Tail behavior of a sum of two dependence and heavy-tailed distributions

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March 21, 2018

Abstract

We consider the problem of a sum of two dependent and heavy tailed distributions through the C-convolution. The C-convolution provides the distribution of the sum of two random variables whose dependence structure is described by a copula function. Moreover, to investigate the role of heavy tails we use three different marginal distributions characterized by this property: Cauchy, Levy and Pareto. We show that the tail behavior of the C-convolution measured by level-q quantiles for $q = 0.01, 0.05$ (left tail) and $q = 0.95, 0.99$ (right tail) is strongly affected by the copula function which links the marginals and by the tail heaviness of marginals themselves.

Mathematics Subject Classification (2010): 62E17, 60E99

Keywords: heavy tails, copula function, C-convolution.

1 Introduction

In this paper we consider the sum of two dependent and heavy tailed distributions through the C-convolution operator which is a generalization of the standard convolution which recovers the distribution of the sum if the summands are independent. The C-convolution, denoted by $\overset{C}{*}$ and introduced by Cherubini et al. (2011), determines the distribution of the sum in the case where the dependence structure is modelled by a copula function. Through the whole paper we will denote by F_x and F_y the marginal distributions of X and Y respectively and by C the copula function linking X and Y. The C-convolution recovers the distribution of the sum $X + Y$ which will be denoted by F_{x+y}.

Our aim is to compare the level-q quantile of the distribution of the sum, $F^{-1}_{x+y}(q)$, with the sum of level-q quantiles of the marginal distributions, $F^{-1}_x(q)+F^{-1}_y(q)$ to investigate the tail behavior of the sum $X+Y$. Level-q quantiles are commonly used as risk measures to analyze portfolio returns in the case where the portfolio components have heavy tailed distributions. In fact, when $q = 0.01$ or $q = 0.05$ $F^{-1}_{x+y}(q)$ can be thought of as the level-q *Value-at-Risk* of the sum of two dependent sources of risk. Ibragimov and Walden (2011) and Embrechts et al. (2009) addressed the problem of aggregation of independent risk in a Value-at-Risk framework. Extensions to dependence using copulas are studied in Embrechts et al. (2009) and Chen et al. (2012). Kao et al. (2009) propose a two-stage approach for estimating value-at-risk when the conditional distribution of asset returns simultaneously reflect stochastic volatility and heavy-tailedness. An interesting and related paper is that of Ibragimov et al. (2015) where the authors consider the problem of portfolio risk diversification in a Value-at-Risk framework with heavy-tailed risks and arbitrary dependence captured by a copula function. In particular they use the power law for modelling the tails and investigate whether the benefits of diversification persist when the risks in consideration are allowed to have extremely heavy tails with tail indices less than one and when their copula describes wide classes of dependence structures. The simulation results are similar to those proposed in our paper but we use specific marginal distributions which are close but not identical to that used in the paper of Ibragimov et al. (2015).

We show that the tail subadditivity (the level-q quantile of the distribution of the sum is less than the sum of level-q quantiles of the marginals) and the tail superadditivity (the level-q quantile of the distribution of the sum is greater than the sum of level-q quantiles of the marginals) are significantly affected by the tails heaviness of F_x and F_y and by their dependence structure. We proceed by analyzing the effect of four different copula families, Gaussian, t-copula, Frank and Clayton combined with three different heavy tailed marginal distributions, Cauchy, Levy and Pareto.

Furthermore, we will show that under extreme positive dependence the ratio approaches to 1 if $q = 0.01$ or $q = 0.05$ for all selected copulas and marginals. On the other hand, if $q = 0.95$ or $q = 0.99$ the ratio is much greater than 1 for marginals with extreme heavy tails, i.e., Levy and Pareto with parameter $\theta \geq 2$ even if the level of dependence is high. In general we can see that the degree of heavy-tailedness of marginals has more impact on the right tail of the distribution of the sum.

The paper is organized as follows. In section 2 we present heavy tailed distributions used in this paper. In section 3 we will introduce the C-convolution we will briefly discuss its properties. Section 4 shows the results. Section 5 concludes.

2 Heavy-tailed marginal distributions

It has become common in financial applications to use univariate distributions with heavy tails to model extreme events (see among others Embrechts et al. (2007), Ibragimov (2009), McCulloch (1997) and Jansen et al. (1991)). In particular, a class of distributions which assumes that the tail index characterizing the heaviness of the tails obeys a power law is studied in this paper. When the tail decay is a power law we speak about power law family of distributions and the law is usually written as

$$\mathbb{P}(|X| > x) \sim x^{-\gamma},$$

where γ is the tail index. Power law distributions permit modelling rates of tail decay that are slower than the exponential decay of the Gaussian distribution. The tail index determines the probability of observing large fluctuations; a smaller tail index means slower rate of tail decay. When $\gamma < 2$ the second moment of the distribution is infinite whereas when $\gamma < 1$ the first moment is infinite. We will say that a distribution has extremely heavy tails if $\gamma < 1$.

Many well known distributions can be viewed as a special cases of power law. In particular, in this paper we will consider three continuous distributions: Cauchy, Levy and Pareto, which are extensively used in applications in finance to model the tails in financial stock returns (see among others Jondeau and Rockinger (2003), Nolan (2003) and Osu and Ohakwe (2011)).

The Cauchy distribution is important as an example of a pathological case. It looks similar to a normal distribution but mean and standard deviation of the Cauchy distribution are undefined. The density function is given by

$$f(x) = \frac{1}{\pi\sigma\left[1 + \left(\frac{x-\mu}{\sigma}\right)^2\right]},$$

where $\mu \in \mathbb{R}$ is a location parameter and $\sigma > 0$ is a scale parameter. In this paper we use the standard version of the Cauchy distribution where $\mu = 0$ and $\sigma = 1$ which is equivalent to a Student's t distribution with 1 degree of freedom. The tail index of a standard Cauchy distribution is 1.

The Levy distribution is a distribution whose density function has the form

$$f(x) = \frac{\sigma}{\sqrt{2\pi}} \frac{1}{(x-\mu)^{3/2}} e^{-\frac{1}{2(x-\mu)}},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are the location and scale parameter respectively and $x \in (\mu, +\infty)$. In this paper we use the standard version of a Levy law with $\mu = 0$ and $\sigma = 1$. The tail index is $\gamma = \frac{1}{2}$ so that a Levy distribution has extremely heavy tails. As a consequence the mean and the variance are both infinite.

The Pareto distribution is a skewed, heavy-tailed distribution characterized by two parameters, a scale parameter σ and a shape parameter θ. Its density function is

$$f(x) = \frac{\theta\sigma^\theta}{x^{1+\theta}},$$

where $\theta > 0$, $\sigma > 0$ and $x > \sigma$. In our application we fix $\sigma = 1$. As we can see, the reason that the Pareto distribution is heavy-tailed is that the density decreases at a power rate rather than an exponential rate and this decreasing depends on the value of θ. Therefore, the existence of the mean and the variance depends on θ. More precisely, we have

$$\mathbb{E}[X] = \frac{\theta}{\theta - 1}$$

for $\theta > 1$ and

$$Var(X) = \frac{\theta}{(\theta-1)^2(\theta-2)}$$

for $\theta > 2$. Moreover, the distribution is positively skewed and $skew(X) \to 2$ when $\theta \to \infty$ Similarly, $kurt(X) \to 9$ as $\theta \to \infty$.

We may note that both cumulative distribution function, F, and its inverse, the quantile function, F^{-1}, may be expressed in closed form for each of the selected distributions in this paper.

3 The C-convolution

The C-convolution operator (introduced by Cherubini et al. (2011)) is an operator which determines the distribution function of a sum of two dependent and continuous random variables X and Y when the dependence structure between X and Y is modelled by a copula function C. Copulas are one of the most flexible tool to modelled the dependence structure between two or more random variables and their crucial characteristic is that they are used to separate the specification of marginal distributions from the dependence structure. For a detailed and complete discussion about copulas the reader can see Nelsen (2006), Cherubini et al. (2004), Cherubini et al. (2012) and Joe (1997). In particular, for the aims of this paper a synthetic but exhaustive discussion of copula functions can be found in chapter 2 of Cherubini et al. (2012). Given the copula function C linking X and Y and the marginal distributions F_x and F_y we must come up with the probability distribution of the sum $X + Y$, say F_{x+y} by

$$F_{x+y}(z) = F_x \overset{C}{*} F_y(z) = \int_0^1 D_1C\left(w, F_y(z - F_x^{-1}(w))\right) dw \tag{1}$$

where $D_1C(u, v)$ denotes the partial derivative with respect to first argument of $C(u, v)$, i.e., $\frac{\partial C(u,v)}{\partial u}$. In this paper we focus on the C-convolution characterized by four copula families linking X and Y: Gaussian, t-copula, Frank and Clayton. Each of them is characterized by properties that depend on one or more parameters.

- The Gaussian copula is an elliptical copula characterized by one parameter, the correlation coefficient $\rho \in [-1, 1]$, and it has no tail dependence. Its cdf is constructed from a multivariate normal distribution over $\mathbb{R}^2$ by using the probability integral transform

$$C_G(u, v; \rho) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{\frac{2\rho st - s^2 - t^2}{2(1-\rho^2)}} dsdt,$$

The Gaussian copula is used to model elliptical dependencies.

- The t-copula is an elliptical copula with two parameters, the correlation coefficient ρ and the degrees of freedom ν, which has both the lower and the upper tail dependence (the lower and the upper tail dependence coefficients are the same and given by $\lambda_L = \lambda_U = 2 - 2t_{\nu+1}(\sqrt{\nu+1}\sqrt{1-\rho}/\sqrt{1+\rho})$). The cdf of a t copula is given by

$$C_T(u, v; \rho, \nu) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{s^2 + t^2 - 2\rho st}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}} dsdt,$$

  where t_ν is the univariate Student's t distribution with ν degrees of freedom.

- The *Frank* copula is a symmetric archimedean copula whose cdf is given by

$$C_F(u, v, \alpha) = -\frac{1}{\alpha} \log\left(1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{e^{-\alpha} - 1}\right),$$

  where the parameter $\alpha \in (-\infty, +\infty)/\{0\}$. The Frank copula does not exhibit tail dependence.

- The *Clayton* copula is an archimedean copula, exhibiting greater dependence in the negative tails than in the positive. In fact, it is characterized by lower tail dependence (the lower tail dependence coefficient is given by $\lambda_L = 2^{-1/\alpha}$). The copula cdf is given by

$$C_C(u, v; \alpha) = (u^{-\alpha} + v^{-\alpha} - 1)^{-\frac{1}{\alpha}},$$

  where the parameter $\alpha \in (-1, +\infty]/\{0\}$.

It is clear that the properties of the C-convolution depend both on the marginals and on the copula linking them. In particular, we study the heaviness of tails of F_{x+y} when F_x and F_y are both heavy-tailed distributions. We notice that to determine the distribution of the sum F_{x+y} we need i) the cumulative distribution function of Y, F_y; ii) the quantile function of X, F_x^{-1} and iii) the partial derivative of the copula function with respect to the first argument, $D_1C(u, v)$. As regards iii) we can easily determine the partial derivatives of the four selected copula functions (see for example Cherubini et al. (2004) and Nelsen (2006)). Furthermore, as mentioned in section 2 each of selected marginal distributions has both the cumulative distribution function and the quantile function in closed form. In fact, for a standard Cauchy distribution we have for $x \in \mathbb{R}$ and $q \in [0, 1]$

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x), \quad F^{-1}(q) = \tan\left[\pi\left(q - \frac{1}{2}\right)\right].$$

The standard Levy distribution has cumulative distribution function and quantile function given by

$$F(x) = 2\left[1 - \Phi\left(\frac{1}{\sqrt{x}}\right)\right], \quad F^{-1}(q) = \frac{1}{\left[\Phi^{-1}\left(1 - \frac{q}{2}\right)\right]^2},$$

where Φ is the standard Gaussian cumulative distribution function, $x \in (0, \infty)$ and $q \in [0, 1)$.

Finally, in the case of Pareto marginals the cumulative distribution function and the quantile function are

$$F(x) = 1 - \frac{1}{x^\theta}, \quad F^{-1}(q) = \frac{1}{(1-q)^{\frac{1}{\theta}}},$$

where $x \in [1, \infty)$ and $q \in [0, 1)$.

For our purposes, we need to compute the quantile function of the C-convolution, F_{x+y}^{-1}. Unfortunately, neither F_{x+y} nor F_{x+y}^{-1} can be written explicitly even though F_y, F_x^{-1} and D_1C are known, and therefore we will compute them numerically. In fact, the C-convolution has not a closed form except for a small number of cases. For a detailed discussion on the C-convolution and its closure properties see the book of Cherubini et al. (2012).

4 Results

In this section we present the results of our application. We will show the effect on the tails of the distribution of the sum F_{x+y}, obtained via C-convolution, using the following ratio

$$r(q) = \frac{F_{x+y}^{-1}(q)}{F_x^{-1}(q) + F_y^{-1}(q)},$$

where F^{-1} denotes the quantile function and q will be equal to 1%, 5% (left tail) and 95%, 99 (right tail). In fact, $r(q)$ compares the level-q quantile of the sum with the sum of level-q quantiles of marginals. It is clear that there are three cases of interest: $r(q) > 1$, which means that the quantile function of the sum is superadditive in the tails, $r(q) < 1$, which means that the quantile function of the sum is subadditive in the tails and $r(q) = 1$, which means that the quantile function of the sum is simply additive in the tails. We can observe that since, without loss of generality, we will use two identical marginals the ratio $r(q)$ is also equal to $\frac{F_{x+y}^{-1}(q)}{2F_x^{-1}(q)}$. As mentioned in the previous sections the value of $r(q)$ is affected by at least three factors:

- the copula function which describes the dependence structure;
- the level of dependence;
- the tail heaviness of the marginals.

We discussed copula functions we use in section 3. The level of dependence is provided by the Kendall's τ coefficient which is within $[-1, 1]$. For the selected copula functions there exists a a one-to-one relationship between the desired level of Kendall's τ and the value of the copula parameter (see for example Nelsen (2006) and Cherubini et al. (2004)). The marginal distributions are Cauchy, Levy and Pareto and their main properties have already been discussed in section 2.

Table 2 and table 4 report the values of $r(q)$ in the case where $X \overset{d}{=} Y \sim Cauchy(0, 1)$. For the sake of simplicity we study the values for different but positive levels of dependence. We recall that Cauchy r.vs. have symmetric tails so that we have $r(0.01) = r(0.99)$ and $r(0.05) = r(0.95)$ if the copulas which links X and Y has no tail dependence (Gaussian or Frank) or has symmetric tail dependence (t-copula). Only in the case of Clayton copula (which has lower tail dependence) we observe $r(0.01) \neq r(0.99)$ and $r(0.05) \neq r(0.95)$. Precisely, because of this we can note that $r(q)$ is systematically higher in the right tail than in the left tail and this positive difference is more pronounced with the increase of the Kendall's τ coefficient. Moreover, we can observe that the quantile function of the C-convolution is superadditive under all levels of dependence in the case of Gaussian and Frank copulas. Differently, when we use the t-copula (table 4) the value of $r(q)$ depends on the degrees of freedom. In particular, $r(q) < 1$ (both for q=0.01 and for $q = 0.05$) when $\nu = 3$ and $\tau \leq 0.50$ and it is just above 1 for $\tau = 0.75$ and $\tau = 0.90$. When $\nu = 5$ and $\nu = 10$ the ratio exceeds 1 albeit slightly.

Table 3 and table 4 report the values of $r(q)$ in the case of Levy marginals. We can see that the ratio is much higher than in the case of Cauchy marginals and this underlines a stronger evidence of superadditivity of the quantile function of the C-convolution for all copula families. However, the effect is stronger for low values of the Kendall's τ. Moreover, superadditivity is greater in the right tail rather than in the left tail. If $\tau \geq 0.50$ Gaussian copula and t-copula show lower values of $r(q)$ compared with the Frank copula and the Clayton copula.

Tables 5-11 report the results relative to Pareto marginals when the tail parameter θ takes values 0.1, 0.5, 1, 2 and 2.5. In this case, we consider that, as the Pareto distribution has asymmetric tails (and the tail behavior depends on θ as discussed in section 2), the most interesting values of $r(q)$ are those obtained in the right tail, i.e. for $q = 0.95$ and $q = 0.99$. In fact, the tail parameter θ seems not to affect the value of $r(q)$ in the left tail and this is true for any copula and under any level of dependence. For example, we observe that if $\tau = 0.5$ and we consider the Frank copula we have $r(0.01) = 1.0235$ for $\theta = 0.1$ and $r(0.01) = 1.0253$ for $\theta = 2.5$. Considering other copulas or other levels of the Kendall's τ the evolution of values of $r(q)$ for $q = 0.01$ and $q = 0.05$ is actually

	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
Frank	1.0000	1.0152	1.0375	1.0749	1.1250	1.1740
	1.0000	1.0390	1.0899	1.1531	1.1771	1.1129
	1.0000	1.0390	1.0899	1.1531	1.1771	1.1129
	1.0000	1.0152	1.0375	1.0749	1.1250	1.1740
Gaussian	1.0000	1.0247	1.0505	1.0598	1.0271	1.0053
	1.0000	1.0421	1.0764	1.0719	1.0274	1.0050
	1.0000	1.0421	1.0764	1.0719	1.0274	1.0050
	1.0000	1.0247	1.0505	1.0598	1.0271	1.0053
Clayton	1.0000	1.0228	1.0075	0.9999	1.0000	1.0000
	1.0000	1.0223	1.0094	0.9985	0.9998	1.0000
	1.0000	1.0322	1.0710	1.1246	1.1761	1.1723
	1.0000	1.0121	1.0266	1.0493	1.0883	1.1443

Table 1: Values of the ratio $r(q)$ for $q = 0.01$, $q = 0.05$, $q = 0.95$ and $q = 0.99$. Standard Cauchy marginals.

moderate. Things drastically change if we consider the right tail ($q = 0.95$ and $q = 0.99$). In fact, if $\theta \leq 0.5$ the ratio is systematically lesser than 1 for any copula and under any level of dependence. But when $\theta \geq 1$ $r(q)$ rapidly grows, especially if $\tau \in [0.25, 0.75]$. The impact is particularly evident for the Frank and Clayton copulas (tables 5-9), whereas in the case of the t-copula the greater the number of degrees of freedom the greater the impact of θ on $r(q)$ (tables 10 and 11).

Figure 1 shows the dynamics of $r(q)$ as a function of $\tau \in [-1, 1]$ in the case where marginal distributions are both Cauchy and the dependence structure is given by a gaussian or by a t-copula. We can see that the ratio is increasing with τ and it is much lesser than one under extreme negative dependence for both $q = 0.01$ and $q = 0.05$. Moreover, we notice that for any fixed level of the Kendall's τ the ratio is slightly higher if we use the Gaussian copula rather than the t-copula.

Figures 2-5 display the same dynamics when the marginal distributions are Pareto and the tail parameter is 0.1 (figures 2 and 3) and 2 (figures 4 and 5). In the former case ($\theta = 0.1$) we have two opposite dynamics. If $q = 0.01$, $r(q)$ is decreasing with τ and for negative levels of dependence the ratio exceeds 1. As before, the ratio is greater in the case of Gaussian copula. If $q = 0.99$, $r(q)$ is increasing and smaller than 1 and tends to 1 under extreme positive dependence. If we consider the case $\theta = 2$ (figures 4 and 5) both dynamics are decreasing and $r(q)$ is high (much greater than one) under negative dependence. Nevertheless, if $q = 0.01$ $r(q)$ approaches to 1 as soon as $\tau > 0$ whereas if $q = 0.99$ the ratio is close ton 1 only under extreme positive values of the Kendall's τ. Figure 6 shows the dynamics in the same graph to better understand the above intuition. We report, as an example, the case where the copula is Gaussian and the margins are Pareto with parameter $\theta = (0.1, 1, 2)$.

5 Concluding remarks

This paper presented some results about the sum of two dependent and heavy-tailed distributions F_x and F_y which can be Cauchy, Levy and Pareto. The dependence structure is modelled by selected copula families, Gaussian, t-copula, Frank and Clayton. The distribution of the sum, i.e. F_{x+y} is provided by the C-convolution operator denoted by $\overset{C}{*}$. The aim of the paper is to investigate the tail behavior of F_{x+y} through the ratio $r(q) = \frac{F^{-1}_{x+y}(q)}{2F^{-1}_x(q)}$ where $q = 0.01, 0.05$ (left tail) and $q = 0.95, 0.99$ (right tail). $r(q)$ compares the level-q quantile of the distribution of the sum with the sum of the level-q quantiles of the marginal heavy tailed distributions. We showed that the right tail and the left tail of F_{x+y} are both affected by the dependence structure and by the tail heaviness of marginals. More precisely, the quantile function of the sum, F^{-1}_{x+y}, tends to be superadditive in the tails if F_x and F_y are Levy which have tail index lesser than 1 (extreme heavy

	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
Frank	2.0018	1.8244	1.6344	1.4213	1.2361	1.1272
	1.9979	1.7890	1.5507	1.2969	1.1063	1.0332
	2.0291	1.9873	1.9453	1.9503	1.8000	1.4940
	2.0285	2.1174	2.0196	2.0141	1.9724	1.9075
Gaussian	2.0018	1.7642	1.4886	1.1909	1.0409	1.0054
	1.9979	1.7731	1.5097	1.2066	1.0577	1.0107
	2.0291	1.9500	1.8738	1.5805	1.1998	1.0409
	2.0285	1.9175	1.9019	1.6616	1.2679	1.0730
Clayton	2.0018	1.5241	1.1739	1.0313	1.0040	0.9980
	1.9979	1.6458	1.2942	1.0621	1.0114	1.0008
	2.0291	2.0344	1.9937	1.9708	1.9032	1.5937
	2.0285	2.1093	2.0367	2.0629	1.9559	1.9161

Table 2: Values of the ratio $r(q)$ for $q = 0.01$, $q = 0.05$, $q = 0.95$ and $q = 0.99$. Standard Levy marginals.

Cauchy margins	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
$\nu = 3$	0.8803	0.9204	0.9656	0.9994	1.0029	1.0006
	0.8752	0.9214	0.9730	1.0099	1.0079	1.0017
	0.8752	0.9214	0.9730	1.0099	1.0079	1.0017
	0.8803	0.9204	0.9656	0.9994	1.0029	1.0006
$\nu = 5$	0.9373	0.9711	1.0047	1.0193	1.0085	1.0016
	0.9275	0.9705	1.0197	1.0332	1.0143	1.0027
	0.9275	0.9705	1.0197	1.0332	1.0143	1.0027
	0.9373	0.9711	1.0047	1.0193	1.0085	1.0016
$\nu = 10$	0.9764	1.0048	1.0315	1.0372	1.0156	1.0032
	0.9662	1.0077	1.0466	1.0518	1.0202	1.0037
	0.9662	1.0077	1.0466	1.0518	1.0202	1.0037
	0.9764	1.0048	1.0315	1.0372	1.0156	1.0032
Levy margins	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
$\nu = 3$	1.6245	1.4637	1.2846	1.1236	1.0268	1.0012
	1.8760	1.6688	1.4321	1.1770	1.0473	1.0085
	1.7843	1.7391	1.6184	1.3664	1.1261	1.0383
	1.8241	1.6623	1.5305	1.5248	1.1702	1.0572
$\nu = 5$	1.7452	1.5692	1.3396	1.1343	1.0391	1.0069
	1.9326	1.7105	1.4603	1.1860	1.0408	1.0118
	1.8375	1.7964	1.6677	1.4345	1.1144	0.9984
	1.9474	1.8957	1.7018	1.4865	1.1597	1.0026
$\nu = 10$	1.8717	1.6529	1.4215	1.1709	1.0455	1.0104
	1.9609	1.7416	1.4735	1.1977	1.0519	1.0017
	1.9372	1.8914	1.7496	1.5188	1.2235	1.0138
	1.9020	1.8541	1.9178	1.4861	1.2485	1.0548

Table 3: Values of the ratio $r(q)$ for $q = 0.01$, $q = 0.05$, $q = 0.95$ and $q = 0.99$. t-copula with three different values of degrees of freedom $\nu = (3, 5, 10)$.

$\theta = 0.1$	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
Frank	1.0639	1.0515	1.0381	1.0235	1.0127	1.0081
	1.1221	1.0978	1.0699	1.0349	1.0149	1.0064
	0.8448	0.8669	0.9001	0.9522	0.9987	1.0134
	0.7593	0.7805	0.8191	0.8641	0.9227	0.9615
Gaussian	1.0639	1.0471	1.0281	1.0104	1.0024	1.0004
	1.1221	1.0965	1.0637	1.0272	1.0058	1.0011
	0.8448	0.8679	0.9035	0.9539	0.9880	0.9981
	0.7593	0.7912	0.8440	0.9238	0.9799	0.9977
Clayton	1.0639	1.0304	1.0092	1.0018	1.0002	1.0000
	1.1221	1.0808	1.0374	1.0077	1.0010	1.0000
	0.8448	0.8581	0.8794	0.9201	0.9728	1.0111
	0.7593	0.7708	0.7893	1.8264	0.8814	0.9424

Table 4: Values of the ratio $r(q)$ for $q = 0.01$, $q = 0.05$, $q = 0.95$ and $q = 0.99$. Pareto margins with parameter $\theta = 0.1$.

$\theta = 0.5$	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
Frank	1.0657	1.0525	1.0387	1.0239	1.0129	1.0079
	1.1299	1.1034	1.0734	1.0395	1.0154	1.0062
	0.8932	0.9143	0.9469	1.0006	1.0461	1.0503
	0.7968	0.8117	0.8368	0.8850	0.9502	0.9984
Gaussian	1.0657	1.0480	1.0284	1.0105	1.0024	1.0004
	1.1299	1.1020	1.0668	1.0282	1.0070	1.0011
	0.8932	0.9139	0.9433	0.9779	0.9953	0.9993
	0.7968	0.8207	0.8638	0.9332	0.9825	0.9972
Clayton	1.0657	1.0309	1.0093	1.0015	1.0002	1.0000
	1.1299	1.0850	1.0388	1.0079	1.0011	1.0000
	0.8932	0.9056	0.9260	0.9672	1.0231	1.0560
	0.7968	0.8045	0.8181	0.8486	0.9028	0.9743

Table 5: Values of the ratio $r(q)$ for $q = 0.01$, $q = 0.05$, $q = 0.95$ and $q = 0.99$. Pareto margins with parameter $\theta = 0.5$.

$\theta = 1$	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
Frank	1.0674	1.0538	1.0394	1.0242	1.0130	1.0080
	1.1404	1.1108	1.0777	1.0415	1.0162	1.0066
	1.0863	1.1017	1.1259	1.1638	1.1785	1.1416
	1.0257	1.0335	1.0474	1.0779	1.1259	1.1622
Gaussian	1.0674	1.0491	1.0289	1.0106	1.0024	1.0004
	1.1404	1.1093	1.0708	1.0295	1.0073	1.0012
	1.0863	1.0974	1.1037	1.0790	1.0285	1.0052
	1.0257	1.0376	1.0555	1.0608	1.0272	1.0053
Clayton	1.0674	1.0314	1.0094	1.0015	1.0002	1.0000
	1.1404	1.0906	1.0407	1.0082	1.0011	1.0000
	1.0863	1.0952	1.1104	1.1419	1.1810	1.1722
	1.0257	1.0296	1.0365	1.0537	1.0900	1.1448

Table 6: Values of the ratio $r(q)$ for $q = 0.01$, $q = 0.05$, $q = 0.95$ and $q = 0.99$. Pareto margins with parameter $\theta = 1$.

$\theta = 2$	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
Frank	1.0712	1.0564	1.0410	1.0240	1.0133	1.0082
	1.1641	1.1274	1.0876	1.0459	1.0178	1.0073
	1.9963	1.9901	1.9737	1.9145	1.7333	1.4877
	1.9999	1.9995	1.9984	1.9935	1.9713	1.9145
Gaussian	1.0712	1.0514	1.0299	1.0109	1.0025	1.0007
	1.1641	1.1256	1.0795	1.0324	1.0079	1.0013
	1.9963	1.9641	1.8590	1.5503	1.1827	1.0351
	1.9999	1.9897	1.9357	1.6877	1.2566	1.0513
Clayton	1.0712	1.0325	1.0096	1.0016	1.0002	1.0000
	1.1641	1.1028	1.0448	1.0089	1.0012	1.0000
	1.9963	1.9994	1.9901	1.9714	1.8919	1.6331
	1.9999	1.9997	1.9995	1.9983	1.9914	1.9558

Table 7: Values of the ratio $r(q)$ for $q = 0.01$, $q = 0.05$, $q = 0.95$ and $q = 0.99$. Pareto margins with parameter $\theta = 2$.

$\theta = 2.5$	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
Frank	1.0733	1.0578	1.0418	1.0253	1.0135	1.0083
	1.1775	1.1366	1.0929	1.0482	1.0186	1.0076
	2.7840	2.7593	2.7063	2.5536	2.1805	1.7474
	2.8196	2.8154	2.8058	2.7770	2.6967	2.5611
Gaussian	1.0733	1.0526	1.0307	1.0110	1.0025	1.0007
	1.1775	1.1346	1.0843	1.0339	1.0082	1.0013
	2.7840	2.7111	2.4987	1.9299	1.3127	1.0588
	2.8196	2.7941	2.6770	2.1808	1.4339	1.0881
Clayton	1.0733	1.0334	1.0097	1.0016	1.0002	1.0000
	1.1775	1.1095	1.0470	1.0092	1.0013	1.0000
	2.7840	2.7738	2.7530	2.6887	2.4912	1.9926
	2.8196	2.8175	2.8136	2.8023	2.7657	2.6426

Table 8: Values of the ratio $r(q)$ for $q = 0.01$, $q = 0.05$, $q = 0.95$ and $q = 0.99$. Pareto margins with parameter $\theta = 2.5$.

$\theta = \mathbf{0.1}$	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
$\nu = 3$	1.0358	1.0258	1.0152	1.0058	1.0014	1.0002
	1.1083	1.0840	1.0542	1.0228	1.0058	1.0010
	0.8409	0.8621	0.8961	0.9479	0.9858	0.9977
	0.7953	0.8257	0.8731	0.9402	0.9846	0.9975
$\nu = 5$	1.0453	1.0328	1.0192	1.0072	1.0017	1.0003
	1.1140	1.0891	1.0580	1.0244	1.0062	1.0010
	0.8422	0.8639	0.8983	0.9497	0.9866	0.9978
	0.7815	0.8124	0.8621	0.9344	0.9831	0.9973
$\nu = 10$	1.0541	1.0394	1.0232	1.0086	1.0020	1.0002
	1.1182	1.0929	1.0609	1.0258	1.0065	1.0010
	0.8435	0.8658	0.9006	0.9516	0.9872	0.9979
	0.7708	0.8022	0.8532	0.9293	0.9816	0.9970
$\theta = \mathbf{0.5}$	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
$\nu = 3$	1.0363	1.0261	1.0154	1.0059	1.0014	1.0002
	1.1145	1.0882	1.0565	1.0235	1.0060	1.0010
	0.8838	0.8995	0.9241	0.9608	0.9887	0.9981
	0.8227	0.8435	0.8790	0.9369	0.9823	0.9970
$\nu = 5$	1.0461	1.0332	1.0194	1.0073	1.0017	1.0005
	1.1209	1.0939	1.0606	1.0253	1.0064	1.0011
	0.8876	0.9045	0.9301	0.9658	0.9907	0.9985
	0.8139	0.8354	0.8727	0.9339	0.9816	0.9970
$\nu = 10$	1.0552	1.0401	1.0235	1.0087	1.0029	1.0003
	1.1255	1.0980	1.0637	1.0267	1.0067	1.0011
	0.8907	0.9092	0.9362	0.9712	0.9927	0.9988
	0.8064	0.8290	0.8685	0.9327	0.9816	0.9970
$\theta = \mathbf{1}$	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
$\nu = 3$	1.0369	1.0265	1.0155	1.0059	1.0014	1.0002
	1.1227	1.0938	1.0595	1.0245	1.0062	1.0011
	1.0510	1.0511	1.0471	1.0310	1.0104	1.0019
	1.0141	1.0149	1.0148	1.0106	1.0037	1.0007
$\nu = 5$	1.0471	1.0338	1.0197	1.0073	1.0017	1.0003
	1.1300	1.1001	1.0640	1.0264	1.0066	1.0011
	1.0656	1.0682	1.0660	1.0458	1.0158	1.0028
	1.0221	1.0253	1.0280	1.0226	1.0086	1.0016
$\nu = 10$	1.0566	1.0429	1.0239	1.0088	1.0020	1.0003
	1.1353	1.1048	1.0674	1.0280	1.0069	1.0011
	1.0769	1.0829	1.0837	1.0607	1.0214	1.0039
	1.0269	1.0338	1.0419	1.0385	1.0157	1.0030

Table 9: Values of the ratio $r(q)$ for $q = 0.01$, $q = 0.05$, $q = 0.95$ and $q = 0.99$. Pareto marginals with parameter $\theta = (0.1, 0.5, 1)$ and t-copula with three different values of degrees of freedom $\nu = (3, 5, 10)$.

$\theta = \mathbf{2}$	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
$\nu = 3$	1.0383	1.0273	1.0159	1.0060	1.0015	1.0002
	1.1402	1.1062	1.0659	1.0266	1.0067	1.0011
	1.8091	1.7404	1.6130	1.3595	1.1205	1.0232
	1.8164	1.7494	1.6212	1.3704	1.1264	1.0246
$\nu = 5$	1.0492	1.0351	1.0202	1.0075	1.0018	1.0003
	1.1506	1.1140	1.0713	1.0288	1.0072	1.0012
	1.8853	1.8241	1.6947	1.4201	1.1414	1.0268
	1.9021	1.8479	1.7294	1.4580	1.1602	1.0310
$\nu = 10$	1.0595	1.0426	1.0246	1.0090	1.0021	1.0003
	1.1675	1.1199	1.0754	1.0306	1.0073	1.0012
	1.9446	1.8945	1.7723	1.4790	1.1616	1.0304
	1.9633	1.9281	1.8308	1.5531	1.1913	1.0391
$\theta = \mathbf{2.5}$	$\tau = 0$	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
$\nu = 3$	1.0390	1.0277	1.0161	1.0060	1.0015	1.0002
	1.1515	1.1130	1.0654	1.0277	1.0070	1.0012
	2.4410	2.3086	2.0630	1.6139	1.2048	1.0403
	2.4695	2.3424	2.1021	1.6474	1.2201	1.0440
$\nu = 5$	1.0503	1.0357	1.0205	1.0075	1.0018	1.0003
	1.1621	1.1216	1.0752	1.0301	1.0074	1.0012
	2.5794	2.4582	2.2112	1.7142	1.2379	1.0459
	2.6310	2.5258	2.2980	1.7986	1.2769	1.0549
$\nu = 10$	1.0610	1.0434	1.0249	1.0090	1.0021	1.0003
	1.1700	1.1282	1.0798	1.0319	1.0078	1.0013
	2.6880	2.5848	2.3467	1.8116	1.2705	1.0517
	2.7475	2.6750	2.4834	1.9635	1.3422	1.0683

Table 10: Values of the ratio $r(q)$ for $q = 0.01$, $q = 0.05$, $q = 0.95$ and $q = 0.99$. Pareto marginals with parameter $\theta = (2, 2.5)$ and t-copula with three different values of degrees of freedom $\nu = (3, 5, 10)$.

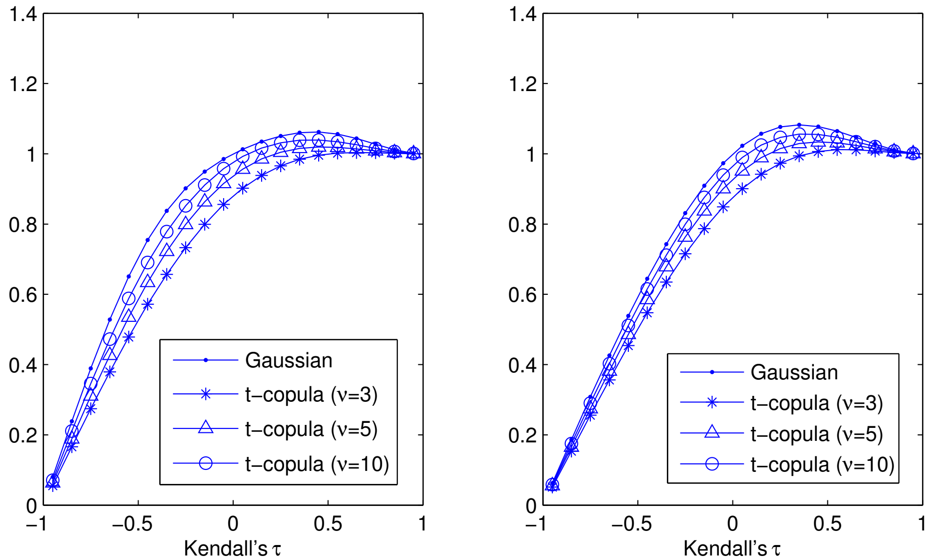

Figure 1: Dynamics of the ratio $r(q)$ for any level of the Kendall's τ and $q = 0.01$ (left) and $q = 0.05$ (right). Cauchy margins.

tails), whereas in the case of Pareto margins the effect is more significant in the right tails and if the tail parameter θ is high. As regards the dependence structure, Frank and Clayton copulas seem to affect the tail superadditivity more than the Gaussian copula. t-copula with low degrees of freedom induces tail subbaditivity mostly in the case where F_x and F_y are Cauchy. Generally, if the level of dependence increases the ratio $r(q)$ tends to decrease both in the right and in the left tails. Through the whole paper we have looked at marginal distributions having the same tail index but this restriction can easily be relaxed. In general, the tail behavior of the sum will be dominated by the marginal distribution with the lowest tail index.

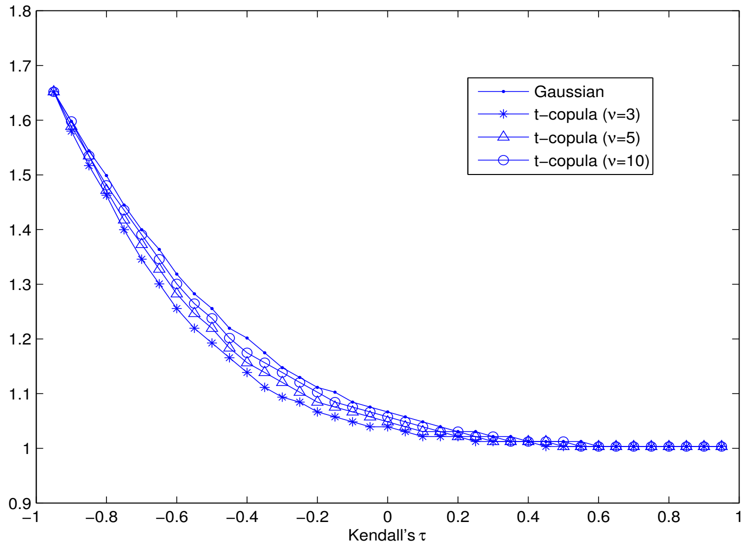

Figure 2: Dynamics of the ratio $r(q)$ for any level of the Kendall's τ and $q = 0.01$. Marginal distributions are Pareto with $\theta = 0.1$.

References

[1] Chen D., Mao T., Pan X., Hu T. (2012): "Extreme value behavior of aggregate dependent risks", *Insurance: Mathematics and Economics*, 50, 99 108.

[2] Cherubini U., Luciano E., Vecchiato W. (2004) *Copula Methods in Finance*, London, John Wiley, (John Wiley Series in Finance).

[3] Cherubini, U., Mulinacci S., Romagnoli S. (2011) "A Copula-based Model of Speculative Price Dynamics in Discrete Time", Journal of Multivariate Analysis, 102, 1047-1063

[4] Cherubini U., Gobbi F., Mulinacci S., Romagnoli S. (2012): "Dynamic Copula Methods in Finance", John Wiley & Sons.

[5] Embrechts P., Kluppelberg C., Mykosch T. (1997): "Modeling External Events for Insurance and finance", Berlin: Springer-Verlag.

[6] Embrechts P., Neslehova J., Wuethrich M.V. (2009): "Additivity properties for Value-at-Risk under Archimedean dependence and heavy-tailedness", *Insurance: Mathematics and Economics*, 44, 164 169.

[7] Jansen D.W., Vries C.G.D. (1991): "On the frequency on large stock returns: Putting booms and busts into perspective", *The Review of Economics and Statistics*, 73, 18-24.

[8] Joe H. (1997): Multivariate Models and Dependence Concepts, Chapman & Hall, London

[9] Jondeau E., Rockinger M. (2003) "Testing for differences in the tails of stock-market returns", *Journal of empirical Finance*, 10, 559-581.

[10] Ibragimov R., Jaffee D., Walden J. (2009): "Non-diversification traps in markets for catastrophic risk", *Review of Financial Studies*, 22, 959993.

[11] Ibragimov R., Jaffee D., Walden J. (2011): "Diversification Disasters", *Journal of Financial Economics*, 99, 333348.

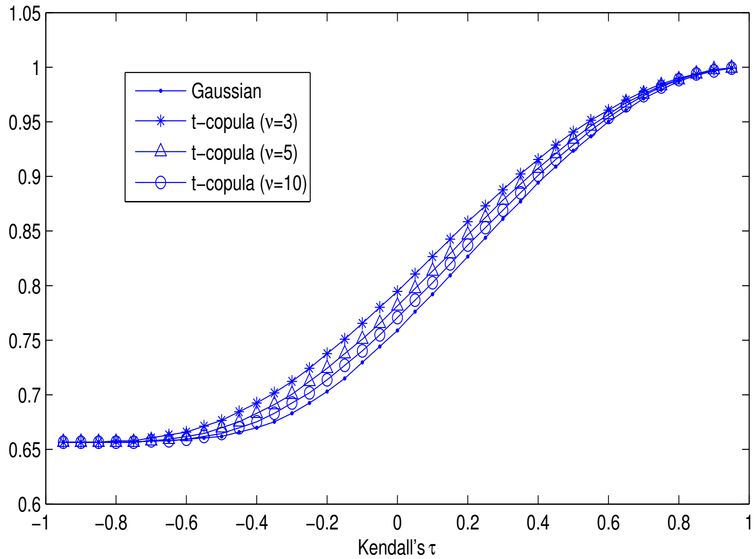

Figure 3: Dynamics of the ratio $\frac{F^{-1}_{X+Y}(q)}{F^{-1}_{X}(q)+F^{-1}_{Y}(q)}$ for any level of the Kendall's τ and $q = 0.99$. Marginal distributions are Pareto with $\theta = 0.1$.

[12] Ibragimov R., Mo J., Prokhorov A. (2015): "Fat tails and copulas: limits of diversification revisited", http://hdl.handle.net/2123/13799

[13] Ibragimov R., Walden J. (2007): "The limits of diversification when losses may be large", *Journal of Banking and Finance*, 31, 25512569.

[14] Kao T.C., Changchen C.C., Lin C.H., Chen H.M. (2009): "A two-stage approach to allowing time-varying volatility and heavytailed distribution for value-at-risk estimation", *Journal of Statistics and Management Systems*, 12(4), 707-720.

[15] McCulloch J.H. (1997): "Measuring Tail Thikness to Estimate the Stable Index Alpha: A Critique", *Journal of Business and Economic Statistics*, 15, 7481.

[16] Nelsen R.(2006): An Introduction to Copulas, Springer

[17] Nolan J.P. (2003): "Modeling Financial Data with Stable Distributions", Handbook of Heavy Tailed Distributions in Finance: Handbooks in Finance.

[18] Osu B.O., Ohakwe J. (2011): "Financial Risk Assessment with Cauchy Distribution under a Simple Transformation of dividing with a Constant", *Theoretical Mathematics and Applications*, vol.1, no.1, 73-89.

[19] Spiegel, M. R. (1992): "Theory and Problems of Probability and Statistics", New York: McGraw-Hill.

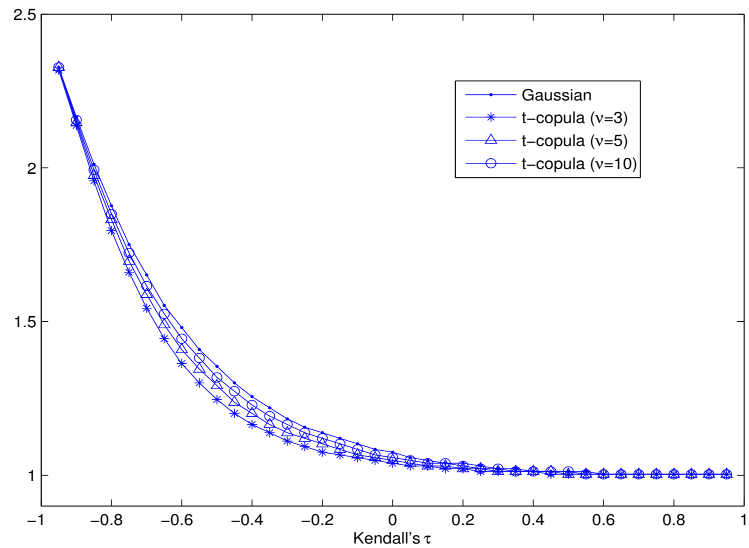

Figure 4: Dynamics of the ratio $r(q)$ for any level of the Kendall's τ and $q = 0.01$. Marginal distributions are Pareto with $\theta = 2$.

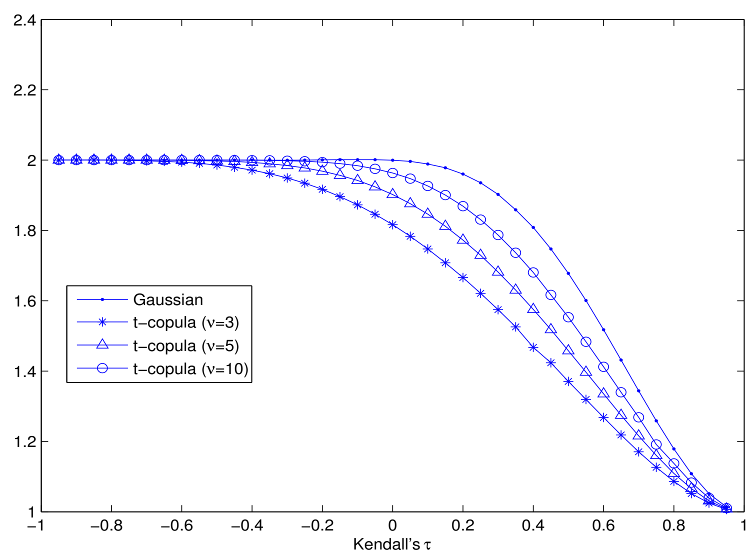

Figure 5: Dynamics of the ratio $r(q)$ for any level of the Kendall's τ and $q = 0.99$. Marginal distributions are Pareto with $\theta = 2$.

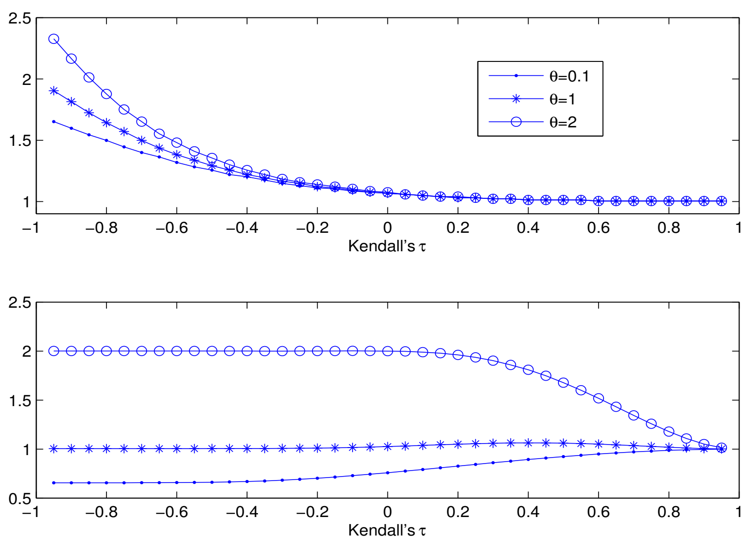

Figure 6: Dynamics of the ratio $r(q)$ for any level of the Kendall's τ and $q = 0.01$ (bottom) and $q = 0.99$ (down). Marginal distributions are Pareto with $\theta = (0.1, 1, 2)$ and the dependence structure is Gaussian.