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# From Condorcet's paradox to Arrow: yet another simple proof of the impossibility theorem

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**Abstract**
Condorcet exposed a limitation of majority-based pairwise comparison by showing that, for specific preference profiles over three alternatives, it leads to a contradiction. Arrow's theorem is often introduced as a generalization of this finding. A novel formulation of the proof is presented that strictly adheres to this logic, establishing that for any non-dictatorial social choice rule, a contradiction-generating preference profile always exists and can be identified using a straightforward procedure.

## 1 Introduction

More than 70 years after its publication, there are many different proofs available of Arrow's impossibility theorem. Some of them are particularly concise and compact (e.g. Barberá 1980; Fishburn 1970; Suzumura 1988, ), and some have been explicitly advanced for "pedagogical" purposes, i.e. with the goal of reducing the amount of abstract deductive reasoning in order to make the theorem more accessible to students and casual readers (Denicolò 1996; Dardanoni 2001).[1] This paper aims to be included in the latter group.

The proposed approach closely mirrors the way the limits of social choice are commonly introduced to students, beginning with the majority voting (Condorcet's) paradox. Typically, majority voting is presented as a map from configurations of individual pairwise rankings to social pairwise rankings. It is then demonstrated that, for certain profiles of individual preferences, majority fails to produce a social order because the resulting pairwise social rankings lead to a logical contradiction (i.e., they violate transitivity).

[1] Most proofs of Arrows theorem rely on one of the two following strategies: either they define a decisive voter set which is progressively shrunk by reverse induction until it include only a dictator (Arrow 1963); or they define a pivotal voter and show that such voter must be a dictator (Barberá 1980; Geanakoplos 2005).

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Arrow's result is often introduced as a generalisation of Condorcet's paradox to a generic map from profiles of individual preferences to social preferences. In this respect, it would be conceptually consistent to develop a formal proof of the impossibility theorem that mirrors the logic used to illustrate the voting paradox. This is what we do in this paper. Namely, after showing that the IIA condition restricts social choice rules to those representable as maps from individual pairwise to social pairwise rankings, we prove that for any such map, there always exists a profile of individual preferences for which the map fails to produce a well-defined (i.e., transitive) social order.

In the case of three individuals, the result can be established by directly examining the entire set of possible social rules: for each rule, identifying the profile that leads to a contradiction (a "contradicting" profile) is straightforward. This proof "by exhaustion" is feasible because, under Arrow's condition, the set of admissible social welfare functions is reduced to a manageable size.

With more than three individuals, after demonstrating that a contradicting profile always exists (which amounts to proving the theorem), we present a simple procedure to easily identify such a profile. Notably, the proof for the general case is direct for any $n \geq 3$ and does not rely on induction.

The analysis is made easier by the restriction to the case of linear (or strict) individual preferences.[2] Limiting pairwise preferences to two values ('$\succ$' and '$\prec$') enables us to represent each profile and social rule as a binary sequence and its corresponding decimal representation. Consequently, some conditions can be translated into arithmetic operations on integers and binaries (including bitwise operations on integers, commonly supported by most programming languages).

The paper is organized as follows. Section 2 introduces the necessary notation, Sect. 3 presents Arrow's conditions and the theorem within our formal setup, Sect. 4 details the proof and the final section shows how the proof can be translated into a procedure to identify a "contradicting profile" for any possible social rule.

## 2 Preference profiles: notation and setup

Consider a society with $n$ individuals with preferences represented by a strict (or linear) order over a set of three alternatives $x$, $y$ and $z$. The assumption of only 3 alternatives is not a real limitation, as any inconsistency among 3 alternatives carries on to a larger set.[3] For each of the (three) possible pairwise comparisons, it is either $x \succ y$ or $x \prec y$.

[2] This makes our analysis close to the approach of aggregation introduced by Wilson (1975) and more recently revived by the analysis of binary evaluation by Dokow and Holzman (2010).

[3] Because the proof is based on the identification of a contradicting profile for each (non-dictatorial) social welfare function, it suffices to demonstrate that such profile exists within a subset of the set of possible individual preference profiles. Thus, with a greater number of alternatives, attention can be narrowed to the subset of preference profiles where individuals disagree only in their ranking of three alternatives (while unanimously agreeing on the ranking of all remaining alternatives), treating these three alternatives as the exclusive focus of analysis.

**Table 1** Matrix representation of a preference profile

| | | $w^1$ 3 | $w^2$ 8 | $w^3$ 14 |
|---|---|---|---|---|
| individual | 1 | 0 | 1 | 1 |
| | 2 | 0 | 0 | 1 |
| | 3 | 1 | 0 | 1 |
| | 4 | 1 | 0 | 0 |

We will indicate the preferences with 0 or 1 as follows:

$$x \xrightarrow[\substack{0=\prec\\1=\succ}]{} y \xrightarrow[\substack{0=\prec\\1=\succ}]{} z \xrightarrow[\substack{0=\prec\\1=\succ}]{} x \tag{1}$$

Therefore, preferences are fully described by an ordered 3-ple of 1 s and 0 s. For example, (1, 1, 0) means $x \succ y \succ z$, while (0, 0, 1) is $z \succ y \succ x$, and so on. Note that (1, 1, 1) and (0, 0, 0) are not valid preferences, as they violate transitivity, so the set $R$ of preference orders includes the following 3-ples:

$$\begin{array}{llll} (0,0,1) & z \succ y \succ x & (0,1,0) & y \succ x \succ z \\ (0,1,1) & y \succ z \succ x & (1,0,0) & x \succ z \succ y \\ (1,0,1) & z \succ x \succ y & (1,1,0) & x \succ y \succ z. \end{array} \tag{2}$$

A profile of preferences for the $n$ individuals can be described by a $n \times 3$ matrix where row $i$ represents individual $i$'s preferences and each column describes how the $n$ individuals compare the pair of alternatives identified by $k = 1, 2, 3$. Each column can be represented concisely by a 3-ple of integers $(w^1, w^2, w^3)$, where $w^k$ corresponds to the decimal representation of the column binary vector of the same matrix (Table 1). We will indicate by $w^k_{[i]}$ $i$th digit of the binary representation of $w^k$, so that $w^k_{[i]}$ is the value (0 or 1) at the $i$th row and $k$th column of the matrix.

Define $\delta_n = 2^n - 1$ (hence: $\delta_3 = 7, \delta_4 = 15$, and so on); $w^k$ takes on values in the set $I_n = \{0, 1, 2, \ldots \delta_n\}$ of cardinality $|I_n| = 2^n$, with a minimum $w^k = 0$ corresponding to the binary $n$-ple $(0, 0, \ldots, 0)$ and a maximum $w^k = \delta_n$ corresponding to the binary $n$-ple $(1, 1, \ldots, 1)$. The values $w^k = 0$ and $w^k = \delta_n$ indicate unanimous agreement of the individuals about the pairwise comparison $k$. For reference, the possible configurations of preferences over a pair of alternatives corresponding to $w \in I_n$ for $n = 3$ and $n = 4$ are listed in Table 2.

We will indicate by $W_n \subset I_n \times I_n \times I_n$ the set of 3-ples $(w^1, w^2, w^3)$ corresponding to individual profiles. It is important to emphasize that, although each $w^k$ can take any value in $I_n$, not all 3-ples $(w^1, w^2, w^3)$ with $w^k \in I_n$ correspond to a preference profile. The reason is that the we must exclude 3-ple that, for some $i$, satisfy $w^1_{[i]} = w^2_{[i]} = w^3_{[i]}$. Clearly, the cardinality of $W_n$ is $|W_n| = 6^n$.[4]

[4] Each element in $W_n$ corresponds to a possible arrangement of $n$ 3-ples among those listed in (2), describing a profile of individual preferences.

**Table 2** Possible values of $w^k$ for $n = 3$ and $n = 4$

| $I_3$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| individual 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| individual 2 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| individual 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

| $I_4$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| individual 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| individual 2 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| individual 3 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| individual 4 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

## 3 Social welfare functions and Arrow's theorem

Given a society of $n$ individuals, a social welfare function (SWF) $S : W_n \to R$ associates to each $(w^1, w^2, w^3) \in W_n$ a (social) preference order, where the latter is represented by a binary 3-ple different from $(0, 0, 0)$ and $(1, 1, 1)$. Note that this definition of SWF incorporates the condition of universality of domain.[5]

Crucial to the analysis will be the function $s_k : I_n \to \{0, 1\}$, which we will refer to as *pairwise comparison rule* (PCR). To each vector $w^k \in I_n$ of pairwise individual preferences over two alternatives, this function associates a pairwise preference that we can interpret as the social preference over those alternatives. Clearly, a 3-ple of pairwise comparison rules $(s_1, s_2, s_3)$ which for each $(w^1, w^2, w^3) \in W_n$ maps to an element in $R$ is a SWF. Restricting admissible SWFs to the subset of SWFs that can be expressed as a 3-ple of pairwise comparison rules is equivalent to imposing one of Arrow's conditions.

**Definition** (IIA) A social welfare function satisfies the condition of Independence of irrelevant alternatives (IIA) if and only if it can be represented by a 3-ple of pairwise comparison rules $(s_1, s_2, s_3)$.

Another condition is consistency with Pareto (or unanimity), which we can define with reference to a PCR:

**Definition** (P) A pairwise comparison rule $s_k$ satisfies the Pareto condition if and only if $s_k(0) = 0$ and $s_k(\delta_n) = 1$.

For a SWF satisfying IIA, a further restriction on the admissible PCR follows from the fact that each $s_k$ satisfies P:

**Theorem 1** *If* $(s_1, s_2, s_3)$ *with each* $s_k$ *satisfying P is a SWF, then, for all* $j, k \in \{1, 2, 3\}$ *and for all* $m \in I_n$*, it is*

[5] The restriction that also social preferences must be linear is not a real limitation, as it can be proved that linear individual preferences do not allow social indifference (Denicolò 1996).

*a)* $s_j(m) = s_k(m)$*;*
*b)* $s_j(m) = 1 - s_k(\delta_n - m)$.

***Proof*** Consider that, for any $m, m' \in I_n$ with $m + m' = \delta_n$, it is $m_{[i]} = 1 - m'_{[i]}$.[6] Therefore, $(m, m', \delta_n)$, $(m, m', 0)$, $(\delta_n, m', m)$ and $(0, m', m)$ all belong to $W_n$, i.e., they are all preference profiles. Because of P, it is $s_k(\delta_n) = 1$ and $s_k(0) = 0$. Then, whatever the value of $s_2(m')$, it must be $s_1(m) = s_3(m) = 1 - s_2(m')$, otherwise it will be $(s_1, s_2, s_3) = (1, 1, 1)$ or $(s_1, s_2, s_3) = (0, 0, 0)$ for one of the four preference profiles above, contradicting the assumption that $(s_1, s_2, s_3)$ is a SWF. With a similar argument we can prove that $s_1(m) = s_2(m) = 1 - s_3(m')$, from which both *a*) and *b*) follow. □

Property *a*) corresponds to Neutrality, which essentially amounts to non-discrimination among alternatives.[7] Namely, we require that if all individuals rank a pair of alternatives the same way they rank another pair, the social ranking for the two pairs must be the same.

Neutrality allows us to get rid of the index $k$ in $s_k$ and write simply $s(w^k)$; thus, a SWF is fully identified by a single pairwise comparison rule $s : I_n \to \{0, 1\}$, i.e. by a binary $2^n$-ple $(s(0), s(1), \ldots, s(\delta_n))$.

We can define dictatoriality with reference to $s$[8]:

**Definition** (D) A pairwise comparison rule $s$ is dictatorial if and only if if there exists $i \in \{1, \ldots, n\}$ such that, for all $m \in I_n$, we have $s(m) = m_{[i]}$.

We can now state:

**Theorem** (Arrow's impossibility theorem) *Consider* $s : I_n \to \{0, 1\}$ *satisfying P. Then* $(s, s, s)$ *is a SWF if and only if* $s$ *is dictatorial.*

The proof that when $s$ is dictatorial $(s, s, s)$ is a SWF is straightforward, given that an individual preference profile $(w^1_{[i]}, w^2_{[i]}, w^3_{[i]})$ must differ from $(0, 0, 0)$ and $(1, 1, 1)$,

The proof that no other pairwise comparison rule will make a SWF amounts to proving that, for any $s$ that is not dictatorial, there exists a profile $(w^1, w^2, w^3) \in W_n$ for which $s(w^1) = s(w^2) = s(w^3)$. We will refer to such a profile as a *contradicting* profile for $s$.

## 4 Proof of the theorem

The possible specifications of a function $s : I_n \to \{0, 1\}$ are $2^n$. However, the requirement that $s(m) = 1 - s(\delta_n - m)$ reduces to half the degrees of freedom in the specification of $s$ (as shown in Table 3 for $n = 3$), while condition P forces $s(0) = 0$ and $s(\delta_n) = 1$.

[6] For example, with $n = 7$ and $\delta_7 = 127$, consider 100 and 27. Their binary representations, respectively 1100100 and 0011011, are complementary one another.

[7] Blau (1972) proved neutrality holds for SWFs satisfying both IIA and P (Ubeda 2003, see also).

[8] Consistently to our previous notation, we indicate by $m_{[i]}$ $i$th digit of the binary representations of $m \in I_n$. For example, with $n = 7$ and $m = 50$, whose binary representation 0110010, we have $m_{[1]} = 0$, $m_{[2]} = 1$, $m_{[3]} = 1$, etc.

**Table 3** An example of $s$ satisfying condition $s(m) = 1 - s(\delta_n - m)$ with $n = 3$

| $m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $s(m)$ | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |

Hence, the set $S_n$ of all functions $s : I_n \to \{0, 1\}$ satisfying P and IIA has cardinality $|S_n| = 2^{2^{n-1}-1}$. With $n = 3$ individuals, the number of possible functions reduces to only $|S_3| = 8$.[9] Before delving into a general proof, we will demonstrate the theorem in this simpler case.[10]

### 4.1 Proof for the case of $n = 3$

When $n = 3$, the limited number of admissible PCR allows us to prove the theorem by exhaustion.

The set $S_3$ is shown in Table 4. Pairwise comparison rules can be identified by an integer (corresponding to the decimal representation of the first four binary digits), but we also name them with mnemonics that refer to the underlying collective decision rules. D1, D2, and D3 correspond to the dictatorial rules, where social preferences coincide with the preferences of one of the three individuals. MA identifies the majority rule, while MI is a "minority rule" where, in case there is no unanimity, social preferences reflect the preference of a minority (and contradict that of a majority) of individuals. Under rules N1, N2, and N3, social preferences always contradict the preferences of one of the individuals (i.e. the individual is an "inverse dictator"), except in the case of unanimous agreement.

We see that (3, 5, 6) is a contradicting profile for the majority rule MA; indeed, (3, 5, 6) corresponds to a profile generating the Condorcet paradox. The same profile excludes MI.[11]

In a similar fashion, N1, N2, and N3 must be rejected because the 3-ples (1, 2, 7), (1, 4, 7) and (2, 4, 7) are, respectively, contradicting profiles.

[9] However, we must be aware that this number grows very fast with $n$: it is $2^7 = 128$ with $n = 4$, and $2^{15} = 32768$ with $n = 5$, but arrives to $2^{31}$ (i.e. more than 2 billions) with $n = 6$. On the other hand, a comparison of $|S_3|$ with the set of possible SWFs for a society of three, i.e. all functions from $W_3$ to $\{1, \ldots, 6\}$, whose numerosity for $n = 3$ is $6^{216} > 1.2 \cdot 10^{168}$, is suggestive of how powerful the two conditions P and IIA are.

[10] Given our focus on the relation with majority rule, we consider the case $n = 2$ uninteresting for our analysis. However, the proof is particularly easy in this case. $I_2$ contains only (0, 0), (0, 1), (1, 0) and (1, 1), while $S_2$ includes only two functions. It is straightforward to verify that such functions imply the social ranking is identical to either one individual's ranking or the other's, making them dictatorial.

[11] There is usually more than one profile leading to a contradiction; for example, we have that (1, 2, 4) implies (0, 0, 0). In general, for each profile giving (1, 1, 1) we could find a profile giving (0, 0, 0) simply by taking $w'_k = \delta_n - w^k$ for each $k$.

**Table 4** The set $S_3$

| | | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $w =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| | $i$ =1 | | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| | 2 | | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| | 3 | | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| $S_3$ | 0 | D1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $S_3$ | 1 | MA | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 |
| $S_3$ | 2 | N3 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| $S_3$ | 3 | D2 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| $S_3$ | 4 | N2 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| $S_3$ | 5 | D3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| $S_3$ | 6 | MI | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| $S_3$ | 7 | N1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |

On the other hand, it is not possible to find a contradicting profile for rules D1, D2 and D3.

### 4.2 General proof for $n \geq 3$

We now prove that a contradicting profile exists for every non-dictatorial $s$ whatever the size $n$ of the society. The prove also provides a simple procedure to identify the profile that contradicts each PCR.

Let $I_n^s \equiv \{m \in I_n \mid s(m) = 1\}$ and, for $m \in I_n$, let $A(m) \equiv \{i \mid m_{[i]} = 1\}$ indicate the set of 1 s in the binary representation of $m$.

For any $s$, either of the following must be true:

1) there exists $m \in I_n^s$ such that $|A(m)| = 1$ (the numerosity of $A(m)$ is one);
2) there exists $m \in I_n^s$ such that $1 < |A(m)| < n$ and for no other $q \in I_n^s$ it is $A(q) \subset A(m)$.

We prove that, in both cases, if $s$ is not dictatorial, $m$ is part of a contradicting profile for $s$.[12]

1) Consider $m \in I_n^s$ with $m_{[i]} = 1$ and $m_{[j]} = 0$ for all $j \neq i$. For $s$ not to be dictatorial, there must exist $m' \in I_n^s$ such that $m'_{[i]} = 0$, but this implies that $(w^1, w^2, w^3) = (m, m', \delta_n)$ is a contradicting profile. To see that $(m, m', \delta_n)$ is indeed a preference profile, consider that $(m_{[i]}, m'_{[i]}, (\delta_n)_{[i]}) = (1, 0, 1)$ while $m_{[j]} = 0$ and $(\delta_n)_{[j]} = 1$ for $j \neq i$.

[12] In this proof, we identify a contradicting profile with $s(w^k) = 1$ for $k = 1, 2, 3$. In fact, from any such profile, by considering $w'_k = \delta_n - w^k$, it will always be possible to derive another profile $(w'_1, w'_2, w'_3)$ with $s(w'_k) = 0$ for $k = 1, 2, 3$.

**Table 5** The values in $I_7$ for which $s = 1$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15, | 23, | 27, | 29, | 30, | 31, | 39, | 43, | 45, | 46, | 47, | 51, | 53, | 54, | 55, | 57, |
| 58, | 59, | 60, | 61, | 62, | 63, | 71, | 75, | 77, | 78, | 79, | 83, | 85, | 86, | 87, | 89, |
| 90, | 91, | 92, | 93, | 94, | 95, | 99, | 101, | 102, | 103, | 105, | 106, | 107, | 108, | 109, | 110, |
| 111, | 113, | 114, | 115, | 116, | 117, | 118, | 119, | 120, | 121, | 122, | 123, | 124, | 125, | 126, | 127. |

2) Consider $m \in I_n^s$ and $1 < |A(m)| < n$, where for no other $q \in I_n^s$ we have $A(q) \subset A(m)$. Because $|A(m)| > 1$, there exist $m', m'' \in I_n$ different from $\delta_n$ such that $m'_{[i]} = m''_{[i]} = 1$ for $i \notin A(m)$ and $m''_{[i]} = 1 - m'_{[i]}$ for $i \in A(m)$.

Consider $q = \delta_n - m'$. Since $q_{[i]} = 1 - m'_{[i]}$ for all $i$, it will be $A(q) \subset A(m)$. It follows that $q \notin I_n^s$ and, as a consequence, $m' \in I_n^s$. With a similar argument, we conclude that $m'' \in I_n^s$. This implies that $(w^1, w^2, w^3) = (m, m', m'')$ is a contradicting profile for $s$. To see that it is indeed a preference profile, consider that $(m_{[i]}, m'_{[i]}, m''_{[i]}) = (1, m'_{[i]}, 1 - m'_{[i]})$ for $i \in A(m)$, while $(m_{[i]}, m'_{[i]}, m''_{[i]}) = (0, 1, 1)$ for $i \notin A(m)$.

This ends the proof.

## 5 Identifying the contradicting profile: examples

The proof directly provides a procedure to identify a contradicting profile for every possible non-dictatorial PCR $s$. Let $s$ be characterized by $I_n^s$, i.e. by the list of integers in $I_n$ which $s$ maps to one. We can describe the procedure as follows:

*a*) consider the lowest value $m$ in $I_n^s$;

*b*) if such value is $m = 2^r$ for some $r = 0, 1, 2, \ldots, n-1$ (such that $m_{[i]} = 1$ if and only if $i = n - r - 1$), then look for a value $m' > m$ in $I_n^s$ such that $m'_{[n-r-1]} = 0$: the contradicting profile is $(w^1, w^2, w^3) = (m, m', \delta_n)$;[13]

*c*) if $m \neq 2^r$ for any $r = 0, 1, \ldots, n-1$, take two integers $m'$ and $m''$ larger than $m$ and such that $m + m' + m'' = 2\delta_n$ (because the lowest $m$ in $I_n^s$ is no larger than $\delta_n/2$, we can take for example $m' = \delta_n - m + 1$ and $m'' = \delta_n - 1$). The contradicting profile is $(w^1, w^2, w^3) = (m, m', m'')$.

The procedure is best illustrated by an example. In a society of 7 individuals, the set $I_7$ has numerosity $2^7 = 128$ and there are $|S| = 2^{2^6-1} = 2^{63}$ possible PCRs, each described by a sequence of 128 zeros and ones.

Among these rules, to pick a familiar one, consider the majority rule, i.e. the rule that assigns $s(m) = 1$ if and only if $|A(m)| \geq 4$. The values in $I_7 = \{0, 1, \ldots, 127\}$ for which $s(m) = 1$ are listed in Table 5.[14]

[13] Admittedly, this step may become demanding when the numerosity of $I_n$ is high and we have no clue about the value $m'$. On a computer, it is possible to check the condition $m'_{[n-r-1]} = 0$ using the bitwise operation AND, provided by most programming languages: in Python and C++, bitwise AND on integers is performed by the operator `&` so that `q&m == 0` if and only if $A(m) \cap A(q) = \emptyset$.

[14] In Python, we can generate the list of values using the following one-liner script, which identifies all integers in $I_7$ whose binary representation has a a majority of 1s:

**Table 6** The profile (15, 113, 126)

| | | 15 | 113 | 126 |
|---|---|---|---|---|
| $i =$ | 1 | 0 | 1 | 1 |
| | 2 | 0 | 1 | 1 |
| | 3 | 0 | 1 | 1 |
| | 4 | 1 | 0 | 0 |
| | 5 | 1 | 0 | 0 |
| | 6 | 1 | 0 | 1 |
| | 7 | 1 | 1 | 0 |

**Table 7** The values in $I_7$ for which $s = 1$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4, | 5, | 7, | 9, | 11, | 13, | 14, | 16, | 18, | 20, | 21, | 24, | 26, | 27, | 28, | 29, |
| 31, | 34, | 36, | 39, | 40, | 41, | 42, | 43, | 44, | 47, | 49, | 50, | 51, | 54, | 57, | 58, |
| 59, | 60, | 61, | 63, | 65, | 71, | 72, | 74, | 75, | 79, | 81, | 82, | 89, | 90, | 92, | 94, |
| 95, | 97, | 102, | 104, | 105, | 108, | 110, | 112, | 115, | 117, | 119, | 121, | 124, | 125, | 126, | 127. |

Because the lowest value is $m = 15$, which is not a power of 2, we consider $m' = \delta_7 - m + 1 = 127 - 15 + 1 = 113$ and $m'' = 2\delta_7 - m' - m = 254 - 113 - 15 = 126$. We see that (15, 113, 126) is a contradicting profile.

In Table 6 we can verify that this 3-ple is a preference profile by writing the three values as binaries (in column) and checking that no row is (0, 0, 0) or (1, 1, 1).

In fact, let $w^1 = 15$ describe how individuals rank $x$ and $y$, $w^2 = 114$ how individuals rank $y$ and $z$, and $w^3 = 126$ how individuals rank $z$ and $x$. The profile describes the case in which: 3 individuals have preferences $y \succ z \succ x$, 2 individuals have preferences $x \succ z \succ y$, one individual has preferences $z \succ x \succ y$, and one individual has preferences $x \succ y \succ z$. This preference profile produces the Condorcet paradox, as each alternative always wins against another alternative with a majority of 4 to 3.

To offer a second example, we follow a different route: with $n = 7$, we randomly generate a $s \in S_7$, where the set $I_7^s = \{m \in I_7 \mid s(m) = 1\}$ is given in Table 7.

In this case, we see that the lowest $m$ with $s(m) = 1$ is 4, that is, $2^2$ (binary: 0000100). Following the step $b$) in the procedure, we look for $m' \in I_7^s$ such that $m'_{[5]} = 0$,[15] This condition is not satisfied by 5 and 7, but it is satisfied by $m = 9$ (binary: 0001001). Hence, the contradicting profile is (4, 9, 127).

```
[n for n in range(2**7) if sum(list(map(int,":07b".format(n))))>3]
```

[15] This search can be made easy using a computer: with Python, the set of $m \in I_7^s$ satisfy the condition is found with the command `[n in S for n&m==0]` where `S` is the list of values in Table 7 and `m=4`. The command produces the following list: 9, 11, 16, 18, 24, 26, 27, 34, 40, 41, 42, 43, 49, 50, 51, 57, 58, 59, 65, 72, 74, 75, 81, 82, 89, 90, 97, 104, 105, 112, 115, 121. Any of this values, together with 4 and 127, makes a contradicting profile for $s$.

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