A free boundary singular transport equation as a formal limit of a discrete dynamical system

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A FREE BOUNDARY SINGULAR TRANSPORT EQUATION AS A FORMAL LIMIT OF A DISCRETE DYNAMICAL SYSTEM

GIOVANNI BELLETTINI, ALESSANDRO BETTI, AND MAURIZIO PAOLINI

ABSTRACT. We study the continuous version of a hyperbolic rescaling of a discrete game, called open mancala. The resulting PDE turns out to be a singular transport equation, with a forcing term taking values in $\{0, 1\}$, and discontinuous in the solution itself. We prove existence and uniqueness of a certain formulation of the problem, based on a nonlocal equation satisfied by the free boundary dividing the region where the forcing is one (active region) and the region where there is no forcing (tail region). Several examples, most notably the Riemann problem, are provided, related to singularity formation. Interestingly, the solution can be obtained by a suitable vertical rearrangement of a multi-function. Furthermore, the PDE admits a Lyapunov functional.

1. INTRODUCTION

The mancala game is a game that, in its open and idealized version, can be described in terms of two moves as follows. Suppose to have an infinite sequence of holes on the half-line $(0, +\infty)$, each hole containing zero or more seeds (always in finite number). Suppose also that the first hole is nonempty, that nonempty holes are consecutive and in finite number. The first half-mancala move consists in taking all seeds in the first hole (the left-most nonempty hole, numbered 1), and sow them in the subsequent holes, one seed per hole. In this way the first hole becomes empty; the second half-mancala move consists in a left-translation of one position of all subsequent holes, so that the new left-most hole becomes nonempty, and the two half-moves can therefore be repeated. Of course, the total mass (i.e., the total number of seeds) is preserved. The open mancala game is modeled as a discrete dynamical system, for which several interesting questions can be posed, such as the classification of periodic configurations, the optimal number of moves necessary to reach a periodic configuration, and so on, see for instance [6] and references therein. The aim of this paper is to analyse a continuous version of this dynamical system, which turns out to be modeled by a new type of transport equation.

After having introduced a natural discrete time, a suitable hyperbolic rescaling of the discrete game and a simple algebraic manipulation (see Section 2 for the details) lead to

$$\frac{u_{ih}^{(k+1)h} - u_{ih}^{kh}}{h} = \frac{u_{ih}^{(k+1)h} - u_{(i-1)h}^{(k+1)h}}{h} + 1_{\left\{2h \leq \cdot \leq u_h^{kh} + h\right\}}(ih), \qquad (1.1)$$

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where ih, $i \geq 1$ integer, denotes the discrete i-th space position, kh, $k \geq 0$ integer, denotes the discrete k-th time, h is the space grid (equal to the time grid), u_{jh}^{lh} stands for the number of seeds at (jh, lh), and 1_A is the characteristic function of the set $A \subset [0, +\infty)$. The discrete system (1.1) is coupled with

$$u_{ih}^0 = u_0(ih), \qquad i \geq 1, \tag{1.2}$$

u_0 being the given initial configuration.

It is then natural to consider the following hyperbolic pde, considered as a formal limit of (1.1), (1.2) as $h \to 0^+$:

$$\begin{cases} u_t(t,x) = u_x(t,x) + 1_{(0,u(t,0))}(x), \\ u(0) = u_0. \end{cases} \tag{1.3}$$

Notice carefully the $\{0,1\}$-valued discontinuous forcing term on the right-hand side which, given any $t \geq 0$, depends on the space right trace $u(t,0)$ of $u(t,\cdot)$ at $x = 0$; this space dependent forcing depends on u in a nonlinear and nonlocal way, and results the PDE in (1.3), to our best knowledge, into a new type of partial differential equation. Observe also that, *formally*, the total mass is conserved:

$$\frac{d}{dt} \int_0^{+\infty} u(t,x) \; dx = \int_0^{+\infty} u_t(t,x) \; dx$$
$$= \int_0^{+\infty} \left(u_x(t,x) + 1_{(0,u(t,0))}(x) \right) \; dx = -u(t,0) + \int_0^{+\infty} 1_{(0,u(t,0))}(x) \; dx = 0.$$

The physical interpretation of (1.3) can be given in terms of mass transportation, as follows. Imagine to have an horizontal conveyor belt transporting, at uniform speed, a certain amount of sand toward the left; once a slice of sand of height κ reaches the (left) boundary $x = 0$ and falls down, it is uniformly redistributed horizontally for a length κ on the conveyor, and the process continues.

From the mathematical point of view, we are facing a homogeneous linear transport equation in a "tail" region

$$T(u) := \{ (t,x) : t \geq 0, \; x > u(t,0) \},$$

where the forcing term is suppressed, and an inhomogeneous transport equation in an "active" region

$$A(u) := \{ (t,x) : t \geq 0, \; 0 < x < u(t,0) \},$$

where the forcing term equals one. Tail and active regions are separated by an interface, a curve in time-space in our setting, which is the (generalized) graph of $u(\cdot, 0)$ over $\mathbb{R}^+$, and that can be considered as a sort of free boundary.

We anticipate here that, interestingly, some similarities with the typical phenomena of entropy solutions of nonlinear first order conservation laws (say the Burger's equation, just to fix ideas) appear also for solutions of (1.3), such as possible creation of decreasing jumps, and the validity of a unilateral Lipschitz condition [3]. Typically, for Burgers' equation, increasing jumps disappear instantaneously, while in the present model they are progressively eroded while travelling toward the origin[1] at unit speed (see Figs. 7 and 8). The most surprising phenomenon happening in the free boundary of (1.3) is what we have described through an *affine transformation* (equation (4.16)) followed by a *vertical rearrangement* (equation (4.17)), a

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[1]After reaching the origin, they possibly disappear.

procedure that reminds the "equal area rule" of conservation laws (see [7, pag. 42], and [2]) and that will be carefully analysed in Sections 4.1 and 8. Roughly, it turns out that the free boundary is the (generalized) graph G_ℓ of a $\mathrm{BV}_{\mathrm{loc}}$ function ℓ which has to satisfy a nonlocal equation (see (1.6)) that can be described in two steps: first the affine transformation forces the creation of a multi-valued graph, in correspondence of certain previous parts of the graph of ℓ having slope less than -1 (called critical slopes). In a second step, (1.6) reduces the multigraph to the graph of a single-valued function by a geometric principle based on a vertical rearrangement (a Steiner symmetrization). In turn, from such a genuine function, the solution u can be directly recovered (Remark 5.1). The remarkable operation of rearrangement is always necessary at a critical time (Definition 4.2), i.e., a time corresponding to the presence of a critical slope, and typically such times are unavoidable.

In order to define and analyse a solution to (1.3), it is convenient to change variables in the obvious way,

$$\Phi\colon (t, x) \in \overline{\mathrm{I}}_{\mathrm{quad}} \to (\tau, \xi) \in \overline{\mathrm{II}}_{\mathrm{oct}}, \qquad \tau := t, \quad \xi := t + x, \tag{1.4}$$

where $\overline{\mathrm{I}}_{\mathrm{quad}} := [0, +\infty) \times [0, +\infty)$ is the closure of the open first quadrant $\mathrm{I}_{\mathrm{quad}}$, $\overline{\mathrm{II}}_{\mathrm{oct}} := \{(\tau, \xi) \in [0, +\infty) \times \mathbb{R}, \xi \geq \tau\}$ is the closure of the open second octant $\mathrm{II}_{\mathrm{oct}}$, and formally consider (1.3) in the new coordinates, which after integration reads as

$$\widehat{u}(\tau, \xi) = u_0(\xi) + \int_0^\tau 1_{(\xi, +\infty)} \left(\widehat{u}(s, s) + s\right) ds, \qquad (\tau, \xi) \in \mathrm{II}_{\mathrm{oct}}, \tag{1.5}$$

where $\widehat{u}(s, s)$ is the right limit of $\widehat{u}(s, \cdot)$ at any $s > 0$.

Switching our perspective from (1.3) to (1.5), we are interested in the global existence, uniqueness, and qualitative properties of a solution $\widehat{u}$: the leading idea is to try to identify a plane curve[2] potentially representing the free boundary, and then reconstruct from it the values of $\widehat{u}$ in $\mathrm{II}_{\mathrm{oct}}$. It turns out that this strategy is successfull (see Sections 3, 4 for detailed statements), the free boundary being identified by the graph of the function ℓ in (1.6) below[3]; proving existence of a solution to (1.6) is the central part of the proof.

Denote by $|\cdot|$ the one-dimensional Lebesgue measure in $\mathbb{R}$.

Theorem 1.1. *Let $u_0 \in \mathrm{BV}_{\mathrm{loc}}([0, +\infty))$ be nonnegative. The nonlocal equation*

$$\ell(t) = u_0(t) + |\{ s \in [0, t] : \ell(t - s) > s \}| \qquad \forall t \in [0, +\infty), \tag{1.6}$$

in the nonnegative unknown $\ell \in \mathrm{BV}_{\mathrm{loc}}([0, +\infty))$, has a solution, which is unique provided u_0 is positive in a right neighbourhood of the origin. In addition, ℓ cannot have increasing jumps, unless u_0 has, while it may have decreasing jumps. Moreover, the function

$$v(\tau, \xi) := u_0(\xi) + |\{ s \in [0, \tau] : \ell(s) + s > \xi \}| \qquad \forall (\tau, \xi) \in \mathrm{II}_{\mathrm{oct}} \tag{1.7}$$

is the global unique solution to (1.5) having the generalized graph G_ℓ of ℓ as free boundary, it satisfies the mass conservation, and $v(\tau, \cdot)$ satisfies a unilateral Lipschitz condition, provided u_0 does.

Theorem 1.1 essentially says that the construction of a solution of (1.5) is equivalent to solve the nonlocal equation (1.6) describing the free boundary: this is not immediate, and requires some recursive argument. As already said, equation (1.6)

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[2]Heuristically, the graph of u as a function of time, on the line $\{x = 0\}$.

[3]How to guess the equation for ℓ is obtained in formula (3.11).

has a geometric meaning (see Remark 4.1 and Section) and has also various equivalent formulations (see the end of Section 4): one of them, see (4.21), is based on the generalized inverse of the vertical rearrangement of the function $L_\ell(t) := \ell(t) + t$, and may be used for the explicit construction of ℓ in specific examples.

The main difficulties in the proof of Theorem 1.1 are due to the fact that, at a point $(\overline{\tau}, \overline{\xi})$, a characteristic line $\{\xi = \overline{\xi}\}$ passing through $(\overline{\tau}, \overline{\xi})$, may intersect several (possibly infinitely many) times both the subgraph and the epigraph of ℓ, before ending on the vertical axis $\{\tau = 0\}$ where keeping the value of u_0; this happens in correspondence of critical slopes of ℓ, corresponding (roughly) to regions where L_ℓ becomes strictly decreasing and L_ℓ loses monotonicity. As already said, in correspondence of these critical values function ℓ and the solution v exhibit interesting features.

It is worthwhile to remark that equation (1.3) admits a Lyapunov functional, which is suggested by looking at the discrete Lyapunov functional known for the open discrete mancala [6], see (2.4) and Section 6.

It is immediate to devise a class of stationary solutions for the flow, i.e., $u(x) = (\sqrt{2m} - x)1_{[0,\sqrt{2m}]}$ (Example 7.1), $m \geq 0$ being the initial mass. In an interesting example (the Riemann problem corresponding to the initial datum $u_0 = 1_{[0,1)}$, discussed in Section 8) it turns out that the Lyapunov functional[4] is constant unless in proximity of a critical time, after which it is strictly decreasing for a while. The Riemann problem is a nice example that illustrates a rather complex solution starting from a very simple initial condition, and that shades light on the main phenomena behind equation (1.3), in particular the vertical rearrangement. As already explained, the main point is to construct the function ℓ, and this can be done recursively, taking advantage of the fact that ℓ must be polygonal[5] , and constructing the sequences of slopes connecting pairs of local minima and local maxima. The explicit construction of ℓ, though not in closed form, is given by recurrence, and is rather intricated, due to the unavoidable presence of (a countable number of) critical slopes, and therefore of a multivalued[6] graph; the most involved case is when two vertical rearragements need to be performed in adjacent segments (see the first picture in Fig. 5). It is proven in Section 8 that the corresponding stationary solution is reached in an infinite time.

The content of the paper is the following. In Section 2, after briefly recalling the moves in the open mancala game, we describe the rescaling leading to (1.1). In Section 3 we consider the (t, x) formulation of the PDE, and its transformed version in (τ, ξ) variables, leading to the notion of integral solution (Definition 3.3). In Section 4 we introduce the equation solved by ℓ, the graph of which will be the free boundary. Global existence and uniqueness of ℓ, as well as some qualitative properties, are proven in Theorem 4.1. The geometric meaning of equation (1.6) in terms of the affine transformation and the vertical rearrangement is explained in Remark 4.1 and Section 4.1. The last part of Section 4 is concerned with a more general class of initial data, in particular those that may vanish at zero. In Section 5 we prove the existence and uniqueness of an integral solution to (1.3), in

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[4]A Lipschitz function in time.

[5]One can see that the flow starting from a polygonal initial condition (i.e., a piecewise affine possibly discontinuous function, see Definition 3.1) generates a piecewise affine solution, a useful fact for the explicit computation of solutions.

[6]In this case there cannot be more than three values.

particular the solvability of (1.7). In Section 6 we study the Lyapunov functional associated to (1.3). Section 7 contains some initial examples. Section 8 is essentially devoted to the explicit construction of ℓ and the solution for the Riemann problem, which are rather involved. The construction of ℓ is given in Theorem 8.1, on the basis of the algorithm described in (8.3), (8.6), and (8.7). How to directly recover, a computable way, the integral solution from the knowledge of ℓ is explained in Remark 8.3, see also Fig. 6. A movie of the solution is illustrated in Figs. 7, 8. A few final examples are described in Section 8.2.

We conclude this introduction by remarking that the rigorous asymptotic analysis of the discrete model as $h \to 0^+$ is out of the scope of the present paper, and will be investigated elsewhere.

2. Motivation: the discrete dynamical system and its rescaling

Let $\mathbb{N}$ be the nonnegative integers, and $\mathbb{N}^*$ be the positive integers. Following [6], a configuration λ is a map $\lambda \colon \mathbb{N}^* \to \mathbb{N}$, and a mancala configuration is a configuration λ such that $\operatorname{supp}(\lambda) := \{\, i \in \mathbb{N}^* : \lambda_i > 0 \,\}$ is connected, $\operatorname{supp}(\lambda) = \{1, 2, \ldots, \operatorname{len}(\lambda)\}$, with $\operatorname{len}(\lambda) \in \mathbb{N}^*$ called the length of the configuration. The *mass* of a mancala configuration λ is defined as $|\lambda| := \sum_{i=1}^{+\infty} \lambda_i = \sum_{i=1}^{\operatorname{len}(\lambda)} \lambda_i$. We denote with Λ the set of all mancala configurations, and with Λ_m the set of all mancala configurations with mass $m \geq 0$.

The *open mancala game* is a discrete dynamical system associated with a function $\mathcal{M} \colon \Lambda \to \Lambda$; the action of $\mathcal{M}$ on a mancala configuration λ is defined by

$$\mathcal{M}(\lambda)_j := \begin{cases} \lambda_{j+1} + 1 & \text{if } 1 \leq j \leq \lambda_1, \\ \lambda_{j+1} & \text{if } j > \lambda_1. \end{cases} \tag{2.1}$$

Conventionally we also assume $\mathcal{M}(000\cdots) = 000\cdots$.

For clarity, it is convenient to split $\mathcal{M}$ into two elementary "half-moves": the sowing move $\mathcal{S}$, which consists in taking all seeds in the leftmost hole ($j = 1$) and redistribute them between the next holes by putting one seed for each hole until the seeds are over,

$$\mathcal{S}(\lambda)_j := \begin{cases} 0 & \text{if } j = 1, \\ \lambda_j + 1 & \text{if } 2 \leq j \leq \lambda_1 + 1, \\ \lambda_j & \text{if } j > \lambda_1 + 1, \end{cases}$$

and the left-shift move[7] $\mathcal{L}$, which shifts to the left all holes by one,

$$\mathcal{L}(\lambda)_j := \lambda_{j+1}, \qquad j \geq 1.$$

The composite map $\mathcal{L} \circ \mathcal{S}$ maps mancala configurations into mancala configurations and indeed we have $\mathcal{L} \circ \mathcal{S}(\lambda) = \mathcal{M}(\lambda) \colon \Lambda_m \to \Lambda_m$.

It is natural to introduce an integer parameter $\kappa \geq 0$, standing for the discrete time; given a configuration $\lambda^\kappa \in \Lambda_m$, the sowing half-move is

$$\begin{cases} \lambda_1^{\kappa+\frac{1}{2}} := 0 \\ \lambda_j^{\kappa+\frac{1}{2}} := \lambda_j^\kappa + 1 & \text{if } 2 \leq j \leq \lambda_1^\kappa + 1, \\ \lambda_j^{\kappa+\frac{1}{2}} := \lambda_j^\kappa & \text{if } j > \lambda_1^\kappa + 1, \end{cases} \tag{2.2}$$

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[7]This is done for convenience in order to obtain a reference system following the configuration, and corresponds to a right translation of the observer.

and the left-shift half-move is

$$\lambda_j^{\kappa+1} := \lambda_{j+1}^{\kappa+\frac{1}{2}}.$$

The composition of the two half-moves gives therefore the move

$$\begin{cases} \lambda_{j-1}^{\kappa+1} = \lambda_j^\kappa + 1 & \text{if } 2 \le j \le \lambda_1^\kappa + 1, \\ \lambda_{j-1}^{\kappa+1} = \lambda_j^\kappa & \text{if } j > \lambda_1^\kappa + 1, \end{cases}$$

i.e.,

$$\lambda_{j-1}^{\kappa+1} = \lambda_j^\kappa + 1_{I^\kappa(\lambda)}(j), \qquad \text{where } I^\kappa(\lambda) := \{\, j \in \mathbb{N} : 2 \le j \le \lambda_1^\kappa + 1 \,\}, \qquad (2.3)$$

with $1_{I^\kappa(\lambda)}(j) = 1$ if $j \in I^\kappa(\lambda)$ and 0 else.

Remark 2.1 (**Conservation of mass**). It is clear that mass is conserved:

$$m = \sum_{j=0}^{+\infty} \lambda_j^\kappa \qquad \forall \kappa \ge 0.$$

Remark 2.2 (**A discrete Lyapunov functional**). In the discrete setting a Lyapunov functional, mimicking a gravitational potential [6], can be defined as

$$\mathfrak{L}(\lambda) = \sum_{\substack{(i,j) \in \mathbb{N}^* \times \mathbb{N}^* \\ i \le j < i + \lambda_i}} j. \qquad (2.4)$$

Now, we suitably rescale the discrete dynamical system (2.3).

2.1. **Rescaling.** From (2.3) it immediately follows

$$\lambda_j^{\kappa+1} - \lambda_j^\kappa = \lambda_j^{\kappa+1} - \lambda_{j-1}^{k+1} + 1_{I^\kappa(\lambda)}(j). \qquad (2.5)$$

Now we rescale j, κ and λ. If $0 < h \ll 1$, we imagine a grid G_h in time-space $[0, +\infty) \times [0, +\infty)$ of the form $\{\, (kh, ih) : k, h \in \mathbb{N}^* \,\}$. Next we substitute[8] in (2.5) kh in place of κ, ih in place of j, and we write $\lambda = u/h$. The constraint $j \in I^\kappa(\lambda)$ is transformed into

$$ih \in \{\, nh \in \mathbb{N}h : 2h \le nh \le (\lambda_1^\kappa + 1)h \,\} = \{\, nh \in \mathbb{N}h : 2h \le nh \le u_h^{kh} + h \,\}.$$

We get from (2.5)

$$\frac{u_{ih}^{(k+1)h} - u_{ih}^{kh}}{h} = \frac{u_{ih}^{(k+1)h} - u_{(i-1)h}^{(k+1)h}}{h} + 1_{\left\{ 2h \le \cdot \le u_h^{kh} + h \right\}}(ih) \qquad (2.6)$$

for $i \ge 1$, $k \ge 0$. We couple this rescaled dynamical system with the initial condition

$$u_{ih}^0 = u_0(ih), \qquad i \ge 1. \qquad (2.7)$$

Our aim is to study a continuous version of (2.6), (2.7). If we interpret the quantities u_{ih}^{kh} as the values that a map $(t, x) \in [0, +\infty) \times [0, +\infty) \mapsto u(t, x)$ takes on the points of the grid G_h (i.e. $u_{ih}^{kh} = u(kh, ih)$) then the equation (2.6) suggests that u formally satisfies the PDE in (1.3).

Remark 2.3. We have

$$m = \frac{1}{h} \sum_{i=1}^{+\infty} u_{ih}^{kh},$$

and the series is a finite sum.

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[8]Notice the hyperbolic rescaling: time and space are rescaled the same way.

3. The continuous problem: a singular transport equation

1_A stands for the characteristic function of the set $A \subset \mathbb{R}^k$, $k = 1, 2$, i.e., $1_A(x) = 0$ if $x \in A$ and $1_A(x) = 0$ if $x \in \mathbb{R}^k \setminus A$. $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}$. All functions we consider are nonnegative and Lebesgue measurable.

We denote by $\mathrm{BV}_{\mathrm{loc}}([0, +\infty))$ the class of all functions v of finite pointwise variation in $[0, a]$, for any $a > 0$. We recall [1] that v is bounded in $[0, a]$, there exists finite $\lim_{x \to 0^+} v(x) =: v(0)$, and v admits finite right and left limits at any $x \in (0, +\infty)$. We also let [4]

$$\mathrm{subgr}_+(v) := \{ (t, x) \in [0, +\infty) \times [0, +\infty) : v(t) > x \} \tag{3.1}$$

be the subgraph of v in $[0, +\infty) \times [0, +\infty)$; its reduced boundary in $(0, +\infty) \times (0, +\infty)$ is given by the generalized graph of v, i.e., the graph of v with the addition of vertical segments joining the left and the right limits at the jump points.

Definition 3.1 (Polygonal function). *We say that $v \in \mathrm{BV}_{\mathrm{loc}}([0, +\infty))$ is polygonal if its generalized graph consists, in any compact set $K \subset [0, +\infty)$, of a finite number of segments (vertical segments are allowed[9]).*

Recall that if $v \colon [0, +\infty) \to [0, +\infty)$, then the function $\lambda \in [0, +\infty) \mapsto |\{ x \in [0, +\infty) : v(x) > \lambda \}|$ is nonincreasing and right-continuous.

We start the study of the PDE in (1.3); from now on, we always assume:

$$u_0 \in \mathrm{BV}_{\mathrm{loc}}([0, +\infty)), \qquad u_0 \text{ right continuous and nonnegative.} \tag{3.2}$$

3.1. (τ, ξ)-formulation. Duly motivated by (2.6) we consider problem (1.3), for which we need a rigorous notion of solution. Let be given a nonnegative function v defined everywhere on $\mathrm{I}_{\mathrm{quad}}$, and suppose that for any $t > 0$ there exists $\lim_{x \to 0^+} v(t, x) =: v(t, 0) \in [0, +\infty)$.

Definition 3.2 (Active and tail regions; free boundary). *We call*

$$A(v) := \{ (t, x) : t > 0, \, 0 < x < v(t, 0) \}, \qquad T(v) := \{ (t, x) : t > 0, \, x > v(t, 0) \},$$

the active region[10] and the tail region (of v), respectively. We also define the free boundary $\Gamma(v)$ as

$$\Gamma(v) := \{ (t, x) \in \mathrm{I}_{\mathrm{quad}} : x = v(t, 0) \}. \tag{3.3}$$

The problem that we want to study reads *formally* as

$$\begin{cases} u_t(t, x) = u_x(t, x) + 1_{(0, u(t, 0))}(x) & \text{if } (t, x) \in (0, +\infty) \times (0, +\infty), \\ u(0, x) = u_0(x) & \text{if } x \in [0, +\infty). \end{cases} \tag{3.4}$$

Observe that

$$1_{(0, u(t, 0))}(x) = 1_{(x, +\infty)}(u(t, 0)) \qquad \forall x > 0$$

and that the PDE reads as

$$u_t(t, x) = \begin{cases} u_x(t, x) & \text{if } x > u(t, 0), \text{ i.e., } (t, x) \in T(u), \\ u_x(t, x) + 1 & \text{if } x < u(t, 0), \text{ i.e., } (t, x) \in A(u). \end{cases}$$

A natural sense in which we can interpret (3.4) is obtained using transformation Φ in (1.4). Given a function $v \colon \mathrm{I}_{\mathrm{quad}} \to [0, +\infty)$, define

$$\widehat{v}(\tau, \xi) := v(\Phi^{-1}(\tau, \xi)) = v(t, x) \qquad \forall(\tau, \xi) \in \mathrm{II}_{\mathrm{oct}}. \tag{3.5}$$

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[9]v can be discontinuous, with a finite number of jump points in K.

[10]The active region coincides with $\mathrm{subgraph}_+(v(\cdot, 0))$.

Then formally the PDE in (3.4) transforms into

$$\begin{cases} \widehat{u}_\tau(\tau, \xi) = 1_{(0, \widehat{u}(\tau, \tau))}(\xi - \tau) = 1_{(\tau, \widehat{u}(\tau, \tau) + \tau)}(\xi) & \text{if } (\tau, \xi) \in \text{II}_\text{oct}, \\ \widehat{u}(0, \xi) = u_0(\xi) & \text{if } \xi \in [0, +\infty). \end{cases} \quad (3.6)$$

Active (resp. tail) region $A(u)$ (resp. $T(u)$) transforms into

$$A(\widehat{u}) = \{\, (\tau, \xi) : \tau > 0, \ \tau < \xi < \widehat{u}(\tau, \tau) + \tau \,\}, \quad T(\widehat{u}) = \{\, (\tau, \xi) : \tau > 0, \ \xi > \widehat{u}(\tau, \tau) + \tau \,\},$$

and the set $\Gamma(u)$ in (3.3) transforms into

$$\Gamma(\widehat{u}) = \{\, (\tau, \xi) : \xi > \tau, \ \xi = \widehat{u}(\tau, \tau) + \tau \,\}. \quad (3.7)$$

We are now in a position to define what we mean by a solution of (3.4).

Definition 3.3 (Integral solution). *We say that a locally bounded function $\widehat{u} \colon \text{II}_\text{oct} \to [0, +\infty)$ defined everywhere is an integral solution to (1.3) if the following conditions hold:*

   (i) *for any $\tau > 0$*

$$\exists \lim_{\xi \to \tau^+} \widehat{u}(\tau, \xi) =: \widehat{u}(\tau, \tau) \in [0, +\infty);$$

   (ii) *for any $(\tau, \xi) \in \text{II}_\text{oct}$,*

$$\widehat{u}(\tau, \xi) = u_0(\xi) + \int_0^\tau 1_{(s, \widehat{u}(s, s) + s)}(\xi) \, ds. \quad (3.8)$$

Some comments are in order.

   (a) if $\widehat{u}$ is an integral solution then, for any $\xi > 0$, the function $\tau \in (0, \xi) \mapsto \widehat{u}(\tau, \xi)$ is nondecreasing and one-Lipschitz.
   (b) For any $(\tau, \xi) \in \text{II}_\text{oct}$ we have[11]

$$\int_0^\tau 1_{(s, \widehat{u}(s, s) + s)}(\xi) \, ds = \int_0^1 |\{\, s \in [0, \tau] : 1_{(s, \widehat{u}(s, s) + s)}(\xi) > \lambda \,\}| \, d\lambda$$

$$= \int_0^1 |\{\, s \in [0, \tau] : s < \xi < \widehat{u}(s, s) + s \,\}| \, d\lambda = |\{\, s \in [0, \tau] : s < \xi < \widehat{u}(s, s) + s \,\}|$$

$$= |\{\, s \in [0, \tau] : \widehat{u}(s, s) + s > \xi \,\}|,$$

$$(3.9)$$

   so that (3.8) is equivalent to

$$\widehat{u}(\tau, \xi) = u_0(\xi) + |\{\, s \in [0, \tau] : \widehat{u}(s, s) + s > \xi \,\}|. \quad (3.10)$$

   In particular, since u_0 is right-continuous, passing to the limit as $\xi \to \tau^+$ and using (i),

$$\widehat{u}(\tau, \tau) = u_0(\tau) + |\{\, s \in [0, \tau] : \widehat{u}(s, s) + s > \tau \,\}| \quad (3.11)$$

   for all $\tau \in (0, +\infty)$.

We conclude this section with a notation: given $\bar{t} > 0$, we let $\mathfrak{h}_{\bar{t}} = \{\, (t, s) : s \geq 0, \ s = h_{\bar{t}}(t) \,\}$ be the half-line pointing up-left at 45° and passing through $(\bar{t}, 0)$, defined by $h_{\bar{t}}(t) = \bar{t} - t$.

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[11]In the first equality of (3.9) we use that, if $f \colon [0, +\infty) \to [0, +\infty)$ is measurable and $I \subseteq [0, +\infty)$, then $\int_I f \, ds = \int_{(0, +\infty)} |I \cap \{f > \lambda\}| \, d\lambda$.

4. Solving the equation for the free boundary

In this section we solve an auxiliary problem (see Theorem 4.1) which, in view of the results in Section 5, is essentially equivalent to find an integral solution of (1.3): the idea is to look for an expression of what, a posteriori, will be the curve $\Gamma(\widehat{u})$ and then to reconstruct $\widehat{u}$ itself (and hence u) by the method of characteristics, distinguishing the active and the tail regions; here a difficulty arises since characteristic lines may pass from the active region to the tail region several times. Recall our assumption (3.2) on u_0, and keep in mind (3.11).

Definition 4.1 (**The function ℓ**). *We say that the graph of a function $\ell = \ell[u_0] \colon [0,+\infty) \to [0,+\infty)$ represents the free boundary of problem* (1.3) *if*

$$\ell(t) = u_0(t) + |\{\, s \in [0,t] : \ell(t-s) > s \,\}| \qquad \forall t \in [0,+\infty). \tag{4.1}$$

In what follows, we shall frequently use the obvious equality

$$|\{\, s \in [0,t] : \ell(t-s) > s \,\}| = |\{\, s \in [0,t] : \ell(s) + s > t \,\}|. \tag{4.2}$$

Notice that, since u_0 is right-continuous, also ℓ is right-continuous; moreover

$$\ell(0) = u_0(0).$$

Remark 4.1 (**Geometric meaning**). Formula (4.1) has a geometric meaning: recalling the notation in the end of Section 3, consider the half-line $\mathfrak{h}_t = \{(t-s,s) : s > 0\}$. Next, take the intersection I_t of $\mathfrak{h}_t$ with $\mathrm{subgraph}_+(\ell) = \{(t,s) \in (0,+\infty) \times (0,+\infty) : \ell(t) > s\}$, so that

$$I_t = \{\, (t,s) : s \in [0,t],\ \ell(t-s) > s \,\}.$$

Denote by $\pi_2 \colon [0,+\infty) \times [0,+\infty) \to \{0\} \times [0,+\infty)$ the orthogonal projection on the vertical s-axis. Then

$$\ell(t) = u_0(t) + |\pi_2(I_t)|, \qquad t \geq 0. \tag{4.3}$$

In particular, if u_0 is polygonal, then ℓ is polygonal[12].

Proving that (4.1) has a solution is nontrivial; in order to do that we start to show that *if* (4.1) *has a solution* and u_0 has a suitable behaviour close to the origin, then ℓ cannot go to zero at any point of $[0,+\infty)$.

Proposition 4.1 (**Lower bound**). *Suppose that ℓ solves* (4.1) *and furthermore*

$$\exists \alpha > 0 : u_0(t) \geq \alpha \qquad \forall t \in [0,\alpha). \tag{4.4}$$

Then

$$\ell(t) \geq \alpha \qquad \forall t \in [0,+\infty). \tag{4.5}$$

Assumption (4.4) says that we can pick a square $Q = (0,\alpha) \times (0,\alpha)$ contained in $\mathrm{subgraph}_+(\ell)$, and (4.5) says that we can slide Q horizontally remaining inside $\mathrm{subgraph}_+(\ell)$, and this is crucial in what follows; dropping this assumption is discussed in Theorem 4.2 and in Section 4.4.

Proof. Suppose by contradiction that (4.5) is false, and define $t_0 := \inf\{\, t \in [0,+\infty) : \ell(t) < \alpha \,\}$. From (4.4) and (4.1) it follows

$$t_0 \geq \alpha. \tag{4.6}$$

---

[12]Recall that jumps are not excluded.

Fix now $\varepsilon \in (0, \alpha)$, that will be selected later (see (4.9)). Pick $t \in (t_0, t_0 + \varepsilon)$, and write $t = t_0 + \eta$, with $\eta \in (0, \varepsilon)$. In particular, $\eta < \alpha$. Set

$$E := \{\, s \in [0, \eta) : \ell(t - s) > s \,\}, \qquad F := \{\, s \in [\eta, \alpha) : \ell(t - s) > s \,\}.$$

From (4.1) and recalling that u_0 is nonnegative it follows

$$\begin{aligned} \ell(t) &\geq |\{\, s \in [0, t] : \ell(t - s) > s \,\}| \\ &= |E| + |\{\, s \in [\eta, t] : \ell(t - s) > s \,\}| \geq |E| + |F|, \end{aligned} \tag{4.7}$$

where in the last inequality we have used (4.6), so that $t > t_0 \geq \alpha$.

Claim 1: We have $|F| = \alpha - \eta$.

Indeed, if $s \in F$, in particular $s \geq \eta$, and so $t - s = t_0 + \eta - s \leq t_0$. Hence, recalling the definition of t_0, we have $\ell(t - s) \geq \alpha$. But for $s \in F$ we have $s < \alpha$, and so $\ell(t - s) > s$, which implies $|F| = \alpha - \eta$.

From (4.7), claim 1 and $\eta \in (0, \varepsilon)$ it follows

$$\ell(t) \geq \alpha - \varepsilon \qquad \forall t \in (t_0, t_0 + \varepsilon). \tag{4.8}$$

Claim 2: If $\varepsilon < \alpha/2$ then $|E| = \eta$.

We start to observe that for any $s \in E$ we have $t_0 \leq t - s < t_0 + \varepsilon$. Indeed, $t - s = t_0 + \eta - s < t_0 + \varepsilon$ since $\eta - s < \varepsilon - s \leq \varepsilon$; on the other hand, $t - s = t_0 + \eta - s > t_0$ since $\eta - s > 0$, from the definition of E. Therefore we are allowed to replace t with $t - s$ in (4.8), and we get

$$\ell(t - s) \geq \alpha - \varepsilon \qquad \forall s \in E.$$

Due to our choice of ε, we have $\alpha - \varepsilon > \alpha/2$, and $\alpha/2 > s$, since $\alpha/2 > \varepsilon > \eta$. Hence all $s \in [0, \eta)$ are points of E, and the claim follows.

From (4.7) and claims 1 and 2 we deduce, provided

$$\varepsilon \in (0, \alpha/2), \tag{4.9}$$

that

$$\ell(t) \geq \eta + \alpha - \eta = \alpha \qquad \forall t \in (t_0, t_0 + \varepsilon),$$

which contradicts the definition of t_0. $\qquad\square$

Lemma 4.1. *Under assumption* (4.4), ℓ *is a solution of* (4.1) *if and only if*

$$\ell(t) = \begin{cases} u_0(t) + t & \forall t \in [0, \alpha), \\ u_0(t) + \alpha + |\{\, s \in [\alpha, t] : \ell(t - s) > s \,\}| & \forall t \in [\alpha, +\infty). \end{cases} \tag{4.10}$$

Proof. Suppose that ℓ satisfies (4.1). According to Proposition 4.1, ℓ must satisfy (4.5). Hence, for any $t \in [\alpha, +\infty)$,

$$\begin{aligned} \ell(t) &= u_0(t) + |\{\, s \in [0, \alpha) : \ell(t - s) > s \,\}| + |\{\, s \in [\alpha, t] : \ell(t - s) > s \,\}| \\ &= u_0(t) + \alpha + |\{\, s \in [\alpha, t] : \ell(t - s) > s \,\}| \end{aligned}$$

since, from (4.1), if $s \in [0, \alpha)$, then $\ell(t - s) \geq \alpha > s$. The case $t \in [0, \alpha)$ is a direct consequence of (4.5) and (4.1).

The proof of the converse implication is similar. $\qquad\square$

Theorem 4.1 (Global existence and uniqueness of ℓ). *Let u_0 be as in (3.2), and suppose that (4.4) holds. Then (4.1) admits a unique solution $\ell \in \mathrm{BV}_{\mathrm{loc}}([0,+\infty))$. Furthermore ℓ satisfies (4.5), $\ell(t) \leq \|u_0\|_\infty + t$ for all $t \in [0,+\infty)$, and*

$$\ell(t+\tau) - \ell(t) \leq u_0(t+\tau) - u_0(t) + \tau, \qquad t > 0, \ \tau \geq 0. \qquad (4.11)$$

In particular, suppose $\mathrm{spt}(u_0) \subset [0,a]$ and $t > a$; then ℓ cannot have increasing jumps, as well as increasing Cantor parts, in $(a,+\infty)$.

Proof. We construct ℓ as follows: first we set

$$\ell(t) := u_0(t) + t \qquad \forall t \in [0,\alpha]. \qquad (4.12)$$

Next, keeping in mind (4.10), we define

$$\ell(t) := u_0(t) + \alpha + |\{\, s \in [\alpha,t] : \ell(t-s) > s \,\}| \qquad \forall t \in [\alpha, 2\alpha]. \qquad (4.13)$$

Note that ℓ is well-defined in the interval $[\alpha, 2\alpha]$, since if $t \in [\alpha, 2\alpha]$ and $s \in [\alpha, t]$, then $t - s \in [0,\alpha]$, and we can use (4.12).

We now repeat the argument inductively for $t \in [k\alpha, (k+1)\alpha]$ for any integer $k \geq 2$; for instance, if $k = 2$, for any $t \in [2\alpha, 3\alpha]$ we set

$$\begin{aligned}
\ell(t) :=& u_0(t) + \alpha + |\{\, s \in [\alpha,t] : \ell(t-s) > s \,\}| \\
=& u_0(t) + \alpha + |\{\, s \in [\alpha, 2\alpha] : \ell(t-s) > s \,\}| \\
& + |\{\, s \in (2\alpha, t] : \ell(t-s) > s \,\}|,
\end{aligned} \qquad (4.14)$$

where we notice that if $s \in [\alpha, 2\alpha]$ then $t - s \in [0, 2\alpha]$, so that

$$\begin{aligned}
|\{\, s \in [\alpha, 2\alpha] : \ell(t-s) > s \,\}| =& |\{\, s \in [\alpha, 2\alpha] : t-s \in [0,\alpha], \ell(t-s) > s \,\}| \\
& + |\{\, s \in [\alpha, 2\alpha] : t-s \in (\alpha, 2\alpha], \ell(t-s) > s \,\}|,
\end{aligned}$$

and if $s \in [2\alpha, t]$ then $t - s \in [0,\alpha]$; thus (4.14) is well-defined by (4.12) and (4.13)[13].

In this way we construct a globally defined function $\ell : [0,+\infty) \to (0,+\infty)$ satisfying (4.10) and (4.5). Hence the existence of a solution to (4.1) follows from Lemma 4.1. To prove that ℓ is unique, it is sufficient to show that it is unique in $[0,\alpha)$; since $\ell \geq u_0 \geq \alpha$ in $[0,\alpha]$, it follows from (4.13) that $\ell(t) = u_0(t) + t$ for any $t \in [0,\alpha)$, as desired.

From (4.12) and the assumption (3.2) it follows $\ell \in \mathrm{BV}([0,\alpha])$; then the $\mathrm{BV}_{\mathrm{loc}}$-regularity of ℓ follows observing that the term $|\{\, s \in [0,t] : \ell(t-s) > s \,\}|$ in (4.1) can be written as difference of two nondecreasing functions. More specifically, using also (4.2),

$$\begin{aligned}
|\{\, s \in [0,t] : \ell(t-s) > s \,\}| =& t - |\{\, s \in [0,t] : \ell(t-s) \leq s \,\}| \\
=& t - |\{\, \tau \in [0,t] : \ell(\tau) + \tau \leq t \,\}|,
\end{aligned}$$

---

[13]For any $t \in [k\alpha, (k+1)\alpha]$ we set

$$\begin{aligned}
\ell(t) :=& u_0(t) + \alpha + |\{\, s \in [\alpha,t] : \ell(t-s) > s \,\}| \\
=& u_0(t) + \alpha + \sum_{j=1}^{k-1} |\{\, s \in [j\alpha, (j+1)\alpha] : \ell(t-s) > s \,\}| \\
& + |\{\, s \in [k\alpha, t] : \ell(t-s) > s \,\}|,
\end{aligned} \qquad (4.15)$$

where, if $s \in [j\alpha, (j+1)\alpha]$ then $t-s \in [(k-1-j)\alpha, (k+1-j)\alpha]$, so that

$$\begin{aligned}
|\{\, s \in [j\alpha, (j+1)\alpha] : \ell(t-s) > s \,\}| =& |\{\, s \in [j\alpha, (j+1)\alpha] : t-s \in [(k-1-j)\alpha, (k-j)\alpha], \ell(t-s) > s \,\}| \\
& + |\{\, s \in [j\alpha, (j+1)\alpha] : t-s \in ((k-j)\alpha, (k-1+j)\alpha], \ell(t-s) > s \,\}|,
\end{aligned}$$

and if $s \in [2k\alpha, t]$ then $t - s \in [0,\alpha]$; thus (4.14) is well-defined by recursion.

and the last term on the right-hand side is nondecreasing in t. Clearly, the bound $\ell(t) \leq \|u_0\|_\infty + t$ for all $t \in [0, +\infty)$, follows from (4.1).

It remains to prove that ℓ satisfies the one-sided Lipschitz condition (4.11). Let $\overline{t} > 0$ and $\tau > 0$; since

$$\{ t \in [0,\overline{t}] : h_{\overline{t}+\tau}(t) \leq \ell(t) \} \subseteq \{ t \in [0,\overline{t}] : h_{\overline{t}}(t) < \ell(t) \},$$

we have v

$$\begin{aligned}
&|\{ t \in [0,\overline{t}] : h_{\overline{t}+\tau}(t) \leq \ell(t) \} \cup \{ t \in (\overline{t},\overline{t}+\tau] : h_{\overline{t}+\tau}(t) \leq \ell(t) \}| \\
={}&|\{ t \in [0,\overline{t}] : h_{\overline{t}+\tau}(t) \leq \ell(t) \}| + |\{ t \in (\overline{t},\overline{t}+\tau] : h_{\overline{t}+\tau}(t) \leq \ell(t) \}| \\
\leq{}&|\{ t \in [0,\overline{t}] : h_{\overline{t}}(t) < \ell(t) \}| + |\{ t \in (\overline{t},\overline{t}+\tau] : h_{\overline{t}+\tau}(t) \leq \ell(t) \}| \\
\leq{}&|\{ t \in [0,\overline{t}] : h_{\overline{t}}(t) < \ell(t) \}| + \tau.
\end{aligned}$$

Whence (4.11) follows Remark 4.1, since the Lebesgue measures of the ortoghonal projection of $I_{\overline{t}+\tau}$ and $I_{\overline{t}}$ on the t-axis, are the same as the Lebesgue measure of the corresponding projections on the s-axis. $\qquad\square$

Remark 4.2 (**Shorter steps**)**.** Suppose (4.4), and take $\beta \in (0, \alpha)$; in particular $\ell(t) \geq \beta$ for any $t \in [0, \beta]$. Then the constructive proof in Theorem 4.1 (see (4.13), (4.14), (4.15)) obtained replacing α with β leads to the same ℓ.

The following useful result has a straightfoward proof.

Lemma 4.2 (**Monotonicity**)**.** Suppose that u_{01}, u_{02} are two nonnegative functions in $\mathrm{BV}_{\mathrm{loc}}([0, +\infty))$, both satisfying (4.4). Let ℓ_1, ℓ_2 be the corresponding solutions given by Theorem 4.1. Then

$$u_{01} \leq u_{02} \implies \ell_1 \leq \ell_2.$$

Proof. Let $\alpha > 0$ be such that (4.4) holds both for u_{01} and u_{02}. From our assumption and (4.10) it immediately follows that $\ell_1 \leq \ell_2$ in $[0, \alpha]$. Then the same inequality holds for $t \in [\alpha, 2\alpha]$, as a consequence of the inclusion $\{ s \in [\alpha, t] : \ell_1(t - s) > s \} \subseteq \{ s \in [\alpha, t] : \ell_2(t - s) > s \}$ and (4.10). Then the assertion follows, recalling the recurrence proof of Theorem 4.1. $\qquad\square$

We conclude this section with a crucial definition which, as we shall see, is related to the vertical rearrangement.

Definition 4.2 (**Critical segments and critical times**)**.** *Any segment in the graph of ℓ having slope in $[-\infty, -1)$ is called a critical segment, and the corresponding slope is called a critical slope. The t-coordinate of the left extremum of a critical segment is called a critical time*[14].

Note carefully that any critical segment in the graph of ℓ forces the existence of a range of $t > 0$ for which $\mathfrak{h}_t \cap \mathrm{subgraph}_+(\ell)$ is *not connected*.

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[14]If the segment has slope $-\infty$, its projection on the t-axis is one point, still called a critical time.

4.1. **Transformation $\mathcal{R}$ and rearrangement.** In this short section we deepen the geometric meaning of formula (4.3); this shades light on the meaning of ℓ, and is useful to construct examples (see Sections 7 and 8).

Let us introduce the affine transformation

$$\mathcal{R}\colon \bar{\mathrm{I}}_{\mathrm{quad}} \to \bar{\mathrm{I}}_{\mathrm{quad}}, \qquad \mathcal{R}(t,s) := (t+s, s). \tag{4.16}$$

It moves a point to the right (same s, larger t) of an amount equal to its vertical s-component[15]. Then there is a peculiar relation between $\mathrm{subgraph}_+(\ell)$ (which is a locally finite perimeter set [4]) and its $\mathcal{R}$-transformed: the former is a kind of rearrangement of the latter.

Specifically, consider the vertical rearrangement $F^\sharp$ of a locally finite perimeter set $F \subset [0,+\infty) \times [0,+\infty)$, defined as follows: for any $\bar{t} \in [0,+\infty)$, if $F_{\bar{t}} := \{t = \bar{t}\} \cap F$, and $F_{\bar{t}}^\sharp := \{\bar{t}\} \times [0, |F_{\bar{t}}|]$, then

$$F^\sharp := \bigcup_{t \geq 0} F_t^\sharp,$$

which is still [5, Theorem 14.4] a locally finite perimeter subset of $[0,+\infty) \times [0,+\infty)$, having the same Lebesgue measure as F. Suppose for svimplicity that u_0 has compact support, say $\mathrm{spt}(u_0) \subset [0,c]$. Then the validity of (4.3) for all $t \in [c,+\infty)$ can be equivalently expressed as

$$\mathrm{subgraph}_+(\ell) \cap \Big([c,+\infty) \times [0,+\infty) \Big) = \mathcal{R}(\mathrm{subgraph}_+(\ell))^\sharp \cap \Big([c,+\infty) \times [0,+\infty) \Big). \tag{4.17}$$

Then it is immediately seen from (4.17) that, provided $k\alpha \geq c$, we have

$$\begin{aligned}
\sup_{t \in [(k+1)\alpha, (k+2)\alpha]} \ell(t) &\leq \sup_{t \in [k\alpha, (k+1)\alpha]} \ell(t) \\
\inf_{t \in [(k+1)\alpha, (k+2)\alpha]} \ell(t) &\geq \inf_{t \in [k\alpha, (k+1)\alpha]} \ell(t)
\end{aligned} \tag{4.18}$$

so that the oscillation of ℓ in $[k\alpha, (k+1)\alpha]$ is nonincreasing in k.

It is interesting to observe that the strict inequalities hold in (4.18) under certain conditions on u_0: for instance, suppose that u_0, beside having support contained in $[0,c]$, is polygonal. If there is a critical segment in the generalized graph of ℓ whose projection on the horizontal t-axis is contained in $[k\alpha, (k+1)\alpha]$, then the strict inequalities hold in (4.18). Finally, segments with slope -1 are transformed via $\mathcal{R}$ to vertical segments, and a critical segment, together with its left-adjacent, are transformed into a polygonal which is not a graph with respect to t. This construction will be empolyed in the examples, in particular the Riemann problem considered in Section 8.

4.2. **The functions L_ℓ and S_ℓ.** It is convenient to introduce the following functions, that have been implicitly used in the end of the proof of Theorem 4.1.

Definition 4.3 (L_ℓ and S_ℓ)**.** *Let u_0 and $\ell = \ell[u_0]$ be as in Definition 4.1. We set*

$$L_\ell(t) := \ell(t) + t \qquad \forall t \geq 0$$

and

$$S_\ell(s) := \begin{cases} 0 & \text{if } s \in [0, \ell(0)], \\ |\{\, \sigma \in [0,s] : L_\ell(\sigma) \leq s \,\}| & \text{if } s > \ell(0). \end{cases} \tag{4.19}$$

---

[15]In particular, it transforms a square $((k-1)\alpha, k\alpha) \times (0,\alpha)$ into the square $(k\alpha, (k+1)\alpha) \times (0,\alpha)$, $k \in \mathbb{N}$.

Remark 4.3. The function S_ℓ is nondecreasing in $[0, +\infty)$, and for any $t > 0$

$$t - S_\ell(t) = |\{\, s \in [0, t] : \ell(s) > t - s \,\}| = |\{\, s \in [0, t] : \ell(t - s) > s \,\}|. \qquad (4.20)$$

Hence formula (4.1) is equivalent to

$$\ell(t) = u_0(t) + t - S_\ell(t) \qquad \forall t \in [0, +\infty). \qquad (4.21)$$

Before concluding this section, we summarize some of the obtained results.

Remark 4.4. The function ℓ can be described in various equivalent ways:

(i) algebraically, using equation (4.1) or, equivalently, equation (4.10) or also equation (4.21);
(ii) geometrically, computing the Lebesgue measure of the projection on one of the coordinate axes of the intersection of the up-left half-lines at 45° with $\text{subgraph}_+(\ell)$ (see formula (4.3)), or using the transformation $\mathcal{R}$ and the vertical rearrangement, as explained in Section 4.1.

4.3. **Examples.** It is worth showing some explicit computations of ℓ; more involved and interesting examples will be illustrated in Sections 7 and 8. The next example is particularly simple, and concerns an initial condition not with compact support; curiously enough, this example involves the triangular numbers

$$T_k := \sum_{i=0}^{k} k = \frac{k(k+1)}{2}.$$

Example 4.1 (Constant initial condition, I). Let

$$u_0 = 1_{[0, +\infty)},$$

which has infinite mass. From (4.12) and (4.13) one checks that $\ell \colon [0, +\infty) \to [1, +\infty)$ is the Lipschitz piecewise affine function that interpolates the points $(T_k, k+1)$, so that

$$\ell(t) = \frac{t}{k+1} + \frac{k}{2} + 1, \qquad t \in [T_k, T_{k+1}], \ k \in \mathbb{N}, \qquad (4.22)$$

see Fig. 1. Hence $L_\ell \colon [0, +\infty) \to [1, +\infty)$ is also Lipschitz piecewise affine, $L_\ell(T_k) = T_{k+1}$, and reads as

$$L_\ell(t) = \frac{k+2}{k+1}(t - T_k) + T_{k+1}, \quad t \in [T_k, T_{k+1}], \quad k \in \mathbb{N}. \qquad (4.23)$$

Functions ℓ and L_ℓ are strictly increasing and surjective.

Equivalently, ℓ can be constructed recursively using S_ℓ as follows. Since $\alpha = 1$, from (4.12) we obtain $\ell(t) = 1 + t$ (and $L_\ell(t) = 1 + 2t$) for $t \in [0, 1]$; from (4.19) we have $S_\ell = 0$ in $[0, 1]$, and $S_\ell(s) = (s - 1)/2$ for any $s \in [1, 3] = L_\ell([0, 1])$. Since we know the function S_ℓ in the interval $[1, 3]$ we can find, using (4.21), L_ℓ in $[1, 3]$. We have $\ell(t) = (t + 3)/2$ and $L_\ell(t) = 3(t + 1)/2$ for $t \in [1, 3]$ and so on. In general, $S_\ell \colon [0, +\infty) \to [0, +\infty)$,

$$S_\ell(s) = \frac{k+1}{k+2}(s - T_{k+1}) + T_k, \quad s \in [T_{k+1}, T_{k+2}], \quad k \geq 0,$$

and is the inverse of L_ℓ and interpolates the values $S_\ell(T_k) = T_{k-1}$.

The next interesting example will be completely solved in Section 8, and it is useful in order to understand how to construct ℓ.
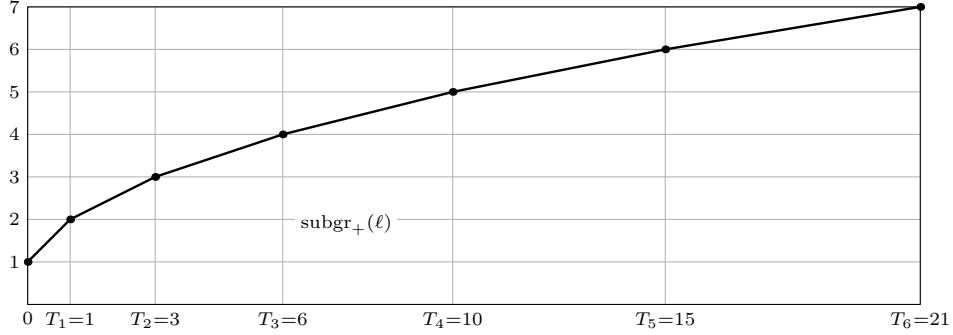
FIGURE 1. Graph of ℓ for Example 4.1. Time t is horizontal.

Example 4.2 (**Riemann problem in** $[0, 11 + 3/2]$)**.** Let

$$u_0 = 1_{[0,1)}. \tag{4.24}$$

Let us calculate $\ell = \ell[u_0]$, which we know to be *polygonal*, since u_0 is polygonal (Remark 4.1). We shall construct ℓ in the time interval $[0, 11 + 3/2]$. Taking $\alpha = 1$, Lemma 4.1 tells us that

$$\ell(t) = u_0(t) + t = 1 + t, \qquad t \in (0, 1). \tag{4.25}$$

It is convenient to add the vertical segment $\{0\} \times [0, 1] = \{0\} \times [0, u_0(0)]$ in the generalized graph of ℓ. In $[0, 1) \times [0, +\infty)$ the set $\mathrm{subgraph}_+(\ell)$ is enclosed between the segment $\{(t, 1 + t) : t \in [0, 1)]\}$ and the horizontal axis and, by Proposition 4.1,

$$\mathrm{subgraph}_+(\ell) \supseteq (0, +\infty) \times (0, 1). \tag{4.26}$$

Hence, for $t \in [1, 2]$, the intersection $I_t = \mathfrak{h}_t \cap \mathrm{subgraph}_+(\ell)$ of the half-line $\mathfrak{h}_t$ with $\mathrm{subgraph}_+(\ell)$ is the segment joining $(t, 0)$ with $(t/2 - 1/2, t/2 + 1/2)$, and so (Remark 4.1)

$$\ell(t) = \frac{t}{2} + \frac{1}{2}, \qquad \forall t \in [1, 2]. \tag{4.27}$$

Therefore the left limit of ℓ at 1 is $2 > \ell(1) = 1$, and ℓ has a decreasing jump at $t = 1$; it is convenient to draw the generalized graph of ℓ in $[0, 2]$, in particular the vertical segment $\{1\} \times [1, 2]$, see Fig. 4[16].

Now, using (4.25), (4.27) and (4.26), we see that for $t \in [2, 3]$ the intersection $I_t = \mathfrak{h}_t \cap \mathrm{subgraph}_+(\ell)$ is *not connected*, and splits into two segments: one joining $(t, 0)$ with $(t/2 - 1/2, t/2 + 1/2)$, and another one joining $(1, t - 1)$ with $((2/3)t - 1/3, t/3 + 1/3)$. Hence

$$\ell(t) = \frac{t}{2} + \frac{1}{2} + \frac{t}{3} + \frac{1}{3} - (t - 1) = \frac{-t + 11}{6} \qquad \forall t \in [2, 3].$$

For times $t \in [3, 11)$ the intersection I_t turns out to be connected, and ℓ can be found by iteratively moving on the right (recall the affine transformation in Section 4.1) the point $(2, 3/2)$ by $3/2$ (its height), and the point $(3, 4/3)$ by $4/3$ (its height) till the transformed points have the same first coordinate, i.e., when $2 + (3/2)n = 3 + (4/3)n$ for $n \in \mathbb{N}$, giving $n = 6$ and so $t = 2 + (3/2)6 = 11$. The graph of ℓ in $[3, 11)$ is plotted in Fig. 4.

---

[16]We have furthermore $S_\ell = 0$ on $[0, 1]$, $S_\ell(x) = (x - 1)/2$ for $x \in [1, 2)$, $S_\ell(x) = (x - 1)/2 + 2x/3 - 4/3 = 5x/6 - 11/6$ for $x \in [2, 7/2]$.

Time $t = 11$ is similar to $t = 1$, when ℓ has a decreasing jump discontinuity, and time $37/3 = 11 + 4/3$ is critical. For $t \in (11, 37/3)$ we find, by a direct computation similar to the one made in [2, 3], $\ell(t) = (1 + t)/9$, while for $t \in (37/3, 25/2 = 11 + 3/2)$ the intersection I_t consists of two segments: a direct computation gives $\ell(t) = (1+t)/9 - (t-11) + (1+t)/10 = -(71/90)t + 1009/90$. Now since the solution of $37/3 + (40/27)n = 25/2 + (27/20)n$ is $n = 1 = \lfloor 90/71 \rfloor$, we have $t = 1009/71$.

The complete recursive construction of ℓ is complicated, and will be done in Section 8.

4.4. Initial conditions vanishing at the origin.

If we drop assumption (4.4) we have the following result.

Proposition 4.2. *Let u_0 be bounded and satisfying* (3.2). *Then there exists a function ℓ satisfying* (4.11) *and*

$$\ell(t) = u_0(t) + |s \in [0, t] : \ell(t - s) \geq s| \qquad \forall t \in [0, +\infty). \qquad (4.28)$$

Proof. For any $\delta > 0$ let ℓ_δ be the unique solution of (4.1) given by Theorem 4.1, with $u_0 + \delta$ in place of u_0. Recall from Lemma 4.2 that, if $0 < \delta_1 \leq \delta_2$, then $\ell_{\delta_1} \leq \ell_{\delta_2}$. Fix a decreasing sequence $(\delta_n) \subset (0, +\infty)$ converging to zero; by monotonicity and since each ℓ_{δ_n} is positive,

$$\exists \lim_{n \to +\infty} \ell_{\delta_n}(t) =: \ell(t) \in [0, +\infty) \qquad \forall t \in [0, +\infty).$$

We know from Theorem 4.1 that

$$\ell_{\delta_n}(t) = u_0(t) + \delta_n + |\{s \in [0, t] : \ell_{\delta_n}(t - s) > s\}| \qquad \forall t \in [0, +\infty), \forall n \in \mathbb{N}, \quad (4.29)$$

so it is sufficient to prove that

$$\lim_{n \to +\infty} |E_n^t| = |E^t| \qquad \forall t \in [0, +\infty), \qquad (4.30)$$

where

$$E_n^t := \{s \in [0, t] : \ell_{\delta_n}(t - s) > s\} \supseteq E^t := \{s \in [0, t] : \ell(t - s) \geq s\}.$$

But $E^t = \cap_{n \in \mathbb{N}} E_n^t$, and so (4.28) follows passing to the limit in (4.29) as $n \to +\infty$. Also, (4.11) follows passing to the limit, as $n \to +\infty$, in equation (4.11) written with ℓ_{δ_n} in place of ℓ and $u_0 + \delta_n$ in place of u_0. $\qquad \square$

The next result in particular ensures that if u_0 is zero on a right interval of $t = 0$ and then becomes strictly positive, and if ℓ exists, then also ℓ must vanish in that interval; this allows to use a restarting procedure for computing ℓ, via Theorem 4.1.

Proposition 4.3. *Let $\kappa > 0$, $a > 0$, and let γ be the positive root of $\gamma^2 + \kappa\gamma - \kappa = 0$. If*

$$u_0(t) \leq \kappa t \qquad \forall t \in [0, a], \qquad (4.31)$$

then a solution to (4.1) *satisfies*

$$\ell(t) \leq \kappa t + \gamma t \qquad \forall t \in [0, a]. \qquad (4.32)$$

Conversely, if

$$u_0(t) \geq \kappa t \qquad \forall t \in [0, a], \qquad (4.33)$$

then

$$\ell(t) \geq \kappa t + \gamma t \qquad \forall t \in [0, a]. \qquad (4.34)$$

Proof. Suppose (4.31). From (4.1) it follows $\ell(t) \le \kappa t + t$ for any $t \in [0, a]$. Hence

$$\{s \in [0, t] : \ell(t - s) > s\} \subseteq \{s \in [0, t] : s < \kappa(t - s) + t - s\} \qquad \forall t \in [0, a].$$

The inequality $s < \kappa(t - s) + t - s$ implies $s < \frac{\kappa + 1}{\kappa + 2} t =: \gamma_1 t$, so from (4.1) we have the improved estimate

$$\ell(t) \le \kappa t + \gamma_1 t \qquad \forall t \in [0, a].$$

We now iterate the argument, and obtain that

$$\ell(t) \le \kappa t + \gamma_n t \qquad \forall t \in [0, a], \ \forall n \in \mathbb{N}^*,$$

with

$$\gamma_n := \frac{\kappa + \gamma_{n-1}}{1 + \kappa + \gamma_{n-1}}, \qquad n \ge 2.$$

The decreasing sequence (γ_n) converges to the positive solution $\gamma = \frac{-\kappa + \sqrt{\kappa^2 + 4\kappa}}{2}$ of the equation $\gamma = \frac{\kappa + \gamma}{1 + \kappa + \gamma}$ or equivalently $\gamma^2 + \kappa\gamma - \kappa = 0$, and (4.32) follows.

Conversely, suppose (4.33). Then, from (4.1) it follows $\ell(t) \ge \kappa t$ for any $t \in [0, a]$. Hence

$$\{s \in [0, t] : \ell(t - s) > s\} \supseteq \{s \in [0, t] : s < \kappa(t - s)\}.$$

The inequality $s < \kappa(t - s)$ implies $s < \frac{\kappa}{\kappa + 1} t =: \alpha_1 t$, so that from (4.1) we have the improved estimate

$$\ell(t) > \kappa t + \alpha_1 t \qquad \forall t \in [0, a].$$

We now iterate the argument, and obtain that

$$\ell(t) > \kappa t + \alpha_n t \qquad \forall t \in [0, a], \forall n \in \mathbb{N}^*,$$

with

$$\alpha_n := \frac{\kappa + \alpha_{n-1}}{1 + \kappa + \alpha_{n-1}}, \qquad n \ge 2.$$

The increasing sequence (α_n) converges to γ, and (4.34) follows. $\qquad \square$

5. Construction of a solution

In this section we prove existence and uniqueness of an integral solution (Definition 3.3). The next result shows that the function ℓ studied in Section 4 captures the relevant information to solve (1.3).

Theorem 5.1 (Existence and uniqueness of an integral solution). *Let u_0 satisfy (3.2) and (4.4). Let $\ell = \ell[u_0] : [0, +\infty) \to [0, +\infty)$ be the solution to (4.1). Then the function $v : \overline{\Pi}_{\text{oct}} \to [0, +\infty)$ defined as*

$$v(\tau, \xi) := u_0(\xi) + |\{s \in [0, \tau] : \ell(s) + s > \xi\}| \qquad \forall (\tau, \xi) \in \overline{\Pi}_{\text{oct}}, \qquad (5.1)$$

is the unique integral solution of (1.3), and

$$v(\tau, \tau) = \ell(\tau) \qquad \forall \tau \in [0, +\infty). \qquad (5.2)$$

Moreover:

(i) *for any $\xi \in [0, +\infty)$ the function $v(\cdot, \xi)$ is one-Lipschitz;*
(ii) *$v(\tau, \cdot) \in \text{BV}_{\text{loc}}([\tau, +\infty))$ for any $\tau > 0$;*
(iii) *if u_{0i}, $i = 1, 2$, satisfies (3.2) and (4.4), and if v_i stands for the expression on the right-hand side of (5.1) with u_0 replaced by u_{0i}, then*

$$u_{01} \le u_{02} \Rightarrow v_1 \le v_2; \qquad (5.3)$$

(iv) *if there exists $C > 0$ such that for all $\xi \in [0, +\infty)$*

$$\frac{u_0(\xi + h) - u_0(\xi)}{h} \leq C, \qquad h > 0, \tag{5.4}$$

then for all $(\tau, \xi), (\tau, \xi + h) \in \overline{\Pi}_{\mathrm{oct}}$,

$$\frac{v(\tau, \xi + h) - v(\tau, \xi)}{h} \leq C, \qquad h > 0; \tag{5.5}$$

(v) *if $u_0 \in L^1((0, +\infty))$ is bounded then $v(\tau, \cdot) \in L^1((\tau, +\infty))$ and v satisfies the conservation of mass*[17]

$$\int_\tau^{+\infty} v(\tau, \xi) d\xi = \int_0^{+\infty} u_0(\xi) \, d\xi \qquad \forall \tau \in [0, +\infty). \tag{5.6}$$

A comment on the expression of v in (5.1) is in order. Fix $\overline{\xi} > 0$; as usual in linear transport equations, we look at the characteristic line $\{\xi = \overline{\xi}\}$, and for a solution we should take the value $u_0(\overline{\xi})$ for (τ, ξ) in the tail region, and $u_0(\overline{\xi}) + \tau$ for (τ, ξ) in the active region; if the (generalized) graph G_{L_ℓ} of $L_\ell = \mathrm{id} + \ell$ is a graph with respect to the vertical ξ-axis, $v(\tau, \xi) = u_0(\xi)$ in the tail region, while the second addendum on the right-hand side of (5.1) activates when (τ, ξ) is in the active region. However, in general G_{L_ℓ} needs not be a graph with respect to the ξ-axis (see for instance Fig. 4); in case that a characteristic line intersects G_{L_ℓ} (once or) more then once, we have to add to $u_0(\overline{\xi})$, both in the tail region and in the active region, the Lebesgue measure of the intersection of $\{\xi = \overline{\xi}\}$ with the subgraph of L_ℓ, i.e., the time spent by the characteristic line in the active region.

Proof. Nonnegativity of v is immediate, since u_0 is nonnegative.
(i) v is one-Lipschitz in τ, since clearly, for any $\xi \geq 0$ and any $h \in \mathbb{R}$ with $\xi + h \geq 0$,

$$|v(\tau + h, \xi) - v(\tau, \xi)| = \Big| |\{s \in [0, \tau + h] : \ell(s) + s > \xi\}|$$
$$- |\{s \in [0, \tau] : \ell(s) + s > \xi\}| \Big| \leq |h|.$$

(ii) $v(\tau, \cdot) \in \mathrm{BV}_{\mathrm{loc}}([\tau, +\infty))$, since $u_0 \in \mathrm{BV}_{\mathrm{loc}}([0, +\infty))$ and the second addendum on the right-hand side of (5.1) is nonincreasing (and right-continuous) if considered as a function of ξ.

Let $\tau > 0$; passing to the limit as $\xi \to \tau^+$ in (5.1) and using the right-continuity of the right-hand side, gives

$$\lim_{\xi \to \tau^+} v(\tau, \xi) = v(\tau, \tau) = u_0(\tau) + |\{s \in [0, \tau] : \ell(s) + s > \tau\}| = \ell(\tau),$$

where the last equality follows from (4.1), and (5.2) follows.
From (5.1) and (5.2) it follows

$$v(\tau, \xi) = u_0(\xi) + |\{s \in [0, \tau] : v(s, s) + s > \xi\}|,$$

i.e., we have that (3.10) holds, which is equivalent to (3.8).
Concerning uniqueness, suppose that w is another integral solution of (1.3); from (3.10) it follows

$$w(\tau, \tau) = u_0(\tau) + |\{s \in [0, \tau] : w(s, s) + s > \tau\}| \qquad \forall \tau \in [0, +\infty).$$

---

[17]Note that $v(\cdot, \xi)$ is increasing; this is not in contradiction with the conservation of mass (5.6), which is required in the time-decreasing interval $(\tau, +\infty)$.

Thus $w(\tau, \tau)$ satisfies the same equation as $\ell(\tau)$; from the uniqueness property stated in Theorem 4.1, we get $w(\tau, \tau) = \ell(\tau)$ for any $\tau \in [0, +\infty)$. Thus

$$v(\tau, \tau) = w(\tau, \tau) \qquad \forall \tau \in [0, +\infty),$$

and so $v = w$ from (5.1) and (5.2).

(iii) If $u_{01} \leq u_{02}$, then (5.3) follows straightforwardly from (5.1) and (4.1).

(iv) It immediately follows from (5.4), (5.1), and the inclusion $\{s \in [0, \tau] : s + \ell(s) > \xi + h\} \subseteq \{s \in [0, \tau] : s + \ell(s) > \xi\}$.

(v) Given $\tau > 0$, $v(\tau, \cdot) \in L^1((\tau, +\infty))$, since ℓ is bounded on $[0, T]$ for any $T > 0$, and so the integral $\int_\tau^{+\infty} |\{s \in [0, \tau] : \ell(s) + s > \xi\}| \, d\xi$ actually reduces to an integral on a bounded interval. Using the boundedness of u_0 and that $v(\cdot, \xi)$ is Lipschitz, it follows that $\int_\tau^{+\infty} v(\tau, \xi) \, d\xi$ is locally absolutely continuous in τ, hence differentiable on a set $I \subseteq (\tau, +\infty)$ of full measure. Since $\ell \in \mathrm{BV}_{\mathrm{loc}}([0, +\infty))$, almost every point of I is a continuity point of ℓ. Fix such a point τ: writing the limits of the incremental quotients around τ we get

$$\frac{d}{d\tau} \int_\tau^{+\infty} v(\tau, \xi) \, d\xi = -v(\tau, \tau) + \int_\tau^{+\infty} v_\tau(\tau, \xi) \, d\xi$$

$$= -v(\tau, \tau) + \int_\tau^{+\infty} 1_{(\tau, v(\tau,\tau)+\tau)}(\xi) \, d\xi$$

$$= -v(\tau, \tau) + \tau + v(\tau, \tau) - \tau = 0.$$

$\square$

The next observation can be used to quickly deduce v from the knowledge of ℓ, and it is used to find the solution of the Riemann problem, see the pictures in Figs. 7, 8 and also Remark 8.3.

Remark 5.1 (**Finding v from ℓ**). If v is as in (5.1), it is immediate to check that

$$v(\tau, \xi) = \ell(\xi) - |\{s \in [\tau, \xi] : \ell(s) + s > \xi\}|$$
$$= \ell(\xi) - (\xi - \tau) + |\{s \in [\tau, \xi] : \ell(s) \leq \xi - s\}| \qquad \forall (\tau, \xi) \in \overline{\Pi}_{\mathrm{oct}}. \qquad (5.7)$$

Indeed, from (4.1) and (4.2),

$$\ell(\xi) = u_0(\xi) + |\{s \in [0, \xi] : \ell(s) + s > \xi\}|$$
$$= u_0(\xi) + |\{s \in [0, \xi] : \ell(s) + s > \xi\}|$$
$$\quad + (\xi - \tau) - |\{s \in [\tau, \xi] : \ell(s) + s \leq \xi\}| - |\{s \in [\tau, \xi] : \ell(s) + s > \xi\}|$$
$$= u_0(\xi) + |\{s \in [0, \tau] : \ell(s) + s > \xi\}| + (\xi - \tau) - |\{s \in [\tau, \xi] : \ell(s) + s \leq \xi\}|$$
$$= v(\tau, \xi) + |\{s \in [\tau, \xi] : \ell(s) + s > \xi\}|,$$

where the last equality follows from (5.1).

Without passing to coordinates (τ, ξ) a possible definition of solution (less transparent than Definition 3.3) is the following.

Definition 5.1 (**Distributional solution**). *We say that a nonnegative function $v \in L^1_{\mathrm{loc}}(\mathrm{I}_{\mathrm{quad}})$ defined everywhere is a distributional solution to (3.4) if for any $t > 0$ there exists $\lim_{x \to 0^+} u(t, x) =: u(t, 0) \in [0, +\infty)$, and for any $\varphi \in C^1_c([0, +\infty) \times$*

$(0, +\infty))$ *we have*

$$
\int_0^{+\infty} \int_0^{+\infty} -u(\varphi_t - \varphi_x) \ dxdt - \int_0^{+\infty} \int_0^{+\infty} 1_{(0, u(t,0))}(x)\varphi(t,x) \ dxdt
$$
$$
+ \int_0^{+\infty} u_0(x)\varphi(0,x) \ dx = 0.
$$
(5.8)

Proposition 5.1 (**Integral solutions and distributional solutions**). *Let* $\widehat{u} = \widehat{u}(\tau, \xi)$ *be an integral solution of* (3.4) *as in Theorem 5.1. Then* $u := \widehat{u}(\Phi) = u(t,x)$ *is a distributional solution of* (3.4). *Conversely, let* $u = u(t,x) \in \mathrm{BV}_{\mathrm{loc}}(\mathrm{I}_{\mathrm{quad}})$ *be a locally bounded distributional solution of* (3.4). *Then* $\widehat{u} := u(\Phi^{-1})$ *satisfies (i) of Definition 3.3 and (ii) for any* $\tau > 0$ *and for almost every* ξ *with* $(\tau, \xi) \in \mathrm{II}_{\mathrm{oct}}$.

Proof. Let $\varphi = \varphi(t,x) \in \mathcal{C}_c^1([0, +\infty) \times (0, +\infty))$, and set $\widehat{\varphi} := \varphi(\Phi^{-1})$.
    Suppose that $\widehat{u}$ is an integral solution. From (5.1) we have

$$
u(t,x) = u_0(t+x) + |\{s \in [0,t] : \ell(s) + s > x + t\}|, \qquad (t,x) \in \mathrm{I}_{\mathrm{quad}}.
$$

Hence

$$
-\int_{\mathrm{I}_{\mathrm{quad}}} u(\varphi_t - \varphi_x) \ dtdx = -\int_{\mathrm{I}_{\mathrm{quad}}} u_0(t+x)(\varphi_t(t,x) - \varphi_x(t,x)) \ dtdx
$$
$$
-\int_{\mathrm{I}_{\mathrm{quad}}} |\{s \in [0,t] : \ell(s) + s > x + t\}|(\varphi_t(t,x) - \varphi_x(t,x)) \ dtdx
$$
$$
= -\int_0^{+\infty} u_0(\xi)\left(\int_0^\xi \widehat{\varphi}_\tau(\tau, \xi) \ d\tau\right) d\xi
$$
$$
-\int_0^{+\infty} \left(\int_0^\xi |\{s \in [0, \tau] : \ell(s) + s > \xi\}| \widehat{\varphi}_\tau(\tau, \xi) \ d\tau\right) d\xi =: \mathrm{I} + \mathrm{II}.
$$

It is immediate that

$$
\mathrm{I} = \int_0^{+\infty} \widehat{\varphi}(0, \xi) u_0(\xi) \ d\xi = \int_0^{+\infty} \varphi(0, x) u_0(x) \ dx.
$$

Moreover, using (3.9),

$$
|\{s \in [0, \tau] : \ell(s) + s > \xi\}| = \int_0^\tau 1_{(s, \ell(s)+s)}(\xi) \ ds \qquad \forall \xi \geq \tau,
$$

and hence

$$
\mathrm{II} = -\int_0^{+\infty} \left(\int_0^\xi \left(\int_0^\tau 1_{(s, \ell(s)+s)}(\xi) \ ds\right) \widehat{\varphi}_\tau(\tau, \xi) \ d\tau\right) d\xi
$$
$$
= \int_0^{+\infty} \left(\int_0^\xi 1_{(\tau, \ell(\tau)+\tau)}(\xi)\widehat{\varphi}(\tau, \xi) \ d\tau\right) d\xi = \int_{\mathrm{I}_{\mathrm{quad}}} 1_{(0, u(t,0))}(x)\varphi(t,x) \ dtdx,
$$

where the second equality follows integrating by parts, and the last equality follows from (5.2). This proves that u is a distributional solution of (3.4).

Conversely, suppose that $u \in \mathrm{BV}_{\mathrm{loc}}(\mathrm{I}_{\mathrm{quad}}) \cap L^{\infty}_{\mathrm{loc}}(\mathrm{I}_{\mathrm{quad}})$ is a distributional solution of (3.4). From (5.8) it follows

$$-\int_0^{+\infty}\int_0^{\xi} \widehat{u}(\tau,\xi)\widehat{\varphi}_\tau(\tau,\xi)\,d\tau d\xi - \int_0^{+\infty}\int_0^{\xi} 1_{(\tau,\widehat{u}(\tau,\tau)+\tau)}(\xi)\widehat{\varphi}(\tau,\xi)\,d\tau d\xi$$
$$+ \int_0^{+\infty} u_0(\xi)\widehat{\varphi}(0,\xi)\,d\xi = 0.$$

Taking φ with compact support in $\mathrm{II}_{\mathrm{oct}}$, it follows that the distributional partial derivative $D_\tau \widehat{u}$, which is a measure with locally finite total variation in $\mathrm{II}_{\mathrm{oct}}$, satisfies

$$D_\tau \widehat{u}(\tau,\xi) = 1_{(\tau,\widehat{u}(\tau,\tau)+\tau)}(\xi) \tag{5.9}$$

in the sense of measures in $\mathrm{II}_{\mathrm{oct}}$. Since the right-hand side of (5.9) is a locally integrable function (taking values in $\{0,1\}$), $D_\tau \widehat{u}$ coincides with its absolutely continuous part, the density of which we denote by $\nabla_\tau \widehat{u}$. For any $\xi \in [0,+\infty)$ consider the slice $\widehat{u}^{(\xi)} : [0,\xi] \to \mathbb{R}$ of $\widehat{u}$, i.e., the restriction of u in $\overline{\mathrm{I}}_{\mathrm{quad}}$ to the horizontal line passing through ξ. By [1, Proposition 4.35] for almost every $\xi \in [0,+\infty)$ we have $\widehat{u}^{(\xi)} \in \mathrm{BV}_{\mathrm{loc}}([0,\xi])$, its distributional derivative $\dot{\widehat{u}}^{(\xi)}$ is absolutely continuous, and $\dot{\widehat{u}}^{(\xi)}(\tau) = \nabla_\tau \widehat{u}(\tau,\xi)$ for almost every $\tau \in [0,\xi]$. We deduce

$$\dot{\widehat{u}}^{(\xi)}(\tau) = 1_{(\tau,\widehat{u}(\tau,\tau)+\tau)}(\xi) \qquad \text{for a.e. } \tau \in [0,\xi].$$

Integrating, (3.8) follows for almost any $\xi \geq 0$ and for almost every $\tau \in [0,\xi]$, and hence for any $\tau \in [0,\xi]$. $\qquad \square$

6. A Lyapunov functional

The functional $\mathfrak{L} : D(\mathfrak{L}) \subset L^2(\mathbb{R}) \to [0,+\infty)$, corresponding to the continuous version of (2.4), is

$$\mathfrak{L}(u) := \int_0^{+\infty}\int_x^{x+u(x)} y\,dy\,dx,$$

where the domain $D(\mathfrak{L})$ of $\mathfrak{L}$ consists of all *nonnegative* $u \in L^2((0,+\infty))$ such that $xu \in L^1((0,+\infty))$, and the height y has the meaning of a potential energy. It can be evaluated as

$$\mathfrak{L}(u) = \frac{1}{2}\int_0^{+\infty}\left(u^2(x) + 2xu(x)\right)\,dx = \frac{1}{2}\|u\|_{L^2}^2 + (x,u)_{L^2}$$

and is a strictly convex functional.

Proposition 6.1 (Stationary solutions). *Fix $m > 0$. Then the solution of the variational problem*

$$\min\left\{\mathfrak{L}(u) : u \in D(\mathfrak{L}), \int_0^{+\infty} u(x)\,dx = m\right\},$$

is

$$u(x) = \max\left\{\sqrt{2m} - x, 0\right\} \qquad \forall x \in [0,+\infty).$$

Proof. Given $u \in D(\mathfrak{L})$, introduce the function $\varphi = \sqrt{u}$, so that

$$u(x) = \varphi^2(x) \quad \text{for every} \quad x \geq 0.$$

In this way the pointwise constraint $u(x) \geq 0$ is trivially satisfied for every $x \geq 0$ and the variational problem reads as follows:

$$\min_{\varphi} \quad \frac{1}{2} \int_0^{+\infty} \left(\varphi^4(x) + 2x\varphi^2(x) \right) \, dx \quad \text{subject to} \int_0^{+\infty} \varphi^2(x) \, dx = m.$$

The Euler-Lagrange equation associated to the above variational problem is

$$4\varphi^3(x) + 4x\varphi(x) - 2\lambda\varphi(x) = 0 \iff \varphi(x)\left(\varphi^2(x) + x - \frac{\lambda}{2} \right) = 0,$$

where λ denotes the Lagrange multiplier of the mass constraint. Such a condition is clearly satisfied if

$$\varphi \equiv 0 \quad \text{or} \quad \varphi^2(x) = \frac{\lambda}{2} - x \quad \forall x \in [0, +\infty),$$

or, equivalently

$$u \equiv 0 \quad \text{or} \quad u(x) = \frac{\lambda}{2} - x \quad \forall x \in [0, +\infty).$$

We have thus determined that a stationary point of the variational problem has the form

$$u(x) = \max\left\{ \frac{\lambda}{2} - x, 0 \right\} \qquad \forall x \in [0, +\infty).$$

In order to determine λ we impose the mass constraint on u, obtaining

$$m = \int_0^{+\infty} u(x) dx = \int_0^{\lambda/2} \left(\frac{\lambda}{2} - x \right) dx = \left(\frac{\lambda^2}{4} - \frac{\lambda^2}{8} \right) = \frac{\lambda^2}{8},$$

which implies $\lambda = 2\sqrt{2m}$. Plugging it in the expression for u and recalling that $\mathfrak{L}$ is strictly convex, we get the statement. $\qquad\square$

6.1. $\mathfrak{L}$ decreases along a solution.

Let $u_0 \in D(\mathfrak{L}) \cap L^\infty((0, +\infty))$ satisfy (4.4) and denote by $\widehat{u}$ the solution given by Theorem 5.1. In coordinates (τ, ξ) we have

$$\mathfrak{L}(\widehat{u}(\tau)) = \int_\tau^{+\infty} \int_{\xi-\tau}^{\xi-\tau+\widehat{u}(\tau,\xi)} y \, dy d\xi = \frac{1}{2} \int_\tau^{+\infty} \left(\widehat{u}(\tau,\xi)^2 + 2(\xi - \tau)\widehat{u}(\tau,\xi) \right) d\xi,$$

where $\widehat{u}(\tau)(\cdot) := \widehat{u}(\tau, \cdot)$, and one checks that $\widehat{u}(\tau) \in D(\mathfrak{L})$ for any $\tau \geq 0$. We claim that for all $\sigma, \tau \in [0, +\infty)$ with $\sigma < \tau$ we have

$$\mathfrak{L}(\widehat{u}(\tau)) + \int_\sigma^\tau \int_{r+\ell(r)}^{+\infty} \widehat{u}(r, \xi) \, d\xi \, dr = \mathfrak{L}(\widehat{u}(\sigma)), \tag{6.1}$$

where we recall that $\ell(r) = \widehat{u}(r, r)$, with ℓ given by Theorem 4.1.

Since $\widehat{u}(\cdot, \xi)$ is Lipschitz, the function $\tau \to \mathfrak{L}(\widehat{u}(\tau))$ is locally absolutely continuous. Hence, at each of its differentiability points which are also continuity points

for ℓ (hence, at almost every $\tau \geq 0$) we have, using also (5.2),

$$\frac{d}{d\tau}\mathfrak{L}(\widehat{u}(\tau)) = -\frac{1}{2}\ell(\tau)^2 + \int_\tau^{+\infty}[\widehat{u}(\tau,\xi)+\xi-\tau]\widehat{u}_\tau(\tau,\xi)\,d\xi - \int_\tau^{+\infty}\widehat{u}(\tau,\xi)\,d\xi$$

$$= -\frac{1}{2}\ell(\tau)^2 + \int_\tau^{+\infty}[\widehat{u}(\tau,\xi)+\xi-\tau]1_{(\tau,\tau+\ell(\tau))}(\xi)\,d\xi - \int_\tau^{+\infty}\widehat{u}(\tau,\xi)\,d\xi$$

$$= -\frac{1}{2}\ell(\tau)^2 - \int_{\tau+\ell(\tau)}^{+\infty}\widehat{u}(\tau,\xi)\,d\xi + \int_\tau^{\tau+\ell(\tau)}(\xi-\tau)\,d\xi$$

$$= -\int_{\tau+\ell(\tau)}^{+\infty}\widehat{u}(\tau,\xi)\,d\xi \leq 0.$$

(6.2)

The nonnegative (double) integral term in (6.1) can be considered as a dissipated quantity by the system, when passing from time σ to a later time τ.

7. Examples

Given u_0, our strategy to construct a solution of (1.3) is as follows: first compute ℓ using one of the methods illustrated in Section 4 (see Remark 4.4); next, using (5.1), compute an integral solution in variables (τ,ξ) and, whenever convenient, re-express it in variables (t,x).

Example 7.1 (Stationary solutions). Let $m \geq 0$,

$$u_0(x) := \max\left\{\sqrt{2m}-x,0\right\} \qquad \forall x \geq 0, \tag{7.1}$$

so that $m = \int_{[0,+\infty)} u_0\,dx$. Then

$$u(t,x) := u_0(x), \qquad (t,x) \in [0,+\infty) \times [0,+\infty) \tag{7.2}$$

is a stationary solution of (1.3), and $\ell(t) = \sqrt{2m}$ for any $t \geq 0$,

$$\Gamma(u) = \{(t,\sqrt{2m}) : t \geq 0\}.$$

In (τ,ξ)-variables, $u(\tau,\xi) = u_0(\xi-\tau)$ is a travelling wave.

The next example concerns the initial condition in Example 4.1: it shows that maxima can increase, and also that smoothness can be lost.

Example 7.2 (Constant initial condition, II). Let
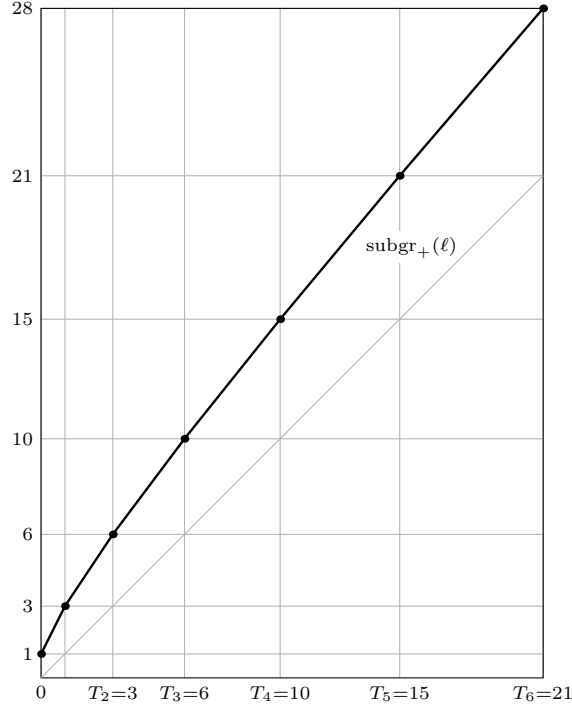
$$u_0 = 1_{[0,+\infty)}, \tag{7.3}$$

which has infinite mass. Recall that ℓ and L_ℓ have been computed in (4.22) and (4.23), see Figs. 1 and 2.

For (t,x) in the interior of $\mathrm{II}_{\mathrm{oct}} \setminus \mathrm{subgraph}_+(\ell)$ we have $u(t,x) = 1$. Observe that

$$(t,x) \in \mathrm{subgraph}_+(\ell) \cap ([T_k,T_{k+1}] \times [0,+\infty)) \Rightarrow t+x \in [T_k,T_{k+1}] \cup [T_{k+1},T_{k+2}],$$

and we have $t+x \in [T_k,T_{k+1}]$ when $0 \leq x \leq -t+T_{k+1}$ while $t+x \in [T_{k+1},T_{k+2}]$ when $-t+T_{k+1} \leq x < \ell(t)$. Equivalently, in (τ,ξ) variables, we can split $\mathrm{subgraph}_+(\ell) \cap ([T_k,T_{k+1}] \times [0,+\infty))$ as the union of two disjoint regions:

$$S_{\mathrm{I}} := \{(\tau,\xi) \in \overline{\mathrm{II}}_{\mathrm{oct}} : T_k \leq \tau \leq T_{k+1}, \tau \leq \xi \leq T_{k+1}\},$$

$$S_{\mathrm{II}} := \left\{(\tau,\xi) \in \overline{\mathrm{II}}_{\mathrm{oct}} : T_k \leq \tau \leq T_{k+1}, T_{k+1} < \xi < L_\ell(\tau) = \tau + \frac{\tau}{k+1} + \frac{k}{2} + 1\right\}.$$

FIGURE 2. Graph of L_ℓ for Example 7.2.

For $(\overline{\tau}, \overline{\xi}) \in S_{\mathrm{I}}$, in order to find the time spent by the characteristic line $\{\xi = \overline{\xi}\}$ in subgraph$_+(\ell)$ before reaching the vertical axis, we need to compute the τ-coordinate of the intersection point of $\{\xi = \overline{\xi}\}$ with graph$(L_\ell)_{|[T_{k-1},T_k]}$; with $\overline{\xi} = L_\ell(\tau) = \frac{k+1}{k}\tau + \frac{k+1}{2}$ (see (4.23)) we get

$$\tau = \frac{k}{k+1}\Big(\overline{\xi} - \frac{k+1}{2}\Big).$$

The time spent is therefore $\overline{\tau} - \tau = \overline{\tau} - \frac{k}{k+1}(\overline{\xi} - \frac{k+1}{2})$ and so, recalling (5.1) and (7.3),

$$v(\overline{\tau}, \overline{\xi}) = 1 + \overline{\tau} - \frac{k}{k+1}\Big(\overline{\xi} - \frac{k+1}{2}\Big) \qquad \forall (\overline{\tau}, \overline{\xi}) \in S_{\mathrm{I}}. \tag{7.4}$$

For $(\overline{\tau}, \overline{\xi}) \in S_{\mathrm{II}}$, we need the τ-coordinate of the intersection point of $\{\xi = \overline{\xi}\}$ with graph$(L_\ell)_{|[T_k,T_{k+1}]}$; with $\overline{\xi} = L_\ell(\tau) = \frac{k+2}{k+1}\tau + \frac{k+2}{2}$ we get

$$\tau = \frac{k+1}{k+2}\Big(\overline{\xi} - \frac{k+2}{2}\Big).$$

The time spent is therefore $\overline{\tau} - \tau = \overline{\tau} - \frac{k+1}{k+2}\Big(\overline{\xi} - \frac{k+2}{2}\Big)$, and so

$$v(\overline{\tau}, \overline{\xi}) = 1 + \overline{\tau} - \frac{k+1}{k+2}\Big(\overline{\xi} - \frac{k+2}{2}\Big) \qquad \forall (\overline{\tau}, \overline{\xi}) \in S_{\mathrm{II}}. \tag{7.5}$$

Note (see Fig. 3) that for any $k \in \mathbb{N}$,

  (i) for any $\overline{\tau} \in (T_k, T_{k+1})$, the derivative of $v(\overline{\tau}, \cdot)$ has two jumps, corresponding to the intersection of $\{\tau = \overline{\tau}\}$ with $\{\xi = T_{k+1}\}$ and with graph$(L_\ell)_{|(T_k,T_{k+1})}$;

(ii) for any $\overline{\xi} \in (T_k, T_{k+1})$, the derivative of $v(\cdot, \overline{\xi})$ has one jump, corresponding to the intersection of $\{\xi = \overline{\xi}\}$ with $\text{graph}(L_\ell)_{|(T_k, T_{k+1})}$.

Going back to (t, x)-coordinates, from (7.4) and (7.5) we have, for any $k \geq 0$ and any $t \in [T_k, T_{k+1}]$,

$$u(t, x) = \begin{cases} \frac{t}{k+1} - \frac{k}{k+1}x + 1 + \frac{k}{2} & \text{for } 0 \leq x \leq -t + T_{k+1}, \\ \frac{t}{k+2} - \frac{k+1}{k+2}x + 1 + \frac{k+1}{2} & \text{for } -t + T_{k+1} < x \leq \frac{t}{k+1} + 1 + \frac{k}{2}, \\ 1 & \text{for } x > \frac{t}{k+1} + 1 + \frac{k}{2}. \end{cases} \quad (7.6)$$

Function u is Lipschitz, piecewise affine, it forms an initial plateau, originating from $x = 1$, that moves vertically upwards with speed one, which is next linearly interpolated with the constant one (which does not move, and is eroded), the x-slope of the interpolants being equal to $-1/2$. The two points (both originating from $x = 1$) on the x-axis corresponding to the two corners in the graph of $u(t, \cdot)$, $t \in (0, 1]$ move with unit speed, one toward the left and the other one toward the right; see Fig. 3.

Example 7.3 (**Increasing jump**). Let $u_0(x) = 1_{[0,1)} + 21_{[1,+\infty)}$. For $t \in [0, 1]$ a solution is $u(t, x) = 1_{[1-t,1)}(x) + (2+t)1_{[1,1+t)} + 21_{[1+t,+\infty)}$. In this case, the initial increasing jump travels toward the left at unit constant speed and then disappears, and meanwhile a new decreasing jump is formed during the evolution.

8. The Riemann problem

The next example, which has been initially discussed in Example 4.2, exhibits interesting phenomena (Figs. 7, 8), and shows in particular a solution for which:

(i) the initial jump persists for some time, it moves with unit speed toward the origin, and then disappears;

(ii) at suitable later times new jumps may form (and persist for some time with a similar behaviour as in (i));

(iii) there are countably many critical times, and so vertical rearrangement is necessary a countable number of times;

(iv) the solution converges to a stationary solution in infinite time;

(v) the Lyapunov functional along the solution is Lipschitz and is constant excluding segments which are projection on the t-axis of critical segments, where instead it strictly decreases.

Let

$$u_0 = 1_{[0,1)}. \quad (8.1)$$

In Example 4.2 we have found ℓ in $[0, 11 + 3/2]$; here we give the general rule for finding ℓ (recall that ℓ may have jumps, and over a jump point there is a vertical segment in its generalized graph; correspondingly, we have a minimal and a maximal value of ℓ; for convenience, the generalized graph of ℓ contains the initial vertical segment $\{0\} \times [0, 1]$).

Our aim is to define inductively a polygonal curve that we shall subsequently prove (Theorem 8.1) to be the graph of the function ℓ with the initial condition (8.1).

Inspired by the computations in Example 4.2 let us first define a real sequence $(\alpha_n)_{n \geq 0}$ and a sequence $(\beta_n)_{n \geq 0}$, with $\beta_n \in [-\infty, 0) \cup (0, +\infty)$, that will be shown
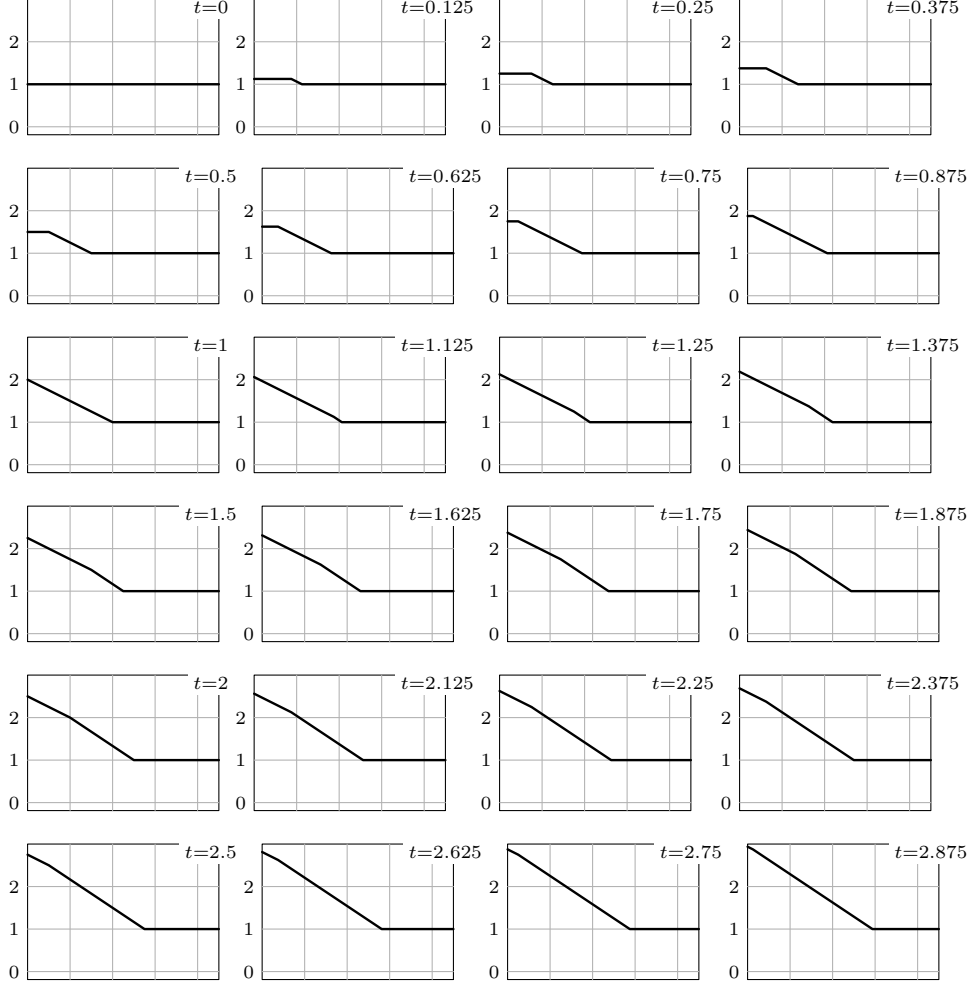
FIGURE 3. Time-slices of the solution u in Example 7.2

to be the slopes of the increasing (resp. decreasing[18]) segments of the graph of ℓ; it is convenient to introduce also a real sequence $(\beta_n^*)_{n\geq 1}$. Specifically:

Definition 8.1 (The sequences (α_n), (β_n^*), (β_n)). *We set*

$$\alpha_0 := 1, \qquad \beta_0 := -\infty, \tag{8.2}$$

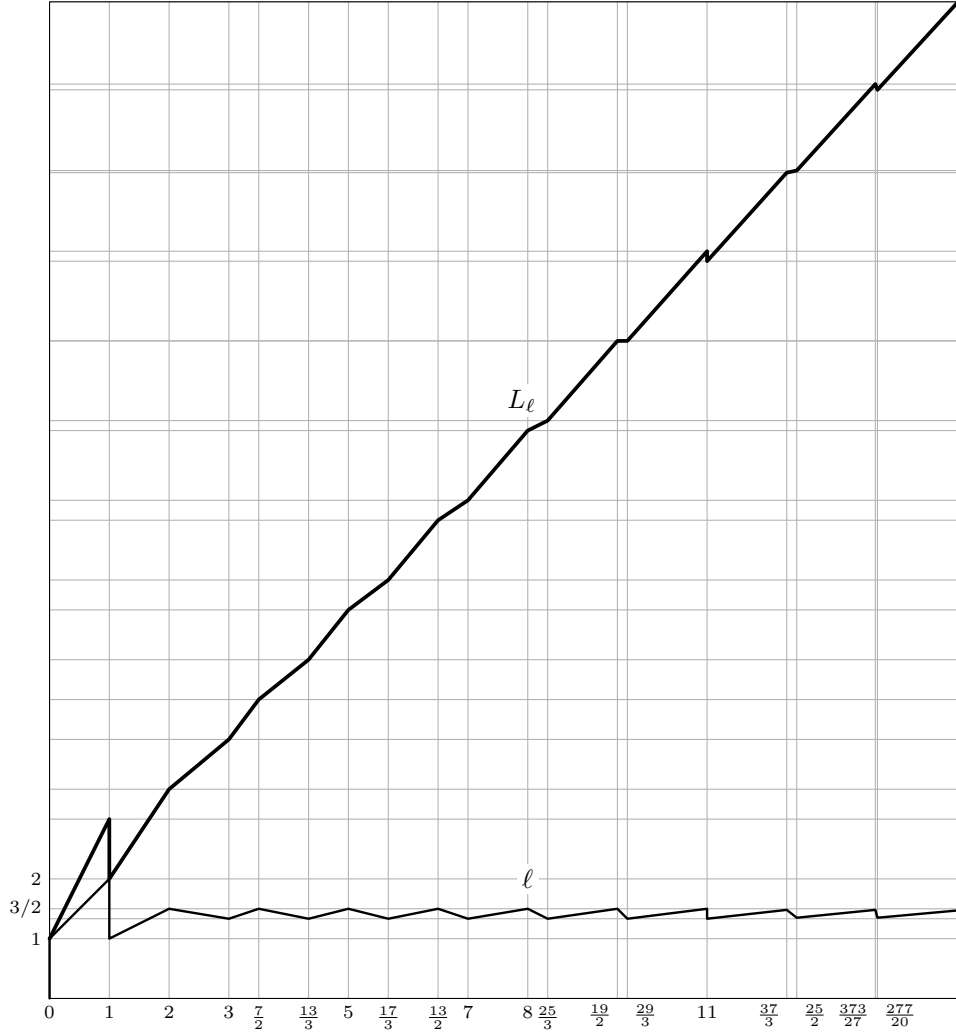[18]In Lemma 8.1 we shall prove that $\beta_n \in [-\infty, 0)$.

FIGURE 4.   The Riemann problem with u_0 in (8.1): graphs of ℓ (the polygonal defined by (8.10)) and of L_ℓ (the bold one). At the local maximum $T_9 = 37/3$ the value of ℓ is $M_9 = 40/27$ which is strictly smaller than the one ($M_8 = 3/2$) at time $T_8 = 11$. At the first local minimum $t_9 > T_9$ the value of ℓ is $m_9 = 27/20$ which is strictly larger than the value ($m_8 = 4/3$) of ℓ at t_8. The first critical time (Definition 4.2) is $t = 1$, and the second one is $t = 37/3$. The distance $t_{n+1} - t_n = 4/3$ between two local minimizers (after $t = 1$) remains constant before the first critical time; next it slightly increases, remaining constant before the second critical time, and so on.

and for any $n \geq 0$,

$$\frac{1}{\alpha_{n+1}} := \frac{1}{\alpha_n} + 1, \qquad i.e.\ \alpha_n = \frac{1}{n+1},$$

$$\frac{1}{\beta_{n+1}^*} := \begin{cases} \frac{1}{\beta_n} + 1 & \text{if } \beta_n \neq -1, 0, \\ 1 & \text{if } \beta_n = -\infty, \\ 0^- & \text{if } \beta_n = -1, \end{cases} \tag{8.3}$$

$$\beta_{n+1} := \begin{cases} \beta_{n+1}^* & \text{if } \beta_{n+1}^* \leq 0, \\ \alpha_{n+1} + \alpha_{n+2} - \beta_{n+1}^* & \text{if } \beta_{n+1}^* > 0, \end{cases}$$
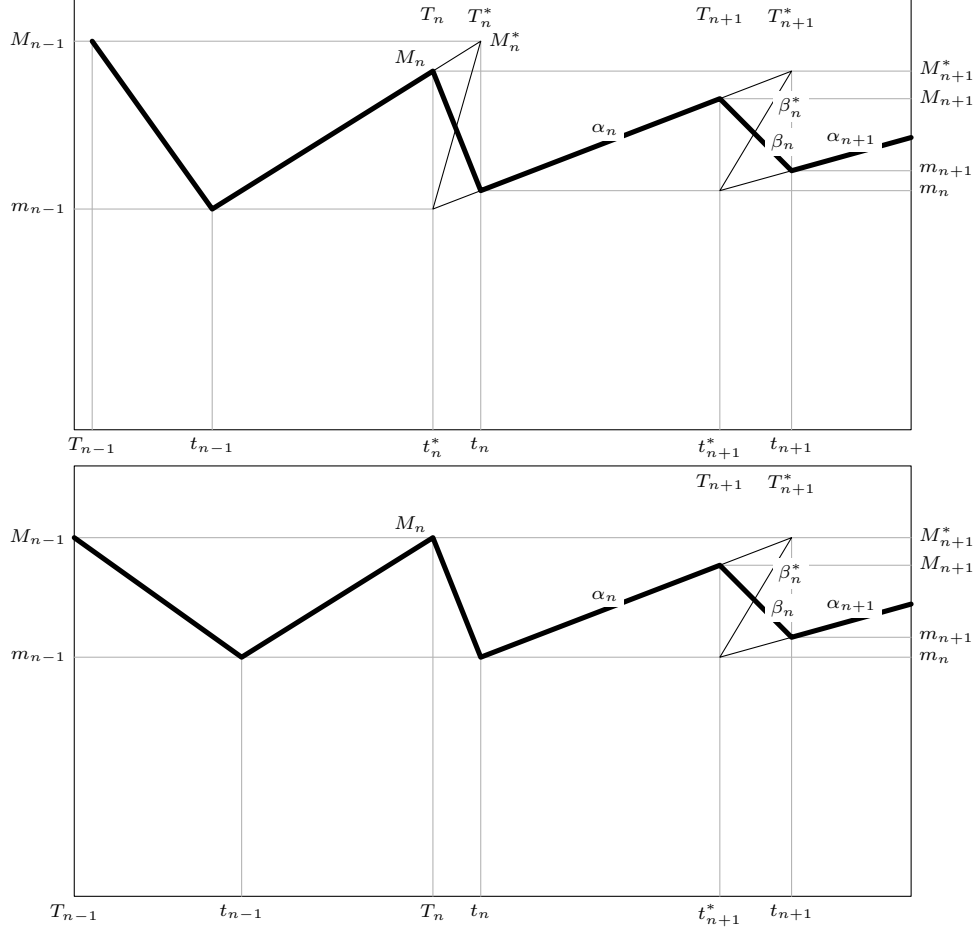
FIGURE 5. Vertical rearrangement takes place in the interval (t_{n+1}^*, T_{n+1}^*) where the thin zig-zag line (which is part of the polygonal $\mathcal{P}^*$) is no-longer a graph. The result of the rearrangement is the bold decreasing segment (which is part of the polygonal $\mathcal{P}$) with negative slope β_n and can be obtained by linear interpolation. In the previous interval (t_n^*, T_n^*) we may have either vertical rearrangement (top figure; the segments over (T_{n-1}, t_{n-1}) and (t_n^*, t_n) are both critical) or no vertical rearrangement (bottom figure; only the segment over (T_n, t_n) is critical).

where the value 0^- indicates that when computing its reciprocal we select the value $-\infty$, i.e., $\beta_{n+1}^* = -\infty$.

For instance, $1/\beta_1^* = 1$, $\beta_1 = \alpha_1 + \alpha_2 - 1 = -1/6$.
Set

$$T_0 := 0, \qquad M_0 := 0, \qquad t_0 := 0, \qquad m_0 := 1. \tag{8.4}$$

Now, we iteratively define nine sequences (the first five sequences could be considered as just auxiliary), as follows:

Definition 8.2 (**The sequences** (t_n^*), (m_n^*), (T_n^*), (M_n^*), (Δ_n^*), (T_n), (M_n), (t_n), (m_n))**.** *For $n \geq 0$ define the quantities:*

$$t_{n+1}^* := t_n + m_n, \quad m_{n+1}^* := m_n, \quad T_{n+1}^* := T_n + M_n, \quad M_{n+1}^* := M_n, \qquad (8.5)$$

$$\Delta_n^* := T_{n+1}^* - t_{n+1}^*.$$

Furthermore:

(i) *if $\Delta_n^* \leq 0$ we set*

$$\begin{aligned}
T_{n+1} &:= T_{n+1}^* + \delta_n, \quad M_{n+1} := M_{n+1}^* + 2\delta_n, \\
t_{n+1} &:= t_{n+1}^*, \qquad\qquad m_{n+1} := m_{n+1}^*,
\end{aligned} \qquad (8.6)$$

where δ_n is 1 if $n = 0$, zero otherwise;

(ii) *if $\Delta_n^* > 0$ we set*

$$\begin{aligned}
T_{n+1} &:= t_{n+1}^*, \quad M_{n+1} := M_{n+1}^* - \Delta_n^* \alpha_n, \\
t_{n+1} &:= T_{n+1}^*, \quad m_{n+1} := m_{n+1}^* + \Delta_n^* \alpha_{n+1}.
\end{aligned} \qquad (8.7)$$

For instance $\Delta_0^* = -1$, $T_1 = 1$, $M_1 = 2$, $t_1 = 1$, $m_1 = 1$, $\Delta_1^* = 1$, $T_2 = 2$, $M_2 = 3/2$, $t_2 = 3$, $m_2 = 4/3$, $T_3 = 7/2$, $M_3 = 3/2$.

Clearly, for all $n \geq 0$

$$0 < T_n \leq T_{n+1}, \quad 0 < t_n \leq t_{n+1}, \quad M_{n+1} \leq M_n \leq 2, \quad m_{n+1} \geq m_n \geq 1.$$

Remark 8.1 (Δ_n^* **as an indicator of rearrangement**)**.** We shall prove in Proposition 8.1 that $T_n \leq t_n$ for all $n \geq 0$; this however does not entail $T_{n+1}^* \leq t_{n+1}^*$. If this is not the case we shall need to perform the so-called vertical rearrangement, as explained next[19]: so if $\Delta_n^* \leq 0$ there will be no vertical rearrangement, whereas if $\Delta_n^* > 0$ there will be vertical rearrangement.

In Proposition 8.2 we shall see that condition $T_{n+2}^* > t_{n+2}^*$ is equivalent to $\beta_{n+1}^* > 0$.

We now use the sequences defined in (8.5), (8.6), (8.7) as coordinates of points in the (t, y)-plane, setting

$$g_n := (t_n, m_n), \quad G_n := (T_n, M_n), \quad g_n^* := (t_n^*, m_n^*), \quad G_n^* := (T_n^*, M_n^*),$$

and observe that from (8.5) that

$$g_{n+1}^* = \mathcal{R}(g_n), \qquad G_{n+1}^* = \mathcal{R}(G_n), \qquad (8.8)$$

where $\mathcal{R}$ is the affine transformation of Section 4.1.

Definition 8.3 (**Polygonals $\mathcal{P}^*$ and $\mathcal{P}$**)**.** *The polygonal $\mathcal{P}^*$ is defined by the points*

$$G_1^*, g_1^*, G_2^*, g_2^*, \ldots, G_{n+1}^*, g_{n+1}^*, \ldots \qquad (8.9)$$

and the polygonal $\mathcal{P}$ is defined by the points

$$G_0, g_0, G_1, g_1, \ldots, G_n, g_n, \ldots \qquad (8.10)$$

We will show that the function having graph $\mathcal{P}$ is the result of the vertical rearrangement applied to the open set enclosed by $\mathcal{P}^*$ and the positive t-axis, after addition of the initial datum u_0. See also Figs. 5 and 6.

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[19]For example, the first rearrangement happens for $n+1 = 2$, leading to $M_{n+1}/m_{n+1} = (3/2)^2 \cdot 1/2 = 9/8 > 1$; the second one is for $n + 1 = 9$, leading to $M_9/m_9 = (10/9)^2 \cdot 8/9 = 800/729 > 1$, and the third one for $n + 1 = 11$, with $M_11/m_11 = (12/11)^2 \cdot 729/800 = 6561/6050 > 1$.

Proposition 8.1 (**Characterization**). *The sequences* (T_n), (t_n), (M_n), (m_n), (M_n^*), (m_n^*) *defined in (8.4) and in Definition 8.2 can be characterized as follows: for all* $n > 0$

$$T_{n+1} = \min\{T_n + M_n, t_n + m_n\}, \qquad t_{n+1} = \max\{T_n + M_n, t_n + m_n\}, \quad (8.11)$$

$$M_{n+1} = \alpha_n(1 + T_{n+1}), \qquad m_{n+1} = \alpha_{n+1}(1 + t_{n+1}), \qquad (8.12)$$

$$M_{n+1}^* = \alpha_n(1 + T_{n+1}^*), \qquad m_{n+1}^* = \alpha_{n+1}(1 + t_{n+1}^*). \qquad (8.13)$$

In particular, $m_{n+1} > 0$, $M_{n+1} > 0$, *and*

$$T_{n+1} \leq t_{n+1}.$$

Proof. Let $n > 0$. If $\Delta_n^* \leq 0$, i.e., $T_n + M_n \leq t_n + m_n$, then $T_{n+1} = T_n + M_n$ and $t_{n+1} = t_n + m_n$, while if $\Delta_n^* > 0$, i.e., $T_n + M_n > t_n + m_n$, then $T_{n+1} = t_n + m_n$ and $t_{n+1} = T_n + M_n$, and (8.11) follows.

For $n \geq 0$ let $y = r_n(t) := \alpha_n(1 + t)$ denote the equation of the line with slope α_n through $(-1, 0)$. With a small abuse of notation we also denote by r_n the line itself. The key observation is that the affine transformation $\mathcal{R}$ maps line r_n onto line r_{n+1}. Given that $g_0, G_1 \in r_0$, we now argue by induction to prove that for any $n \geq 1$ points $g_n, g_n^*, G_{n+1}, G_{n+1}^* \in r_n$. Now suppose $g_n, G_{n+1} \in r_n$, then (8.8) implies that $g_{n+1}^*, G_{n+2}^* \in r_{n+1}$. If $g_{n+1} \neq g_{n+1}^*$, i.e. rearrangement occurs at step n (resp. $G_{n+2} \neq G_{n+2}^*$, i.e. rearrangement occurs at step $n + 1$), equation (8.7) tells us that the segment $g_{n+1}^* g_{n+1}$ (resp. $G_{n+2} G_{n+2}^*$) has slope α_n, so that the "unstarred" points are the result of suitably moving along line r_{n+1} the corresponding "starred" point. This concludes the induction step and the claim follows together with the equivalent assertions (8.12) and (8.13). $\square$

Remark 8.2. It follows from Proposition 8.1, using e.g. $\frac{\alpha_n}{\alpha_{n+1}} = 1 + \alpha_n$ that the sequences (t_n) and (T_n) can be decoupled from the sequences (m_n) and (M_n), and for all $n > 0$

$$T_{n+1} = \min\left\{\alpha_{n-1} + \frac{\alpha_{n-1}}{\alpha_n}T_n, \alpha_n + \frac{\alpha_n}{\alpha_{n+1}}t_n\right\},$$
$$t_{n+1} = \max\left\{\alpha_{n-1} + \frac{\alpha_{n-1}}{\alpha_n}T_n, \alpha_n + \frac{\alpha_n}{\alpha_{n+1}}t_n\right\}. \qquad (8.14)$$

Concerning the values (m_n), (M_n), in the interesting case $\Delta_n^* > 0$ (rearrangement), enforcing (8.5), (8.7), (8.12), (8.13), we obtain

$$\Delta_n^* > 0 \implies \quad M_{n+1} = \frac{\alpha_n}{\alpha_{n+1}}m_n, \quad m_{n+1} = \frac{\alpha_{n+1}}{\alpha_n}M_n \qquad (8.15)$$

leading to the remarkable fact that $m_{n+1}M_{n+1} = m_n M_n$ regardless whether we have rearrangement. Consequently

$$m_n M_n = m_1 M_1 = 2 \qquad \forall n \geq 1. \qquad (8.16)$$

Recalling that (m_n) is non-decreasing and (M_n) is non-increasing, using (8.11) we get

$$m_n \geq 1, \qquad M_n \leq 2,$$
$$t_n \geq t_1 + m_1 + m_2 + \cdots + m_{n-1} \geq t_1 + (n-1)m_1 = n,$$
$$T_n \leq T_1 + M_1 + M_2 + \cdots + M_{n-1} \leq T_1 + (n-1)M_1 = 2n - 1.$$

Lemma 8.1 (**Sign of** β_n). *We have* $\beta_n \in [-\infty, 0)$ *and* $\beta_n \in [-\infty, -1 + \alpha_n + \alpha_{n+1}]$ *for all* $n \geq 0$.

Proof. The result is clearly true for $n = 0$. We first prove $\beta_n \in [-\infty, 0)$ for all $n \in \mathbb{N}$. Suppose by induction that $\beta_n \in [-\infty, 0)$ for some $n \geq 0$. If $\beta_n \in [-1, 0)$ then (8.3) gives $\beta_{n+1} = \beta_{n+1}^* = \frac{\beta_n}{\beta_n + 1} < 0$. Suppose now that $\beta_n < -1$, i.e. $\beta_{n+1}^* \geq 1$. We have that $\alpha_n + \alpha_{n+1}$ is (strictly) decreasing and $\alpha_1 + \alpha_2 = \frac{5}{6}$, so that (8.3), second case in definition of β_{n+1}, entails $\beta_{n+1} = \alpha_{n+1} + \alpha_{n+2} - \beta_{n+1}^* \leq \frac{5}{6} - 1 < 0$.

Now, let us prove $\beta_n \in [-\infty, -1 + \alpha_n + \alpha_{n+1}]$ for all $n \in \mathbb{N}$. We first observe that in case of no vertical rearrangement ($\beta_n^* < 0$) we have $\beta_n < \beta_{n-1}$, so that we need to prove the result only in case of vertical rearrangement ($\beta_n < 0 < \beta_n^*$). Definition (8.3) of β_n^* entails $1/\beta_n^* = 1/\beta_{n-1} + 1 \leq 1$. It follows $\beta_n^* \geq 1$, so that $\beta_n = -\beta_n^* + \alpha_n + \alpha_{n+1} \leq -1 + \alpha_n + \alpha_{n+1}$ (which is < 0 if $n \geq 1$). $\qquad \square$

In connection with the next result, it is useful to notice that a segment/line of slope γ is transformed via $\mathcal{R}$ into a segment/line of slope γ' with $\frac{1}{\gamma'} = \frac{1}{\gamma} + 1$ (with the position $\frac{1}{0} = \infty$).

Proposition 8.2 (**Slopes of polygonals**). *For any $n \geq 0$ we have:*
- (i) *points g_n and G_{n+1} lie of a line having slope α_n;*
- (ii) *the segment with endpoints G_{n+1}^*, g_{n+1}^* has slope β_n^*;*
- (iii) *the segment with endpoints G_{n+1}, g_{n+1} has slope β_n;*
- (iv) $\Delta_n^* > 0 \iff \beta_n^* > 0$;
- (v) $m_{n+1} < M_{n+1}$;
- (vi) $t_n < T_{n+1}$.

Proof. To prove (i)-(iv) we argue by induction. All assertions (i) to (iv) are true for $n = 0$. Suppose that all claims are true for n and let us prove them for $n + 1$. Segment s from g_n^* to G_{n+1}^* is obtained from segment from g_{n-1} to G_n via the affine transformation $\mathcal{R}$, hence it has inverse slope given by $\frac{1}{\alpha_{n-1}} + 1$, so that its slope is α_n. Point G_{n+1} (resp. g_n) either coincides with G_{n+1}^* (resp. g_n^*), the right (resp. left) endpoint of s, or, in case of vertical rearrangement, can be checked to lie on segment s using the first equation in (8.7) (resp. the second equation taking n in place of $n + 1$). In either case the corresponding segment from g_n to G_{n+1} has itself slope α_n. Likewise segment r with endpoints G_{n+1}^*, g_{n+1}^* has inverse slope $\frac{1}{\beta_{n-1}} + 1$, i.e. slope β_n^*. This entails $\Delta_n^* > 0 \iff \beta_n^* > 0$. In case of no vertical rearrangement ($T_{n+1}^* \leq t_{n+1}^*$) we have $\beta_n = \beta_n^*$ and we have finished. In case of vertical rearrangement we have both $\Delta_n^* > 0$ and $\beta_n^* > 0$, the slope of $G_{n+1}g_{n+1}$ can be obtained by computing the vertical displacement of a point that starts from G_{n+1}, moves to G_{n+1}^* along a segment with slope α_n, then to g_{n+1}^* (slope β_n^*), then to g_{n+1} (slope α_{n+1}). The horizontal displacements have size Δ_n^*, $-\Delta_n^*$, Δ_n^* for a combined slope of $\alpha_n + \alpha_{n+1} - \beta_n^*$, consistently with the definition of β_n (Figure 5). This concludes the proof of (i)-(iv).

We are left with (v) and (vi), which we also prove by induction, observing that the claims are true for $n = 0$.

(v). If $\Delta_n^* \leq 0$ we have $m_{n+1} = m_n < M_n = M_{n+1}$. If $\Delta_n^* > 0$ we use that $\beta_n < 0$ (Lemma 8.1) and item (iii): we have $m_{n+1} - M_{n+1} = \beta_n \Delta_n^* < 0$.

(vi). Since $m_{n+1} \geq m_n$ we get $M_{n+1} > m_n$ and from item (i) and the fact that $\alpha_n > 0$ we directly obtain

$$T_{n+1} - t_n = \frac{1}{\alpha_n}(M_{n+1} - m_n) > 0. \tag{8.17}$$

$\qquad \square$

The sequence of points so constructed defines the polygonal $\mathcal{P}$ which is the graph of a function ℓ. We finally prove:

Theorem 8.1. *The function ℓ so defined verifies* (4.1).

Proof. In view of Section 4.1, we only need to compute the vertical rearrangement of the multivalued function having polygonal $\mathcal{P}^*$ as graph. We have vertical rearrangement only in segments (T_{n+1}, t_{n+1}) where $\Delta_n^* > 0$. Since each of the three values varies linearly, the result can be computed by linear interpolation between the values M_{n+1} and m_{n+1}, giving, except for the interval $[0, 1]$, the polygonal $\mathcal{P}$ as graph. The result follows after the addition of u_0. $\square$

In the end, $(t_n)_{n\geq 1}$ is the sequence of positive local minima of ℓ, $m_n := \ell(t_n)$, $T_{n+1} \in (t_n, t_{n+1}]$ is the local maximum of ℓ in $[t_n, t_{n+1}]$, and $M_{n+1} := \ell(T_{n+1})$. Equality $T_{n+1} = t_{n+1}$ holds only at a jump, where $\ell(t_{n+1})$ (resp. $\ell(T_{n+1})$) indicates here the smallest (resp. largest) value of ℓ.

Remark 8.3 (Recovering geometrically the solution from ℓ). Having constructed, although not in closed form, function ℓ, we are in a position to recover from it the solution $v(\tau, \cdot)$ at any given positive time τ. To this aim we can resort to Remark 5.1, which provides a convenient way to achieve our goal. Particularly important is the term $|\{s \in [\tau, \xi] : \ell(s) \leq \xi - s\}|$ in formula (5.7), which can actually be written in a slightly different way, if we consider the image

$$A^* := \mathcal{R}(\mathrm{subgraph}_+(\ell)).$$

A^* is bounded by the polygonal $\mathcal{P}^*$ and the positive τ-axis, it gets intersected with the vertical line lifted from abscissa ξ, and more specifically

$$|\{s \in [\tau, \xi] : \ell(s) \leq \xi - s\}| = |\{(\xi, s) : s \in [0, \xi - \tau]\} \setminus A^*|$$

which, in view of (4.1) and (4.2), can be recognized as a "partial" vertical rearrangement of A^* happening only inside $\overline{\Pi}_{\mathrm{oct}}$.

This is illustrated in Figure 6, containing the evolution of the present Example at specific times. At time $\tau = 0.8$ the "zig-zag" in polygonal $\mathcal{P}^*$, in particular the triangular "void" that is bound to be filled by the vertical rearrangement, is completely below the dashed line with slope 1. This implies that no rearrangement is taking place yet in the reconstruction of $v(\tau, \cdot)$ and we simply use the top segment in the zig-zag of $\mathcal{P}^*$. This gives the same result as re-adding the size of the void triangle intersected with the vertical line at ξ to the quantity $\ell(\xi) - (\xi - \tau)$. At time $\tau = 1.4$ the zig-zag (the "void" triangle) is partially above the dashed line with slope 1. This implies partial vertical rearrangement of only the part of A^* lying above this line, which is the same as re-adding to $\ell(\xi) - (\xi - \tau)$ that part of the vertical segment at ξ that is in the void triangle and below the line at 45 degrees. At time $\tau = 1.8$ the zig-zag is completely above the line at 45 degrees, so that there is no portion of the void triangle below that line and there is no contribution from $\{(\xi, s) : s \in [0, \xi - \tau]\} \setminus A^*$ in the reconstruction of v. Finally, time $\tau = 3.2$ illustrates the situation where there is no vertical rearrangement at all in function ℓ. Clearly in this case we also have no contribution from $\{(\xi, s) : s \in [0, \xi - \tau]\} \setminus A^*$.

It should be noted that the Lyapunov functional decreases strictly exactly when we have only partial rearrangement (e.g. in a neighborhood of time $\tau = 1.4$ in Figure 6).
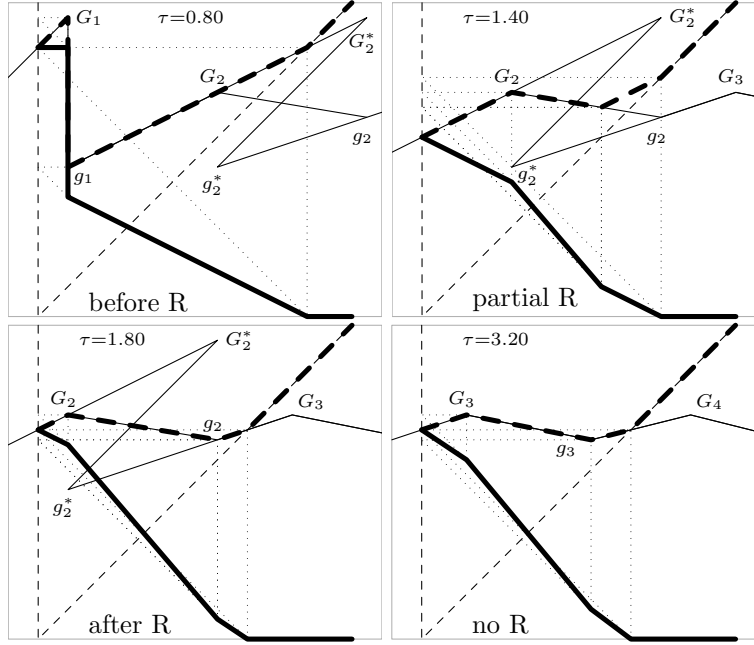
FIGURE 6. Various typical situations in the reconstruction of $v(\tau,\cdot)$ at specific times during the evolution of the Riemann problem. The bold polygonal represents the graph of $v(\tau,\cdot)$, the thin lines display part of the polygonals $\mathcal{P}$ and $\mathcal{P}^*$ (defined by (8.10) and (8.9) respectively), whereas the bold dashed line is the graph of $v(\tau,\xi)+(\xi-\tau)$. R stands for (vertical) rearrangement.

Once ℓ is known, the solution u is uniquely determined via Theorem 5.1 (and using the map Φ^{-1}); for instance (see Figs. 7 and 8):

$$u(t,x) = \begin{cases} 1+t & t \in [0,1),\ 0 \le x < 1-t, \\ -\frac{1}{2}x + \frac{1}{2}t + \frac{1}{2} & t \in [0,1),\ 1-t < x \le 1+t, \\ 0 & t \in [0,1),\ x \ge 1+t. \end{cases} \qquad (8.18)$$

Note that u is discontinuous along $\{(t,x) : t \in [0,1], x+t=1\}$. Also

$$\lim_{s\uparrow 1} u(s,x) = \left(1 - \frac{1}{2}x\right) \vee 0, \qquad x \in (0,+\infty),$$

which is piecewise affine and Lipschitz, and

$$u(t,x) = \begin{cases} \frac{t}{2} - \frac{1}{2}(x-1) & 0 \le x \le 2-t, \\ \frac{11}{6} - \frac{7}{6}x - \frac{t}{6} & 2-t < x \le \frac{t}{2} + \frac{1}{2}, \\ \frac{3}{2} - \frac{x}{2} - \frac{t}{2} & \frac{t}{2} + \frac{1}{2} < x \le 3-t, \\ 0 & x > 3-t, \end{cases} \qquad t \in [1, 5/3]. \qquad (8.19)$$

Recalling (6.2), one checks that $\tau \in [0,+\infty) \to \mathfrak{L}(\widehat{u}(\tau))$ is Lipschitz and non-increasing, strictly decreasing (quadratically) only on those intervals where L_ℓ becomes strictly decreasing. In particular, since $\mathfrak{L}(u_0) = 1$, $\mathfrak{L}(u(1^-,\cdot)) = 1$, it follows
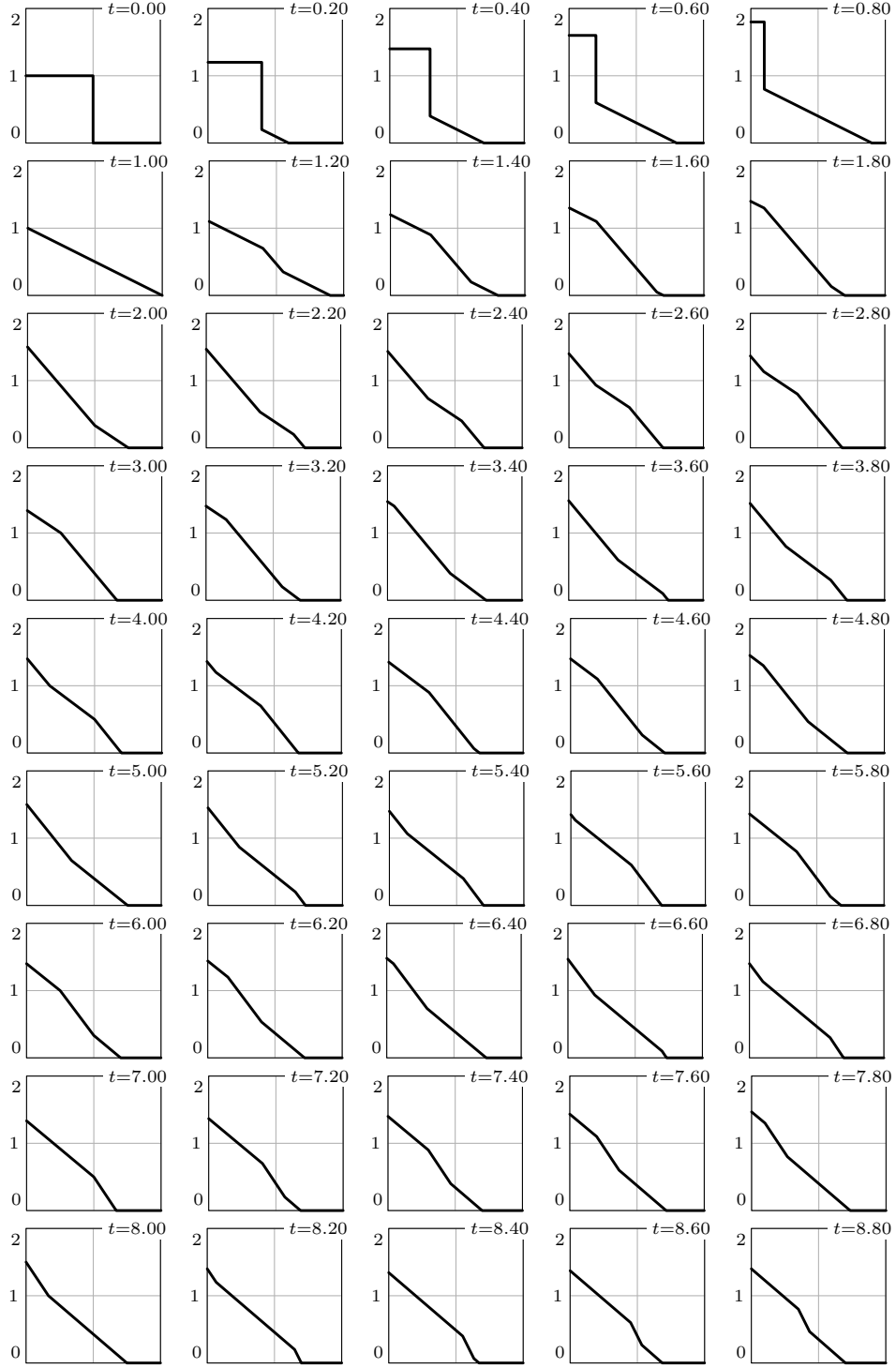
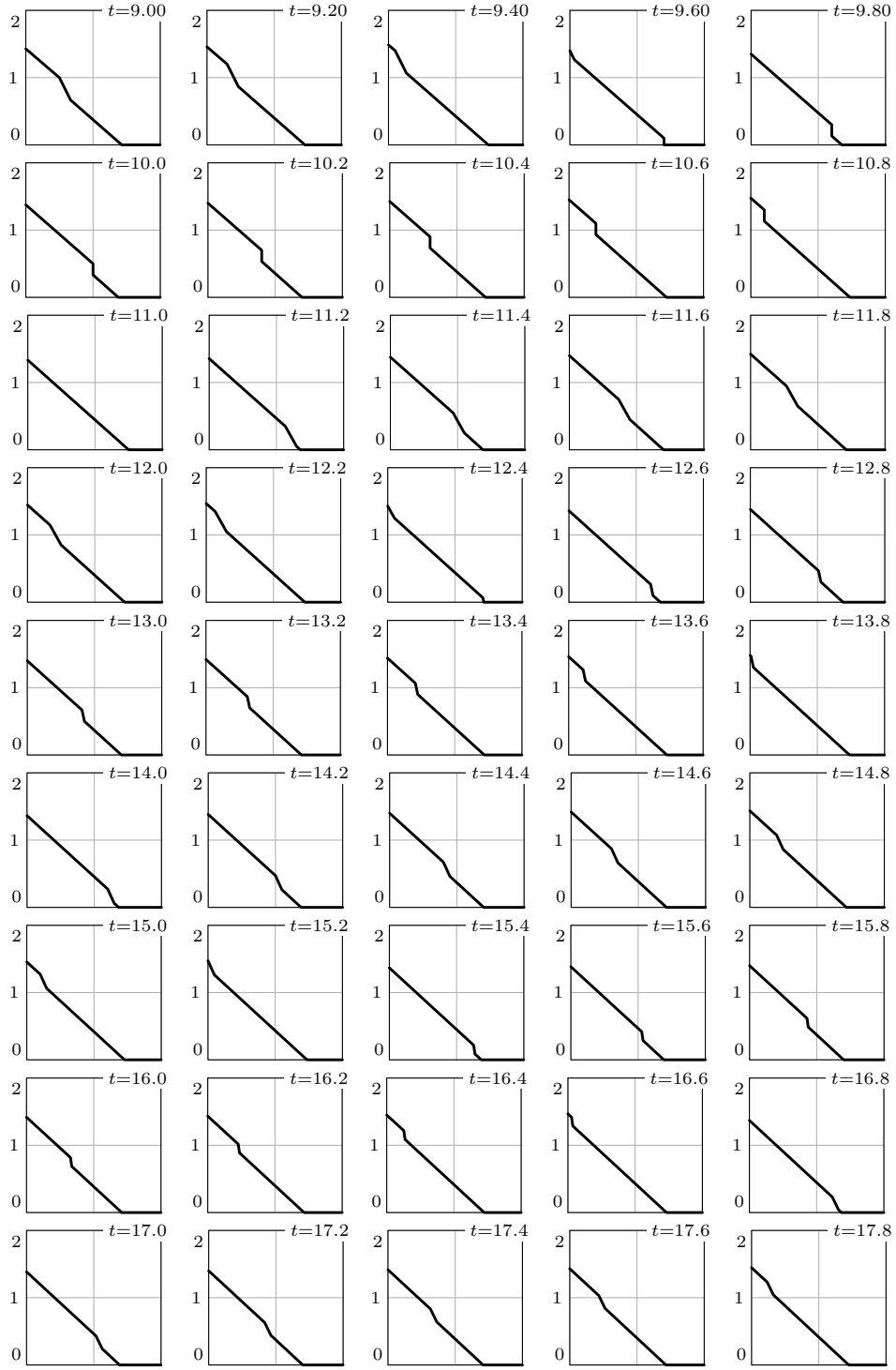FIGURE 7. Time-movie of the solution of the Riemann problem ($u_0 = 1_{[0,1)}$), up to time $t = 8.80$.

FIGURE 8. Time-movie of the solution of the Riemann problem ($u_0 = 1_{[0,1)}$), from time $t = 9.00$ up to time $t = 17.8$.

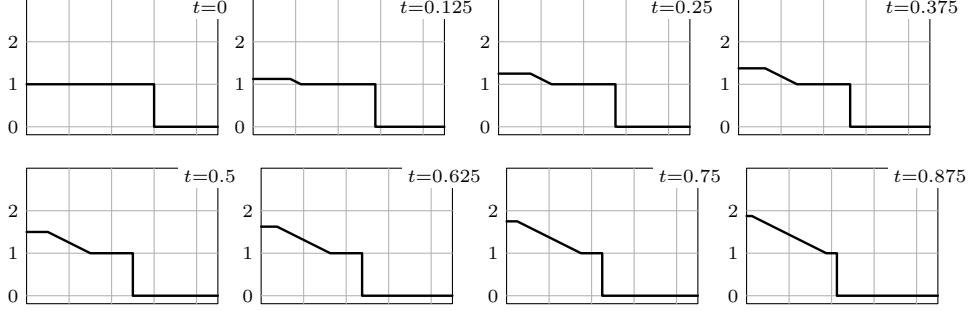FIGURE 9. Time-slices of the solution $u_1(t,x)$ described in Example 8.2 ($a = 3$), for t in the allowed range: the slope of the oblique segment is $-1/2$.

$\mathfrak{L}(u(t,\cdot)) = 1$ for all $t \in [0,1]$. In addition $\mathfrak{L}\left(u\left(\frac{5}{3},\cdot\right)\right) = \frac{17}{18} < 1$. For $t \in (5/3, 11)$ the value $\mathfrak{L}(u(t,\cdot)$ remains constant, and next it decreases slightly in the interval $[11, 11 + 3/20]$.

8.1. Asymptotic properties of the solution.

A direct consequence of the results of the previous section is the convergence of the solution to a stationary configuration (described in Example 7.1) in infinite time.

From (v) of Proposition 8.2 it follows

$$1 < \lambda^- := \lim_{n \to +\infty} m_n \le \lambda^+ := \lim_{n \to +\infty} M_n < 2. \tag{8.20}$$

This immediately leads to the desired convergence property as $t \to +\infty$.

Proposition 8.3 (**Asymptotic convergence to the stationary solution**). *We have*

$$\lambda^+ = \lambda^- = \sqrt{2} \tag{8.21}$$

and $\lim_{t \to +\infty} \|u(t,\cdot) - u_{\text{stat}}(\cdot)\|_{L^\infty([0,+\infty))} = 0$, *where* $u_{\text{stat}}(x) = \max\{\sqrt{2} - x, 0\}$ *is the stationary solution with unit mass.*

Proof. From the definition we have $T_{n+1} \le T^*_{n+1}$. Using $T_n \le t_n$ (Proposition 8.1) we obtain the estimate

$$T_{n+1} - t_n \le T^*_{n+1} - T_n = M_n \le 2.$$

Then (8.21) follows from

$$\lim_{n \to \infty} (M_{n+1} - m_n) = \lim_{n \to \infty} \alpha_n(T_{n+1} - t_n) \le 2 \lim_{n \to \infty} \alpha_n = 0,$$

where we use (8.17).

Hence $\lim_{t \to +\infty} \sup_{s > t} |\ell(s) - \sqrt{2}| = 0$, which, from (5.1), implies the desired convergence of the solution to u_{stat}. $\square$

8.2. Final examples.

Example 8.1 (**Parent of** $1_{[0,1)}$**: creation of a decreasing jump**). Let

$$u_0(x) = (2 - 2x) \vee 0 \qquad \forall x \ge 0.$$

We have $\ell(t) = -t + 2$ for $t \in (0,1]$, and $\ell(t) = t$ for $t \in [1,2]$. Applying transformation $\mathcal{R}$ for $t \in [0,1]$ produces the vertical segment $\{1\} \times [1,2]$. We have

$$u(1, \cdot) = 1_{[0,1)}.$$

The flow for times larger than 1 is then the same as the flow of the Riemann problem.

Example 8.2 (**Riemann problem with a parameter**). Let $u_0 = 1_{[0,a)}$, with $a > 1$. Let T be obtained as the intersection between the lines $\{x - t = 1\}$ and $\{x + t = a\}$, i.e., $T = \frac{a-1}{2}$, and set $T_1 = \min(1, \frac{a-1}{2})$. The solution is given, for any $(t, x) \in [0, T_1] \times (0, +\infty)$ by

$$u(t,x) = \begin{cases} 1 + t & \text{if } x \leq 1 - t, \\ -\frac{x}{2} + \frac{t}{2} + \frac{3}{2} & \text{if } 1 - t \leq x \leq 1 + t, \\ 1 & \text{if } 1 + t \leq x \leq a - t, \\ 0 & \text{if } x + t \geq a. \end{cases}$$

Stationary solutions can be reached also in finite time, as shown in this final example; notice that here the positivity condition (4.4) on u_0 is not satisfied.

Example 8.3 (**Reaching a stationary solution in finite time**). Let u_0 be the linear interpolation of $u_0(0) = 0$, $u_0(1) = 1/2$ and $u_0(2) = 0$. Then

$$\ell(t) = \begin{cases} t & \text{if } t \in [0,1], \\ 1 & \text{if } t > 1, \end{cases}$$

and the corresponding solution u reaches the stationary solution $\max\{1 - x, 0\}$ at time $t = 1$.

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Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, 53100 Siena, Italy, and International Centre for Theoretical Physics ICTP, Mathematics Section, 34151 Trieste, Italy
   Email address: `bellettini@diism.unisi.it`

Université Côte d'Azur, Inria, CNRS, Laboratoire I3S, Maasai team, Nice, France
   Email address: `alessandro.betti@inria.fr`

Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore, 25121 Brescia, Italy
   Email address: `maurizio.paolini@unicatt.it`