



OPEN

Entanglement and quantum correlation measures for quantum multipartite mixed states

Arthur Vesperini^{1,2,3}, Ghofrane Bel-Hadj-Aissa^{1,2,3} & Roberto Franzosi^{1,2,3}✉

Entanglement, and quantum correlation, are precious resources for quantum technologies implementation based on quantum information science, such as quantum communication, quantum computing, and quantum interferometry. Nevertheless, to our best knowledge, a directly or numerically computable measure for the entanglement of multipartite mixed states is still lacking. In this work, (i) we derive a measure of the degree of quantum correlation for mixed multipartite states. The latter possesses a closed-form expression valid in the general case unlike, to our best knowledge, all other known measures of quantum correlation. (ii) We further propose an entanglement measure, derived from this quantum correlation measure using a novel regularization procedure for the density matrix. Therefore, a comparison of the proposed measures, of quantum correlation and entanglement, allows one to distinguish between quantum correlation detached from entanglement and the one induced by entanglement and, hence, to identify separable but non-classical states. We have tested our quantum correlation and entanglement measures, on states well-known in literature: a general Bell diagonal state and the Werner states, which are easily tractable with our regularization procedure, and we have verified the accordance between our measures and the expected results for these states. Finally, we validate the two measures in two cases of multipartite states. The first is a generalization to three qubits of the Werner state, the second is a one-parameter three qubits mixed state interpolating between a bi-separable state and a genuine multipartite state, passing through a fully separable state.

Entanglement has assumed an important role in quantum information theory and in the development of the quantum technologies. It is considered as a valuable resource in quantum cryptography, in quantum computation, in teleportation and in quantum metrology¹. Nevertheless, entanglement remains elusive and the problem of its quantification in the case of a general system, is still open^{2,3}. In the last decades, several approaches have been developed to quantify the entanglement in the variety of states of the quantum realm. However, the rigorous achievements in the explicit quantification of entanglement, are limited to bipartite systems case⁴. Entropy of entanglement is uniquely accepted as measure of entanglement for pure states of bipartite systems⁵, while entanglement of formation⁶, entanglement distillation^{7–9} and relative entropy of entanglement¹⁰ are largely acknowledged as faithful measures for bipartite mixed systems¹¹. An extensive literature is devoted to the study of entanglement in multipartite systems. Over time, different approaches have been proposed including, e.g. in the case of pure states, the study of the equivalence classes in the set of multipartite entangled states^{12,13}, whereas, the study of entanglement in mixed multipartite states have been addressed, e.g., with a Schmidt measure¹⁴ or with a generalisation of concurrence^{15,16}. In the last years, have been proposed entanglement estimation-oriented approaches and derived from a statistical distance¹⁷ concept, as, for instance, the quantum Fisher information^{18–21}.

In the case of pure states, entanglement and correlation are completely equivalent, therefore an appropriate measure of quantum correlation can provide also an entanglement measure. On the contrary, in the case of mixed states, one can observe states that manifest correlations detached from entanglement^{11,22}.

In the present work, (i) we propose a new directly computable measure of quantum correlation for mixed states. Moreover, (ii) we propose a numerically computable entanglement measure for mixed states, which is derived from the quantum correlation measure through a regularization process. In the following we first derive the quantum correlation measure for mixed states and, from the latter, we derive an entanglement measure valid for the same class of states. Finally, we report four examples of the application of quantum correlation and

¹DSFTA, University of Siena, Via Roma 56, Siena 53100, Italy. ²INFN Sezione di Perugia, Perugia 06123, Italy. ³QSTAR & CNR - Istituto Nazionale di Ottica, Largo Enrico Fermi 2, Firenze 50125, Italy. ✉email: roberto.franzosi@unisi.it

entanglement measures. We have considered two well-known classes of states: a general Bell diagonal state and the Werner states. In addition, we have applied the quantum correlation and entanglement measures to Werner state generalization to three qubits, and to a one-parameter three qubits mixed states interpolating between a bi-separable state and a genuine multipartite state, passing through a fully separable state.

Entanglement distance for mixed states

Quantum correlation distance. We consider the Hilbert space $\mathcal{H} = \mathcal{H}^0 \otimes \mathcal{H}^1 \dots \mathcal{H}^{M-1}$ tensor product of M two qubits Hilbert spaces. The Hilbert–Schmidt distance D between two general square matrices, A and B , is given by

$$D(A, B) = \sqrt{\frac{1}{2} \text{tr}[(A - B)^\dagger (A - B)]}. \quad (1)$$

We derive from the latter, the distance between two close density matrices of a quantum state in \mathcal{H} , by

$$d_{\text{dm}}^2(\rho, \rho + d\rho) = \frac{1}{2} \text{tr}[(d\rho)^\dagger (d\rho)]. \quad (2)$$

The Hilbert–Schmidt distance is not the only possible choice, e.g. the Bures' distance represents an appropriate alternative option. The infinitesimal variation $d\rho$ of state ρ is

$$\begin{aligned} d\rho &= \sum_{j=0}^{M-1} d\tilde{U}^j \rho + \rho \sum_{\mu=0}^{M-1} d\tilde{U}^{\mu\dagger} \\ &= -i \sum_{\mu=0}^{M-1} \sum_{j=1}^3 [\sigma_j^\mu \rho] n_j^\mu d\xi^\mu, \end{aligned} \quad (3)$$

where

$$d\tilde{U}^\mu = -i(\sigma_{\mathbf{n}})^\mu d\xi^\mu \quad (4)$$

and with $[\cdot, \cdot]$, we mean the commutator. Here and in the following we use the notation $(\sigma_{\mathbf{n}})^\mu = (\mathbf{n}^\mu \cdot \boldsymbol{\sigma}^\mu)$, and for $\mu = 0, \dots, M-1$, we denote by $\sigma_1^\mu, \sigma_2^\mu$ and σ_3^μ the three Pauli matrices operating on the μ -th qubit, where the index μ labels the spins. We have

$$d_{\text{dm}}^2(\rho, \rho + d\rho) = \sum_{\mu, \nu=0}^{M-1} g_{\mu\nu}(\rho, \mathbf{n}) d\xi^\mu d\xi^\nu, \quad (5)$$

where

$$g_{\mu\nu}(\rho, \mathbf{n}) = \frac{1}{2} \sum_{i,j=1}^3 \text{tr}[\rho \{\sigma_i^\mu, \sigma_j^\nu\} \rho - 2\rho \sigma_i^\mu \rho \sigma_j^\nu] n_i^\mu n_j^\nu, \quad (6)$$

with $\{\cdot, \cdot\}$ we mean the anticommutator. We define the quantum correlation for the state ρ as

$$C(\rho) = \inf_{\{\mathbf{n}^\nu\}_\nu} \text{tr}(g(\rho, \mathbf{n})). \quad (7)$$

Since $C(\rho)$ derives from a distance, we name it quantum correlation distance (QCD). Furthermore, the quantum correlation is the inferior value of the trace of g when the unit vectors are varied, therefore its numerical value is invariant under local unitary transformations. We have

$$\sum_{\mu=0}^{M-1} g_{\mu\mu}(\rho, \mathbf{n}) = M \text{tr}(\rho^2) - \sum_{\mu=0}^{M-1} \sum_{i,j=1}^3 \text{tr}[\rho \sigma_i^\mu \rho \sigma_j^\mu] n_i^\mu n_j^\mu. \quad (8)$$

Finally, by defining the matrices $A^\mu(\rho)$, for $\mu = 0, \dots, M-1$, whose entries are

$$A_{ij}^\mu(\rho) = \text{tr}[\rho \sigma_i^\mu \rho \sigma_j^\mu], \quad (9)$$

we obtain the closed-form expression for the QCD of ρ ,

$$C(\rho) = \sum_{\mu=0}^{M-1} \left(\text{tr}(\rho^2) - \lambda_{\text{max}}^\mu(\rho) \right) = \sum_{\mu=0}^{M-1} C_\mu(\rho) \quad (10)$$

where, for $\mu = 0, \dots, M-1$, $\lambda_{\text{max}}^\mu(\rho)$ is the maximum of the eigenvalues of $A^\mu(\rho)$, and $C_\mu(\rho) = \text{tr}(\rho^2) - \lambda_{\text{max}}^\mu(\rho)$ is the QCD of the subsystem μ . The QCD is a directly computable measure of the degree of correlation of ρ . Remarkably, Eq. (10) contains two competing terms. The first term is named Purity, which takes account of the degree of statistical mixing of ρ , its upper bound 1 corresponds to a pure state. The second term ranges between

0 and 1 and derives from the degree of correlation of ρ , with the lower value, 0, corresponding to the higher correlation.

The time complexity of the obtained formula for the QCD is that of $D \times D$ matrix multiplications, that is $o(D^3)$, where D is the dimension of the full Hilbert space. In particular, the QCD possesses a closed formula and do not require any optimisation (other than finding the largest eigenvalue of 3×3 matrices). This is in contrast with other measures of quantum correlation which, to our best knowledge, all require time-costly optimisation procedures, except for some specific classes of states¹¹.

The QCD (10) fulfills the following requirements for a *bona fide* measure of quantum correlation¹¹:

- 1) $C_\mu(\rho) = 0$ if $\rho \in \mathcal{C}_\mu$, i.e. if ρ is classical in the subsystem μ . Indeed, $\forall \rho \in \mathcal{C}_\mu$ we can write $\rho = \sum_j p_j \rho_j^{\mu_C} \otimes |j\rangle\langle j|^\mu$, where the $\{|j\rangle^\mu\}$ form an orthonormal basis in \mathcal{H}^μ , μ_C is the complement of subsystem μ and $\sum_j p_j = 1$. Then $\exists \mathbf{n}^\mu$ such that $\forall j, {}^\mu \langle j | \mathbf{n}^\mu \cdot \boldsymbol{\sigma}^\mu | j \rangle^\mu = \pm 1$, hence $\lambda_{max}^\mu(\rho) = 1$ and $C_\mu(\rho) = 0$. It results $C(\rho) = 0$ if $\rho \in \mathcal{C}$, i.e. if ρ is fully classical.
- 2) $C(U\rho U^\dagger) = C(\rho)$, i.e. it is invariant under local unitary transformations.
- 3) In the case of a pure state $\rho = |\psi\rangle\langle\psi|$, $C(|\psi\rangle\langle\psi|)$ reduces to the measure of entanglement valid for pure states that some of us have derived in former work²³. This confirms that for pure states a genuine correlation measure provides also an entanglement measure.

Entanglement distance. As stated above, for a mixed state, the existence of quantum correlation is not a sufficient condition to guarantee the presence of entanglement. To extract from a given state ρ its entanglement essence, we now propose a procedure of regularization of ρ , repurposing our measure of quantum correlations to catch the true degree of entanglement owned by ρ .

Given a state ρ , we consider all of its possible decomposition $\{p_j, \rho_j\}$, such that

$$\rho = \sum_j p_j \rho_j, \tag{11}$$

where $\sum_j p_j = 1$ and $\text{tr}[\rho_j] = 1$. Also, we consider all the possible local partial transformation on qubit μ :

$$\rho_U^\mu(\{p_j, \rho_j, U_j^\mu\}) = \sum_j p_j U_j^\mu \rho_j U_j^{\mu\dagger}, \tag{12}$$

where, for each j , U_j^μ is an $SU(2)$ local unitary operator acting on qubit μ . We define the entanglement measure for state ρ

$$E(\rho) = \inf_{\{p_j, \rho_j\}} \left\{ \sum_{\mu=1}^{M-1} \inf_{\{U_j^\mu\}} C_\mu(\rho_U^\mu(\{p_j, \rho_j, U_j^\mu\})) \right\}. \tag{13}$$

Since the definition $E(\rho)$ derives from a distance, we named it entanglement distance (ED). Note that, similarly to the QCD, one can define $E_\mu(\rho)$ as the ED of subsystem μ , simply discarding the complement in the sum on μ in (8). The ED (13) fulfills the following requirements for a suitable measure of quantum entanglement:

- i) $E_\mu(\rho) = 0$ if $\rho \in \mathcal{S}_\mu$, that is if ρ is separable in μ . Indeed, it then admits a decomposition $\{p_j, \rho_j\}$, where, for each j , $\rho_j = (\mathbb{1}^\mu + \boldsymbol{\sigma}_{\mathbf{n}_j}^\mu)/2 \otimes \rho_j^{\mu_C}$, where. Thus, it is always possible to determine local partial operators U_j^μ , such that, after transformation (12) it results $\rho_U^\mu = \sum_j p_j |j\rangle\langle j|^\mu \otimes \rho_j^{\mu_C}$ and, from property 1), it follows our statement. It results $E(\rho) = 0$ if $\rho \in \mathcal{S}$, that is if ρ is fully separable.
- ii) Reciprocally, if $E(\rho) = 0$, then ρ is separable. First of all, we note that, for each $\mu = 0, \dots, M-1$, $\lambda_{max}^\mu(\rho) \leq \text{tr}(\rho^2)$. In fact, for each μ and for each unit vector \mathbf{n}^μ it is possible to determine a unitary local operator U , so that $\text{tr}[(\rho(\boldsymbol{\sigma}_{\mathbf{n}})^\mu \rho(\boldsymbol{\sigma}_{\mathbf{n}})^\mu)] = \text{tr}[\tilde{\rho} \sigma_3^\mu \tilde{\rho} \sigma_3^\mu]$, where $\tilde{\rho} = U\rho U^\dagger$. Furthermore $\text{tr}[\tilde{\rho} \sigma_3^\mu \tilde{\rho} \sigma_3^\mu] = \sum_j \tilde{\rho}_{jj}^2 + 2 \sum_{i \neq j} \pm |\tilde{\rho}_{ij}|^2 \leq \sum_j \tilde{\rho}_{jj}^2 + 2 \sum_{i \neq j} |\tilde{\rho}_{ij}|^2 = \text{tr}[\tilde{\rho}^2] = \text{tr}[\rho^2]$. Moreover, for each pair $i \neq j$, $\exists \mu$ such that the term $|\tilde{\rho}_{ij}|^2$ appears in $\text{tr}[(\tilde{\rho} \sigma_3^\mu)^2]$ with a negative sign. Yet, $E(\rho) = 0$ implies that there exist a decomposition of ρ , let's say $\bar{\rho}$, for which

$$\sup_{\mathbf{n}^\mu} \text{tr}[\bar{\rho}(\boldsymbol{\sigma}_{\mathbf{n}})^\mu \bar{\rho}(\boldsymbol{\sigma}_{\mathbf{n}})^\mu] = \text{tr}[\bar{\rho}^2] \tag{14}$$

for each μ . We hence have $|\bar{\rho}_{ij}|^2 = 0$ for each $i \neq j$. But this implies that $\bar{\rho}$ is diagonal and then ρ separable. For a given density matrix decomposition $\{p_j, \rho_j\}$, the minimization on the local unitary partial transformations, entailed by Eq. (13), can be addressed by studying the local minima of $C(\rho(\{p_j, \rho_j, U_j\}))$ under variation of $\{U_j\}$. Nevertheless, it can be proven that such fixed points do correspond only to cases where $E(\rho) = 0$, hence to separable states. Therefore, the minima of (13) in the case of non-separable states, do not correspond to fixed points, but rather to nonlocal (boundary) minima. Remarkably, these fixed points of the minimization procedure (13) can, at least in some cases, be realized by a decomposition $\{p_j, \rho_j\}$ including entangled pure states ρ_j . In particular, for two-qubits states diagonal in the Bell basis (the Bell-diagonal (BD) states, see^{24,25}) the fixed points can always be realized on the eigen-decomposition (hence, where the ρ_j are Bell states). This of course greatly simplify the problem, as the full exploration of the $\{p_j, \rho_j\}$ -space is avoided. It is worth emphasizing that BD states are representative of the larger class of two-qubits states of maximally mixed marginals (that is, for which

$\forall \mu$ and $\forall j$, $\text{tr}[\rho \sigma_j^\mu] = 0$, see²⁴), hence (13) is tractable in the same manner for this class of states. Leaning on numerical evidences, we further conjecture that, for a given state $\rho(\boldsymbol{\gamma})$ depending on parameters $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots)$, the decomposition realizing the minimum (13) is the same in the whole parametric domain of $\boldsymbol{\gamma}$, and can hence be inferred from the fixed points found in the domains where this state is separable, if such a domain exists. This suggests that the minimization over all possible decompositions $\{p_j, \rho_j\}$ might in fact possess *non-trivial* general solutions, depending on the considered class of states. Here, by “non-trivial solutions” of the minimization procedure, we mean solutions which do not require to find the decomposition of ρ in terms of pure product-states $\rho_j = \bigotimes_{\mu} (\mathbb{I}^\mu + (\sigma_{n_j})^\mu)/2$. A subsequent more thorough work on such a classification of the solutions of this procedure could thus lead to an entanglement measure of relatively low computational cost, in particular for systems symmetric under qudit permutations, and with low $\text{rank}(\rho)$.

Applications

Bell diagonal states. As a first and seminal example of application of this procedure, we consider general BD states. They can be expressed as:

$$\begin{aligned} \rho_{BD}(\{p_\alpha\}) &= \sum_{\alpha=1}^4 p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \\ &= \frac{1}{4} \left(\mathbb{I} + \sum_i c_i \sigma_i^0 \sigma_i^1 \right), \end{aligned} \tag{15}$$

where the $|\psi_\alpha\rangle$ are the four Bell states: $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ and $|\phi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$. Furthermore, we have $\forall i, |c_i| \leq 1$, and the c_i are such that the vector (c_1, c_2, c_3) , fully characterizing the state, belongs to the tetrahedron \mathcal{T} of vertices $(-1, 1, 1), (1, -1, 1), (1, 1, -1), (-1, -1, -1)$. The separable BD states belong to the octahedron \mathcal{O} of vertices $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$, corresponding to the condition $\forall \alpha, p_\alpha \leq 2$, and the classical BD states are located on the Cartesian axis $(c_1, 0, 0), (0, c_2, 0), (0, 0, c_3)$ ^{24,25}.

Direct calculation yields the following result for the QCD of general BD states

$$C(\rho_{BD}(\{p_\alpha\})) = 2 \sum_{\alpha=1}^4 p_\alpha^2 - 4 \max_{P\{i,j,k,l\}} \{p_i p_j + p_k p_l\}, \tag{16}$$

where the maximum is taken on all permutations $P\{i, j, k, l\}$ of the indices $\{1, 2, 3, 4\}$. Figure 1 shows the QCD of BD states on a face of \mathcal{T} . We were not able to find a simple analytic solution of the minimization procedure for the most general case of BD states. However, numerical minimization (for these calculations, we have applied a gradient steepest-descent method) provided us with empirical evidence that the procedure (22) also leads for these states to the squared concurrence, as shown in Fig. 2, which represent a face of the tetrahedral domain of BD states. It is interesting to note that the ED, as the concurrence and unlike the QCD, is constant on planes parallel to the boundary faces of the separability region: the ED of any given state indeed equates the QCD of the closest point located on a hinge of \mathcal{T} , hence the closest mixture of only two Bell states.

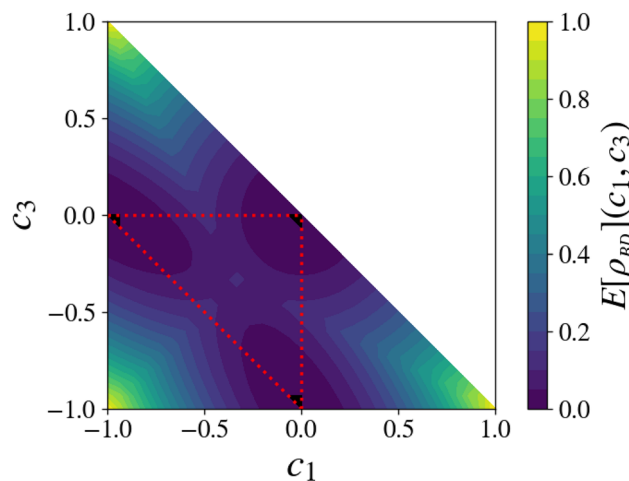


Figure 1. Quantum correlations $C[\rho_{BD}](c_1, c_2 = c_1, c_3)/2$ for a face of the BD state tetrahedron \mathcal{T} , corresponding to a mixture of three Bell states. The red dotted line defines the smaller triangle where the state is separable, according to the PPT criterion^{26,27}. The vertices of the large triangle correspond to pure Bell states. Those of the red dotted triangle, of vanishing QCD, correspond to equal-weight mixtures of two Bell states, which are evidently the three only *classical* states in the represented domain.

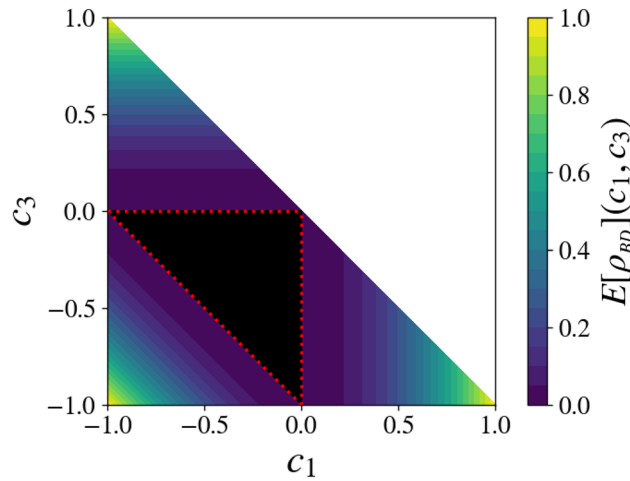


Figure 2. Entanglement distance $E[\rho_{BD}](c_1, c_2 = c_1, c_3)/2$ for a face of the BD state tetrahedron \mathcal{T} , corresponding to a mixture of three Bell states. The red dotted line defines the smaller triangle where the state is separable, according to the PPT criterion^{26,27} and a number of alternative derivations available in the literature (see e.g.²⁴). Values below the threshold of 10^{-3} have been represented in black to emphasize that they correspond to a numerical zero, given the level of precision allowed by such time-costly minimization. The vertices of the large triangle correspond to pure Bell states, and those of the smaller black triangle to equal-weight mixtures of two Bell states.

Werner states. Let us now consider the two-qubit Werner states (WS)²⁸, which stems as a special case of BD state, for which a simple analytical solution for the proposed procedure is available. WS are used as a testbed since they illustrate many features of mixed-states entanglement⁷. Using Eq. (15), they can simply be expressed as

$$\rho_W(p) = \rho_{BD}\left(\frac{p}{3}, \frac{p}{3}, \frac{p}{3}, (1-p)\right). \tag{17}$$

Via direct calculations, one gets for the QCD of the WS

$$C(\rho_W(p)) = 2\left(1 - \frac{4}{3}p\right)^2. \tag{18}$$

WS yields a relatively simple solution to the minimization procedure (22). Indeed, as it can be easily verified, if we set

$$\begin{aligned} U_{|\psi_+\rangle}(\theta) &= U_z^\mu(\theta)U_x^\mu(\pi), \\ U_{|\psi_-\rangle}(\theta) &= U_z^\mu(\pi - \theta)U_x^\mu(\pi), \text{ and} \\ U_{|\phi_+\rangle} &= U_{|\phi_-\rangle} = \mathbb{I}, \end{aligned} \tag{19}$$

with $\mu = 0, 1$ arbitrarily chosen, the fixed points are found for $\theta = \arccos\left(\frac{3}{2p} - 2\right)$. This last expression has a solution if and only if $p \geq 1/2$, which is the parametric region of separability for $\rho_W(p)$ (as can be verified by application of the positive partial trace criterion, see²⁷). Hence, $E(\rho_W) = 0$ for $p \geq 1/2$. For $p < 1/2$ numerical minimization yields $E(\rho_W) = 4p^2 - 4p + 1$. This corresponds to $\theta = 0$ uniformly on this whole domain, which is also the value previously determined at $p = 1/2$: hence, the minimum after this point cease to be a fixed point, but keeps the last position in terms of the parameters governing the rotations. One can understand this as the fixed point reaching the boundary of the parametric domain as the geometry of the state is changing continuously, becoming a simple point on a slope, located at this boundary. All together, for Werner states, the result of our entanglement measure exactly equates twice the square of the concurrence⁶, that is

$$E(\rho_W(p)) = 2\Theta(1/2 - p)(1 - 2p)^2, \tag{20}$$

Figure 3 shows $C(\rho_W(p))/2$ versus p , there it is clear that the only state with no quantum correlation, i.e. *classical state* according to the conventional terminology¹¹, is the one corresponding to the value $p = 3/4$, whereas the maximally quantum-correlated state is that of $p = 0$. On the other hand, the state is entangled only in the region $p < 1/2$, and separable otherwise, a well-known fact that can be easily checked by application of the positive partial transpose (PPT) criterion^{26,27}. Alternatively, one can find, in the separable region, the expression of ρ_W convex combination of (non-orthogonal) product states, using a more involved calculation resorting to the so-called Bloch representation.

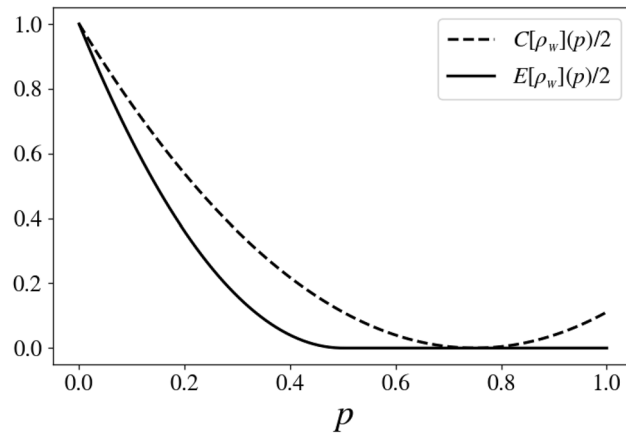


Figure 3. $C[\rho_W](p)/2$ and $E[\rho_W](p)/2$ versus p for state (17). It is clear that the state $\rho_W(p = 0)$ is, as expected, the maximally-entangled, and that the states $\rho_W(p > 1/2)$ are fully-separable, as can be verified using the PPT criterion^{26,27}. This plot emphasizes that separable states can contain quantum correlation (i.e. not be classical). Note that, here $E[\rho_W](p)/2 = C_2^2[\rho_W](p)$, that is, the ED equates twice the squared concurrence for 2-qubits Werner states.

Generalized Werner states. Let us now consider as a multipartite example the following one-parameter density matrix

$$\rho_{W_3}(p) = p|GHZ_+\rangle\langle GHZ_+| + \frac{(1-p)}{8}\mathbb{I}_8, \tag{21}$$

where $|GHZ_+\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$, \mathbb{I}_8 is the identity operator of the three-qubits Hilbert space and $0 \leq p \leq 1$. This is a generalization of the Werner states to three qubits, termed generalized Werner states²⁹⁻³¹. The states $\rho_{W_3}(p)$ are known to be fully separable for $0 \leq p \leq 1/5$ ^{29,30,32} and genuinely multipartite entangled states in the region $3/7 < p \leq 1$ ³³. In the region $1/5 < p \leq 3/7$ the states $\rho_{W_3}(p)$ are bi-separable yet inseparable under any fixed bipartition³³. Via direct calculations, one gets

$$C(\rho_{W_3}(p)) = 3p^2. \tag{22}$$

Numerical minimization provided the values for the ED shown in Fig. 4. There, we report in dotted line the QCD per qubit and continuous line the ED per qubit for the states $\rho_{W_3}(p)$. Figure 4 clearly shows that $ED(\rho_{W_3}(p)) > 0$ only for $p > 3/7$, that is when the states are generally entangled. As for the region $1/5 < p \leq 3/7$ where ED should not be zero according to (ii), we got numerical zero which we assume corresponds to very weak, but finite values. We interpreted this as a consequence of the fact that, in this region, the states $\rho_{W_3}(p)$ are not separable under any fixed bipartition, thus assuming the decomposition of the form $\sum_j \rho_j^1 \otimes \rho_j^{23} + \rho_j^2 \otimes \rho_j^{13} + \rho_j^3 \otimes \rho_j^{12}$. Hence the regularization procedure reaches easily small values for the ED.

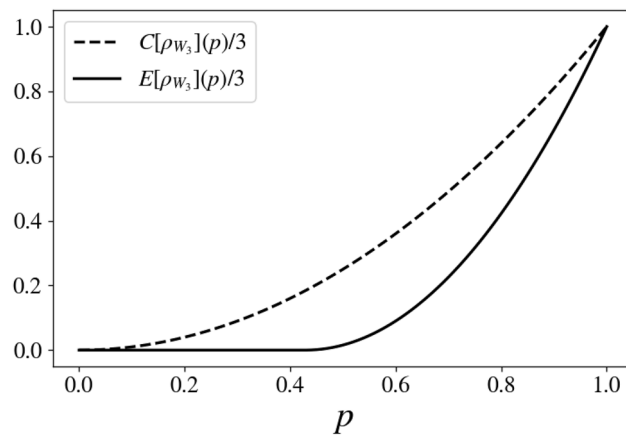


Figure 4. $C[\rho_{W_3}](p)/3$ (dotted line) and $E[\rho_{W_3}](p)/3$ (continuous line) versus p for state (21). It is clear that the state $\rho_{W_3}(p = 1)$ is, as expected, the maximally entangled, and that the states $\rho_{W_3}(p > 3/7)$ are not separable. The latter are genuinely three-partite entangled states.

Three qubit states interpolating between bi-separable and genuine entangled states. Let consider a further multipartite example, that is the one-parameter density matrix

$$\rho_3(p) = w_+ |GHZ_+\rangle\langle GHZ_+| + w_2 |\psi_2\rangle\langle\psi_2| + w \frac{(1-p)}{8} \mathbb{I}_8, \quad (23)$$

where

$$\begin{aligned} w_+ &= p[1 - 4p(1 - p)], \\ w_2 &= (1 - p)[1 - 4p(1 - p)], \\ w &= 4p(1 - p), \end{aligned} \quad (24)$$

$|\psi_2\rangle = |0\rangle(|00\rangle + |11\rangle)/\sqrt{2}$ and $0 \leq p \leq 1$. For $p = 0$, $\rho_3(p = 0)$ is a pure bi-separable state, for $p = 1/2$, $\rho_3(p = 1/2)$ is a maximally mixed state of three qubits and for $p = 1$, $\rho_3(p = 1)$ is a pure maximally entangled state. Via direct calculations, one gets

$$C(\rho_3(p)) = \frac{(1 - 2p)^4}{2} \left[5 - 10p + 11p^2 - (1 - p)\sqrt{1 - 2p(1 - p)} \right]. \quad (25)$$

Using numerical minimization, we have obtained the results for the ED shown in Fig. 5. In this figure, we report in dotted line the QCD per qubit and in continuous line the ED per qubit, for the states $\rho_3(p)$. Figure 5 shows that $E(\rho_3(p)) > 0$ for $0 \leq p \lesssim 0.18$ and for $0.81 \lesssim p \leq 1$. Furthermore, the maximum value for ED per qubit in the region $0 \leq p \lesssim 0.18$ is located at $p = 0$ and has the value $2/3$. $2/3$ is the maximum value for ED per qubit, in the case of bi-separable three qubits states. This confirms that the states of this region are stably bi-separable and that the state $|\psi_2\rangle\langle\psi_2|$ has the maximum local degree of entanglement. The maximum value for ED per qubit in the region $0.81 \lesssim p \leq 1$ is located at $p = 1$ and has value 1. Therefore, the states of this region are not separable and, at least close to $p = 1$, are certainly genuinely entangled. For $0.18 < p < 0.81$ the entanglement is numerically null, thus suggesting the states of this region are separable or bi-separable yet inseparable under any fixed bipartition, hence not genuinely three-partite entangled states. Remarkably, the QCD is null only for the state corresponding to $p = 1/2$, which is the maximally mixed one.

Summary

The increasing interest in quantum information experimental applications, and the consequent demand for the development of skills in quantum state manipulation, has made pressing the development of effective measures of correlation and entanglement, valid for the general case of mixed multipartite states. Also, such measures are expected to be easily computable. For multipartite systems, a broad range of measures has covered pure states and mixed states, among which a Schmidt measure and a generalization of concurrence have been proposed. Nevertheless, the application of these measures to general multipartite mixed states still shows some issues. The main aim of the present work is to propose alternative measures for correlation and entanglement based on a geometric framework. Remarkably, the latter fact, makes the validity of our measures dimension independent.

Our goal in this work has been to derive (i) a directly computable and genuine quantum correlation measure and (ii) a numerically computable entanglement measure, from the geometric properties of the projective Hilbert space describing a quantum multipartite system. For mixed states, the quantum correlation is not a faithful measure of entanglement. In our derivation, to extract from a given state ρ its entanglement essence, we have defined a regularization procedure for the density matrix that allows our measure of quantum correlation to catch the true degree of entanglement owned by ρ . The calculation of the entanglement involves a minimization procedure that,

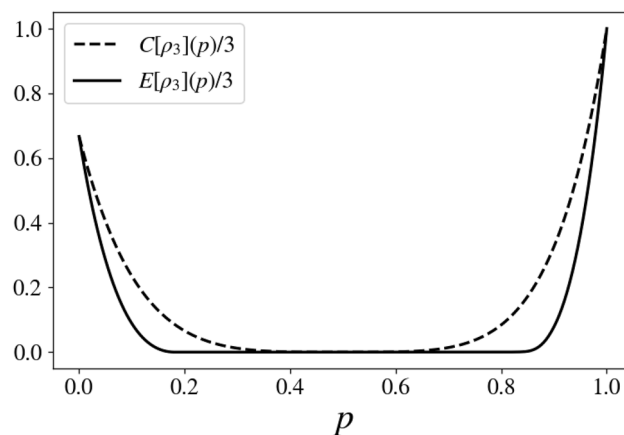


Figure 5. $C[\rho_3(p)]/3$ (dotted line) and $E[\rho_3(p)]/3$ (continuous line) versus p for state (23). It is clear that the state $\rho_3(p = 1)$ is, as expected, the maximally entangled one, and that the states $\rho_3(p > 0.81)$ or $\rho_3(p < 0.18)$ are not separable.

in the general case requires numeric simulations. In the latter sense, the entanglement corresponds to the inferior value of the correlation of a density matrix ρ when local decompositions and local unitary transformations are operated on it. We have proved that the entanglement and quantum-correlation measures derived do satisfy the requirements for suitable measures of these quantities. To test our quantum correlation and entanglement measures, we have applied them to two classes of mixed two-qubit states of which are well-known the entanglement properties, the Bell diagonal states and the Werner states, and we have verified the accordance between our measures and the expected results. Furthermore, we have applied the quantum correlation and entanglement measures to Werner state generalization to three qubits, and to a one-parameter family of three qubits mixed states. These latter interpolate between a bi-separable state and a genuine multipartite state, passing through a fully separable state. Also in these cases of multipartite states, then we have verified a satisfactory agreement between the behaviours deduced by our measures and the ones expected or already known in the literature.

Data availability

The datasets used and/or analysed during the current study available from the corresponding author on reasonable request.

Received: 6 December 2022; Accepted: 3 February 2023

Published online: 17 February 2023

References

- Gühne, O. & Toth, G. Entanglement detection. *Phys. Rep.* **474**, 1–75. <https://doi.org/10.1016/j.physrep.2009.02.004> (2009).
- Sperling, J. & Walmsley, I. A. Entanglement in macroscopic systems. *Phys. Rev. A* **95**, 062116. <https://doi.org/10.1103/PhysRevA.95.062116> (2017).
- Giovannetti, V., Mancini, S., Vitali, D. & Tombesi, P. Characterizing the entanglement of bipartite quantum systems. *Phys. Rev. A* **67**, 022320. <https://doi.org/10.1103/PhysRevA.67.022320> (2003).
- Horodecki, R., Horodecki, P., Horodecki, M. & Horodecki, K. Quantum entanglement. *Rev. Mod. Phys.* **81**, 865–942. <https://doi.org/10.1103/RevModPhys.81.865> (2009).
- Popescu, S. & Rohrlich, D. Thermodynamics and the measure of entanglement. *Phys. Rev. A* **56**, R3319–R3321. <https://doi.org/10.1103/PhysRevA.56.R3319> (1997).
- Wootters, W. K. Entanglement of formation of an arbitrary state of two qubits. *Phys. Rev. Lett.* **80**, 2245–2248. <https://doi.org/10.1103/PhysRevLett.80.2245> (1998).
- Bennett, C. H., DiVincenzo, D. P., Smolin, J. A. & Wootters, W. K. Mixed-state entanglement and quantum error correction. *Phys. Rev. A* **54**, 3824–3851. <https://doi.org/10.1103/PhysRevA.54.3824> (1996).
- Bennett, C. H. *et al.* Purification of noisy entanglement and faithful teleportation via noisy channels. *Phys. Rev. Lett.* **76**, 722–725. <https://doi.org/10.1103/PhysRevLett.76.722> (1996).
- Horodecki, M., Horodecki, P. & Horodecki, R. Mixed-state entanglement and distillation: Is there a “bound” entanglement in nature? *Phys. Rev. Lett.* **80**, 5239–5242. <https://doi.org/10.1103/PhysRevLett.80.5239> (1998).
- Vedral, V., Plenio, M. B., Rippin, M. A. & Knight, P. L. Quantifying entanglement. *Phys. Rev. Lett.* **78**, 2275–2279. <https://doi.org/10.1103/PhysRevLett.78.2275> (1997).
- Adesso, G., Bromley, T. R. & Cianciaruso, M. Measures and applications of quantum correlations. *J. Phys. A: Math. Theor.* **49**, 473001. <https://doi.org/10.1088/1751-8113/49/47/473001> (2016).
- Dür, W., Vidal, G. & Cirac, J. I. Three qubits can be entangled in two inequivalent ways. *Phys. Rev. A* **62**, 062314. <https://doi.org/10.1103/PhysRevA.62.062314> (2000).
- Briegel, H. J. & Raussendorf, R. Persistent entanglement in arrays of interacting particles. *Phys. Rev. Lett.* **86**, 910–913. <https://doi.org/10.1103/PhysRevLett.86.910> (2001).
- Eisert, J. & Briegel, H. J. Schmidt measure as a tool for quantifying multiparticle entanglement. *Phys. Rev. A* **64**, 022306. <https://doi.org/10.1103/PhysRevA.64.022306> (2001).
- Coffman, V., Kundu, J. & Wootters, W. K. Distributed entanglement. *Phys. Rev. A* **61**, 052306. <https://doi.org/10.1103/PhysRevA.61.052306> (2000).
- Carvalho, A. R. R., Mintert, F. & Buchleitner, A. Decoherence and multipartite entanglement. *Phys. Rev. Lett.* **93**, 230501. <https://doi.org/10.1103/PhysRevLett.93.230501> (2004).
- Braunstein, S. L. & Caves, C. M. Statistical distance and the geometry of quantum states. *Phys. Rev. Lett.* **72**, 3439–3443. <https://doi.org/10.1103/PhysRevLett.72.3439> (1994).
- Pezzè, L. & Smerzi, A. Entanglement, nonlinear dynamics, and the Heisenberg limit. *Phys. Rev. Lett.* **102**, 100401. <https://doi.org/10.1103/PhysRevLett.102.100401> (2009).
- Hyllus, P. *et al.* Fisher information and multiparticle entanglement. *Phys. Rev. A* **85**, 022321. <https://doi.org/10.1103/PhysRevA.85.022321> (2012).
- Tóth, G. Multipartite entanglement and high-precision metrology. *Phys. Rev. A* **85**, 022322. <https://doi.org/10.1103/PhysRevA.85.022322> (2012).
- Scali, S. & Franzosi, R. Entanglement estimation in non-optimal qubit states. *Ann. Phys.* **411**, 167995. <https://doi.org/10.1016/j.aop.2019.167995> (2019).
- Ollivier, H. & Zurek, W. H. Quantum discord: A measure of the quantumness of correlations. *Phys. Rev. Lett.* **88**, 017901. <https://doi.org/10.1103/PhysRevLett.88.017901> (2001).
- Cocchiarella, D. *et al.* Entanglement distance for arbitrary m -qudit hybrid systems. *Phys. Rev. A* **101**, 042129. <https://doi.org/10.1103/PhysRevA.101.042129> (2020).
- Horodecki, R. & Horodecki, M. Information-theoretic aspects of inseparability of mixed states. *Phys. Rev. A* **54**, 1838–1843. <https://doi.org/10.1103/PhysRevA.54.1838> (1996).
- Aaronson, B., Lo Franco, R. & Adesso, G. Comparative investigation of the freezing phenomena for quantum correlations under nondissipative decoherence. *Phys. Rev. A* **88**, 012120. <https://doi.org/10.1103/PhysRevA.88.012120> (2013).
- Peres, A. Separability criterion for density matrices. *Phys. Rev. Lett.* **77**, 1413–1415. <https://doi.org/10.1103/PhysRevLett.77.1413> (1996).
- Horodecki, M., Horodecki, P. & Horodecki, R. Separability of mixed states: Necessary and sufficient conditions. *Phys. Lett. A* **223**, 1–8. [https://doi.org/10.1016/S0375-9601\(96\)00706-2](https://doi.org/10.1016/S0375-9601(96)00706-2) (1996).
- Werner, R. F. Quantum states with Einstein–Podolsky–Rosen correlations admitting a hidden-variable model. *Phys. Rev. A* **40**, 4277–4281. <https://doi.org/10.1103/PhysRevA.40.4277> (1989).
- Pittenger, A. O. & Rubin, M. H. Note on separability of the Werner states in arbitrary dimensions | this work was supported in part by the national security agency 1. *Opt. Commun.* **179**, 447–449. [https://doi.org/10.1016/S0030-4018\(00\)00612-X](https://doi.org/10.1016/S0030-4018(00)00612-X) (2000).

30. Dür, W. & Cirac, J. I. Classification of multiqubit mixed states: Separability and distillability properties. *Phys. Rev. A* **61**, 042314. <https://doi.org/10.1103/PhysRevA.61.042314> (2000).
31. Eltschka, C. & Siewert, J. Entanglement of three-qubit greenberger-horne-zeilinger-symmetric states. *Phys. Rev. Lett.* **108**, 020502. <https://doi.org/10.1103/PhysRevLett.108.020502> (2012).
32. Schack, R. & Caves, C. M. Explicit product ensembles for separable quantum states. *J. Mod. Opt.* **47**, 387–399. <https://doi.org/10.1080/09500340008244050> (2000).
33. Gühne, O. & Seevinck, M. Separability criteria for genuine multiparticle entanglement. *New J. Phys.* **12**, 053002. <https://doi.org/10.1088/1367-2630/12/5/053002> (2010).

Acknowledgements

We acknowledge support from the RESEARCH SUPPORT PLAN 2022 - Call for applications for funding allocation to research projects curiosity driven (F CUR) - Project “Entanglement Protection of Qubits’ Dynamics in a Cavity” - EPQDC and the support by the Italian National Group of Mathematical Physics (GNFM-INdAM). We also acknowledge discussions with Lorenzo Capra. We also acknowledge the support by Fondo Open Access 2023 of the University of Siena.

Author contributions

The authors have equally contributed to the manuscript. They have all read and approved its final version.

Competing Interests

The authors declare no competing interests.

Additional information

Correspondence and requests for materials should be addressed to R.F.

Reprints and permissions information is available at www.nature.com/reprints.

Publisher’s note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

© The Author(s) 2023