.

Prior to give the precise statement of our main results, we firstly give the main notations that will be used throughout the paper.

In what follows, Re and Im stand for the real and the imaginary part of a complex number, $\langle\cdot,\cdot\rangle$ denotes the usual scalar product on $L^2((0,1);\mathbb{C})$ i.e.

$$\langle u, v\rangle = \mathrm{Re}\int_0^1 u(x)\overline{v(x)}\,dx, \quad \forall u,v \in L^2((0,1);\mathbb{C}),$$

and the notation $A \asymp B$ means that there exist two constants $C_1, C_2 > 0$, such that $C_1 A \leq B \leq C_2 A$.

Finally, we recall the following technical lemma, whose proof is a simple adaptation of [5, Theorem 1.1 and Lemma 2.1] to the complex case.

Lemma 1. *Let $\mu(\alpha)$ be as in (3). Then, for all $\alpha \in [0,2)$ and for all $u \in C_c^\infty((0,1);\mathbb{C})$, we have*

$$\int_0^1 x^2|u_x(x)|^2\,dx \leq C_\alpha \int_0^1 \left(x^\alpha |u_x(x)|^2 - \mu(\alpha)\frac{|u(x)|^2}{x^{2-\alpha}}\right) dx \tag{9}$$

and

$$\int_0^1 |u(x)|^2\,dx \leq C'_\alpha \int_0^1 \left(x^\alpha |u_x(x)|^2 - \mu(\alpha)\frac{|u(x)|^2}{x^{2-\alpha}}\right) dx, \tag{10}$$

where

$$C_\alpha = \begin{cases} 1, & \text{if } \; 0 \leq \alpha < 1, \\ 1 + \frac{4(1-\alpha)(\alpha-3)}{(2-\alpha)^2}, & \text{if } \; 1 \leq \alpha < 2 \end{cases} \tag{11}$$

and

$$C'_\alpha = \begin{cases} \min\left(\frac{4}{(1-\alpha)(3-\alpha)}, \frac{16}{(2-\alpha)^2}\right), & \text{if } \; 0 \leq \alpha < 1, \\ \frac{16}{(2-\alpha)^2}, & \text{if } \; 1 \leq \alpha < 2. \end{cases} \tag{12}$$

2. Preliminary results

In order to study well posedness and controllability properties for (1), we shall need some basic properties of the corresponding homogeneous problem (5).

Before going into further details, we first introduce the functional setting associated to the degenerate/singular problems (see [12] or [39]). For any $\mu \le \mu(\alpha)$, we consider the Hilbert space $H^{1,\mu}_{\alpha}((0,1);\mathbb{C})$ given by

$$H^{1,\mu}_{\alpha}((0,1);\mathbb{C}) := \Big\{ u \in L^2((0,1);\mathbb{C}) \cap H^1_{loc}((0,1];\mathbb{C}) \text{ such that } \int_0^1 \Big(x^\alpha |u_x(x)|^2 - \frac{\mu}{x^{2-\alpha}} |u(x)|^2 \Big)\, dx < +\infty \Big\}$$

endowed with the scalar product

$$\langle u, v\rangle_{H^{1,\mu}_{\alpha}(0,1)} := \operatorname{Re} \int_0^1 \Big(u(x)\overline{v(x)} + x^\alpha u_x(x)\overline{v_x(x)} - \frac{\mu}{x^{2-\alpha}} u(x)\overline{v(x)} \Big)\, dx,$$

for all $u, v \in H^{1,\mu}_{\alpha}((0,1);\mathbb{C})$.

The previous scalar product obviously induces the related respective norm

$$\|u\|_{H^{1,\mu}_{\alpha}(0,1)} := \Big(\int_0^1 \Big(|u(x)|^2 + x^\alpha |u_x(x)|^2 - \frac{\mu}{x^{2-\alpha}} |u(x)|^2 \Big)\, dx \Big)^{\frac{1}{2}},$$

for all $u \in H^{1,\mu}_{\alpha}((0,1);\mathbb{C})$.

According to [39], the trace at $x = 0$ of any $u \in H^{1,\mu}_{\alpha}((0,1);\mathbb{C})$ makes sense as soon as $\alpha < 1$. This leads us to introduce the following space:

(i) For $0 \le \alpha < 1$, we define

$$H^{1,\mu}_{\alpha,0}((0,1);\mathbb{C}) := \big\{ u \in H^{1,\mu}_{\alpha}((0,1);\mathbb{C}) \mid u(0) = u(1) = 0 \big\}.$$

(ii) For $1 < \alpha < 2$, we change the definition of $H^{1,\mu}_{\alpha,0}(0,1)$ in the following way

$$H^{1,\mu}_{\alpha,0}((0,1);\mathbb{C}) := \big\{ u \in H^{1,\mu}_{\alpha}((0,1);\mathbb{C}) \mid u(1) = 0 \big\}.$$

Let us mention that in both cases, $H^{1,\mu}_{\alpha,0}((0,1);\mathbb{C})$ may be seen as the completion of $C^\infty_c((0,1);\mathbb{C})$ with respect to the norm $\|\cdot\|_{H^{1,\mu}_{\alpha}(0,1)}$; thus (4), (9) and (10) also hold true in $H^{1,\mu}_{\alpha,0}((0,1);\mathbb{C})$. Moreover, thanks to (10), one can see that $H^{1,\mu}_{\alpha,0}((0,1);\mathbb{C})$ is a Hilbert space with respect to the inner scalar product

$$\langle u, v\rangle_{H^{1,\mu}_{\alpha,0}(0,1)} := \operatorname{Re} \int_0^1 \Big(x^\alpha u_x(x)\overline{v_x(x)} - \frac{\mu}{x^{2-\alpha}} u(x)\overline{v(x)} \Big)\, dx, \quad \forall u, v \in H^{1,\mu}_{\alpha,0}((0,1);\mathbb{C}),$$

and associated norm

$$\|u\|_{H^{1,\mu}_{\alpha,0}(0,1)} := \left(\int_0^1 \Big(x^\alpha |u_x(x)|^2 - \frac{\mu}{x^{2-\alpha}} |u(x)|^2 \Big)\, dx \right)^{\frac{1}{2}}, \quad \forall u \in H^{1,\mu}_{\alpha,0}((0,1);\mathbb{C}),$$

which is equivalent to $\|\cdot\|_{H^{1,\mu}_\alpha(0,1)}$ on $H^{1,\mu}_{\alpha,0}((0,1);\mathbb{C})$.

Next, we will indicate with $H^{-1,\mu}_\alpha((0,1);\mathbb{C})$ the dual of $H^{1,\mu}_{\alpha,0}((0,1);\mathbb{C})$ with respect to the pivot space $L^2((0,1);\mathbb{C})$, endowed with the natural norm

$$\|f\|_{H^{-1,\mu}_\alpha} := \sup_{\|g\|_{H^{1,\mu}_{\alpha,0}}=1} \langle f, g\rangle_{H^{-1,\mu}_\alpha, H^{1,\mu}_{\alpha,0}}.$$

In order to simplify the notations, in the sequel, we denote by $L^2(0,1)$, $H^{1,\mu}_{\alpha,0}(0,1)$, and $H^{-1,\mu}_\alpha(0,1)$ the spaces $L^2((0,1);\mathbb{C})$, $H^{1,\mu}_{\alpha,0}((0,1);\mathbb{C})$, and $H^{-1,\mu}_\alpha((0,1);\mathbb{C})$, respectively.

Remark 2. It is classical that, even though $H^{1,\mu}_{\alpha,0}(0,1)$ is a Hilbert space, one generally does not identify $H^{-1,\mu}_\alpha(0,1)$ with $H^{1,\mu}_{\alpha,0}(0,1)$. One rather identifies $L^2(0,1)$ with its dual, so that $H^{-1,\mu}_\alpha(0,1)$ becomes a subspace of $\mathcal{D}'(0,1)$ containing $L^2(0,1)$. In particular, if $u \in H^{1,\mu}_{\alpha,0}(0,1)$ and $v \in L^2(0,1)$, then

$$\langle v, u\rangle_{H^{-1,\mu}_\alpha, H^{1,\mu}_{\alpha,0}} = \mathrm{Re}\int_0^1 v(x)\overline{u(x)}\,dx.$$

Further, we define

$$H^{2,\mu}_\alpha(0,1) := \left\{ u \in H^{1,\mu}_\alpha(0,1) \cap H^2_{\mathrm{loc}}((0,1]) \mid (x^\alpha u_x)_x + \frac{\mu}{x^{2-\alpha}} u \in L^2(0,1) \right\}.$$

In the following lemma, we collect useful properties of the above functional spaces which play an important role in oder to evaluate boundary terms, see [4, Proposition 2.5] and [5, Lemma 4 and 5].

In the rest of the paper, we use the following notation: $H^{i,\mu=0}_\alpha(0,1)$ and $H^{1,\mu=0}_{\alpha,0}(0,1)$ denote the spaces $H^{i,\mu}_\alpha(0,1)$ and $H^{1,\mu}_{\alpha,0}(0,1)$ when $\mu = 0$.

Proposition 2. *Assume that $0 \le \alpha < 2$. Then the following properties hold true:*

1. *For every $u \in H^{1,\mu=0}_\alpha(0,1)$*

$$\lim_{x\downarrow 0} x|u(x)|^2 = 0, \tag{13}$$

thus

$$\lim_{x\downarrow 0} xu(x)\overline{v(x)} = 0, \tag{14}$$

for every $u, v \in H^{1,\mu=0}_\alpha(0,1)$.

2. *For every $u \in H^{2,\mu=0}_\alpha(0,1)$*

$$\lim_{x\downarrow 0} x^{\alpha+1}|u'(x)|^2 = 0. \tag{15}$$

Moreover, for all $u \in H^{2,\mu=0}_\alpha(0,1)$ and for all $v \in H^{1,\mu=0}_\alpha(0,1)$ such that $v(0) = 0$, if $\alpha \in [0,1[$ then

$$\lim_{x\downarrow 0} x^\alpha u'(x)\overline{v(x)} = 0. \tag{16}$$

3. *Assume $0 \le \alpha < 1$. Then, for all $u \in H^{2,\mu=0}_\alpha(0,1)$ such that $u(0) = 0$, one has*

$$x^{\alpha-1}|u(x)|^2 \to 0 \quad \textit{as } x \to 0^+. \tag{17}$$

4. *Assume* $1 < \alpha < 2$. *Then, for all* $u \in H^{1,\mu=0}_{\alpha,0}(0,1)$

$$x^{\alpha-1}|u(x)|^2 \to 0 \qquad \text{as } x \to 0^+. \tag{18}$$

Finally, for all $\mu \le \mu(\alpha)$, we define the operator

$$A^\mu_\alpha u := (x^\alpha u_x)_x + \frac{\mu}{x^{2-\alpha}} u$$

with domain depending on the value of α:

$$D(A^\mu_\alpha) := \left\{u \in H^{1,\mu}_{\alpha,0}(0,1) \cap H^2_{\text{loc}}((0,1]) \mid A^\mu_\alpha u \in H^{1,\mu}_{\alpha,0}(0,1)\right\},$$

if $0 \le \alpha < 1$, and

$$D(A^\mu_\alpha) := \left\{u \in H^{1,\mu}_{\alpha,0}(0,1) \cap H^2_{\text{loc}}((0,1]) \mid A^\mu_\alpha u \in H^{1,\mu}_{\alpha,0}(0,1) \text{ and } (x^\alpha u_x)(0) = 0\right\},$$

if $1 < \alpha < 2$.

Remark 3.

1. Notice that, if $u \in D(A^\mu_\alpha)$, then u satisfies the Dirichlet boundary conditions $u(0) = 0$ and $u(1) = 0$ in the first case. Also, it is proved in [39] that if $u \in D(A^\mu_\alpha)$, then $x^\alpha u_x \in W^{1,1}(0,1)$ and thus the condition $(x^\alpha u_x)(0) = u(1) = 0$ in the second case makes sense, as well.
2. Thanks to the definition of $D(A^\mu_\alpha)$, we can apply the results in Proposition 2 to give a sense and to evaluate the boundary terms involving u_t appearing in the proof of Lemma 5.

We have the following properties of $(A^\mu_\alpha; D(A^\mu_\alpha))$.

Proposition 3. *Assume that* (2) *holds. Then* iA^μ_α *is a maximal dissipative operator on* $H^{1,\mu}_{\alpha,0}(0,1)$.

Proof. Let $u \in D\left(A^\mu_\alpha\right)$. We have

$$\int_0^1 \left(x^\alpha v_x \overline{u_x} - \frac{\mu}{x^{2-\alpha}} v\overline{u}\right) dx = -\int_0^1 \left((x^\alpha \overline{u_x})_x + \frac{\mu}{x^{2-\alpha}}\overline{u}\right) v dx,$$

for every $v \in C^\infty_c(0,1)$. Since in both cases $H^{1,\mu}_{\alpha,0}(0,1)$ is the closure of $C^\infty_c(0,1)$ with respect to the norm induced by $\langle\cdot,\cdot\rangle_{H^{1,\mu}_\alpha(0,1)}$ (see [39, page 768]), one can deduce that the above inequality holds for every $v \in H^{1,\mu}_{\alpha,0}(0,1)$. Applying this inequality with $v = A^\mu_\alpha u \in H^{1,\mu}_{\alpha,0}(0,1)$, we obtain that

$$\begin{aligned}
\langle iA^\mu_\alpha u, u\rangle_{H^{1,\mu}_{\alpha,0}} &= \operatorname{Re}\left[i \int_0^1 \left(x^\alpha (A^\mu_\alpha u)_x \overline{u_x} - \frac{\mu}{x^{2-\alpha}} A^\mu_\alpha u \overline{u}\right) dx\right] \\
&= \operatorname{Re}\left[-i \int_0^1 A^\mu_\alpha u \left((x^\alpha \overline{u_x})_x + \frac{\mu}{x^{2-\alpha}}\overline{u}\right) dx\right] \\
&= \operatorname{Re}\left[-i \int_0^1 |A^\mu_\alpha u|^2 \, dx\right] = 0.
\end{aligned}$$

Therefore, iA^μ_α is dissipative.

In order to show that iA^μ_α is maximal dissipative, it remains to check that $I - iA^\mu_\alpha$ is surjective. Equivalently, given any $f \in H^{1,\mu}_{\alpha,0}(0,1)$, we have to prove that there exists $u \in D\left(A^\mu_\alpha\right)$ such that

$$u - iA^\mu_\alpha u = f. \tag{19}$$

For this, note that for all $u, v \in H^{1,\mu}_{\alpha,0}(0,1)$

$$\langle u, v\rangle_{1,0} := \int_0^1 \left(x^\alpha u_x(x)\overline{v_x(x)} - \frac{\mu}{x^{2-\alpha}} u(x)\overline{v(x)}\right) dx$$

defines another scalar product in $H^{1,\mu}_{\alpha,0}(0,1)$ with the corresponding norm $\|\cdot\|_{H^{1,\mu}_{\alpha,0}(0,1)}$. Hence, $H^{1,\mu}_{\alpha,0}(0,1)$ endowed with the scalar product $\langle\cdot,\cdot\rangle_{1,0}$ is also a Hilbert space.

Now, we consider the sesquilinear form $\Gamma : H^{1,\mu}_{\alpha,0}(0,1) \times H^{1,\mu}_{\alpha,0}(0,1) \to \mathbb{C}$ given by

$$\Gamma(u,z) = \int_0^1 \left(i\,\bar{u}z + x^\alpha \overline{u_x} z_x - \frac{\mu}{x^{2-\alpha}}\bar{u}z\right) dx, \quad \forall u, z \in H^{1,\mu}_{\alpha,0}(0,1).$$

We have

$$\begin{aligned}
\operatorname{Re}\Gamma(u,u) &= \operatorname{Re}\int_0^1 \left(i\,|u|^2 + x^\alpha |u_x|^2 - \frac{\mu}{x^{2-\alpha}}|u|^2\right) dx \\
&= \int_0^1 \left(x^\alpha |u_x|^2 - \frac{\mu}{x^{2-\alpha}}|u|^2\right) dx \\
&= \|u\|^2_{H^{1,\mu}_{\alpha,0}(0,1)},
\end{aligned}$$

and thus $\Gamma(\cdot,\cdot)$ is coercive. Moreover, applying the Cauchy-Schwarz inequality, for all $u, z \in H^{1,\mu}_{\alpha,0}(0,1)$, we have

$$\begin{aligned}
|\Gamma(u,z)| &\leq \|u\|_{L^2(0,1)}\|z\|_{L^2(0,1)} + \|u\|_{H^{1,\mu}_{\alpha,0}(0,1)}\|z\|_{H^{1,\mu}_{\alpha,0}(0,1)} \\
&\text{(by (10))} \\
&\leq (C'_\alpha + 1)\,\|u\|_{H^{1,\mu}_{\alpha,0}(0,1)}\|z\|_{H^{1,\mu}_{\alpha,0}(0,1)}
\end{aligned}$$

and then $\Gamma(\cdot,\cdot)$ is continuous.

Next, we introduce the linear form $\ell : H^{1,\mu}_{\alpha,0}(0,1) \to \mathbb{C}$ given by

$$\ell(z) = i\int_0^1 \bar{f} z dx, \quad \forall z \in H^{1,\mu}_{\alpha,0}(0,1).$$

Using again the Cauchy-Schwarz inequality and in view of (10), we see that

$$\begin{aligned}
|\ell(z)| &\leq \|f\|_{L^2(0,1)}\|z\|_{L^2(0,1)} \\
&\leq \sqrt{C'_\alpha}\|f\|_{L^2(0,1)}\|z\|_{H^{1,\mu}_{\alpha,0}(0,1)}.
\end{aligned}$$

That is, ℓ is a continuous linear functional on $H^{1,\mu}_{\alpha,0}(0,1)$. Therefore, by the complex form of the Lax-Milgram Theorem (see [33, Lemma 1.3]), there exists a unique solution $u \in H^{1,\mu}_{\alpha,0}(0,1)$ of

$$\Gamma(u,z) = \ell(z), \quad \forall z \in H^{1,\mu}_{\alpha,0}(0,1). \tag{20}$$

In addition, since $C_c^\infty(0,1) \subset H^{1,\mu}_{\alpha,0}(0,1)$, from (20), we have

$$\int_0^1 \left(i\bar{u}z + x^\alpha \overline{u_x} z_x - \frac{\mu}{x^{2-\alpha}} \bar{u} z \right) dx = i \int_0^1 \bar{f} z dx, \quad \forall z \in C_c^\infty(0,1).$$

This gives

$$i\bar{u} - A^\mu_\alpha \bar{u} = i\bar{f}. \tag{21}$$

Multiplying (21) by i and taking its complex conjugate, one can see that identity (19) holds. It remains to prove that $u \in D(A^\mu_\alpha)$. Since $u \in H^{1,\mu}_{\alpha,0}(0,1)$, we have $u \in H^1_{loc}((0,1])$. Thus, in order to prove that $u \in H^2_{\text{loc}}((0,1])$, it suffices to show that $u_{xx} \in L^2_{loc}((0,1])$. To this aim, let $\varepsilon > 0$ and observe that

$$\begin{aligned}\int_\varepsilon^1 |u_{xx}|^2 dx &= \int_\varepsilon^1 \left(x^{-\alpha}((x^\alpha u_x)_x - \alpha x^{\alpha-1} u_x)\right)^2 dx \\ &= \int_\varepsilon^1 \left(x^{-\alpha}\left((x^\alpha u_x)_x + \frac{\mu}{x^{2-\alpha}} u - \frac{\mu}{x^{2-\alpha}} u - \alpha x^{\alpha-1} u_x\right)\right)^2 dx\end{aligned}$$

which is finite since $(x^\alpha u_x)_x + \frac{\mu}{x^{2-\alpha}} u \in L^2(0,1)$, $u \in L^2(0,1)$ and $u_x \in L^2_{loc}((0,1])$. Therefore, $u_{xx} \in L^2_{loc}((0,1])$. Now, we prove that $(x^\alpha u_x)(0) = 0$ when $\alpha \in (1,2)$. Coming back to (21) and integrating by parts, we have

$$\int_0^1 \left(i\bar{u}z + x^\alpha \overline{u_x} z_x - \frac{\mu}{x^{2-\alpha}} \overline{u} z \right) dx - [x^\alpha \overline{u_x} z]_{x=0}^{x=1} = i \int_0^1 \bar{f} z dx \quad \forall z \in H^{1,\mu}_{\alpha,0}(0,1).$$

This, combined with (20), gives

$$[x^\alpha \overline{u_x} z]_{x=0}^{x=1} = 0 \quad \forall z \in H^{1,\mu}_{\alpha,0}(0,1).$$

Recalling that $u \in H^2_{loc}((0,1])$, the term $(x^\alpha \overline{u_x})(1)$ makes sense. Since $z(1) = 0$, then

$$\lim_{x \downarrow 0} x^\alpha \overline{u_x} z = 0, \quad \forall z \in H^{1,\mu}_{\alpha,0}(0,1).$$

Now, define $z(x) := 1 - x$ for all $x \in (0,1)$; then $z \in H^{1,\mu}_{\alpha,0}(0,1)$ and $\lim_{x\downarrow 0} z(x) = 1$, thus

$$\lim_{x \downarrow 0} x^\alpha \overline{u_x} = 0.$$

In conclusion, $u \in D(A^\mu_\alpha)$ and solves (19). □

Therefore, from standard semigroup theory, we get the following well posedness result:

Theorem 4. *Let $T > 0$ be given and assume (2). Given $u_0 \in H^{1,\mu}_{\alpha,0}(0,1)$, problem (5) has a unique solution*

$$u \in C\left([0,+\infty), H^{1,\mu}_{\alpha,0}(0,1)\right) \cap C^1\left([0,+\infty), H^{-1,\mu}_{\alpha}(0,1)\right).$$

Moreover,

$$\|u(t)\|_{L^2(0,1)} = \|u_0\|_{L^2(0,1)}, \quad \forall\, t \in [0,T]. \tag{22}$$

If $u_0 \in D\left(A^\mu_\alpha\right)$, then

$$u \in C\left([0,+\infty), D\left(A^\mu_\alpha\right)\right) \cap C^1\left([0,+\infty), H^{1,\mu}_{\alpha,0}(0,1)\right)$$

and

$$\|u(t)\|_{H^{1,\mu}_{\alpha,0}(0,1)} = \|u_0\|_{H^{1,\mu}_{\alpha,0}(0,1)}, \quad \forall\, t \in [0,T]. \tag{23}$$

Proof. Likewise [16, Proposition 3.5.13], from the skew-adjointness of iA^μ_α, we deduce the desired existence, uniqueness and regularity results (see also [15, Proposition 2.1.1]). Let us prove the other facts. Suppose that $u_0 \in D(A^\mu_\alpha)$ so that u is a classical solution of (5) in the sense that $u \in C\left([0,T], D(A^\mu_\alpha)\right) \cap C^1\left([0,T], H^{1,\mu}_{\alpha,0}(0,1)\right)$ (see [16, Theorem 3.2.3]). Then, multiplying (5) by $i\overline{u}$ and integrating over $(0,1)$, we obtain

$$\begin{aligned}
0 &= \int_0^1 i\overline{u(t,x)}\left\{iu_t(t,x) + \left(x^\alpha u_x\right)_x(t,x) + \frac{\mu}{x^{2-\alpha}}u(t,x)\right\} dx \\
&= \int_0^1 \left\{-\overline{u(t,x)}u_t(t,x) - i\left(x^\alpha |u_x|^2 - \frac{\mu}{x^{2-\alpha}}|u|^2\right)\right\} dx + \left[ix^\alpha u_x(t,x)\overline{u(t,x)}\right]_{x=0}^{x=1}.
\end{aligned}$$

Moreover, according to (16), we see that the boundary terms vanish. Then, taking the real part, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(0,1)} = 0,$$

which guarantees the conservation of the L^2-norm of u:

$$\|u(t)\|_{L^2(0,1)} = \|u_0\|_{L^2(0,1)}, \text{ for every } t \in [0,T].$$

On the other hand, multiplying (5) by $\overline{u_t}$ and integrating over $(0,1)$, we have

$$\begin{aligned}
0 &= \int_0^1 \overline{u_t(t,x)}\left\{iu_t(t,x) + \left(x^\alpha u_x\right)_x(t,x) + \frac{\mu}{x^{2-\alpha}}u(t,x)\right\} dx \\
&= \int_0^1 \left\{i|u_t|^2 + \overline{u_t(t,x)}(x^\alpha u_x)_x(t,x) + \mu\frac{\overline{u_t(t,x)}u(t,x)}{x^{2-\alpha}}\right\} dx \\
&= \int_0^1 \left\{i|u_t|^2 - \left(x^\alpha u_x\overline{u_{tx}} - \mu\frac{\overline{u_t}u}{x^{2-\alpha}}\right)\right\} dx + \left[x^\alpha u_x(t,x)\overline{u_t(t,x)}\right]_{x=0}^{x=1}.
\end{aligned}$$

Noting that the boundary terms vanish because of the boundary conditions and thanks to (16) in both the weakly and strongly degenerate cases, we obtain

$$\int_0^1 \left(x^\alpha u_x \overline{u_{tx}} - \mu \frac{u\overline{u_t}}{x^{2-\alpha}} \right) dx = i \int_0^1 |u_t|^2 \, dx \in i\mathbb{R}.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^{1,\mu}_{\alpha,0}(0,1)} = \mathrm{Re} \int_0^1 \left(x^\alpha u_x \overline{u_{tx}} - \mu \frac{u\overline{u_t}}{x^{2-\alpha}} \right) dx = 0$$

and we deduce that

$$\|u(t)\|_{H^{1,\mu}_{\alpha,0}(0,1)} = \|u_0\|_{H^{1,\mu}_{\alpha,0}(0,1)}, \text{ for every } t \in [0,T].$$

Finally, an approximation argument allows us to extend the conclusion to mild solutions. □

3. Boundary observability

By employing the nowadays classical multiplier method, we begin by establishing an identity which is the crucial starting point to prove the desired direct and inverse inequalities for the adjoint system (5).

Lemma 5. *Let $T > 0$ be given and assume (2). If u is a classical solution of (5), then we have*

$$\frac{1}{2}\int_0^T |u_x(t,1)|^2 \, dt = \frac{2-\alpha}{2} T \|u_0\|^2_{H^{1,\mu}_{\alpha,0}(0,1)} + \frac{1}{2}\left[\mathrm{Im} \int_0^1 x\overline{u_x} u \, dx \right]_{t=0}^{t=T}. \tag{24}$$

Proof. Multiplying (5) by $x\overline{u_x} + \frac{1}{2}\overline{u}$ and integrating over Q, we obtain

$$\begin{aligned} 0 &= \int_Q iu_t(x\overline{u_x} + \frac{1}{2}\overline{u}) \, dx \, dt + \int_Q (x^\alpha u_x)_x (x\overline{u_x} + \frac{1}{2}\overline{u}) \, dx \, dt \\ &\quad + \int_Q (x\overline{u_x} + \frac{1}{2}\overline{u}) \frac{\mu}{x^{2-\alpha}} u \, dx \, dt \\ &:= \mathcal{I} + \mathcal{J} + \mathcal{K}. \end{aligned} \tag{25}$$

We proceed integrating by parts the first two terms on the right-hand side of this equality as follows. For the first integral, we have

$$\begin{aligned} \mathcal{I} &= \left[\int_0^1 ixu\overline{u_x} \, dx \right]_{t=0}^{t=T} - \int_Q ixu\overline{u_{tx}} \, dx \, dt + \frac{1}{2}\left[\int_0^T ixu_t\overline{u} \, dt \right]_0^1 \\ &\quad - \frac{1}{2}\int_Q ixu_{tx}\overline{u} \, dx \, dt - \frac{1}{2}\int_Q ixu_t\overline{u_x} \, dx \, dt \end{aligned}$$

$$= \left[\int_0^1 ixu\overline{u_x}\,dx\right]_{t=0}^{t=T} - \int_Q ixu\overline{u_{tx}}\,dx\,dt + \frac{1}{2}\left[\int_0^T ixu_t\overline{u}\,dt\right]_0^1$$
$$- \frac{1}{2}\int_Q ixu_{tx}\overline{u}\,dx\,dt - \frac{1}{2}\left[\int_0^1 ixu\overline{u_x}\,dx\right]_{t=0}^{t=T} + \frac{1}{2}\int_Q ixu\overline{u_{tx}}\,dx\,dt.$$

Then, using the boundary conditions together with the fact that $u_t = iA^\mu_\alpha u \in H^{1,\mu}_{\alpha,0}(0,1)$, by (14) it follows that

$$\mathcal{I} = \frac{i}{2}\left[\int_0^1 xu\overline{u_x}\,dx\right]_{t=0}^{t=T} - \frac{i}{2}\int_Q x(u\overline{u_{tx}} + u_{tx}\overline{u})\,dx\,dt. \tag{26}$$

Moreover, after suitable integrations by parts, we also have

$$\mathcal{J} = \left[\int_0^T x^{\alpha+1}|u_x|^2\,dt\right]_0^1 - \int_Q x^\alpha u_x(x\overline{u_x})_x\,dx\,dt$$
$$+ \frac{1}{2}\left[\int_0^T x^\alpha u_x\overline{u}\,dt\right]_0^1 - \frac{1}{2}\int_Q x^\alpha u_x\overline{u_x}\,dx\,dt$$
$$= \left[\int_0^T x^{\alpha+1}|u_x|^2\,dt\right]_0^1 - \int_Q x^\alpha|u_x|^2\,dx\,dt$$
$$- \int_Q x^{\alpha+1}u_x\overline{u_{xx}}\,dx\,dt + \frac{1}{2}\left[\int_0^T x^\alpha u_x\overline{u}\,dt\right]_0^1$$
$$- \frac{1}{2}\left[\int_0^T x^{\alpha+1}|u_x|^2\,dt\right]_0^1 + \frac{1}{2}\int_Q x(x^\alpha u_x\overline{u_x})_x\,dx\,dt.$$

In view of the boundary conditions and (15), we obtain

$$\begin{aligned}\mathcal{J} = \frac{1}{2}\int_0^T |u_x(t,1)|^2\,dt + \frac{\alpha-2}{2}\int_Q x^\alpha|u_x|^2\,dx\,dt\\ - \frac{1}{2}\int_Q x^{\alpha+1}(u_x\overline{u_{xx}} - \overline{u_x}u_{xx})\,dx\,dt.\end{aligned} \tag{27}$$

Inserting (26) and (27) into (25) and taking the real parts, we have

$$\frac{1}{2}\int_0^T |u_x(t,1)|^2\,dt = \frac{2-\alpha}{2}\int_Q x^\alpha|u_x|^2\,dx\,dt - \operatorname{Re}\mathcal{K} + \frac{1}{2}\left[\operatorname{Im}\int_0^1 x\overline{u_x}u\,dx\right]_{t=0}^{t=T}. \tag{28}$$

On the other hand, after an integration by parts and making use of (17) and (18), we deduce that

$$\operatorname{Re}\mathcal{K} = \frac{2-\alpha}{2}\int_Q \frac{\mu}{x^{2-\alpha}}|u|^2\,dx\,dt. \tag{29}$$

Then the identity (24) follows inserting (29) into (28) and using (23). $\square$

With the help of Lemma 5, we can now prove the main result of this section.

Proposition 6. *Let $T > 0$ be given and assume (2). Then, there exist some constants $c_1, c_2 > 0$ such that, for every $u_0 \in H^{1,\mu}_{\alpha,0}(0,1)$, the solution u of (5) satisfies*

$$\int_0^T |u_x(t,1)|^2\,dt \le c_1\|u_0\|^2_{H^{1,\mu}_{\alpha,0}(0,1)} \tag{30}$$

and

$$\|u_0\|^2_{H^{1,\mu}_{\alpha,0}(0,1)} \le c_2\int_0^T |u_x(t,1)|^2\,dt. \tag{31}$$

Proof. By the Young inequality, due to (9) and (10), there exists a positive constant $C = C(\alpha)$ such that, for all $t \in [0,T]$, we have

$$\begin{aligned}\left|\operatorname{Im}\int_0^1 x\overline{u_x(t,x)}u(t,x)\,dx\right| &\le \frac{1}{2}\int_0^1 |u(t,x)|^2\,dx + \frac{1}{2}\int_0^1 x^2|u_x(t,x)|^2\,dx\\ &\le C\int_0^1\left(x^\alpha|u_x(t,x)|^2 - \frac{\mu(\alpha)}{x^{2-\alpha}}|u(t,x)|^2\right)dx\\ &\le C\|u(t)\|^2_{H^{1,\mu}_{\alpha,0}(0,1)},\end{aligned}$$

since $\|\cdot\|_{H^{1,\mu(\alpha)}_{\alpha,0}} \le \|\cdot\|_{H^{1,\mu}_{\alpha,0}}$ $(\forall\,\mu \le \mu(\alpha))$. Then (23) yields

$$\left|\operatorname{Im}\int_0^1 x\overline{u_x(t,x)}u(t,x)\,dx\right| \le C\|u_0\|^2_{H^{1,\mu}_{\alpha,0}(0,1)}, \quad \forall\, t \in [0,T]. \tag{32}$$

Using this inequality in (24), we then deduce that there exists a positive constant $c_1 = c_1(T,\alpha)$ such that

$$\int_0^T |u_x(t,1)|^2\,dt \le c_1\|u_0\|^2_{H^{1,\mu}_{\alpha,0}(0,1)}.$$

Let us now prove the inverse inequality (31). We split the proof in two main steps. First, applying Young's inequality, for all $\varepsilon > 0$, we have

$$\left|\operatorname{Im}\int_0^1 x\overline{u_x}u\,dx\right| \le C_{\alpha,\varepsilon}\int_0^1 |u|^2\,dx + \frac{\varepsilon}{C_\alpha}\int_0^1 x^2|u_x|^2\,dx,$$

where C_α denotes the constant in (9). Using (9) together with (22) and (23), one has

$$\left|\mathrm{Im}\int_0^1 x\overline{u_x}u\,dx\right| \le C_{\alpha,\varepsilon}\|u_0\|^2_{L^2(0,1)} + \varepsilon\|u_0\|^2_{H^{1,\mu}_{\alpha,0}(0,1)}. \tag{33}$$

Thus, choosing $\varepsilon < \frac{2-\alpha}{2}T$, by (24) and (33), we deduce that

$$\left(\frac{2-\alpha}{2}T-\varepsilon\right)\|u_0\|^2_{H^{1,\mu}_{\alpha,0}(0,1)} \le \frac{1}{2}\int_0^T |u_x(t,1)|^2\,dt + C_{\alpha,\varepsilon}\|u_0\|^2_{L^2(0,1)}. \tag{34}$$

In a second step, to complete the proof, it is enough to prove that there exists a constant $K > 0$ such that

$$\|u_0\|^2_{L^2(0,1)} \le K\int_0^T |u_x(t,1)|^2\,dt. \tag{35}$$

Following [29], we argue by contradiction via a compactness-uniqueness argument. Let us assume that (35) is not satisfied. This implies that there exists a sequence $\{u_n\}$ of solutions of (5) such that

$$\|u_n(0)\|_{L^2(0,1)} = 1, \quad \forall\, n \in \mathbb{N} \tag{36}$$

and

$$\int_0^T |u_{n,x}(t,1)|^2\,dt \to 0 \text{ as } n \to +\infty. \tag{37}$$

From (34) we deduce that $\{u_n(0)\}$ is bounded in $H^{1,\mu}_{\alpha,0}(0,1)$ and, using Theorem 4, we get

$$\{u_n\} \text{ is bounded in } L^\infty(0,T;H^{1,\mu}_{\alpha,0}(0,1)) \cap W^{1,\infty}(0,T;H^{-1,\mu}_{\alpha}(0,1)).$$

Hence, extracting a subsequence (that we will still denote by $\{u_n\}$) we have

$$\begin{cases} u_n \to u, & \text{in } L^\infty\left(0,T;H^{1,\mu}_{\alpha,0}(0,1)\right) \quad \text{weakly }^\star,\\ (u_n)_t \to u_t, & \text{in } L^\infty\left(0,T;H^{-1,\mu}_{\alpha}(0,1)\right) \quad \text{weakly }^\star. \end{cases}$$

The function $u \in L^\infty(0,T;H^{1,\mu}_{\alpha,0}(0,1)) \cap W^{1,\infty}(0,T;H^{-1,\mu}_{\alpha}(0,1))$ is clearly a solution of (5). Moreover, by the compactness of the embedding (see [36, section 8])

$$L^\infty(0,T;H^{1,\mu}_{\alpha,0}(0,1)) \cap W^{1,\infty}(0,T;H^{-1,\mu}_{\alpha}(0,1)) \to C\left([0,T];L^2(0,1)\right)$$

and, by (36), we deduce

$$\|u(0)\|_{L^2(0,1)} = 1. \tag{38}$$

On the other hand, (37) implies

$$u_x(t,1) = 0 \text{ on } (0,T). \tag{39}$$

Applying the standard unique continuation method (see [37]), it results that (5) combined with (39) implies $u \equiv 0$, which is in contradiction with (38). Indeed, unique continuation results may be applied far from the origin where the coefficients of the Schrödinger operator $i\partial_t \cdot + \partial_x(x^\alpha \partial_x \cdot) + \frac{\mu}{x^{2-\alpha}} \cdot$ are analytic in time (actually, they are independent of time and bounded in space). Then one can apply Holmgreen's unique continuation (see [37, section 5.2]) that may be justified as described in [40, Theorem 6.1] (see also [14, Remark 4.1]), to get $u = 0$, a.e. in $(\varepsilon, 1)$ for any $\varepsilon > 0$. Thus, we will conclude that $u \equiv 0$ in $(0, 1)$. This proves (35). Finally, by (34) and (35), the desired inverse inequality (31) follows. □

4. Boundary controllability

Prior to the formulation of the exact controllability theorem we have to give a sense to the solution of the system (1) which has non homogeneous Dirichlet data on a part of the boundary. To this aim, we need to make some necessary preparation. First of all, let us consider the degenerate/singular Schrödinger equation with homogeneous boundary conditions and a source term:

$$\begin{cases} iy_t + (x^\alpha y_x)_x + \frac{\mu}{x^{2-\alpha}} y = h, & (t,x) \in Q, \\ \begin{cases} y(t,0) = 0, & \text{if } 0 \le \alpha < 1, \\ (x^\alpha y_x)(t,0) = 0, & \text{if } 1 < \alpha < 2, \end{cases} & t \in (0,T), \\ y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0, & x \in (0,1), \end{cases} \tag{40}$$

with $y_0 \in H^{1,\mu}_{\alpha,0}(0,1)$ and $h \in L^1\left(0,T; H^{1,\mu}_{\alpha,0}(0,1)\right)$.

The following result is a consequence of Proposition 3 and [16, Lemmas 4.1.1 and 4.1.5].

Theorem 7. *Assume (2). Given $h \in L^1\left(0,T; H^{1,\mu}_{\alpha,0}(0,1)\right)$ and $y_0 \in H^{1,\mu}_{\alpha,0}(0,1)$, the system (40) admits a unique solution*

$$y \in C([0,T], H^{1,\mu}_{\alpha,0}(0,1)).$$

In addition, we have

$$\|y\|_{C([0,T],H^{1,\mu}_{\alpha,0}(0,1))} \le C \left(\|y_0\|_{H^{1,\mu}_{\alpha,0}(0,1)} + \|h\|_{L^1\left(0,T;H^{1,\mu}_{\alpha,0}(0,1)\right)} \right). \tag{41}$$

In the following, we will give a sharp trace regularity result for problem (40).

Lemma 8. *Assume (2) and consider y the unique solution of (40) corresponding to the initial data $y_0 \in H^{1,\mu}_{\alpha,0}(0,1)$. Then*

$$y_x(t,1) \in L^2(0,T).$$

Moreover, there exists $C_T > 0$ such that

$$\int_0^T |y_x(t,1)|^2 \, dt \le C_T \left(\|y_0\|^2_{H^{1,\mu}_{\alpha,0}(0,1)} + \|h\|^2_{L^1\left(0,T;H^{1,\mu}_{\alpha,0}(0,1)\right)} \right) \tag{42}$$

and y satisfies the identity

$$\frac{1}{2}\int_0^T |y_x(t,1)|^2\,dt = \frac{2-\alpha}{2}T\|y_0\|^2_{H^{1,\mu}_{\alpha,0}(0,1)} + \frac{1}{2}\left[\operatorname{Im}\int_0^1 x\overline{y_x}y\,dx\right]_{t=0}^{t=T} + \operatorname{Re}\int_Q x\overline{y_x}h\,dx\,dt + \frac{1}{2}\operatorname{Re}\int_Q \overline{y}h\,dx\,dt. \tag{43}$$

Proof. Similar computations as in (24) lead to the identity (43). Then, inequality (42) follows from (43) and the energy inequality (41). □

As a consequence, in a second step, we are going to prove well posedness of the non-homogeneous boundary value problem (1) with zero initial data. For this purpose, we introduce the following definition of a solution by transposition in the spirit of [22,28].

Definition 1. Let $f \in L^2(0,T)$. We say that y is a solution by transposition of the problem

$$\begin{cases} iy_t + (x^\alpha y_x)_x + \frac{\mu}{x^{2-\alpha}}y = 0, & (t,x)\in Q, \\ \begin{cases} y(t,0)=0, & \text{if } 0\le\alpha<1, \\ (x^\alpha y_x)(t,0)=0, & \text{if } 1<\alpha<2, \end{cases} & t\in(0,T), \\ y(t,1)=f, & t\in(0,T), \\ y(0,x)=0, & x\in(0,1), \end{cases} \tag{44}$$

when $y \in L^\infty\left(0,T; H^{-1,\mu}_\alpha(0,1)\right)$ and, for each $h_1 \in L^1\left(0,T; H^{1,\mu}_{\alpha,0}(0,1)\right)$, one has

$$\int_0^T \langle y(t), h_1(t)\rangle_{H^{-1,\mu}_\alpha, H^{1,\mu}_{\alpha,0}}\,dt = \langle f, w_x(t,1)\rangle_{L^2(0,T)}, \tag{45}$$

where w is the solution of the backward Schrödinger equation

$$\begin{cases} iw_t + (x^\alpha w_x)_x + \frac{\mu}{x^{2-\alpha}}w = h_1, & (t,x)\in Q, \\ \begin{cases} w(t,0)=0, & \text{if } 0\le\alpha<1, \\ (x^\alpha w_x)(t,0)=0, & \text{if } 1<\alpha<2, \end{cases} & t\in(0,T), \\ w(t,1)=0, & t\in(0,T), \\ w(T,x)=0, & x\in(0,1). \end{cases} \tag{46}$$

Remark 4. Observe that Lemma 8 can be applied to the backward Schrödinger equation (46). Indeed, system (46) can be reduced to (40) by changing t in $T-t$. In particular, we have that the solution w of (46) satisfies $w_x(t,1) \in L^2(0,T)$ so that the above definition makes sense.

We now state the following theorem that concerns the existence and uniqueness of solution to the new system (44) using the method of transposition.

Theorem 9. *Assume (2) and let $f \in L^2(0,T)$. Then the system (44) has a unique solution y belonging to the space $C\left([0,T]; H^{-1,\mu}_\alpha(0,1)\right)$ in the sense of transposition. Moreover, the operator $f \mapsto y$ is linear and continuous from $L^2(0,T)$ into $C\left([0,T]; H^{-1,\mu}_\alpha(0,1)\right)$.*

Proof. Let us define a linear form $\mathcal{L}$ on $L^1\left(0,T;H^{1,\mu}_{\alpha,0}(0,1)\right)$ by

$$\mathcal{L}(h_1) = \langle f, w_x(t,1)\rangle_{L^2(0,T)},$$

where w is the unique solution to the adjoint system (46) with given source term h_1. The map $\mathcal{L}$ is well-defined because of the hidden regularity as mentioned in the remark above.

Using Cauchy-Schwarz's inequality, from the estimate (42) for the solution w of (46), we obtain

$$\begin{aligned} |\mathcal{L}(h_1)| &\le \|f\|_{L^2(0,T)} \left(\int_0^T |w_x(t,1)|^2 \, dt\right)^{\frac{1}{2}} \\ &\le C\|f\|_{L^2(0,T)} \|h_1\|_{L^1\left(0,T;H^{1,\mu}_{\alpha,0}(0,1)\right)}, \end{aligned} \tag{47}$$

so that $\mathcal{L}$ is continuous on $L^1\left(0,T;H^{1,\mu}_{\alpha,0}(0,1)\right)$. Therefore, from the Riesz representation Theorem, there exists a unique $y \in L^\infty\left(0,T;H^{-1,\mu}_{\alpha}(0,1)\right)$ that satisfies (45) for every $f \in L^1\left(0,T;H^{1,\mu}_{\alpha,0}(0,1)\right)$. Moreover, the continuity of $\mathcal{L}$ reads as

$$\|y\|_{L^\infty\left(0,T;H^{-1,\mu}_{\alpha}(0,1)\right)} \le C\|f\|_{L^2(0,T)}. \tag{48}$$

Thus, the map $f \mapsto y$ is continuous from $L^2(0,T)$ into $L^\infty\left(0,T;H^{-1,\mu}_{\alpha}(0,1)\right)$.

It remains to prove that y actually belongs to $C\left([0,T];H^{-1,\mu}_{\alpha}(0,1)\right)$. We take a sequence of smooth approximation $f_n \to f$ strongly in $L^2(0,T)$. The problem (44) with boundary condition f_n admits a smooth solution y_n, which is also a transposition solution. In particular, $y_n \in C\left([0,T];H^{-1,\mu}_{\alpha}(0,1)\right)$ (see for example [27]) and the estimate (48) implies that y is the limit of y_n in $L^\infty\left(0,T;H^{-1,\mu}_{\alpha}(0,1)\right)$. Since $C\left([0,T];H^{-1,\mu}_{\alpha}(0,1)\right)$ is a closed subspace of $L^\infty\left(0,T;H^{-1,\mu}_{\alpha}(0,1)\right)$, this implies $y \in C\left([0,T];H^{-1,\mu}_{\alpha}(0,1)\right)$. Thus the proof is complete. □

After these preparations, we deduce the well posedness of the full initial boundary value problem (1).

Definition 2. Let $f \in L^2(0,T)$ and $y_0 \in H^{-1,\mu}_{\alpha}(0,1)$. We say that y is a solution by transposition of the problem (1) when $y \in L^\infty\left(0,T;H^{-1,\mu}_{\alpha}(0,1)\right)$ and, for each $h_1 \in L^1\left(0,T;H^{1,\mu}_{\alpha,0}(0,1)\right)$, one has

$$\int_0^T \langle y(t), h_1(t)\rangle_{H^{-1,\mu}_{\alpha},H^{1,\mu}_{\alpha,0}} \, dt = \langle f, w_x(t,1)\rangle_{L^2(0,T)} + i\langle y_0, w(0)\rangle_{H^{-1,\mu}_{\alpha},H^{1,\mu}_{\alpha,0}}, \tag{49}$$

where w is the solution of the backward Schrödinger equation (46).

Theorem 10. *Assume (2). For every $f \in L^2(0,T)$ and every $y_0 \in H^{-1,\mu}_{\alpha}(0,1)$, the system (1) has a unique weak solution y belonging to the space $C\left([0,T],H^{-1,\mu}_{\alpha}(0,1)\right)$ in the sense of transposition and the operator defined by*

$$(y_0, f) \mapsto y,$$

is linear and continuous from $H^{-1,\mu}_{\alpha}(0,1) \times L^2(0,T)$ into $C\left([0,T];H^{-1,\mu}_{\alpha}(0,1)\right)$.

Proof. Observe that, for a given $y_0 \in L^2(0,1)$, the system (40) with $h = 0$ admits a unique solution $y \in C([0,T], L^2(0,1))$ which satisfies

$$\|y\|_{C([0,T],L^2(0,1))} \le C\|y_0\|_{L^2(0,1)}.$$

In fact, this solution is also a transposition solution. This is the consequence of an integration by parts if y is smooth enough, and the general case follows by a standard density argument. This fact, Theorem 9 and linearity imply the thesis. □

Now we can prove the null controllability theorem.

Theorem 11. *Let $T > 0$ be arbitrary and assume (2). Then, given $y_0 \in H_\alpha^{-1,\mu}(0,1)$, there exists a control $f \in L^2(0,T)$ such that the corresponding solution of problem (1) (in the sense of transposition) satisfies*

$$y(T,x) = 0, \quad \text{for all } \ x \in (0,1). \tag{50}$$

Proof. We employ the Hilbert Uniqueness Method (HUM) introduced by J.L. Lions in [28]. Given $u_0 \in H^{1,\mu}_{\alpha,0}(0,1)$, by the direct inequality (30), we know that the solution u of (5) satisfies:

$$u_x(t,1) \in L^2(0,T).$$

Now, let us introduce the following system:

$$\begin{cases} iy_t + (x^\alpha y_x)_x + \frac{\mu}{x^{2-\alpha}} y = 0, & (t,x) \in Q, \\ y(t,1) = u_x(t,1), & t \in (0,T), \\ \begin{cases} y(t,0) = 0, & \text{if } 0 \le \alpha < 1, \\ (x^\alpha y_x)(t,0) = 0, & \text{if } 1 < \alpha < 2, \end{cases} & t \in (0,T), \\ y(T) = 0, & x \in (0,1). \end{cases} \tag{51}$$

By Theorem 9, problem (51) has a unique solution y, satisfying $y_0 := y(0,x) \in H_\alpha^{-1,\mu}(0,1)$. Hence the linear map

$$\Lambda : H^{1,\mu}_{\alpha,0} \to H^{-1,\mu}_\alpha, \quad u_0 \mapsto -iy_0$$

is continuous from $H^{1,\mu}_{\alpha,0}$ into $H^{-1,\mu}_\alpha$. It is evident that, if Λ is surjective, then the null controllability problem for (1) is solved with a control of the form $f(t) = u_x(t,1)$, where u is the solution of (5) with initial data $u_0 = \Lambda^{-1}(-iy_0)$.

Multiplying equation (51) by $\overline{u}$, integrating by parts over Q and taking the real parts, it follows that:

$$\langle -iy_0, u_0 \rangle_{H^{-1,\mu}_\alpha, H^{1,\mu}_{\alpha,0}} = \int_0^T |u_x(t,1)|^2 \, dt.$$

Equivalently,

$$\langle \Lambda u_0, u_0 \rangle_{H^{-1,\mu}_\alpha, H^{1,\mu}_{\alpha,0}} = \int_0^T |u_x(t,1)|^2 \, dt.$$

By Proposition 6, for every $T > 0$ and $u_0 \in H^{1,\mu}_{\alpha,0}(0,1)$, we have

$$\int_0^T |u_x(t,1)|^2 \, dt \asymp \|u_0\|^2_{H^{1,\mu}_{\alpha,0}(0,1)}.$$

Therefore, for every $T > 0$, one has:

$$\langle \Lambda u_0, u_0 \rangle_{H^{-1,\mu}_\alpha, H^{1,\mu}_{\alpha,0}} \asymp \|u_0\|^2_{H^{1,\mu}_{\alpha,0}(0,1)}.$$

Then, thanks to the Lions-Lax-Milgram Lemma (see [27]), Λ is an isomorphism from $H^{1,\mu}_{\alpha,0}(0,1)$ onto $H^{-1,\mu}_\alpha(0,1)$ and this completes the proof of Theorem 11. □

5. Exponential stabilization

This section is devoted to the study of boundary stabilization for the degenerate and singular linearly damped Schrödinger equation:

$$iu_t + (x^\alpha u_x)_x + \frac{\mu}{x^{2-\alpha}} u = 0, \quad \text{in } (0,+\infty) \times (0,1), \tag{52}$$

with

$$\begin{cases} u_t(t,1) + u_x(t,1) + \beta u(t,1) = 0, & t > 0, \\ \begin{cases} u(t,0) = 0, & \text{if } 0 \le \alpha < 1, \\ (x^\alpha u_x)(t,0) = 0, & \text{if } 1 < \alpha < 2, \end{cases} & t > 0, \\ u(0,x) = u_0(x), & 0 \le x \le 1, \end{cases} \tag{53}$$

where $\mu \in \mathbb{R}$ and $\beta \geq 0$ is given.

In order to study problem (52)-(53), we now make the following assumption:

Assumption 12. We assume that the parameters α and μ satisfy:

$$\alpha \in [0,2) \setminus \{1\} \text{ and } \mu < \mu(\alpha). \tag{54}$$

Remark 5. In the controllability problem, we have assumed (2) which include both cases: the subcritical potential $\mu < \mu(\alpha)$ and the critical one $\mu = \mu(\alpha)$. But for the stabilization problem, we only treat the first one, i.e. $\mu < \mu(\alpha)$. The reason relies on the fact that we need the Hardy-Poincaré inequalities given in Lemma 1 valid in the space $H^{1,\mu}_\alpha(0,1)$ instead of $H^{1,\mu}_{\alpha,0}(0,1)$. In the subcritical case, we can work on the space $H^{1,\mu=0}_\alpha(0,1)$ (see [39]), where we prove the Hardy-Poincaré inequality (55). Also, to deal with the critical case, similar results to the ones stated in Proposition 15 will be needed to be proved in the space $W^{1,\mu}_\alpha(0,1)$ instead of $W^{1,\mu=0}_\alpha(0,1)$ (see the next subsection for the definition of these two spaces).

5.1. *Preliminary results and well posedness*

We start introducing the functional setting needed to treat our problem. Let us denote by $W^{1,\mu}_\alpha(0,1)$ the space $H^{1,\mu}_\alpha(0,1)$ itself if $\alpha \in (1,2)$ and, if $\alpha \in [0,1)$, the closed subspace of $H^{1,\mu}_\alpha(0,1)$ consisting of all the functions $u \in H^{1,\mu}_\alpha(0,1)$ such that $u(0) = 0$. Moreover, we set

$$W^{2,\mu}_\alpha(0,1) = H^{2,\mu}_\alpha(0,1) \cap W^{1,\mu}_\alpha(0,1).$$

Notice that $W^{2,\mu}_\alpha(0,1) = H^{2,\mu}_\alpha(0,1)$ when $\alpha \in (1,2)$.

In the Hilbert space $W^{1,\mu}_\alpha(0,1)$ we consider the following scalar product

$$\langle u,v\rangle_{W^{1,\mu}_\alpha} = \mathrm{Re}\left(\int_0^1 \left(u(x)\overline{v(x)} + x^\alpha u_x(x)\overline{v_x(x)} - \frac{\mu}{x^{2-\alpha}}u(x)\overline{v(x)}\right)dx + \beta u(1)\overline{v(1)}\right),$$

for all $u, v \in W^{1,\mu}_\alpha(0,1)$, and the associated norm

$$\|u\|_{W^{1,\mu}_\alpha(0,1)} = \left(\int_0^1 \left(|u(x)|^2 + x^\alpha |u_x(x)|^2 - \frac{\mu}{x^{2-\alpha}}|u(x)|^2\right)dx + \beta|u(1)|^2\right)^{\frac{1}{2}},$$

for all $u \in W^{1,\mu}_\alpha(0,1)$.

Let us also set

$$(u,v)_{W^{1,\mu}_\alpha(0,1)} = \mathrm{Re}\left(\int_0^1 \left(x^\alpha u_x(x)\overline{v_x(x)} - \frac{\mu}{x^{2-\alpha}}u(x)\overline{v(x)}\right)dx + \beta u(1)\overline{v(1)}\right),$$

for all $u, v \in W^{1,\mu}_\alpha(0,1)$, and its corresponding norm

$$|u|_{W^{1,\mu}_\alpha(0,1)} := \left(\int_0^1 \left(x^\alpha |u_x(x)|^2 - \frac{\mu}{x^{2-\alpha}}|u(x)|^2\right)dx + \beta|u(1)|^2\right)^{\frac{1}{2}},$$

for all $u \in W^{1,\mu}_\alpha(0,1)$.

We first show the following Hardy-type inequality.

Lemma 13. *Let $\alpha \in [0,2)$. There exists a constant D_α such that, for every $u \in W^{1,\mu=0}_\alpha(0,1)$, we have*

$$\frac{(1-\alpha)^2}{4}\int_0^1 \frac{|u(x)|^2}{x^{2-\alpha}}dx \le \int_0^1 x^\alpha |u_x(x)|^2 dx + D_\alpha |u(1)|^2. \tag{55}$$

More precisely,

$$D_\alpha := \max\left\{0, \frac{\alpha-1}{2}\right\}.$$

Proof. Let $u \in W^{1,\mu=0}_\alpha(0,1)$. We can assume that u is a real function, since the result can be easily extended to the complex case using the fact that $|u|^2 = (\mathrm{Re}\, u)^2 + (\mathrm{Im}\, u)^2$.

It is well known that (55) is valid for all $\alpha \in [0,1)$ (see for example [3, Proposition 2.1]). Let us prove the result in the case $\alpha \in [1,2)$. For all $x \in (0,1)$, we have that

$$\begin{aligned} 0 &\le \int_x^1 \left(s^{\frac{\alpha}{2}}u'(s) - \frac{1-\alpha}{2}\frac{u(s)}{s^{\frac{2-\alpha}{2}}}\right)^2 ds \\ &= \int_x^1 s^\alpha |u'(s)|^2 ds + \frac{(1-\alpha)^2}{4}\int_x^1 \frac{|u(s)|^2}{s^{2-\alpha}}ds - \frac{1-\alpha}{2}\int_x^1 \frac{1}{s^{1-\alpha}}\left(u^2(s)\right)' ds \end{aligned}$$

$$= \int_x^1 \left(s^\alpha |u'(s)|^2 - \mu(\alpha) \frac{|u(s)|^2}{s^{2-\alpha}} \right) ds + \frac{\alpha-1}{2}|u(1)|^2 - \frac{\alpha-1}{2} x^{\alpha-1}|u(x)|^2$$
$$\leq \int_x^1 \left(s^\alpha |u'(s)|^2 - \mu(\alpha) \frac{|u(s)|^2}{s^{2-\alpha}} \right) ds + \frac{\alpha-1}{2}|u(1)|^2,$$

where we recall that $\mu(\alpha)$ is defined in (3).

Thus we get

$$\frac{(1-\alpha)^2}{4} \int_x^1 \frac{|u(s)|^2}{s^{2-\alpha}} \, dx \leq \int_x^1 s^\alpha |u'(s)|^2 \, dx + D_\alpha |u(1)|^2.$$

Therefore, taking the limit as $x \downarrow 0$, we obtain the announced result. □

In the subcritical case $\mu < \mu(\alpha)$, thanks to (55), one can easily prove that $|\cdot|_{W^{1,\mu}_\alpha(0,1)}$ is equivalent to the norm $|\cdot|_{W^{1,\mu=0}_\alpha}$, and hence $W^{1,\mu}_\alpha(0,1) = W^{1,\mu=0}_\alpha(0,1)$. To be more precise, in the subcritical case, one can prove the following result.

Lemma 14. *Assume Hypothesis 12 and consider $\beta \geq 0$. Then there exist two constants $C^1_{\alpha,\mu} > 0$ and $C^2_{\alpha,\mu} > 0$ such that, for every $u \in W^{1,\mu=0}_\alpha(0,1)$*

$$C^1_{\alpha,\mu} |u|^2_{W^{1,\mu=0}_\alpha(0,1)} \leq |u|^2_{W^{1,\mu}_\alpha(0,1)} \leq C^2_{\alpha,\mu} |u|^2_{W^{1,\mu=0}_\alpha(0,1)}. \tag{56}$$

More precisely,

$$C^1_{\alpha,\mu} = 1 - \frac{\max(0,\mu)}{\mu(\alpha)}, \quad C^2_{\alpha,\mu} = 1 - \frac{\min(0,\mu)}{\mu(\alpha)}.$$

Next, we recall some preliminary results that will be very useful to tackle well posedness and stabilization issues for system (52)-(53) (see [4, Proposition 4.3]). First, let us set

$$\|u\|_{\alpha,\mu=0} = \left(\int_0^1 \left(|u(x)|^2 + x^\alpha |u_x(x)|^2 \right) dx \right)^{\frac{1}{2}}, \quad \forall\, u \in W^{1,\mu=0}_\alpha(0,1).$$

Then, we have the following two results.

Proposition 15. *Assume Hypothesis 12. Then, for every $u \in W^{1,\mu=0}_\alpha(0,1)$*

$$\|u\|^2_{L^2(0,1)} \leq 2|u(1)|^2 + \tilde{C}_\alpha \int_0^1 x^\alpha |u_x(x)|^2 \, dx, \tag{57}$$

where

$$\tilde{C}_\alpha = \min\left\{4, \frac{2}{2-\alpha}\right\}.$$

Moreover, assuming that $\beta > 0$, we have

$$|u(1)| \leq \frac{1}{\sqrt{\beta}} |u|_{W^{1,\mu=0}_\alpha(0,1)}, \quad \forall\, u \in W^{1,\mu=0}_\alpha(0,1). \tag{58}$$

Proposition 16. *Assume Hypothesis 12 and consider $\beta \geq 0$. We have*

$$|u|^2_{W^{1,\mu=0}_\alpha(0,1)} \geq c_{\alpha,\beta}\|u\|^2_{L^2(0,1)}, \quad \forall\, u \in W^{1,\mu=0}_\alpha(0,1), \tag{59}$$

where

$$c_{\alpha,\beta} = \min\left\{\frac{1}{\tilde{C}_\alpha}, \frac{\beta}{2}\right\}.$$

Moreover, we also have

$$\frac{c_{\alpha,\beta}}{c_{\alpha,\beta}+1}\left(\|u\|^2_{\alpha,\mu=0} + \beta|u(1)|^2\right) \leq |u|^2_{W^{1,\mu=0}_\alpha(0,1)} \leq \gamma_{\alpha,\beta}\|u\|^2_{\alpha,\mu=0}, \quad \forall\, u \in W^{1,\mu=0}_\alpha(0,1), \tag{60}$$

where

$$\gamma_{\alpha,\beta} = \max\left\{2\beta, 1 + \frac{2\beta}{2-\alpha}\right\}.$$

In view of (56) and (60), we have the equivalence below.

Corollary 17. *Assume Hypothesis 12 and consider $\beta \geq 0$. Then the two norms $\|\cdot\|_{W^{1,\mu=0}_\alpha(0,1)}$ and $|\cdot|_{W^{1,\mu}_\alpha(0,1)}$ are equivalent in $W^{1,\mu=0}_\alpha(0,1)$.*

We are now ready to study the well posedness of problem (52)-(53). For this, we consider the linear unbounded operator $A_\beta : D(A_\beta) \subset W^{1,\mu=0}_\alpha(0,1) \to W^{1,\mu=0}_\alpha(0,1)$ given by

$$A_\beta u := (x^\alpha u_x)_x + \frac{\mu}{x^{2-\alpha}}u,$$

for all $u \in D(A_\beta)$, where the domain $D(A_\beta)$ is defined in the following way:

$$D(A_\beta) = \left\{u \in W^{1,\mu=0}_\alpha(0,1) : A_\beta u \in W^{1,\mu=0}_\alpha(0,1) \text{ and } u_x(1) + \beta u(1) = -i(A_\beta u)(1)\right\}.$$

Remark 6. Observe that:

1. In view of [39, Proposition 1], if $u \in D(A_\beta)$ then, in addition to the boundary conditions $u(0) = 0$ in the weakly degenerate case, also the condition $(x^\alpha u_x)(0) = 0$ in the strongly degenerate case makes sense, as well.
2. Because of the classical Sobolev embedding Theorem, $u_x(1), (A_\beta u)(1)$, and $\beta u(1)$ are well defined for all $u \in W^{1,\mu}_\alpha(0,1)$.
3. Let us also note that the identity

   $$u_x(1) + \beta u(1) = -i(A_\beta u)(1)$$

   has to be understood in the following variational sense:

   $$\int_0^1 \left(x^\alpha u_x \overline{z_x} - \frac{\mu}{x^{2-\alpha}}u\overline{z}\right)dx + \int_0^1 \left((x^\alpha u_x)_x\,\overline{z_x} + \frac{\mu}{x^{2-\alpha}}u\overline{z}\right)dx + \beta u(1)\overline{z(1)} + i(A_\beta u)(1)\overline{z(1)} = 0,$$

   for all $z \in W^{1,\mu}_\alpha(0,1)$.

The next result holds.

Proposition 18. *Assume Hypothesis 12 and consider $\beta > 0$. Then iA_β is a maximal dissipative operator in $W^{1,\mu=0}_\alpha(0,1)$.*

Proof. Let $u \in D(A_\beta)$. Then, by using (16), we have

$$\begin{aligned}
(iA_\beta u, u)_{W^{1,\mu}_\alpha(0,1)} &= \mathrm{Re}\left[i\int_0^1 \left(x^\alpha (A_\beta u)_x \overline{u_x} - \frac{\mu}{x^{2-\alpha}} A_\beta u \overline{u} \right) dx + \beta (iA_\beta u)(1)\overline{u(1)} \right] \\
&= \mathrm{Re}\left[[(iA_\beta u)x^\alpha \overline{u_x}]_0^1 - i\int_0^1 A_\beta u \left((x^\alpha \overline{u_x})_x + \frac{\mu}{x^{2-\alpha}} \overline{u} \right) dx + \beta (iA_\beta u)(1)\overline{u(1)} \right] \\
&= \mathrm{Re}\left[-i\int_0^1 |A_\beta u|^2\, dx + (iA_\beta u)(1)\overline{u_x(1)} + \beta (iA_\beta u)(1)\overline{u(1)} \right] \\
&= \mathrm{Re}\left[-i\int_0^1 |A_\beta u|^2\, dx - |(A_\beta u)(1)|^2 \right] = -|(A_\beta u)(1)|^2\, dx \le 0.
\end{aligned}$$

Therefore, iA_β is dissipative.

In order to show that A_β is maximal dissipative, it remains to check that $I - iA_\beta$ is surjective. Equivalently, given any $f \in W^{1,\mu=0}_\alpha(0,1)$, we have to prove that there exists $u \in D(A_\beta)$ such that

$$u - iA_\beta u = f. \tag{61}$$

To this aim, observe that, by Corollary 17, for all $u, v \in W^{1,\mu=0}_\alpha(0,1)$

$$\langle u, v\rangle_1 := \int_0^1 \left(x^\alpha u_x(x)\overline{v_x(x)} - \frac{\mu}{x^{2-\alpha}} u(x)\overline{v(x)} \right) dx + \beta u(1)\overline{v(1)}$$

defines another scalar product in $W^{1,\mu=0}_\alpha(0,1)$ whose corresponding norm $|\cdot|_{W^{1,\mu}_\alpha(0,1)}$ is equivalent to $\|\cdot\|_{W^{1,\mu=0}_\alpha(0,1)}$. Hence $W^{1,\mu=0}_\alpha(0,1)$ endowed with the scalar product $\langle\cdot,\cdot\rangle_1$ is also a Hilbert space.

Let us consider the sesquilinear form $\Lambda : W^{1,\mu=0}_\alpha(0,1) \times W^{1,\mu=0}_\alpha(0,1) \to \mathbb{C}$ given by

$$\Lambda(u, z) = \int_0^1 \left(i\,\bar{u}z + x^\alpha \overline{u_x} z_x - \frac{\mu}{x^{2-\alpha}} \bar{u}z \right) dx + (\beta+1)\overline{u(1)}z(1), \quad \forall u, z \in W^{1,\mu=0}_\alpha(0,1).$$

We have

$$\begin{aligned}
\mathrm{Re}\,\Lambda(u,u) &= \mathrm{Re}\int_0^1 \left(i\,|u|^2 + x^\alpha |u_x|^2 - \frac{\mu}{x^{2-\alpha}} |u|^2 \right) dx + (\beta+1)|u(1)|^2 \\
&= \int_0^1 \left(x^\alpha |u_x|^2 - \frac{\mu}{x^{2-\alpha}} |u|^2 \right) dx + (\beta+1)|u(1)|^2 \\
&\ge |u|^2_{W^{1,\mu}_\alpha(0,1)} \ge C^1_{\alpha,\mu} |u|^2_{W^{1,\mu=0}_\alpha(0,1)},
\end{aligned}$$

and hence $\Lambda(\cdot,\cdot)$ is coercive. Moreover $\Lambda(\cdot,\cdot)$ is continuous: indeed, applying the Cauchy-Schwarz inequality, for all $u, z \in W^{1,\mu=0}_\alpha(0,1)$, we have

$$\begin{aligned}|\Lambda(u,z)| &\leq \|u\|_{L^2(0,1)}\|z\|_{L^2(0,1)} + |u|_{W^{1,\mu}_\alpha(0,1)}|z|_{W^{1,\mu}_\alpha(0,1)} + (\beta+1)\,|u(1)||z(1)| \\ &\text{(by Proposition 16)} \\ &\leq \frac{1}{c_{\alpha,\beta}}|u|_{W^{1,\mu=0}_\alpha(0,1)}|z|_{W^{1,\mu=0}_\alpha(0,1)} + |u|_{W^{1,\mu}_\alpha(0,1)}|z|_{W^{1,\mu}_\alpha(0,1)} + \frac{\beta+1}{\beta}|u|_{W^{1,\mu=0}_\alpha(0,1)}|z|_{W^{1,\mu=0}_\alpha(0,1)} \\ &\text{(by Lemma 14)} \\ &\leq \left[C^2_{\alpha,\mu} + \frac{1}{c_{\alpha,\beta}} + \frac{\beta+1}{\beta}\right]|u|_{W^{1,\mu=0}_\alpha(0,1)}|z|_{W^{1,\mu=0}_\alpha(0,1)}\end{aligned}$$

and the claim follows.

Now, we introduce the linear form $F : W^{1,\mu=0}_\alpha(0,1) \to \mathbb{C}$ given by

$$F(z) = i\int_0^1 \bar{f} z dx + \overline{f(1)} z(1), \quad \forall z \in W^{1,\mu=0}_\alpha(0,1).$$

Using again the Cauchy-Schwarz inequality, in view of (58) and (59), it is clear that

$$\begin{aligned}|F(z)| &\leq \|f\|_{L^2(0,1)}\|z\|_{L^2(0,1)} + |f(1)||z(1)| \\ &\leq \frac{1}{\sqrt{c_{\alpha,\beta}}}\|f\|_{L^2(0,1)}|z|_{W^{1,\mu=0}_\alpha(0,1)} + \frac{1}{\sqrt{\beta}}|f(1)||z|_{W^{1,\mu=0}_\alpha(0,1)}.\end{aligned}$$

Then, we obtain

$$|F(z)| \leq \left(\frac{1}{\sqrt{c_{\alpha,\beta}}}\|f\|_{L^2(0,1)} + \frac{1}{\sqrt{\beta}}|f(1)|\right)|z|_{W^{1,\mu=0}_\alpha(0,1)},$$

i.e. F is a continuous linear functional in $W^{1,\mu=0}_\alpha(0,1)$.

As a consequence, by the complex form of the Lax-Milgram Theorem (see [33, Lemma 1.3]), there exists a unique solution $u \in W^{1,\mu=0}_\alpha(0,1)$ of

$$\Lambda(u,z) = F(z), \quad \forall\, z \in W^{1,\mu=0}_\alpha(0,1). \tag{62}$$

Now, we will prove that $u \in D(A_\beta)$ and solves (61). Since $\mathcal{C}^\infty_c(0,1) \subset W^{1,\mu=0}_\alpha(0,1)$, from (62), we have

$$\int_0^1 \left(i\bar{u}z + x^\alpha \overline{u_x} z_x - \frac{\mu}{x^{2-\alpha}}\bar{u}z\right)dx = i\int_0^1 \bar{f} z dx, \quad \forall\, z \in \mathcal{C}^\infty_c(0,1).$$

This implies that

$$i\bar{u} - A_\beta \bar{u} = i\bar{f}, \tag{63}$$

or equivalently (61).

Moreover, coming back to (63) and thanks to (16), we infer that

$$i\int_0^1 \bar{u} z dx + \int_0^1 \left(x^\alpha \overline{u_x} z_x - \frac{\mu}{x^{2-\alpha}}\bar{u}z\right)dx - \overline{u_x(1)}z(1) = i\int_0^1 \bar{f} z dx, \quad \forall\, z \in W^{1,\mu}_\alpha(0,1).$$

This, combined with (62), gives

$$z(1)\Big(\overline{u_x(1)}+(\beta+1)\overline{u(1)}-\overline{f(1)}\Big)=0 \quad \forall; z\in W^{1,\mu=0}_{\alpha}(0,1).$$

Since the function z defined by $z(x)=x$ for all $x\in(0,1)$ is in $W^{1,\mu=0}_{\alpha}(0,1)$, we deduce that

$$\overline{u_x(1)}+(\beta+1)\overline{u(1)}-\overline{f(1)}=0.$$

Thus,

$$\overline{u_x(1)}+\beta\overline{u(1)}=i(A_\beta\bar{u})(1),$$

which implies that $u_x(1)+\beta u(1)=-i(A_\beta u)(1)$. In conclusion, $u\in D(A_\beta)$ and solves (61). □

Consequently, from semigroup theory, we find the following well posedness result.

Theorem 19. *Assume Hypothesis 12 and consider $\beta>0$. Then, for any $u_0\in W^{1,\mu=0}_{\alpha}(0,1)$, problem (52)-(53) has a unique solution*

$$u\in C\left([0,+\infty),W^{1,\mu=0}_{\alpha}(0,1)\right)\cap C^1\left([0,+\infty),(D\left(A_\beta\right))'\right).$$

If $u_0\in D\left(A_\beta\right)$, problem (52)-(53) has a unique solution

$$u\in C\left([0,+\infty),D\left(A_\beta\right)\right)\cap C^1\left([0,+\infty),W^{1,\mu=0}_{\alpha}(0,1)\right).$$

The last result, which will be crucial to obtain the stabilization of (52)-(53), is given by the following proposition.

Proposition 20. *Assume Hypothesis 12 and consider $\beta>0$. Then, for every $\lambda\in\mathbb{C}$, the variational problem*

$$\int_0^1\left(x^\alpha v_x\overline{z_x}-\frac{\mu}{x^{2-\alpha}}v\overline{z}\right)dx+\beta v(1)\overline{z(1)}=\lambda\overline{z(1)},\quad \forall\, z\in W^{1,\mu=0}_{\alpha}(0,1) \tag{64}$$

admits a unique solution $v\in W^{1,\mu=0}_{\alpha}(0,1)$, which satisfies the following estimates

$$|v|^2_{W^{1,\mu}_{\alpha}(0,1)}\le\frac{|\lambda|^2}{\beta}\ \text{and}\ \|v\|^2_{L^2(0,1)}\le\frac{|\lambda|^2}{\beta c_{\alpha,\beta}C^1_{\alpha,\mu}}. \tag{65}$$

Moreover, $v\in W^{2,\mu=0}_{\alpha}(0,1)$ and solves

$$\begin{cases}-\left(x^\alpha v_x\right)_x-\frac{\mu}{x^{2-\alpha}}v=0,\\ v_x(1)+\beta v(1)=\lambda.\end{cases} \tag{66}$$

Proof. For all $z\in W^{1,\mu=0}_{\alpha}(0,1)$ consider

$$L(z):=\lambda\overline{z(1)}.$$

Clearly, L is a continuous antilinear form. Indeed, by (58), one has

$$|L(z)| \leq \frac{|\lambda|}{\sqrt{\beta}} |z|_{W_\alpha^{1,\mu=0}(0,1)}.$$

Now, we recall that $W_\alpha^{1,\mu=0}(0,1)$ is a Hilbert space for the scalar product $\langle \cdot, \cdot \rangle_1$. Consequently, for all $z \in W_\alpha^{1,\mu=0}(0,1)$, there exists a unique $v \in W_\alpha^{1,\mu=0}(0,1)$ such that

$$\langle v, z \rangle_1 = L(z).$$

It means that, the above variational problem admits a unique solution $v \in W_\alpha^{1,\mu=0}(0,1)$. Moreover, we have

$$|v|^2_{W_\alpha^{1,\mu}(0,1)} = L(v) = \lambda \overline{v(1)} \leq \frac{|\lambda|}{\sqrt{\beta}} |v|_{W_\alpha^{1,\mu}(0,1)}.$$

Thus,

$$|v|^2_{W_\alpha^{1,\mu}(0,1)} \leq \frac{|\lambda|^2}{\beta}.$$

Combining (56)and (59) together with this last estimate, we obtain

$$\|v\|^2_{L^2(0,1)} \leq \frac{|\lambda|^2}{\beta c_{\alpha,\beta} C^1_{\alpha,\mu}}.$$

Proceeding as in the proof of Proposition 18, one can show that $v \in W_\alpha^{2,\mu=0}(0,1)$ and solves (66). □

5.2. Stabilization result

In this subsection we prove the main exponential stabilization result of the paper when condition (54) holds. To this aim, let u be a solution of (52)-(53) and consider its energy, given by

$$\begin{aligned} \mathcal{E}_u(t) :=& \frac{1}{2} \left[\int_0^1 \left(x^\alpha |u_x|^2 - \frac{\mu}{x^{2-\alpha}} |u|^2 \right) dx + \beta |u(t,1)|^2 \right] \\ =& \frac{1}{2} |u(t)|^2_{W_\alpha^{1,\mu}(0,1)}, \quad t \geq 0. \end{aligned} \tag{67}$$

With this definition in hand, we will prove that the energy is nonincreasing.

Theorem 21. *Assume Hypothesis 12 and let u be a classical solution of (52)-(53). Then the energy is nonincreasing, in particular*

$$\frac{d}{dt} \mathcal{E}_u(t) = -|u_t(t,1)|^2 \leq 0, \quad \forall\, t \geq 0. \tag{68}$$

Proof. By multiplying the equation (52) by $\overline{u_t}$, integrating over $(0,1)$ and using (16), one has

$$\begin{aligned} 0 &= \int_0^1 \overline{u_t(t,x)} \left\{ i u_t(t,x) + (x^\alpha u_x)_x (t,x) + \frac{\mu}{x^{2-\alpha}} u(t,x) \right\} dx \\ &= i \int_0^1 |u_t(t,x)|^2 \, dx - \int_0^1 \left(x^\alpha u_x \overline{u_{tx}} - \frac{\mu}{x^{2-\alpha}} \overline{u_t} u \right) dx + \left[x^\alpha u_x(t,x) \overline{u_t(t,x)} \right]_{x=0}^{x=1} \end{aligned}$$

$$= i\int_0^1 |u_t(t,x)|^2\,dx - \int_0^1 \left(x^\alpha u_x\overline{u_{tx}} - \frac{\mu}{x^{2-\alpha}}\overline{u_t}u\right)\,dx + u_x(t,1)\overline{u_t(t,1)}.$$

Taking into account the boundary conditions, we get

$$\int_0^1 \left(x^\alpha u_x\overline{u_{tx}} - \frac{\mu}{x^{2-\alpha}}\overline{u_t}u\right)\,dx + \beta u(t,1)\overline{u_t(t,1)} = i\int_0^1 |u_t(t,x)|^2\,dx - |u_t(t,1)|^2.$$

Hence

$$\frac{d\mathcal{E}_u}{dt}(t) = \mathrm{Re}\left(\int_0^1 \left(x^\alpha u_x\overline{u_{tx}} - \mu\frac{u\overline{u_t}}{x^{2-\alpha}}\right)\,dx + \beta u(t,1)\overline{u_t(t,1)}\right) = -|u_t(t,1)|^2 \le 0,$$

for all $t \ge 0$. □

Since $t \to \mathcal{E}_u(t)$ is nonincreasing, we can then address the question to know how fast this energy decays. For this reason, in the rest of the paper, we will prove an exponential decay result for system (52)-(53).

Theorem 22. *Assume Hypothesis 12 and consider $\beta > 0$. Then for any $u_0 \in W^{1,\mu}_\alpha(0,1)$, the solution of (52)-(53) satisfies the uniform exponential decay*

$$\mathcal{E}_u(t) \le e^{1-t/M_{\alpha,\beta,\mu}}\mathcal{E}_u(0), \quad \forall\, t \in [M_{\alpha,\beta,\mu}, +\infty)\,, \tag{69}$$

where $M_{\alpha,\beta,\mu} > 0$ is given in (84) and is independent of u_0.

Proof. We prove the theorem for regular solutions, the general case will follow by a density argument. We divide the proof into several steps.

Step 1. We begin deriving the following key identity:

$$(2-\alpha)\int_S^T \mathcal{E}_u(t)\,dt = -\frac{1}{2}\,\mathrm{Im}\left[\int_0^1 xu\overline{u_x}\,dx\right]_S^T + \frac{1}{2}\int_S^T h(t)\,dt, \quad \forall\, 0 \le S \le T, \tag{70}$$

where

$$\begin{aligned} h(t) :=&|u_t(t,1)|^2 + (\beta^2+\beta+\mu-\alpha\beta)|u(t,1)|^2 \\ &- \mathrm{Im}(u_t(t,1)\overline{u(t,1)}) + (2\beta-1)\,\mathrm{Re}(u_t(t,1)\overline{u(t,1)}). \end{aligned} \tag{71}$$

For this purpose, we multiply both sides of (52) by $x\overline{u_x} + \frac{1}{2}\overline{u}$ and integrate by parts over $(S,T)\times(0,1)$. Then we have

$$\begin{aligned} 0 =& \int_0^1\int_S^T iu_t\left(x\overline{u_x} + \frac{1}{2}\overline{u}\right)\,dx\,dt + \int_0^1\int_0^T (x^\alpha u_x)_x\left(x\overline{u_x} + \frac{1}{2}\overline{u}\right)\,dx\,dt \\ &+ \int_0^1\int_S^T \frac{\mu}{x^{2-\alpha}}u\left(x\overline{u_x} + \frac{1}{2}\overline{u}\right)\,dx\,dt := \tilde{I} + \tilde{J} + \tilde{K}. \end{aligned} \tag{72}$$

After suitable integration by parts, we obtain

$$\tilde{I} = \frac{i}{2}\left[\int_0^1 xu\overline{u_x}\,dx\right]_{t=S}^{t=T} - \frac{i}{2}\int_0^1\int_S^T x\left(u\overline{u_{tx}} + u_{tx}\overline{u}\right)\,dx\,dt + \frac{i}{2}\int_S^T u_t(t,1)\overline{u(t,1)}\,dt, \tag{73}$$

$$\tilde{J} = \frac{1}{2}\int_S^T |u_x(t,1)|^2\,dt + \frac{\alpha-2}{2}\int_0^1\int_S^T x^\alpha|u_x|^2\,dx\,dt - \frac{1}{2}\int_0^1\int_S^T x^{\alpha+1}\left(u_x\overline{u_{xx}} - \overline{u_x}u_{xx}\right)\,dx\,dt + \frac{1}{2}\int_S^T u_x(t,1)\overline{u(t,1)}\,dt \tag{74}$$

and

$$\tilde{K} = \frac{1}{2}\int_S^T\int_0^1 \frac{\mu}{x^{1-\alpha}}\left(u\overline{u_x} - u_x\overline{u}\right)\,dx\,dt + \frac{\mu}{2}\int_S^T |u(t,1)|^2\,dt + \frac{2-\alpha}{2}\int_S^T\int_0^1 \frac{\mu}{x^{2-\alpha}}|u|^2\,dx\,dt. \tag{75}$$

In light of (73)-(75), by taking the real part of equation (72), we get

$$(2-\alpha)\int_S^T \mathcal{E}_u(t)\,dt = -\frac{1}{2}\operatorname{Im}\left[\int_0^1 xu\overline{u_x}\,dx\right]_S^T + \frac{1}{2}\int_S^T\Big\{-\operatorname{Im}(u_t(t,1)\overline{u(t,1)}) + |u_x(t,1)|^2 + \operatorname{Re}(u_x(t,1)\overline{u(t,1)}) + (\mu+(2-\alpha)\beta)|u(t,1)|^2\Big\}dt.$$

Recalling that $u_x(t,1) = -u_t(t,1) - \beta u(t,1)$, we have

$$-\operatorname{Im}(u_t(t,1)\overline{u(t,1)}) + |u_x(t,1)|^2 + \operatorname{Re}(u_x(t,1)\overline{u(t,1)}) + (\mu+(2-\alpha)\beta)|u(t,1)|^2 = h(t),$$

where h is defined in (71). Hence the conclusion follows.

Step 2. We claim that for every $0 \le S \le T$ and $\delta > 0$,

$$\int_S^T |u(t,1)|^2\,dt \le \delta\left(\frac{1}{c_{\alpha,\beta}C^1_{\alpha,\mu}} + \frac{1}{\beta^3}\right)\int_S^T \mathcal{E}_u(t)\,dt + \frac{1}{2\delta}\left(1+\frac{1}{\beta c_{\alpha,\beta}C^1_{\alpha,\mu}}\right)\mathcal{E}_u(S) + \frac{2}{c_{\alpha,\beta}C^1_{\alpha,\mu}}\left(1+\frac{1}{\beta^2}\right)\mathcal{E}_u(S). \tag{76}$$

Set $\lambda = u(t,1)$ and denote by v the solution of the degenerate/singular elliptic problem (66). We multiply (52) by $\overline{v}$ and integrate the resulting equation over $(S,T)\times(0,1)$. This gives, after appropriate integration by parts, together with (16),

$$
\begin{aligned}
0 &= \int_S^T\int_0^1 \overline{v(t,x)}\left(iu_t(t,x) + (x^\alpha u_x(t,x))_x + \frac{\mu}{x^{2-\alpha}}u(t,x)\right)\,dx\,dt \\
&= \left[\int_0^1 i\overline{v(t,x)}u(t,x)\,dx\right]_{t=S}^{t=T} - \int_S^T\int_0^1 i\overline{v_t(t,x)}u(t,x)\,dx\,dt + \int_S^T \overline{v(t,1)}u_x(t,1)\,dt \\
&- \int_S^T \overline{v_x(t,1)}u(t,1)\,dt.
\end{aligned}
$$

Taking into account the boundary conditions in both systems of v and u, we immediately have

$$
\begin{aligned}
0 &= \left[\int_0^1 i\overline{v(t,x)}u(t,x)\,dx\right]_{t=S}^{t=T} - \int_S^T\int_0^1 i\overline{v_t(t,x)}u(t,x)\,dx\,dt \\
&+ \int_S^T \overline{v(t,1)}\big(u_x(t,1) + \beta u(t,1)\big)\,dt - \int_S^T |u(t,1)|^2\,dt \\
&= \left[\int_0^1 i\overline{v(t,x)}u(t,x)\,dx\right]_{t=S}^{t=T} - \int_S^T\int_0^1 i\overline{v_t(t,x)}u(t,x)\,dx\,dt - \int_S^T \overline{v(t,1)}u_t(t,1)\,dt \\
&- \int_S^T |u(t,1)|^2\,dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_S^T |u(t,1)|^2\,dt = &\left[\int_0^1 i\overline{v(t,x)}u(t,x)\,dx\right]_{t=S}^{t=T} - \int_S^T\int_0^1 i\overline{v_t(t,x)}u(t,x)\,dx\,dt \\
&- \int_S^T \overline{v(t,1)}u_t(t,1)\,dt.
\end{aligned} \tag{77}
$$

We need to estimate the terms on the right-hand side of the previous equality as follows. First, thanks to the second inequality in (65), we have

$$
\|v_t\|^2_{L^2(0,1)} \le \frac{1}{\beta c_{\alpha,\beta} C^1_{\alpha,\mu}}|u_t(t,1)|^2. \tag{78}
$$

Moreover, thanks to the first inequality in (65), we have

$$
\beta|v(t,1)|^2 \le |v|^2_{W^{1,\mu}_\alpha(0,1)} \le \frac{1}{\beta}|u(t,1)|^2,
$$

so that

$$
|v(t,1)|^2 \le \frac{1}{\beta^2}|u(t,1)|^2 \le \frac{2}{\beta^3}\mathcal{E}_u(t). \tag{79}
$$

We repeat the same argument that we have used above to obtain

$$\begin{aligned}\left|\int_0^1 i\overline{v(t,x)}u(t,x)\,dx\right| &\le \frac{1}{2}\int_0^1 |v(t,x)|^2\,dx + \frac{1}{2}\int_0^1 |u(t,x)|^2\,dx \\ &\le \frac{1}{2\beta c_{\alpha,\beta}C^1_{\alpha,\mu}}|u(t,1)|^2 + \frac{1}{2c_{\alpha,\beta}C^1_{\alpha,\mu}}|u|^2_{W^{1,\mu}_\alpha(0,1)} \\ &\le \left(\frac{1}{\beta^2 c_{\alpha,\beta}C^1_{\alpha,\mu}} + \frac{1}{c_{\alpha,\beta}C^1_{\alpha,\mu}}\right)\mathcal{E}_u(t), \quad \forall\, t\in[S,T].\end{aligned} \tag{80}$$

Using Young's inequality and inserting estimates (78)-(80) in (77), we have for any $\delta > 0$

$$\begin{aligned}\int_S^T |u(t,1)|^2\,dt &\le \delta\left(\frac{1}{c_{\alpha,\beta}C^1_{\alpha,\mu}} + \frac{1}{\beta^3}\right)\int_S^T \mathcal{E}_u(t)\,dt \\ &+ \frac{1}{2\delta}\left(1+\frac{1}{\beta c_{\alpha,\beta}C^1_{\alpha,\mu}}\right)\int_S^T |u_t(t,1)|^2\,dt + \frac{2}{c_{\alpha,\beta}C^1_{\alpha,\mu}}\left(1+\frac{1}{\beta^2}\right)\mathcal{E}_u(S).\end{aligned}$$

Using the dissipation relation (68), the claim follows.

Step 3. Now, we establish the existence of a positive constant $M_{\alpha,\beta,\mu}$ such that for all $0 \le S \le T$,

$$\int_S^T \mathcal{E}_u(t)\,dt \le M_{\alpha,\beta,\mu}\mathcal{E}_u(S). \tag{81}$$

Let h be the function given in (71). Using Young's inequality, one has

$$h(t) \le 2|u_t(t,1)|^2 + \eta|u(t,1)|^2, \quad \forall\, t\in(S,T),$$

where $\eta = \beta^2 + |1-\alpha|\beta + |\mu| + \frac{1}{2}(2\beta-1)^2 + \frac{1}{2}$.

Thanks to (76) with $\delta = \dfrac{2-\alpha}{\eta\left(\frac{1}{c_{\alpha,\beta}C^1_{\alpha,\mu}} + \frac{1}{\beta^3}\right)}$ we have

$$\begin{aligned}\frac{1}{2}\int_S^T h(t)\,dt &\le \mathcal{E}_u(S) + \frac{2-\alpha}{2}\int_S^T \mathcal{E}_u(t)\,dt + \frac{\eta^2\left(\frac{1}{c_{\alpha,\beta}C^1_{\alpha,\mu}} + \frac{1}{\beta^3}\right)}{4(2-\alpha)}\left(1+\frac{1}{\beta c_{\alpha,\beta}C^1_{\alpha,\mu}}\right)\mathcal{E}_u(S) \\ &+ \frac{\eta}{c_{\alpha,\beta}C^1_{\alpha,\mu}}\left(1+\frac{1}{\beta^2}\right)\mathcal{E}_u(S).\end{aligned} \tag{82}$$

On the other hand, by using Young's inequality, we write

$$\begin{aligned}\left|\mathrm{Im}\int_0^1 xu\overline{u_x}\,dx\right| &\le \frac{1}{2}\|u\|^2_{L^2(0,1)} + \frac{1}{2}\int_0^1 x^2|u_x|^2\,dx \\ &\le \frac{1}{2}\|u\|^2_{L^2(0,1)} + \frac{1}{2}\int_0^1 x^\alpha|u_x|^2\,dx.\end{aligned}$$

Now, using Lemma 14 and (59), it follows that

$$\left|\operatorname{Im}\int_0^1 x u \overline{u_x}\, dx\right| \le \frac{1}{2c_{\alpha,\beta}}|u|^2_{W^{1,\mu=0}_\alpha(0,1)} + \frac{1}{2}|u|^2_{W^{1,\mu=0}_\alpha(0,1)}$$
$$\le \frac{1}{2c_{\alpha,\beta}C^1_{\alpha,\mu}}|u|^2_{W^{1,\mu}_\alpha(0,1)} + \frac{1}{2C^1_{\alpha,\mu}}|u|^2_{W^{1,\mu}_\alpha(0,1)}.$$

Hence

$$\left|\frac{1}{2}\operatorname{Im}\left[\int_0^1 x u \overline{u_x}\, dx\right]_S^T\right| \le C'\mathcal{E}_u(t), \tag{83}$$

where $C' = \dfrac{1}{C^1_{\alpha,\mu}} + \dfrac{1}{c_{\alpha,\beta}C^1_{\alpha,\mu}}$.

Using (82) and (83) in (70), we deduce that (81) holds with

$$M_{\alpha,\beta,\mu} = \frac{2}{(2-\alpha)}\left[1 + C' + \frac{\eta^2\left(\frac{1}{c_{\alpha,\beta}C^1_{\alpha,\mu}} + \frac{1}{\beta^3}\right)}{4(2-\alpha)}\left(1 + \frac{1}{\beta c_{\alpha,\beta}C^1_{\alpha,\mu}}\right) + \frac{\eta}{c_{\alpha,\beta}C^1_{\alpha,\mu}}\left(1 + \frac{1}{\beta^2}\right)\right]. \tag{84}$$

By invoking [22, Theorem 8.1], this implies the desired stability estimate (69). □

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