Journal of Functional Analysis 290 (2026) 111317

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Nonlocal operators in divergence form and existence theory for integrable data

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ARTICLE INFO

Article history:
Received 12 May 2025
Accepted 11 December 2025
Available online 19 December 2025
Communicated by J. Wei

Keywords:
Nonlocal operator
Divergence form operator
Dirichlet boundary condition

ABSTRACT

We present an existence and uniqueness result for weak solutions of Dirichlet boundary value problems governed by a nonlocal operator in divergence form and in the presence of a datum which is assumed to belong only to $L^1(\Omega)$ and to be suitably dominated.

We also prove that the solution that we find converges, as $s \nearrow 1$, to a solution of the local counterpart problem, recovering the classical result as a limit case. This requires some nontrivial customized uniform estimates and representation formulas, given that the datum is only in $L^1(\Omega)$ and therefore the usual regularity theory cannot be leveraged to our benefit in this framework.

The limit process uses a nonlocal operator, obtained as an affine transformation of a homogeneous kernel, which recovers, in the limit as $s \nearrow 1$, every classical operator in divergence form.

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https://doi.org/10.1016/j.jfa.2025.111317
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1. Introduction

1.1. Nonlocal existence theory for integrable data

A long-established line of research focuses on the construction of solutions for suitable elliptic equations. In the traditional setting, the data of the equation are typically required to belong to some L^p-space with $p > 1$, as this allows for the application of regularity theory and standard functional analysis techniques.

When $p = 1$, these methods are typically inaccessible. For instance, as is well known, regularity theory fails in the presence of merely integrable data (see e.g. Sections 4.1 and 5.1 in [19]). Consequently, no general existence theory for second-order elliptic equations is available in this context.

However, the results presented in [3,4] established an existence theory for equations in divergence form under certain structural assumptions on the equation. These assumptions permit the implementation of a tailored uniform approximation, which produces the desired solution via a limit procedure.

One of the goals of this article is to extend this existence theory to the nonlocal setting. To this end, we consider an interaction kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ such that

$$\frac{c\,\chi_{[0,\varrho)}\big(|x-z|\big)}{|x-z|^{n+2s}} \leqslant K(x,z) \leqslant \frac{C}{|x-z|^{n+2s}}, \tag{1.1}$$

for some c, $C \in (0,+\infty)$, $\varrho \in (0,+\infty]$ and $s \in (0,1)$, where, as usual, $\chi_{[0,\varrho)}$ denotes the characteristic function of $[0,\varrho)$, that is

$$\chi_{[0,\varrho)}\big(|x-z|\big) := \begin{cases} 1 & \text{if } |x-z| \in [0,\varrho), \\ 0 & \text{otherwise.} \end{cases}$$

To be compatible with the divergence structure, it is convenient to assume the symmetry condition

$$K(x,z) = K(z,x) \quad \text{for any } x, z \in \mathbb{R}^n, \tag{1.2}$$

see e.g. [21, Section 2.5] and the references therein for more information about nonlocal operators with a divergence structure.

The nonlocal operator associated with the interaction kernel K can thus be written in the pointwise form

$$\begin{aligned} \mathcal{L}_K u(x) &:= \text{P.V.} \int_{\mathbb{R}^n} \big(u(x)-u(z)\big)\, K(x,z)\, dz \\ &= \text{P.V.} \int_{\mathbb{R}^n} \big(u(x)-u(x-y)\big)\, K(x,x-y)\, dy \\ &:= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon} \big(u(x)-u(x-y)\big)\, K(x,x-y)\, dy, \end{aligned} \tag{1.3}$$

where P.V. stands for the principal value notation.

We observe that the kernel $\mathbb{R}^n \ni y \mapsto K(x,x-y)$ is not assumed to be even in y. This lack of symmetry prevents, in principle, the pointwise calculation in (1.3) when $s \in \big[\frac{1}{2},1\big)$, even for smooth functions. Thus, when $s \in \big[\frac{1}{2},1\big)$, to make sense of the above setting it is customary to assume an "almost symmetry" condition for the kernel with respect to the variable y. Specifically, if one wishes to use the pointwise form of the operator in (1.3) it comes in handy to assume that[1]

$$\sup_{x \in \mathbb{R}^n} \int_{B_1} |y| \, \big|K(x,x+y) - K(x,x-y)\big|\, dy < +\infty, \tag{1.4}$$

see in particular Lemma A.2.

In this paper, however, we will not assume condition (1.4) to prove our existence results in Theorems 1.1 and 1.2, since they rely only on weak formulations. However, it will be convenient to pass through condition (1.4) when dealing with limit properties.

[1] Other technical conditions can be used to replace (1.4), see e.g. [21, Section 2.5.2] for an extensive discussion.

In the rest of this paper, we will denote by Ω an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. Also, we will denote by $H_0^s(\Omega)$ the space of functions in $H^s(\mathbb{R}^n)$ such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$ (see Section 2.1 for the details related to the functional setting).

In this framework, our existence result reads as follows:

Theorem 1.1. *Let $h : \mathbb{R} \to \mathbb{R}$. Suppose that*

$$h \text{ is continuous, odd and strictly increasing} \tag{1.5}$$

and let

$$\gamma := \lim_{t \to +\infty} h(t) \in (0, +\infty]. \tag{1.6}$$

Let $a,\ f \in L^1(\Omega)$. Assume that

$$\text{there exists } Q \in (0, \gamma) \text{ such that } |f(x)| \leqslant Q\, a(x) \text{ for a.e. } x \in \Omega. \tag{1.7}$$

Then, there exists a unique weak solution $u \in H_0^s(\Omega) \cap L^\infty(\Omega)$ of

$$\begin{cases} \mathcal{L}_K u + a h(u) = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.8}$$

In addition, we have that

$$\|u\|_{L^\infty(\Omega)} \leqslant h^{-1}(Q) \tag{1.9}$$

and

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \big(u(x) - u(z)\big)^2 K(x,z)\, dx\, dz \leqslant 2h^{-1}(Q) \left(\|f\|_{L^1(\Omega)} + Q\|a\|_{L^1(\Omega)}\right). \tag{1.10}$$

We stress that the result in Theorem 1.1 is new in the nonlocal setting, even in the special case in which $K(x,z)$ is, up to multiplicative constants, the homogeneous kernel $|x-z|^{-n-2s}$ (in this situation, the operator (1.3) reduces to the fractional Laplacian).

To prove Theorem 1.1, the following result, obtained by a minimization argument, will be also helpful:

Theorem 1.2. *Let $h : \mathbb{R} \to \mathbb{R}$ satisfy (1.5). Let $\eta \in L^\infty(\Omega)$ with $\eta \geqslant 0$ a.e. in Ω, and $\zeta \in L^2(\Omega)$.*

Then, there exists a weak solution $u \in H_0^s(\Omega)$ of

$$\begin{cases} \mathcal{L}_K u + \eta h(u) = \zeta & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.11}$$

1.2. Back to the classical case (from M to A)

The nonlocal result obtained in Theorem 1.1 is also useful to shed new light on the classical case, recovering it in the limit. This limit procedure is not standard and will require, on the one hand, the selection of "the most natural" fractional operator that recovers a given second-order elliptic operator in divergence form, and, on the other hand, a number of ad-hoc inequalities allowing one to implement this asymptotic technique. In this sense, we believe that the methodology introduced here can be advantageous in several situations, not only to recover known results in partial differential equations as a limit case of nonlocal equations (which has also the benefit of providing a unified scenario comprising both classical and fractional problems), but also potentially to obtain new results in partial differential equations by using techniques coming from the fractional world. This approach has been used, for instance, in the regularity theory "by approximation" performed in [10], in the asymptotic regularity theories of nonlocal minimal surfaces carried out in [12,8,13], as well as in the construction of nonlocal catenoids in [15].

To implement our strategy, we recall that second-order operators in divergence form are typically written in the form

$$-\operatorname{div}\big(A(x)\nabla u(x)\big) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right), \tag{1.12}$$

where[2] $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is symmetric, uniformly bounded and uniformly positive definite.

A natural question in this context is whether or not it is possible to introduce a nonlocal operator which recovers the structure of the operator in (1.12) through an asymptotic limit procedure (by focusing on the "simplest" possible structure of this nonlocal operator). This plan of action has been presented in [2, Section 3], where it was proven that the operator in (1.12) can be recovered starting from a "master equation" (see e.g. [11]). However, our approach demands more accuracy on the assumptions needed to make this procedure work, to enable us to offer a unified view of local and nonlocal cases and to recover the local case starting from the nonlocal one through a limit process.

The "simplest possible" approach in this framework consists in[3] taking $\varrho \in (0, +\infty]$ and $M : \mathbb{R}^n \times B_\varrho \to \mathbb{R}^{n\times n}$ and in "modulating" a fractional interaction kernel by means of the matrix-valued function M.

More precisely, we choose

$$K(x,z) := \frac{c_{n,s}\,\chi_{[0,\varrho)}\big(|x-z|\big)}{|M(z,x-z)(x-z)|^{n+2s}}, \tag{1.13}$$

[2] As customary, the entries of a given matrix L will be denoted by L_{ij}.
[3] Of course, when $\varrho = +\infty$, we have that $B_\varrho = \mathbb{R}^n$.

and we suppose that there exist $\alpha \geqslant \beta > 0$ such that, for any $x \in \mathbb{R}^n$, $y \in B_\varrho$ and $\xi \in \mathbb{R}^n$,

$$\beta|\xi| \leqslant |M(x-y,y)\xi| \leqslant \alpha|\xi|. \tag{1.14}$$

To ensure compatibility with the divergence form structure, we suppose that

$$M(x-y,y) = M(x,-y) \quad \text{for any } (x,y) \in \mathbb{R}^n \times B_\varrho. \tag{1.15}$$

This setting produces a nonlocal operator defined as

$$\begin{aligned} \mathcal{L}_{M,\varrho,s}u(x) &:= c_{n,s}\,\text{P.V.} \int\limits_{B_\varrho} \frac{u(x)-u(x-y)}{|M(x-y,y)y|^{n+2s}}\,dy \\ &= c_{n,s}\,\text{P.V.} \int\limits_{B_\varrho(x)} \frac{u(x)-u(z)}{|M(z,x-z)(x-z)|^{n+2s}}\,dz. \end{aligned} \tag{1.16}$$

Here and in the rest of this paper, the constant $c_{n,s}$ is taken to be

$$c_{n,s} := \frac{2^{2s}\,\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{n/2}\,\Gamma(2-s)}\,s(1-s), \tag{1.17}$$

where Γ denotes the Euler Gamma function. The choice of the constant $c_{n,s}$ allows the recovering of the limit cases as $s \searrow 0$ and $s \nearrow 1$, see e.g. [1].

In particular, if M is the identity matrix, the operator in (1.16) reduces to the fractional Laplacian. We also observe that the kernel in (1.13) satisfies the structural condition (1.1), thanks to (1.14), and the symmetry condition in (1.2), thanks to (1.15).

We recall that the idea of considering affine transformations of nonlocal operators is particularly fruitful also regarding the theory of fractional operators in nondivergence form and fully nonlinear equations (compare, for example, our operator $\mathcal{L}_{M,\varrho,s}$ with L_A in [9]), but the role of affine transformations in nonlocal operators with divergence form has not been fully investigated yet.

This nonlocal setting recovers the classical case as $s \nearrow 1$. More explicitly, we have that:

Proposition 1.3. *Let $s \in (0,1)$ and $u \in H^1(\mathbb{R}^n)$. Suppose that $\varrho \in (0,+\infty]$ and $M : \mathbb{R}^n \times B_\varrho \to \mathbb{R}^{n\times n}$ satisfy (1.14) and (1.15). If there exists $r \in (0,\varrho)$ such that*

$$\text{the map } \mathbb{R}^n \times B_r \ni (x,y) \longmapsto M(x,y) \text{ is continuous,} \tag{1.18}$$

then, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\lim_{s\nearrow 1}\frac{c_{n,s}}{2}\iint\limits_{\mathbb{R}^n\times B_\varrho}\frac{(u(x)-u(x-y))(\varphi(x)-\varphi(x-y))}{|M(x-y,y)y|^{n+2s}}\,dx\,dy$$
$$=\sum_{i,j=1}^{n}\int\limits_{\mathbb{R}^n}A_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial\varphi}{\partial x_j}(x)\,dx, \tag{1.19}$$

where[4]

$$A_{ij}(x):=\frac{n}{\omega_{n-1}}\int\limits_{\mathcal{S}^{n-1}}\frac{\psi_i\psi_j}{|M(x,0)\,\psi|^{n+2}}d\mathcal{H}^{n-1}_{\psi} \tag{1.20}$$

and $\psi=(\psi_1,\dots,\psi_n)\in\mathcal{S}^{n-1}$.

In addition, denoting by $A(x)$ *the matrix with entries* $A_{ij}(x)$, *there exist* $\overline{\alpha}\geqslant\overline{\beta}>0$, *depending only on* α, β *and* n, *such that, for any* x, $\xi\in\mathbb{R}^n$,

$$\overline{\beta}|\xi|^2\leqslant A(x)\xi\cdot\xi\leqslant\overline{\alpha}|\xi|^2.$$

Proposition 1.3 can be considered one of the several possible results relating a nonlocal bilinear form to its classical counterpart (see e.g. [23] for general results of this type involving appropriate notions of convergence in functional spaces). In order to compare our result with that of [23], we emphasize that Proposition 1.3 is valid even without the translation invariance assumption on the kernel required in [23, hypothesis (I)].

However, the role of Proposition 1.3 in our framework is to give an explicit description of the limiting case, with precise control over the coefficients of the corresponding classical elliptic operator. The proof of Proposition 1.3 will be presented in Appendix E.

The spirit of this kind of asymptotic results is that, given $M:\mathbb{R}^n\times B_\varrho\to\mathbb{R}^{n\times n}$, which will be used in a nonlocal problem, one constructs a matrix-valued function A as in (1.20) corresponding to a classical second-order equation. This is useful when considering sequences of solutions u_s of fractional equations related to the operator $\mathcal{L}_{M,\varrho,s}$ in order to determine its limit as $s\nearrow 1$. In our setting, in particular, this strategy leads to the following result:

Theorem 1.4. *Let* $\varrho\in(0,+\infty]$ *and* $M:\mathbb{R}^n\times B_\varrho\to\mathbb{R}^{n\times n}$ *be a Lipschitz continuous matrix-valued function such that* (1.14) *and* (1.15) *hold true.*

Let $h:\mathbb{R}\to\mathbb{R}$ *satisfy* (1.5). *Let* a, $f\in L^1(\Omega)$ *be such that* (1.7) *is satisfied.*

Given $s\in(0,1)$, *let* $u_s\in H^s_0(\Omega)\cap L^\infty(\Omega)$ *be the unique weak solution of*

$$\begin{cases}\mathcal{L}_{M,\varrho,s}u+ah(u)=f & \text{in }\Omega,\\ u=0 & \text{in }\mathbb{R}^n\setminus\Omega,\end{cases} \tag{1.21}$$

provided by Theorem 1.1.

[4] As customary, here and in the rest of this paper ω_{n-1} denotes the surface area of the unit sphere in $\mathbb{R}^n$.

Then, there exists a unique weak solution $u_1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *of*

$$\begin{cases} -\operatorname{div}(A\nabla u) + ah(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.22}$$

where A *is as in* (1.20)*, and, for a.e.* $x \in \mathbb{R}^n$*,*

$$\lim_{s \nearrow 1} u_s(x) = u_1(x).$$

In addition, we have that

$$\begin{aligned} &\|u_1\|_{L^\infty(\Omega)} \leqslant h^{-1}(Q) \\ \text{and} \quad &\|\nabla u_1\|^2_{L^2(\Omega)} \leqslant C\, h^{-1}(Q) \left(\|f\|_{L^1(\Omega)} + Q\|a\|_{L^1(\Omega)}\right), \end{aligned} \tag{1.23}$$

where $C > 0$ *depends only on* n*,* α*,* β *and* Ω*.*

The stability property showcased in Theorem 1.4 is important since it highlights a series of uniform estimates which go beyond the standard regularity theory and presents a "unified" vision in which the classical case is actually obtained as a byproduct of a nonlocal construction.

We observe that the existence of a classical solution u_1 for the second-order equation (1.22) was established in [5, Theorem 2.1], working in a local framework. However, in Theorem 1.4 we deduce the existence of this solution as limit of solutions of nonlocal problems. Indeed, the concrete applicability of Theorem 1.4 leverages Theorem 1.1 to ensure the existence of solutions for all nonlocal problems, then it relies on some tailor-made estimates which are independent of the parameter $s \in (0,1)$ in order to pass these solutions to the limit as $s \nearrow 1$, finally concluding that the limit function obtained in this manner is actually a solution of the classical problem (we will see in the forthcoming Theorem 1.7 that this strategy can also be adopted to construct a solution of any classical problem in divergence form).

1.3. Back to the classical case (from A to M)

Interestingly, one can also modify the above strategy in order to provide an independent proof of [5, Theorem 2.1] by relying on a limit procedure. This requires significant additional work, since, while Proposition 1.3 describes the limit matrix-valued function A in terms of a given matrix-valued function M used in a sequence of fractional problems, to recover the classical case by a limit method one needs some "reverse engineering": namely, given any elliptic matrix-valued function A one needs to construct a matrix-valued function M to be used in the sequence of fractional problems. In a sense, one needs to "invert" the relation in (1.20) and, as detailed in the forthcoming equation (1.30), such inversion is not unique (however, the explicit inversion formula that we find allows for "the simplest possible choice" of M to recover A).

To clarify the technicalities involved in this construction, let us first present a result in linear algebra[5] permitting to invert (1.20) at a given point:

Lemma 1.5. *Let $\lambda_1, \dots, \lambda_n > 0$ and $A \in \mathbb{R}^{n\times n}$ be defined as*

$$A := O\Lambda O^T, \tag{1.24}$$

where O, $\Lambda \in \mathbb{R}^{n\times n}$ satisfy

$$OO^T = \mathrm{Id} \qquad \text{and} \qquad \Lambda_{ij} := \lambda_i \delta_{ij}. \tag{1.25}$$

Let $\overline{\lambda} := \lambda_1 \cdots \lambda_n$ and, for any $i \in \{1, \dots, n\}$,

$$\sigma_i := \sqrt{\frac{\overline{\lambda}^{\frac{1}{n+2}}}{\lambda_i}}. \tag{1.26}$$

Let $\Sigma \in \mathbb{R}^{n\times n}$ with entries $\sigma_i \delta_{ij}$ and set

$$N_A := O\Sigma O^T. \tag{1.27}$$

Then,

$$A_{ij} = \frac{n}{\omega_{n-1}} \int\limits_{\mathcal{S}^{n-1}} \frac{\psi_i\, \psi_j}{|N_A\, \psi|^{n+2}}\, d\mathcal{H}^{n-1}_{\psi}. \tag{1.28}$$

We recall that, by the Spectral Theorem (see e.g. [28, Theorem 5S, page 330]), conditions (1.24) and (1.25) just make it explicit that the matrix A is symmetric with eigenvalues $\lambda_1, \dots, \lambda_n$. Furthermore, the positivity of these eigenvalues is equivalent to the positive definiteness of A.

The inversion provided by Lemma 1.5 needs to be adapted to comprise the interaction setting appearing in the fractional case: roughly speaking, while in the classical problem (1.22) the matrix-valued function A depends only on the point x, its nonlocal counterpart makes use of the setting in (1.16) and the corresponding matrix-valued function M depends on both x and y (or both x and z, depending on the preferred form in which one can write the operator). This is done according to the following result:

Theorem 1.6. *Suppose that $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a continuous matrix-valued function. Suppose also that, for all $x \in \mathbb{R}^n$, the matrix $A(x)$ is symmetric and denote by $\lambda_i(x)$ its eigenvalues.*

[5] As customary, given a matrix $N \in \mathbb{R}^{n\times n}$, we denote by N^T the transpose matrix. Also, δ_{ij} denotes the Kronecker symbol.

Assume that there exist $\underset{\sim}{\lambda}, \widetilde{\lambda} > 0$ *such that, for any* $i \in \{1, \dots, n\}$ *and for any* $x \in \mathbb{R}^n$,

$$\underset{\sim}{\lambda} \leqslant \lambda_i(x) \leqslant \widetilde{\lambda}. \tag{1.29}$$

Then, there exists $M : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ *satisfying* (1.14), (1.15) *and* (1.20) *(with* $\varrho = +\infty$*).*

The bounds in (1.14) *depend only on* n, $\underset{\sim}{\lambda}$ *and* $\widetilde{\lambda}$.

The explicit expression of M *has the form*

$$M(x,y) := \frac{N_{A(x)} + N_{A(x+y)}}{2} + H(y), \tag{1.30}$$

where the notation in (1.27) *has been used, and* $H : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ *is any bounded and continuous matrix-valued function, with* $H(y)$ *positive semidefinite uniformly in* y*, satisfying*

$$H(0) = 0 \quad \text{and} \quad H(y) = H(-y) \ \text{ for any } y \in \mathbb{R}^n. \tag{1.31}$$

We point out that condition (1.29) is equivalent to assume that $A(x)$ is bounded and strictly positive definite, uniformly in x. We have however stated this condition explicitly to clarify the dependence of the structural constants on the quantities $\underset{\sim}{\lambda}$ and $\widetilde{\lambda}$.

The representation formula (1.30) is interesting, since, given an elliptic matrix A for the second-order problem in divergence form, it selects the "simplest" possible nonlocal operator in divergence form that recovers the classical case in the limit: in particular, the choice $H := 0$ in (1.30) provides a "natural" one-to-one correspondence between classical and nonlocal problems (the nonlocal case being "more abundant", formula (1.30) allows for a convenient choice).

The construction in Theorem 1.6, together with the estimates for the solutions found through Theorem 1.1 and the convergence result in Proposition 1.3, allows one to recover classical results established in [5, Theorem 2.1] for the local problem with a limit procedure, according to the following result:

Theorem 1.7. *Let* $A : \Omega \to \mathbb{R}^{n\times n}$. *Suppose that* $A(x)$ *is a symmetric matrix and denote by* $\lambda_i(x)$ *its eigenvalues.*

Assume that there exist $\underset{\sim}{\lambda}, \widetilde{\lambda} > 0$ *such that, for any* $i \in \{1, \dots, n\}$ *and for any* $x \in \Omega$,

$$\underset{\sim}{\lambda} \leqslant \lambda_i(x) \leqslant \widetilde{\lambda}.$$

Let $h : \mathbb{R} \to \mathbb{R}$ *satisfy* (1.5). *Let* $a,\ f \in L^1(\Omega)$ *be such that* (1.7) *is satisfied.*

Then, there exists a unique weak solution $u_1 \in H^1_0(\Omega) \cap L^\infty(\Omega)$ *of* (1.22) *satisfying*

$$\begin{aligned} &\|u_1\|_{L^\infty(\Omega)} \leqslant h^{-1}(Q) \\ \text{and} \quad &\|\nabla u_1\|_{L^2(\Omega)} \leqslant C\, h^{-1}(Q)\left(\|f\|_{L^1(\Omega)} + Q\|a\|_{L^1(\Omega)}\right). \end{aligned}$$

Moreover, there exists a sequence $M_\ell : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, *satisfying* (1.14) *and* (1.15) *(for any* $\varrho \in (0,+\infty]$*), and* $s_j \in (0,1)$ *with* $s_j \nearrow 1$ *as* $j \to +\infty$, *such that, for every* $\varrho \in (0,+\infty]$ *and every* ℓ, $j \in \mathbb{N}$, *there exists* $u_{j,\ell} \in H_0^{s_j}(\Omega) \cap L^\infty(\Omega)$ *which is the unique weak solution of*

$$\begin{cases} \mathcal{L}_{M_\ell,\varrho,s_j} u + ah(u) = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

with

$$\lim_{\ell\to+\infty} \lim_{j\to+\infty} u_{j,\ell}(x) = u_1(x).$$

Interestingly, Theorem 1.7 also provides an alternative proof of [5, Theorem 2.1], since it demonstrates the existence of the classical solution u_1 which is obtained via a limit procedure as $s \nearrow 1$.

Our paper is organized as follows. In Section 2 we set up the stage for Theorems 1.1 and 1.2, which are then proved at the end of the section. In Section 3, we show that the solution found through Theorem 1.1 converges, in the limit $s \nearrow 1$, to a solution of the local problem (1.22), thus establishing Theorem 1.4.

In Section 4 we discuss the twofold nature of formula (1.20), proving that it can also be applied to construct M starting from A. This section contains the proof of Theorem 1.6, which allows to recover the classical result, as stated in Theorem 1.7.

The paper ends with some appendices that contain the statements and proofs of some technical results.

2. Existence theory and proof of Theorems 1.1 and 1.2

2.1. Functional analysis preliminaries

In this section we introduce the variational framework needed to deal with problems (1.8) and (1.11) and we provide some preliminary results which will be required to prove Theorems 1.1 and 1.2.

For $s \in (0,1)$, we let

$$[u]_s := \left(c_{n,s} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \right)^{\frac{1}{2}} \tag{2.1}$$

be the Sobolev-Gagliardo seminorm of a measurable function $u : \mathbb{R}^n \to \mathbb{R}$, see e.g. [16].

The positive normalizing constant $c_{n,s}$ is chosen in such a way that it provides consistent limits as $s \searrow 0$ and $s \nearrow 1$, that is

$$\lim_{s\nearrow 1}[u]_s = [u]_1 := \|\nabla u\|_{L^2(\mathbb{R}^n)} \qquad \text{and} \qquad \lim_{s\searrow 0}[u]_s = [u]_0 := \|u\|_{L^2(\mathbb{R}^n)}.$$

As customary, we define the fractional Sobolev space

$$H^s(\mathbb{R}^n) := \left\{u \in L^2(\mathbb{R}^n) : [u]_s < +\infty\right\}$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \left([u]_s^2 + \|u\|^2_{L^2(\mathbb{R}^n)}\right)^{\frac{1}{2}}.$$

Moreover, given an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary, we set

$$H_0^s(\Omega) := \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$$

endowed with the equivalent norm $\|\cdot\|_{H_0^s(\Omega)} = [\cdot]_s$.

It is also useful to observe that the norm induced by the kernel K is equivalent to the fractional Sobolev-Gagliardo norm, as stated in the following result:

Lemma 2.1. *There exists $C_\star > 0$, depending only on n, s, c, ϱ and Ω, such that, for all $u : \mathbb{R}^n \to \mathbb{R}$ with $u = 0$ in $\mathbb{R}^n \setminus \Omega$,*

$$\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz \leqslant C_\star \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(z)|^2 K(x,z)\,dx\,dz.$$

For convenience of the reader, the proof of this result is presented in Appendix D.

With the setting introduced above, the definition of weak solution of problem (1.8) reads as follows:

Definition 2.2. We say that $u \in H_0^s(\Omega)$ is a weak solution of (1.8) if, for any $v \in H_0^s(\Omega) \cap L^\infty(\Omega)$,

$$\frac{1}{2}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \big(u(x)-u(z)\big)\big(v(x)-v(z)\big)K(x,z)\,dx\,dz + \int_\Omega a(x)h(u(x))v(x)\,dx = \int_\Omega f(x)v(x)\,dx.$$

We now let $k \geqslant 0$ and we consider the cutoff function $G_k : \mathbb{R} \to \mathbb{R}$ defined as

$$G_k(t) := \begin{cases} 0 & \text{if } |t| \leqslant k, \\ t-k & \text{if } t > k, \\ t+k & \text{if } t < -k. \end{cases} \tag{2.2}$$

The function G_k enjoys some useful properties, that we now recall, for the facility of the reader.

Proposition 2.3. *Let $k \geqslant 0$ and $G_k : \mathbb{R} \to \mathbb{R}$ be defined as in (2.2). Then, for any t, $r \in \mathbb{R}$, we have that*

$$t\,G_k(t) = |t||G_k(t)|, \tag{2.3}$$

$$G_k(t) \text{ is nondecreasing,} \tag{2.4}$$

$$\text{and} \quad |G_k(t) - G_k(r)| \leqslant |t - r|. \tag{2.5}$$

Proof. We can write (2.2) in a compact way as

$$G_k(t) = \int_0^t \chi_{\mathbb{R}\setminus[-k,k]}(x)\,dx.$$

From this, the property in (2.3) plainly follows.

Also, by inspection of all possible cases, one can check that (2.4) holds true.

Moreover,

$$|G_k(t) - G_k(r)| = \left| \int_r^t \chi_{\mathbb{R}\setminus[-k,k]}(x)\,dx \right| \leqslant |t - r|,$$

which proves (2.5). □

Proposition 2.4. *Let $k \geqslant 0$ and $u \in H_0^s(\Omega)$. Then, $G_k(u)$ belongs to $H_0^s(\Omega)$.*

Proof. From (2.5), we have that $|G_k(u)| \leqslant |u|$, and therefore $G_k(u) \in L^2(\mathbb{R}^n)$.

Moreover, by (2.1) and (2.5), we infer that

$$\begin{aligned}[G_k(u)]_s^2 &= c_{n,s} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|G_k(u(x)) - G_k(u(y))|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\leqslant c_{n,s} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = [u]_s^2,\end{aligned}$$

and therefore $G_k(u) \in H^s(\mathbb{R}^n)$.

Furthermore, in $\mathbb{R}^n \setminus \Omega$ we have that $u = 0$, thus $G_k(u) = 0$. Gathering these pieces of information, we obtain the desired claim. □

2.2. Proof of Theorem 1.2

The proof of Theorem 1.2 is deduced from a minimization argument, whose details are as follows.

We set

$$H(t) := \int_0^t h(\tau)\, d\tau.$$

We consider the functional $J : H_0^s(\Omega) \to \mathbb{R} \cup \{+\infty, -\infty\}$ defined as

$$J(u) := \begin{cases} \dfrac{1}{4} \displaystyle\iint_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(z)|^2 K(x,z)\, dx\, dz + \int_\Omega \eta(x) H(u(x))\, dx - \int_\Omega \zeta(x) u(x)\, dx \\ \qquad\qquad \text{if } \eta(H \circ u) \in L^1(\Omega), \\ +\infty \qquad \text{if } \eta(H \circ u) \notin L^1(\Omega). \end{cases} \tag{2.6}$$

We now check the following properties of J:

Lemma 2.5. *Let $u \in H_0^s(\Omega)$. If $\eta(H \circ u) \in L^1(\Omega)$, then $|J(u)| < +\infty$.*
In particular, $J : H_0^s(\Omega) \to \mathbb{R} \cup \{+\infty\}$.

Proof. If $u \in H_0^s(\Omega)$, then from the upper bound in (1.1) we have that

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(z)|^2 K(x,z)\, dx\, dz \leqslant C \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2s}}\, dx\, dz = \frac{C}{c_{n,s}} [u]_s^2 < +\infty.$$

Moreover,

$$\left| \int_\Omega \eta(x) H(u(x))\, dx \right| \leqslant \int_\Omega \eta(x)\, H(u(x))\, dx < +\infty$$

and, since $u \in L^2(\Omega)$, by the Hölder inequality,

$$\left| \int_\Omega \zeta(x) u(x)\, dx \right| \leqslant \|\zeta\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} < +\infty. \tag{2.7}$$

Combining these considerations with (2.6), we get desired result. □

We also observe that

$$\text{critical points of the functional } J \text{ are weak solutions of (1.11).} \tag{2.8}$$

Lemma 2.6. *The functional J is weakly lower semicontinuous with respect to the topology of $H_0^s(\Omega)$, namely for every sequence $u_k \in H_0^s(\Omega)$ which converges weakly to some $u \in H_0^s(\Omega)$ as $k \to +\infty$ we have that*

$$\liminf_{k\to+\infty} J(u_k) \geqslant J(u).$$

Proof. To prove the desired result, we argue by contradiction and we suppose that there exists a sequence $u_k \in H_0^s(\Omega)$ weakly converging to some $u \in H_0^s(\Omega)$ as $k \to +\infty$ and such that

$$\lim_{k\to+\infty} J(u_k) < J(u). \tag{2.9}$$

In particular,

$$\lim_{k\to+\infty} J(u_k) < +\infty, \tag{2.10}$$

which, in tandem with Lemma 2.5, implies that $\eta(H \circ u_k) \in L^1(\Omega)$ for k sufficiently large.

By the compactness result in [16, Theorem 7.1] we have that there exists a subsequence u_{k_j} strongly converging to u in $L^2(\Omega)$ and a.e. in Ω. Therefore, by Fatou's Lemma,

$$\liminf_{j\to+\infty} \iint_{\mathbb{R}^n\times\mathbb{R}^n} |u_{k_j}(x) - u_{k_j}(z)|^2 K(x,z)\,dx\,dz \geqslant \iint_{\mathbb{R}^n\times\mathbb{R}^n} |u(x) - u(z)|^2 K(x,z)\,dx\,dz.$$

By (1.5), it follows that

$$H(t) \geqslant 0 \quad \text{for any } t \in \mathbb{R}. \tag{2.11}$$

Consequently, since $\eta \geqslant 0$ and H is continuous, it follows from Fatou's Lemma that

$$\liminf_{j\to+\infty} \int_\Omega \eta(x) H(u_{k_j}(x))\,dx \geqslant \int_\Omega \eta(x) H(u(x))\,dx.$$

Furthermore, since $\zeta \in L^2(\Omega)$, the strong convergence of u_{k_j} in $L^2(\Omega)$ implies that

$$\lim_{j\to+\infty} \int_\Omega \zeta(x) u_{k_j}(x)\,dx = \int_\Omega \zeta(x) u(x)\,dx.$$

Gathering these pieces of information and recalling the definition of J in (2.6), we conclude that

$$\begin{aligned}
&\lim_{j\to+\infty} J(u_{k_j})\\
&\geqslant \liminf_{j\to+\infty} \frac{1}{4}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} |u_{k_j}(x)-u_{k_j}(z)|^2K(x,z)\,dx\,dz\\
&\quad+\int_\Omega \eta(x)H(u_{k_j}(x))\,dx-\int_\Omega \zeta(x)u_{k_j}(x)\,dx\\
&\geqslant \frac{1}{4}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(z)|^2K(x,z)\,dx\,dz+\int_\Omega \eta(x)H(u(x))\,dx-\int_\Omega \zeta(x)u(x)\,dx.
\end{aligned}\tag{2.12}$$

Hence, recalling (2.7),

$$\int_\Omega \eta(x)H(u(x))\,dx\leqslant \int_\Omega \zeta(x)u(x)\,dx+\lim_{j\to+\infty} J(u_{k_j})\leqslant \|\zeta\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}+\lim_{j\to+\infty} J(u_{k_j}).$$

We stress that the latter quantity is finite, owing to (2.10), and therefore $\eta(H\circ u)\in L^1(\Omega)$.

As a result, by Lemma 2.5,

$$J(u)=\frac{1}{4}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(z)|^2K(x,z)\,dx\,dz+\int_\Omega \eta(x)H(u(x))\,dx-\int_\Omega \zeta(x)u(x)\,dx.$$

From this, (2.9) and (2.12) we obtain that

$$\begin{aligned}
J(u)&>\lim_{j\to+\infty} J(u_{k_j})\\
&\geqslant \frac{1}{4}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(z)|^2K(x,z)\,dx\,dz+\int_\Omega \eta(x)H(u(x))\,dx-\int_\Omega \zeta(x)u(x)\,dx=J(u).
\end{aligned}$$

This is a contradiction establishing the desired result. □

Lemma 2.7. *The functional J is coercive, namely*

$$\lim_{[u]_s\to+\infty}\frac{J(u)}{[u]_s}=+\infty.$$

Proof. We can suppose that $H\circ u\in L^1(\Omega)$. From Lemma 2.1 and (2.11),

$$\begin{aligned}
J(u)&=\frac{1}{4}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(z)|^2K(x,z)\,dx\,dz+\int_\Omega \eta(x)H(u(x))\,dx-\int_\Omega \zeta(x)u(x)\,dx\\
&\geqslant \frac{\widetilde{C}}{4}[u]_s^2+\int_\Omega \eta(x)H(u)\,dx-\int_\Omega \zeta(x)u(x)\,dx
\end{aligned}$$

$$\geqslant \frac{\widetilde{C}}{4}[u]_s^2 - \int_\Omega |\zeta(x)||u(x)|\,dx, \tag{2.13}$$

for some $\widetilde{C} > 0$.

Now, by the Hölder inequality and [16, Theorem 6.7], there exists a constant $\widehat{C} > 0$, depending only on n, s and Ω, such that

$$\int_\Omega |\zeta(x)||u(x)|\,dx \leqslant \|\zeta\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \leqslant \widehat{C}\|\zeta\|_{L^2(\Omega)}[u]_s.$$

From this and (2.13), we deduce that

$$J(u) \geqslant \frac{\widetilde{C}}{4}[u]_s^2 - \widehat{C}\|\zeta\|_{L^2(\Omega)}[u]_s.$$

Hence,

$$\lim_{[u]_s\to+\infty} \frac{J(u)}{[u]_s} \geqslant \lim_{[u]_s\to+\infty} \left(\frac{\widetilde{C}}{4}[u]_s - \widehat{C}\|\zeta\|_{L^2(\Omega)}\right) = +\infty,$$

as desired. □

With the work done so far, the proof of Theorem 1.2 can be completed as follows.

Proof of Theorem 1.2. Since J is weakly lower semicontinuous (thanks to Lemma 2.6) and coercive (thanks to Lemma 2.7) and the space $H_0^s(\Omega)$ is reflexive, by [29, Theorem 1.2], there exists a minimizer for the functional J, which is a weak solution of problem (1.11) (recall (2.8)). □

2.3. Proof of Theorem 1.1

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. For any $j \in \mathbb{N}$, we set

$$f_j(x) := \frac{f(x)}{1 + \frac{|f(x)|}{j}} \quad \text{and} \quad a_j(x) := \frac{a(x)}{1 + \frac{Q\,a(x)}{j}}. \tag{2.14}$$

We observe that f_j, $a_j \in L^\infty(\Omega)$ for any $j \in \mathbb{N}$ and

$$\|f_j\|_{L^\infty(\Omega)} \leqslant j \quad \text{and} \quad \|a_j\|_{L^\infty(\Omega)} \leqslant \frac{j}{Q}. \tag{2.15}$$

In addition, setting

$$\iota_j(t) := \frac{jt}{j+t},$$

one can check that ι_j is nondecreasing, and therefore from (1.7) we deduce that

$$|f_j(x)| = \frac{|f(x)|}{1+\frac{|f(x)|}{j}} = \iota_j(|f(x)|) \leqslant \iota_j(Qa(x)) = \frac{Qa(x)}{1+\frac{Qa(x)}{j}} = Qa_j(x). \tag{2.16}$$

We now consider the approximated problems

$$\begin{cases} \mathcal{L}_K u + a_j\, h(u) = f_j & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{2.17}$$

and we observe that we are in the setting of Theorem 1.2 with $\eta := a_j$ and $\zeta := f_j$. Consequently, for any $j \in \mathbb{N}$, there exists a weak solution u_j of (2.17), that is, for any $j \in \mathbb{N}$, there exists $u_j \in H_0^s(\Omega)$ such that, for any $v \in H_0^s(\Omega)$,

$$\begin{aligned} &\frac{1}{2} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_j(x)-u_j(z)\big)\big(v(x)-v(z)\big)K(x,z)\,dx\,dz + \int_\Omega a_j(x)h(u_j(x))v(x)\,dx \\ &\qquad = \int_\Omega f_j(x)v(x)\,dx. \end{aligned} \tag{2.18}$$

Now, in light of the assumptions on h in (1.5), we have that the h is invertible in $(-\gamma,\gamma)$. Hence, denoted by h^{-1} this inverse and recalling that $Q \in (0,\gamma)$ (see (1.7)), we have

$$k := h^{-1}(Q) > 0.$$

Recalling the definition in (2.2) and exploiting the assumptions on h in (1.5),

$$G_k(u_j) := \begin{cases} 0 & \text{if } |h(u_j)| \leqslant Q, \\ u_j - k & \text{if } h(u_j) > Q, \\ u_j + k & \text{if } h(u_j) < -Q. \end{cases} \tag{2.19}$$

Moreover, by Proposition 2.4, we can use $G_k(u_j)$ as a test function in (2.18), to obtain that

$$\begin{aligned} &\frac{1}{2} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_j(x)-u_j(z)\big)\big(G_k(u_j(x))-G_k(u_j(z))\big)K(x,z)\,dx\,dz \\ &\qquad + \int_\Omega a_j(x)h(u_j(x))G_k(u_j(x))\,dx = \int_\Omega f_j(x)G_k(u_j(x))\,dx. \end{aligned}$$

We also observe that, thanks to (1.5) and (2.3),

$$h(t)G_k(t) = |h(t)G_k(t)| \quad \text{for any } t \in \mathbb{R},$$

and therefore

$$\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_j(x)-u_j(z)\big)\big(G_k(u_j(x))-G_k(u_j(z))\big)K(x,z)\,dx\,dz + \int_\Omega a_j(x)|h(u_j(x))|\,|G_k(u_j(x))|\,dx = \int_\Omega f_j(x)G_k(u_j(x))\,dx.$$

From this and (2.16), we thus deduce that

$$\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_j(x)-u_j(z)\big)\big(G_k(u_j(x))-G_k(u_j(z))\big)K(x,z)\,dx\,dz + \int_\Omega a_j(x)|h(u_j(x))|\,|G_k(u_j(x))|\,dx \leqslant Q\int_\Omega a_j(x)\,|G_k(u_j(x))|\,dx. \tag{2.20}$$

Moreover, from (2.4) and (2.5) we infer that

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}^n} |G_k(u_j(x))-G_k(u_j(z))|^2K(x,z)\,dx\,dz\\ &\leqslant \iint_{\mathbb{R}^n\times\mathbb{R}^n} \big|G_k(u_j(x))-G_k(u_j(z))\big|\big|u_j(x)-u_j(z)\big|K(x,z)\,dx\,dz\\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \big(G_k(u_j(x))-G_k(u_j(z))\big)\big(u_j(x)-u_j(z)\big)K(x,z)\,dx\,dz.\end{aligned}$$

Combining this with (2.20), we get that

$$\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \big|G_k(u_j(x))-G_k(u_j(z))\big|^2K(x,z)\,dx\,dz + \int_\Omega a_j(x)\big(|h(u_j(x))|-Q\big)|G_k(u_j(x))|\,dx \leqslant 0.$$

Thus, since $G_k(u_j(x)) = 0$ if $|h(u_j(x))| \leqslant Q$ (recall the definition in (2.19)), we conclude that

$$\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \big|G_k(u_j(x))-G_k(u_j(z))\big|^2K(x,z)\,dx\,dz$$

$$+\int\limits_{\Omega\cap\{|h(u_j(x))|>Q\}} a_j(x)\big(|h(u_j(x))|-Q\big)|G_k(u_j(x))|\,dx\leqslant 0.$$

This and Lemma 2.1 entail that $G_k(u_j(x))=0$ for a.e. $x\in\Omega$, and therefore

$$\|u_j\|_{L^\infty(\Omega)}\leqslant h^{-1}(Q). \tag{2.21}$$

We now claim that

$$u_j \text{ is bounded in } H_0^s(\Omega). \tag{2.22}$$

Indeed, by (2.14) and (2.21), and being h strictly increasing due to (1.5),

$$|a_j(x)h(u_j(x))u_j(x)|\leqslant a(x)h(\|u_j\|_{L^\infty(\Omega)})\|u_j\|_{L^\infty(\Omega)}\leqslant a(x)\,Q\,h^{-1}(Q)$$
$$\text{and}\quad |f_j(x)u_j(x)|\leqslant |f(x)|\,h^{-1}(Q).$$

Thus, by using $u_j\in H_0^s(\Omega)$ as a test function in (2.18), we find that

$$\frac12\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n}\big(u_j(x)-u_j(z)\big)^2K(x,z)\,dx\,dz=\int\limits_\Omega f_j(x)u_j(x)\,dx-\int\limits_\Omega a_j(x)h\big(u_j(x)\big)u_j(x)\,dx$$
$$\leqslant h^{-1}(Q)\Big(\|f\|_{L^1(\Omega)}+Q\|a\|_{L^1(\Omega)}\Big). \tag{2.23}$$

This and Lemma 2.1 entail (2.22).

By (2.22), there exists $u\in H_0^s(\Omega)$ such that, up to subsequences,

$$u_j\rightharpoonup u \quad \text{in } H_0^s(\Omega), \tag{2.24}$$
$$\text{and}\quad u_j\to u \quad \text{a.e. in } \Omega. \tag{2.25}$$

Moreover, by (2.21),

$$\|u\|_{L^\infty(\Omega)}\leqslant h^{-1}(Q). \tag{2.26}$$

Also, by (2.23), (2.25) and Fatou's Lemma, we infer that

$$\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n}|u(x)-u(z)|^2K(x,z)\,dx\,dz\leqslant\liminf_{j\to+\infty}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n}|u_j(x)-u_j(z)|^2K(x,z)\,dx\,dz$$
$$\leqslant 2h^{-1}(Q)\Big(\|f\|_{L^1(\Omega)}+Q\|a\|_{L^1(\Omega)}\Big). \tag{2.27}$$

Now, we check that u is a weak solution of problem (1.8) according to Definition 2.2. For this, we take $v\in H_0^s(\Omega)\cap L^\infty(\Omega)$ as a test function in (2.18) and we obtain that

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_j(x)-u_j(z)\big)\big(v(x)-v(z)\big)K(x,z)\,dx\,dz+\int_\Omega a_j(x)h(u_j(x))v(x)\,dx\\&\quad=\int_\Omega f_j(x)v(x)\,dx.\end{aligned}\tag{2.28}$$

The aim is now to pass to the limit as $j\to+\infty$ the equation in (2.28). For this, we use (2.21) and the properties of h in (1.5) to see that

$$\begin{aligned}&|a_jh(u_j)v|\leqslant h(h^{-1}(Q))a\|v\|_{L^\infty(\Omega)}=Qa\|v\|_{L^\infty(\Omega)}\in L^1(\Omega)\\\text{and}\quad&|f_jv|\leqslant f\|v\|_{L^\infty(\Omega)}\in L^1(\Omega).\end{aligned}$$

Hence, we employ the Dominated Convergence Theorem and the weak convergence in (2.24) to pass to the limit as $j\to+\infty$ in (2.28) and deduce that u satisfies

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \big(u(x)-u(z)\big)\big(v(x)-v(z)\big)K(x,z)\,dx\,dz+\int_\Omega a(x)h(u(x))v(x)\,dx=\int_\Omega f(x)v(x)\,dx,$$

as desired.

Furthermore, the estimates in (1.9) and (1.10) follow from (2.26) and (2.27).

We now show that the solution that we found is unique. To this aim, we take u_1, $u_2\in H_0^s(\Omega)\cap L^\infty(\Omega)$ solutions of (1.8). We point out that u_1-u_2 can be used as a test function in the weak formulation of (1.8). In this way, one obtains that

$$\begin{aligned}&\frac12\iint_{\mathbb{R}^n\times\mathbb{R}^n}\big((u_1-u_2)(x)-(u_1-u_2)(z)\big)^2K(x,z)\,dx\,dz\\&\quad+\int_\Omega a(x)\big(h(u_1(x))-h(u_2(x))\big)(u_1(x)-u_2(x))\,dx=0.\end{aligned}$$

From the fact that $a\geqslant 0$, the assumptions on h in (1.5) and Lemma 2.1, we deduce that $u_1=u_2$ a.e. in Ω. □

Remark 2.8. It is tempting to use the Schauder Fixed Point Theorem to prove that problem (1.11) admits a weak solution. While this approach has been previously used in [3–5], some care is needed to make the full argument work. Indeed, given $f\in L^2(\Omega)$ and $s\in(0,1]$, one can define $S:L^2(\Omega)\to H_0^s(\Omega)$ as the solution operator $u=S(f)$ of the problem

$$\begin{cases}(-\Delta)^s u=f & \text{in }\Omega\\ u=0 & \text{in }\mathbb{R}^N\setminus\Omega.\end{cases}\tag{2.29}$$

By either classical or fractional regularity theory, it follows that S is linear and continuous.

Then, one could be tempted to introduce the operator $T : H_0^s(\Omega) \to H_0^s(\Omega)$ defined as

$$T(u) := S(\zeta - \eta\, h(u)). \tag{2.30}$$

With this choice, $T(u)$ would be the solution of the problem (2.29) with $f := \zeta - \eta\, h(u)$.

At a first glance, this seems a convenient setting for the application of the Schauder Fixed Point Theorem, up to showing that T is continuous, compact and leaves a suitable closed ball invariant.

However, even if $\eta \in L^\infty(\Omega)$ and $\zeta \in L^2(\Omega)$, the assumptions on the function h are, in general, not strong enough to guarantee that $\zeta - \eta\, h(u) \in L^2(\Omega)$ (the issues arising when h grows faster than linear at infinity: for example, if $\zeta := 0$, $\eta := 1$, $h(t) := t^3$, $u(x) := |x|^{-\frac{1}{3}}$ and $\Omega \ni 0$, then $\zeta - \eta\, h(u) \notin L^2(\Omega)$). This caveat prevents one from defining the operator T and makes the Schauder Fixed Point Theorem unavailable in this situation (likely, a two-step approach would be needed in cases like this, based first on a truncation of the function h with a strictly increasing and bounded function and on a subsequent limit argument).

Remark 2.9. As kindly mentioned to us by Tomasz Klimsiak, a different proof of Theorem 1.1 can be obtained with methods of probabilistic potential theory.

Below is a sketch of the arguments to be used for this purpose.

Let $G_k(t)$ as in (2.2). The existence of u such that $(u - G_k(u)) \in H_0^s(\Omega)$ for any $k \geqslant 0$ follows from [25, Theorem 4.7] and [25, Propositions 5.9 and 5.5].

Then, the boundedness of u, as stated in (1.9) here, can be obtained from [27, Theorem 3.11]. To see this, one needs to observe that:

(a) Set $F(x, y) := f(x) - a(x)h(y)$. Then $\mathcal{L}_K u = F(x, u)$ in Ω, and $u = 0$ in $\mathbb{R}^n \setminus \Omega$.
(b) Let $w_1 := -h^{-1}(Q)$, $w_2 := h^{-1}(Q)$. Then, $\mathcal{L}_K w_1 = 0$ in Ω, with $w_1 = -h^{-1}(Q)$ in $\mathbb{R}^n \setminus \Omega$, and similarly $\mathcal{L}_K w_2 = 0$ in Ω, with $w_2 = h^{-1}(Q)$ in $\mathbb{R}^n \setminus \Omega$. Also, by assumption (1.7), it follows that $F(\cdot, w_1) \geqslant 0$ and $F(\cdot, w_2) \leqslant 0$. Thus, by [27, Theorem 3.11], one concludes that $w_1 \leqslant u \leqslant w_2$.

The energy estimate in (1.10) follows from [25, Propositions 5.9 and 5.5] with $k = h^{-1}(Q)$.

We also mention that the equivalence between different notions of solutions can be found in [25, Proposition 5.3 and Remark 5.12] and [27, Theorem 3.7]. To establish the equivalence between probabilistic solutions and the ones considered here, one can rely on [26, Theorem 3.5].

3. Asymptotic behavior of nonlocal solutions and proof of Theorem 1.4

In this section we show that the unique solution of problem (1.21), obtained through Theorem 1.1, converges as $s \nearrow 1$ to a function which is the unique solution of prob-

lem (1.22). The existence of this classical solution was obtained in [5, Theorem 2.1], but our proof does not need to assume this result and can be also seen as a different approach to the classical theory, which is now recovered as a byproduct of a limit case. In addition, the stability of the solutions of Theorem 1.1 as $s \nearrow 1$ is interesting in itself, since it relies on a series of uniform estimates which do not come from the standard regularity theory.

For the sake of convenience, for any $s \in (0,1)$, we denote by $u_s \in H_0^s(\Omega) \cap L^\infty(\Omega)$ the unique weak solution of problem (1.21) and by $u_1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ the unique weak solution of problem (1.22).

Now we show that, given $u \in H_0^s(\Omega)$, we can decrease the seminorm of u by performing a mollification procedure.

Lemma 3.1. *Let $s \in (0,1)$ and $u \in H_0^s(\Omega)$. Let $\eta \in C_c^\infty(B_1)$, $\eta \geqslant 0$ with $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$. For any $\varepsilon > 0$, let*

$$\eta_\varepsilon(x) := \varepsilon^{-n}\eta\left(\frac{x}{\varepsilon}\right) \qquad \text{and} \qquad u_\varepsilon(x) := (\eta_\varepsilon * u)(x). \tag{3.1}$$

Then,

$$[u_\varepsilon]_s \leqslant [u]_s.$$

Proof. By definition, we have that $\eta_\varepsilon \in C^\infty(\Omega)$, $\operatorname{supp}(\eta_\varepsilon) \subset B_\varepsilon$ and

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x)\,dx = 1.$$

Hence, recalling the definition in (2.1) and using the Jensen inequality, we get

$$\begin{aligned}
[u_\varepsilon]_s^2 &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\leqslant \iint_{\mathbb{R}^n\times\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|u(x-z)-u(y-z)|^2}{|x-y|^{n+2s}}\eta_\varepsilon(z)\,dz\right) dx\,dy \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \left(\int_{\mathbb{R}^n} \eta_\varepsilon(z)\,dz\right) = [u]_s^2,
\end{aligned}$$

as desired. □

We are now in the position to prove Theorem 1.4.

Proof of Theorem 1.4. We will show that every sequence of solutions of the nonlocal problem (1.21) is precompact and possesses a subsequence converging to the unique

solution of the classical problem (1.22), which in turn implies that all sequences actually converge to the same classical solution, thus yielding the desired result.

We consider a sequence of weak solutions u_{s_j} of (1.21) with s_j approaching 1 as $j \to +\infty$. Without loss of generality, we can assume that $s_j \in (1/2, 1)$.

By (1.9) and (1.10), we have that

$$\|u_{s_j}\|_{L^\infty(\Omega)} \leqslant h^{-1}(Q) \tag{3.2}$$

and

$$c_{n,s_j} \iint\limits_{\mathbb{R}^n \times B_\varrho(x)} \frac{\big(u_{s_j}(x) - u_{s_j}(z)\big)^2}{|M(z, x-z)(x-z)|^{n+2s_j}}\, dx\, dz \leqslant 2h^{-1}(Q)\left(\|f\|_{L^1(\Omega)} + Q\|a\|_{L^1(\Omega)}\right).$$

From this, the assumption (1.14) and Lemma 2.1, we also see that

$$[u_{s_j}]^2_{s_j} \leqslant 2C_{\alpha,\beta,n}h^{-1}(Q)\left(\|f\|_{L^1(\Omega)} + Q\|a\|_{L^1(\Omega)}\right).$$

We now take $\overline{s} \in (1/2, 1)$. By [17, Lemma 2.1], we thereby find that there exists a positive constant C, only depending on n and Ω, such that, when $s_j \in (\overline{s}, 1)$,

$$[u_{s_j}]^2_{\overline{s}} \leqslant C[u_{s_j}]^2_{s_j} \leqslant 2C\, C_{\alpha,\beta,n}h^{-1}(Q)\left(\|f\|_{L^1(\Omega)} + Q\|a\|_{L^1(\Omega)}\right) =: C_1, \tag{3.3}$$

namely the sequence u_{s_j} is bounded in $H_0^{\overline{s}}(\Omega)$, uniformly in j.

Thus, we can extract a subsequence s_{j_k} for which there exists $w \in H_0^{\overline{s}}(\Omega)$ such that, as $k \to +\infty$,

$$u_{s_{j_k}} \rightharpoonup w \text{ in } H^{\overline{s}}(\mathbb{R}^n)$$

$$u_{s_{j_k}} \to w \text{ in } L^2(\mathbb{R}^n), \tag{3.4}$$

$$\text{and} \quad u_{s_{j_k}} \to w \text{ a.e. in } \mathbb{R}^n \tag{3.5}$$

and (see e.g. [7, Theorem 4.9]) there exists $g \in L^2(\Omega)$ such that

$$|u_{s_{j_k}}(x)| \leqslant g(x) \tag{3.6}$$

for any $k \in \mathbb{N}$ and for a.e. $x \in \Omega$.

We will actually show below that w is the solution u_1 of the classical problem (1.22).

By (3.3), (3.5) and Fatou's Lemma, we conclude that

$$[w]^2_{\overline{s}} \leqslant C_1. \tag{3.7}$$

Moreover, we remark that (3.2) entails that

$$\|w\|_{L^\infty(\Omega)} \leqslant h^{-1}(Q). \tag{3.8}$$

We now claim that

$$\|\nabla w\|_{L^2(\mathbb{R}^n)} \leqslant C_1. \tag{3.9}$$

To prove this, we define

$$w_\varepsilon(x) := (\eta_\varepsilon * w)(x),$$

being η_ε as in (3.1). By [20, Appendix C4, Theorem 6], we have that

$$\lim_{\varepsilon \searrow 0} w_\varepsilon(x) = w(x) \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{3.10}$$

Moreover, (3.8) gives that

$$\|w_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leqslant \|w\|_{L^\infty(\mathbb{R}^n)} \leqslant h^{-1}(Q). \tag{3.11}$$

Also, in light of Lemma 3.1 and (3.7),

$$[w_\varepsilon]^2_{\overline{s}} \leqslant [w]^2_{\overline{s}} \leqslant C_1,$$

and therefore

$$\lim_{\overline{s} \nearrow 1} [w_\varepsilon]^2_{\overline{s}} \leqslant \lim_{\overline{s} \nearrow 1} [w]^2_{\overline{s}} \leqslant C_1. \tag{3.12}$$

Moreover, since $w_\varepsilon \in C^\infty_c(\mathbb{R}^n)$, we have that

$$\lim_{\overline{s} \nearrow 1} [w_\varepsilon]^2_{\overline{s}} = \|\nabla w_\varepsilon\|^2_{L^2(\mathbb{R}^n)}.$$

Combining this and (3.12), we deduce that

$$\|\nabla w_\varepsilon\|^2_{L^2(\mathbb{R}^n)} \leqslant C_1.$$

From this, (3.10) and (3.11), for an infinitesimal subsequence ε_ℓ,

$$w_{\varepsilon_\ell} \rightharpoonup w \text{ in } H^1(\mathbb{R}^n).$$

This shows that $w \in H^1(\mathbb{R}^n)$ and

$$\|\nabla w\|^2_{L^2(\mathbb{R}^n)} \leqslant \liminf_{\ell \to +\infty} \|\nabla w_{\varepsilon_\ell}\|^2_{L^2(\mathbb{R}^n)} \leqslant C_1,$$

which establishes (3.9).

From (3.9), since $w \in H^{\overline{s}}_0(\Omega)$, we infer that $w \in H^1_0(\Omega)$.

We now show that w is a weak solution (and, in fact, the unique solution, due to the comparison principle) of (1.22), namely, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$, w satisfies

$$\int_\Omega A(x)\nabla w(x)\cdot\nabla\varphi(x)\,dx = \int_\Omega \big(f(x)-a(x)h(w(x))\big)\varphi(x)\,dx. \tag{3.13}$$

To prove this, we notice that by (1.5) and (1.9), and since $a \in L^1(\Omega)$,

$$|a(x)h(u_{s_{j_k}})\varphi(x)| \leqslant Q\|\varphi\|_{L^\infty(\mathbb{R}^n)}a(x) \in L^1(\Omega).$$

Hence, from this, (3.5) and the Dominated Convergence Theorem, we infer that

$$\lim_{k\to+\infty}\int_\Omega a(x)h(u_{s_{j_k}}(x))\varphi(x)\,dx = \int_\Omega a(x)h(w(x))\varphi(x)\,dx. \tag{3.14}$$

We also observe that

$$|\mathcal{L}_{M,\varrho,s}\varphi(x)| \leqslant C^\sharp\|\varphi\|_{C^2(\mathbb{R}^n)}, \tag{3.15}$$

being $C^\sharp > 0$ independent of s.

Indeed, Corollary B.4 ensures that condition (1.4) is satisfied, allowing us to use Lemma A.2, concluding that

$$|\mathcal{L}_{M,\varrho,s}\varphi(x)| \leqslant C_n\,(C+C_\sharp)\,\|\varphi\|_{C^2(\mathbb{R}^n)},$$

where $C_n > 0$ depends only on n, C is as in (1.1) and $C_\sharp$ denotes the left-hand side of (1.4).

This proves (3.15), up to renaming constants.

On this account, by combining (3.15) with (3.6), we deduce that

$$\big|(u_{s_{j_k}}(x)-w(x))\mathcal{L}_{M,\varrho,s_{j_k}}\varphi(x)\big| \leqslant C^\sharp\,\big(g(x)+|w(x)|\big)\,\|\varphi\|_{C^2(\mathbb{R}^n)},$$

where the latter function belongs to $L^1(\Omega)$.

Accordingly, by (3.5),

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}(u_{s_{j_k}}(x)-w(x))\,\mathcal{L}_{M,\varrho,s_{j_k}}\varphi(x)\,dx = 0.$$

As a consequence, we have that

$$\lim_{k\to+\infty}\frac{c_{n,s_{j_k}}}{2}\iint_{\mathbb{R}^n\times B_\varrho}\frac{(u_{s_{j_k}}(x)-u_{s_{j_k}}(x-y))(\varphi(x)-\varphi(x-y))}{|M(x-y,y)y|^{n+2s_{j_k}}}\,dx\,dy$$

$$= \lim_{k\to+\infty} \int_{\mathbb{R}^n} u_{s_{j_k}}(x)\mathcal{L}_{M,\varrho,s_{j_k}}\varphi(x)\,dx = \lim_{k\to+\infty} \int_{\mathbb{R}^n} w(x)\mathcal{L}_{M,\varrho,s_{j_k}}\varphi(x)\,dx$$

$$= \lim_{k\to+\infty} \frac{c_{n,s_{j_k}}}{2} \iint_{\mathbb{R}^n\times B_\varrho} \frac{(w(x)-w(x-y))(\varphi(x)-\varphi(x-y))}{|M(x-y,y)y|^{n+2s_{j_k}}}\,dx\,dy.$$

From this and Proposition 1.3 (which can be used here because $w \in H^1(\mathbb{R}^n)$, thanks to (3.8) and (3.9)), we deduce that

$$\lim_{k\to+\infty} \frac{c_{n,s_{j_k}}}{2} \iint_{\mathbb{R}^n\times B_\varrho} \frac{(u_{s_{j_k}}(x)-u_{s_{j_k}}(x-y))(\varphi(x)-\varphi(x-y))}{|M(x-y,y)y|^{n+2s_{j_k}}}\,dx\,dy$$

$$= \int_{\mathbb{R}^n} A(x)\nabla w(x)\cdot\nabla\varphi(x)\,dx,$$

being $A(x)$ as in (1.20).

Combining this and (3.14) with the fact that $u_{s_{j_k}}$ is a weak solution of (1.21), we obtain the desired claim in (3.13).

The estimates in (1.23) follow from (3.8) and (3.9). □

4. Tailoring a matrix M satisfying (1.20) and proof of Theorems 1.6 and 1.7

4.1. The algebraic roadmap leading to Theorem 1.6 and proof of Lemma 1.5

We start by explicitly computing the following integral.

Lemma 4.1. *Let $\sigma_1,\dots,\sigma_n > 0$ and $\varphi = (\varphi_1,\dots,\varphi_n) \in \mathcal{S}^{n-1}$.*

Then, for any $i \in \{1,\dots,n\}$,

$$\int_{\mathcal{S}^{n-1}} \frac{\varphi_i^2}{\left(\sum_{k=1}^n \sigma_k^2\varphi_k^2\right)^{\frac{n+2}{2}}}\,d\mathcal{H}^{n-1}_\varphi = \frac{V_{\mathbb{E}^n}}{\sigma_i^2},$$

where

$$V_{\mathbb{E}^n} := \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n+2}{2}\right)\sigma_1\dots\sigma_n}$$

denotes the volume of the n-dimensional hyperellipsoid with semiaxes $\frac{1}{\sigma_1},\dots,\frac{1}{\sigma_n}$.

Proof. Up to reordering indices, we may suppose that $i = 1$. We recall the system of spherical coordinates (see e.g. [6, equation (*) on page 65]) given by

$$\begin{aligned}
\varphi_1 &= \cos x_1,\\
\varphi_j &= \cos x_j \prod_{k=1}^{j-1} \sin x_k, \qquad \text{for all } j \in \{2,\dots,n-2\},\\
\varphi_{n-1} &= \sin x_{n-1} \prod_{k=1}^{n-2} \sin x_k,\\
\varphi_n &= \cos x_{n-1} \prod_{k=1}^{n-2} \sin x_k,
\end{aligned}$$

with $x_1,\dots,x_{n-2} \in [0,\pi]$ and $x_{n-1} \in [0,2\pi]$.

The Jacobian determinant of this transformation (see e.g. [6, page 66]) can be written as

$$J = \prod_{k=1}^{n-2} \sin^k x_{n-1-k}.$$

In addition, for any $j \in \{1,\dots,n-1\}$, we set

$$\alpha_j := \sqrt{\frac{\sum_{k=j+1}^{n} \sigma_k^2 \varphi_k^2}{\prod_{k=1}^{j} \sin^2 x_k}}. \tag{4.1}$$

Using the short notation $x^{(j)} := (x_j,\dots,x_{n-1})$, we point out that

$$\alpha_j \text{ depends only on } x^{(j+1)}. \tag{4.2}$$

In this way, we have that α_1 does not depend on x_1 and consequently

$$\begin{aligned}
\mathcal{I} &:= \int_{\mathcal{S}^{n-1}} \frac{\varphi_1^2}{\left(\sum_{k=1}^{n} \sigma_k^2 \varphi_k^2\right)^{\frac{n+2}{2}}}\, d\mathcal{H}_\varphi^{n-1} = \int_{\mathcal{S}^{n-1}} \frac{\varphi_1^2}{\left(\sigma_1^2 \varphi_1^2 + \sum_{k=2}^{n} \sigma_k^2\, \varphi_k^2\right)^{\frac{n+2}{2}}}\, d\mathcal{H}_\varphi^{n-1}\\
&= \int_{[0,\pi]^{n-2}\times[0,2\pi]} \frac{\cos^2 x_1}{\left(\sigma_1^2 \cos^2 x_1 + \alpha_1^2 \sin^2 x_1\right)^{\frac{n+2}{2}}} \prod_{k=1}^{n-2} \sin^k x_{n-1-k}\, dx^{(1)}.
\end{aligned} \tag{4.3}$$

Now we claim that, for all $m \in \{1,\dots,n-2\}$,

$$\mathcal{I} = \frac{\pi^{\frac{m}{2}}}{2\sigma_1^2 \prod_{k=1}^{m} \sigma_k} \frac{\Gamma\left(\frac{n-m}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \int_{[0,\pi]^{n-m-2}\times[0,2\pi]} \frac{1}{\alpha_m^{n-m}} \prod_{k=1}^{n-m-2} \sin^k x_{n-1-k}\, dx^{(m+1)}. \tag{4.4}$$

Indeed, by [22, 3.642, n.1, page 403], if a, b, ν, $\mu \in (0,+\infty)$ with $2\nu-1 \in 2\mathbb{N}$,

$$\begin{aligned}\int_{[0,\pi]} \frac{\cos^{2\nu-1}\tau\,\sin^{2\mu-1}\tau}{\left(b^2\cos^2\tau+a^2\sin^2\tau\right)^{\mu+\nu}}\,d\tau &= 2\int_0^{\frac{\pi}{2}} \frac{\cos^{2\nu-1}\tau\,\sin^{2\mu-1}\tau}{\left(b^2\cos^2\tau+a^2\sin^2\tau\right)^{\mu+\nu}}\,d\tau\\ &= \frac{\Gamma(\nu)\,\Gamma(\mu)}{a^{2\mu}b^{2\nu}\,\Gamma(\mu+\nu)}.\end{aligned} \tag{4.5}$$

By using this (with $a := \alpha_1$, $b := \sigma_1$, $\mu := (n-1)/2$ and $\nu := 3/2$, on the integration variable $\tau := x_1$), we find that

$$\int_{[0,\pi]} \frac{\cos^2 x_1\,\sin^{n-2}x_1}{\left(\sigma_1^2\cos^2 x_1+\alpha_1^2\sin^2 x_1\right)^{\frac{n+2}{2}}}\,dx_1 = \frac{\sqrt{\pi}}{2\alpha_1^{n-1}\sigma_1^3}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}.$$

Plugging this information into (4.3), we conclude that

$$\mathcal{I} = \frac{\sqrt{\pi}}{2\sigma_1^3}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\int_{[0,\pi]^{n-3}\times[0,2\pi]} \frac{1}{\alpha_1^{n-1}}\prod_{k=1}^{n-3}\sin^k x_{n-1-k}\,dx^{(2)}.$$

This proves (4.4) with $m=1$.

Now we argue recursively, supposing that (4.4) holds true for the index m, with $m \leqslant n-3$, and we establish it for the index $m+1$. To accomplish this goal, we point out that, for all $j \in \{1,\dots,n-3\}$,

$$\begin{aligned}\alpha_j^2 &= \frac{\sum_{k=j+1}^{n}\sigma_k^2\varphi_k^2}{\prod_{k=1}^{j}\sin^2 x_k} = \frac{\sigma_{j+1}^2\varphi_{j+1}^2+\sum_{k=j+2}^{n}\sigma_k^2\varphi_k^2}{\prod_{k=1}^{j}\sin^2 x_k} = \frac{\sigma_{j+1}^2\varphi_{j+1}^2+\alpha_{j+1}^2\prod_{k=1}^{j+1}\sin^2 x_k}{\prod_{k=1}^{j}\sin^2 x_k}\\ &= \sigma_{j+1}^2\cos^2 x_{j+1}+\alpha_{j+1}^2\sin^2 x_{j+1}.\end{aligned}$$

This and the inductive hypothesis yield

$$\begin{aligned}\mathcal{I} = {}& \frac{\pi^{\frac{m}{2}}}{2\sigma_1^2\prod_{k=1}^{m}\sigma_k}\frac{\Gamma\left(\frac{n-m}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\int_{[0,\pi]^{n-m-2}\times[0,2\pi]} \frac{1}{\left(\sigma_{m+1}^2\cos^2 x_{m+1}+\alpha_{m+1}^2\sin^2 x_{m+1}\right)^{\frac{n-m}{2}}}\\ &\times\prod_{k=1}^{n-m-2}\sin^k x_{n-1-k}\,dx^{(m+1)}.\end{aligned} \tag{4.6}$$

Besides, by (4.2), α_{m+1} depends only on $x^{(m+2)}$ and we can thereby utilize (4.5) (here with $a := \alpha_{m+1}$, $b := \sigma_{m+1}$, $\mu := \frac{n-m-1}{2}$ and $\nu := 1/2$) and we see that

$$\int_{[0,\pi]} \frac{\sin^{n-m-2} x_{m+1}}{\left(\sigma_{m+1}^2 \cos^2 x_{m+1} + \alpha_{m+1}^2 \sin^2 x_{m+1}\right)^{\frac{n-m}{2}}}\, dx_{m+1} = \frac{\sqrt{\pi}}{\alpha_{m+1}^{n-m-1}\sigma_{m+1}} \frac{\Gamma\left(\frac{n-m-1}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)}.$$

From this and (4.6) we arrive at

$$\begin{aligned}\mathcal{I} = {} & \frac{\pi^{\frac{m}{2}}}{2\sigma_1^2 \prod_{k=1}^{m} \sigma_k} \frac{\Gamma\left(\frac{n-m}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \frac{\sqrt{\pi}}{\sigma_{m+1}} \frac{\Gamma\left(\frac{n-m-1}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)} \\ & \times \int_{[0,\pi]^{n-m-3}\times[0,2\pi]} \frac{1}{\alpha_{m+1}^{n-m-1}} \prod_{k=1}^{n-m-3} \sin^k x_{n-1-k}\, dx^{(m+2)},\end{aligned}$$

which gives (4.4) for the index $m+1$. This completes the inductive step and establishes (4.4).

We also remark that

$$\alpha_{n-2}^2 = \frac{\sum_{k=n-1}^{n} \sigma_k^2 \varphi_k^2}{\prod_{k=1}^{n-2} \sin^2 x_k} = \frac{\sigma_{n-1}^2 \varphi_{n-1}^2 + \sigma_n^2 \varphi_n^2}{\prod_{k=1}^{n-2} \sin^2 x_k} = \sigma_{n-1}^2 \sin^2 x_{n-1} + \sigma_n^2 \cos^2 x_{n-1}.$$

Thus, we deduce from (4.4) (used with $m := n-2$) that

$$\begin{aligned}\mathcal{I} &= \frac{\pi^{\frac{n-2}{2}}}{2\,\sigma_1^2 \prod_{k=1}^{n-2} \sigma_k} \frac{1}{\Gamma\left(\frac{n+2}{2}\right)} \int_{[0,2\pi]} \frac{1}{\alpha_{n-2}^2}\, dx_{n-1} \\ &= \frac{\pi^{\frac{n-2}{2}}}{2\sigma_1^2 \prod_{k=1}^{n-2} \sigma_k} \frac{1}{\Gamma\left(\frac{n+2}{2}\right)} \int_{[0,2\pi]} \frac{1}{\sigma_{n-1}^2 \sin^2 x_{n-1} + \sigma_n^2 \cos^2 x_{n-1}}\, dx_{n-1}.\end{aligned} \tag{4.7}$$

Also, from (4.5) (used here with $a := \sigma_{n-1}$, $b := \sigma_n$ and $\mu := \nu := 1/2$),

$$\begin{aligned}\int_{[0,2\pi]} \frac{1}{\sigma_{n-1}^2 \sin^2 x_{n-1} + \sigma_n^2 \cos^2 x_{n-1}}\, dx_{n-1} &= 2 \int_{[0,\pi]} \frac{1}{\sigma_{n-1}^2 \sin^2 x_{n-1} + \sigma_n^2 \cos^2 x_{n-1}}\, dx_{n-1} \\ &= \frac{2\pi}{\sigma_{n-1}\sigma_n}.\end{aligned}$$

Combining this and (4.7), the desired result follows. □

Lemma 4.2. *Let $\lambda_1, \dots, \lambda_n > 0$. Let $\overline{\lambda} := \lambda_1 \cdots \lambda_n$ and, for any $i \in \{1, \dots, n\}$,*

$$\sigma_i := \sqrt{\frac{\overline{\lambda}^{\frac{1}{n+2}}}{\lambda_i}}. \tag{4.8}$$

Let also

$$V_n := \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n+2}{2}\right)}$$

denote the volume of the n-dimensional unit ball.

Then, for any i, $j \in \{1, \dots, n\}$,

$$\frac{1}{V_n} \int\limits_{\mathcal{S}^{n-1}} \frac{\varphi_i \varphi_j}{\left(\sum\limits_{k=1}^{n} \sigma_k^2 \varphi_k^2\right)^{\frac{n+2}{2}}} \, d\mathcal{H}_\varphi^{n-1} = \lambda_i \delta_{ij}. \tag{4.9}$$

In addition,

$$\frac{\left(\min\limits_{j\in\{1,\dots,n\}} \lambda_j\right)^{\frac{n}{2(n+2)}}}{\left(\max\limits_{j\in\{1,\dots,n\}} \lambda_j\right)^{\frac{1}{2}}} \leqslant \sigma_i \leqslant \frac{\left(\max\limits_{j\in\{1,\dots,n\}} \lambda_j\right)^{\frac{n}{2(n+2)}}}{\left(\min\limits_{j\in\{1,\dots,n\}} \lambda_j\right)^{\frac{1}{2}}}. \tag{4.10}$$

Proof. We set $\overline{\sigma} := \sigma_1 \dots \sigma_n$ and we deduce from (4.8) that

$$\overline{\sigma} = \frac{\overline{\lambda}^{\frac{n}{2(n+2)}}}{\overline{\lambda}^{\frac{1}{2}}} = \frac{1}{\overline{\lambda}^{\frac{1}{n+2}}}.$$

As a consequence

$$\lambda_i = \frac{\overline{\lambda}^{\frac{1}{n+2}}}{\sigma_i^2} = \frac{1}{\sigma_i^2 \overline{\sigma}}.$$

From this and Lemma 4.1, we obtain that

$$\frac{1}{V_n} \int\limits_{\mathcal{S}^{n-1}} \frac{\varphi_i^2}{\left(\sum\limits_{k=1}^{n} \sigma_k^2 \varphi_k^2\right)^{\frac{n+2}{2}}} \, d\mathcal{H}_\varphi^{n-1} = \frac{V_{\mathbb{E}^n}}{\sigma_i^2 V_n} = \frac{1}{\sigma_i^2 \overline{\sigma}} = \lambda_i,$$

which is the desired identity in (4.9) when $i = j$.

If instead $i \neq j$, both sides of (4.9) vanish, due to the odd symmetry of the integrand in the i-th coordinate.

Moreover, that inequalities in (4.10) plainly follow from (4.8). □

We can now prove Lemma 1.5:

Proof of Lemma 1.5. By (1.24) and (1.25), we have that

$$A_{ij} = \sum_{k,l=1}^{n} O_{ik}\,\Lambda_{kl}\,O^T_{lj} = \sum_{k,l=1}^{n} O_{ik}\lambda_k\delta_{kl}O_{jl}.$$

From this and Lemma 4.2, we deduce that

$$A_{ij} = \frac{n}{\omega_{n-1}} \int\limits_{\mathcal{S}^{n-1}} \frac{\displaystyle\sum_{k,l=1}^{n} O_{ik}\varphi_k\varphi_l O_{jl}}{\left|\displaystyle\sum_{l=1}^{n}\sigma_l^2\varphi_l^2\right|^{\frac{n+2}{2}}}\, d\mathcal{H}^{n-1}_{\varphi}, \tag{4.11}$$

where we have also used the identity $\omega_{n-1} = nV_n$.

Now we perform the change of variable $\psi := O\varphi$. In light of (1.27), we see that

$$\begin{aligned} |N_A\psi|^2 &= (N_A\psi)^T(N_A\psi) = \psi^T N_A^T N_A\psi = \psi^T(O\Sigma O^T)^T(O\Sigma O^T)\psi \\ &= \psi^T O\Sigma^2 O^T\psi = \varphi^T\Sigma^2\varphi = \sum_{l=1}^{n}\sigma_l^2\varphi_l^2. \end{aligned} \tag{4.12}$$

Also,

$$\sum_{k,l=1}^{n} O_{ik}\varphi_k\varphi_l O_{jl} = (O\varphi)_i(O\varphi)_j = \psi_i\psi_j. \tag{4.13}$$

Moreover, we observe that, since $\varphi \in \mathcal{S}^{n-1}$ and O is orthogonal, we have that $d\mathcal{H}^{n-1}_{\psi} = d\mathcal{H}^{n-1}_{\varphi}$.

Accordingly, from this, (4.11), (4.12) and (4.13), we obtain that

$$A_{ij} = \frac{n}{\omega_{n-1}} \int\limits_{\mathcal{S}^{n-1}} \frac{\psi_i\,\psi_j}{|N_A\psi|^{n+2}}\, d\mathcal{H}^{n-1}_{\psi},$$

as desired. □

4.2. Proof of Theorem 1.6

We observe that a matrix-valued satisfying (1.14) and (1.15) can be reconstructed by the mere knowledge of $M(x,0)$, as we now discuss.

Lemma 4.3. *Let $N : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and let $H : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be such that*

$$H(0) = 0 \quad \text{and} \quad H(y) = H(-y) \ \text{ for any } y \in \mathbb{R}^n. \tag{4.14}$$

For any x, $y \in \mathbb{R}^n$, define $M : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ as

$$M(x,y) := \frac{N(x) + N(x+y)}{2} + H(y).$$

Then,

$$M(x,0) = N(x) \quad \text{and} \quad M(x-y,y) = M(x,-y).$$

Proof. In light of (4.14), we have that $M(x,0) = N(x)$ and that

$$M(x-y,y) - M(x,-y) = \frac{N(x-y) + N(x)}{2} + H(y) - \frac{N(x) + N(x-y)}{2} + H(-y) = 0,$$

as desired. □

Lemma 4.4. *Let $\Lambda : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a matrix-valued function with $\Lambda(x)$ diagonal and such that*

$$0 < \inf_{\substack{x\in\Omega \\ j\in\{1,\dots,n\}}} \Lambda_{jj}(x) \leqslant \sup_{\substack{x\in\Omega \\ j\in\{1,\dots,n\}}} \Lambda_{jj}(x) < +\infty. \tag{4.15}$$

Let $O : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be such that $O(x)$ is an orthogonal matrix and define

$$N(x) := O(x)\Lambda(x)O^T(x). \tag{4.16}$$

Then, there exist $\widetilde{\alpha} \geqslant \widetilde{\beta} > 0$ such that, for any x, $\xi \in \mathbb{R}^n$,

$$\widetilde{\beta}|\xi|^2 \leqslant N(x)\xi\cdot\xi \leqslant \widetilde{\alpha}|\xi|^2.$$

Proof. We use the notation $\Lambda_{ij}(x) := \sigma_i(x)\delta_{ij}$. Moreover, for any $x, \xi \in \mathbb{R}^n$, we set

$$\eta(x) := O^T(x)\xi.$$

Since $O(x)$ is orthogonal, we have $|\eta(x)| = |\xi|$ for any $x \in \mathbb{R}^n$. Hence, combining (4.15) and (4.16), we see that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} N(x)\xi\cdot\xi = \xi^T O(x)\Lambda(x)O^T(x)\xi = \eta^T(x)\Lambda(x)\eta(x) = \sum_{i=1}^n \sigma_i(x)\eta_i^2(x) \\ \in \big[\widetilde{\beta}|\eta(x)|^2, \widetilde{\alpha}|\eta(x)|^2\big] = \big[\widetilde{\beta}|\xi|^2, \widetilde{\alpha}|\xi|^2\big], \end{aligned} \tag{4.17}$$

for some $\widetilde{\alpha} \geqslant \widetilde{\beta} > 0$ independent of x. □

Lemma 4.5. *Let N be as in Lemma 4.4. Let M and H be as in Lemma 4.3.*

Assume also that for all x, $\xi \in \mathbb{R}^n$,

$$0 \leqslant H(x)\xi \cdot \xi \leqslant C_0, \tag{4.18}$$

for some $C_0 \in (0, +\infty)$.

Then, there exist $\alpha \geqslant \beta > 0$, depending only on $\widetilde{\alpha}$, $\widetilde{\beta}$, C_0 and n, such that, for all x, y, $\xi \in \mathbb{R}^n$,

$$\beta|\xi|^2 \leqslant M(x-y, y)\xi \cdot \xi \leqslant \alpha|\xi|^2 \tag{4.19}$$

and

$$\beta|\xi| \leqslant |M(x-y, y)\xi| \leqslant \alpha|\xi|. \tag{4.20}$$

Proof. We have that

$$M(x-y, y)\xi \cdot \xi = \frac{1}{2}(N(x-y) + N(x))\xi \cdot \xi + H(y)\xi \cdot \xi. \tag{4.21}$$

Hence, the desired result in (4.19) follows from (4.18) and Lemma 4.4.

We can thereby apply Lemma B.2 and deduce (4.20) from (4.19) (up to renaming α and β). □

We now complete the proof of Theorem 1.6.

Proof of Theorem 1.6. Since $A(x)$ is symmetric, by [28, Theorem 5S, page 330] we have that

$$A(x) := O(x)\Lambda(x)O^T(x), \tag{4.22}$$

where $O(x)$, $\Lambda(x) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfy

$$O(x)O^T(x) = \mathrm{Id} \quad \text{and} \quad \Lambda_{ij}(x) := \lambda_i(x)\delta_{ij} \quad \text{for any } x \in \mathbb{R}^n.$$

Since A is continuous, so are Λ and O, due to Lemmata C.1 and C.2. Hence, if $\Sigma(x)$ and $N_{A(x)}$ are defined as in Lemma 1.5, we have that Σ is a bounded and continuous matrix-valued function (recall also (1.29)), and thus so is $N_{A(x)}$.

Now, by using Lemmata 4.3 and 4.5 we infer that there exists a matrix $M(x, y)$, with $x \in \mathbb{R}^n$ and $y \in B_\varrho$, satisfying (1.14) and (1.15) for any $\varrho \in (0, +\infty]$.

Also, the claim in (1.20) follows from (1.28).

We remark that the assumption that $N_{A(x)}$ and H are continuous allows us to define $M(x, 0)$. □

4.3. Reconstructing the classical solution through the limit $s \nearrow 1$ and proof of Theorem 1.7

We now complete the proof of Theorem 1.7.

Proof of Theorem 1.7. Given $A : \Omega \to \mathbb{R}^{n\times n}$, we extend it in a small neighborhood containing the closure of Ω (e.g., by orthogonal projection) and consider a mollification of all its entries, which will be denoted by A_ℓ, with $\ell \in \mathbb{N}$. In this way, we have that

$$\lim_{\ell\to+\infty} \|A - A_\ell\|_{L^1(\Omega)} = 0$$

and, up to a (not relabeled) subsequence, for a.e. $x \in \Omega$,

$$\lim_{\ell\to+\infty} A_\ell(x) = A(x). \tag{4.23}$$

We observe that, denoting by $\eta_\ell \in C_0^\infty(\mathbb{R}^n)$ the mollifier used in this procedure, for all $\xi \in \mathcal{S}^{n-1}$,

$$A_\ell(x)\xi\cdot\xi = \int_{\mathbb{R}^n} A(y)\xi\cdot\xi\eta_\ell(x-y)\,dy \in \left[\underset{\sim}{\lambda}\int_{\mathbb{R}^n}\eta_\ell(x-y)\,dy,\ \widetilde{\lambda}\int_{\mathbb{R}^n}\eta_\ell(x-y)\,dy\right] = [\underset{\sim}{\lambda},\widetilde{\lambda}]. \tag{4.24}$$

Now, we diagonalize the symmetric matrix $A(x)$ by writing, for all $x \in \Omega$, that

$$A(x) = O(x)\Lambda(x)O^T(x), \tag{4.25}$$

where $\Lambda(x)$ is diagonal and $O(x)$ is orthogonal.

We now utilize Lemma C.2 with $B := A_\ell(x)$. Specifically, by (4.23), for a.e. $x \in \Omega$ we find $\ell_x \in \mathbb{N}$ such that, if $\ell \geqslant \ell_x$, then $\|A(x) - A_\ell(x)\| \leqslant c$, being $c > 0$ the structural constant in Lemma C.2. Accordingly, combining (4.25) and Lemma C.2, for all $\ell \geqslant \ell_x$ we find an orthogonal matrix $O_\ell(x)$ such as $\Lambda_\ell(x) := O_\ell^T(x)A_\ell(x)O_\ell(x)$ is diagonal and

$$\|O(x) - O_\ell(x)\| \leqslant C\|A(x) - A_\ell(x)\|.$$

Actually, by virtue of Lemma C.1, we also know that

$$\lim_{\ell\to+\infty} \|\Lambda_\ell(x) - \Lambda(x)\| = 0.$$

Owing to (4.24), if $\lambda_{1,\ell} \leqslant \cdots \leqslant \lambda_{n,\ell}$ are the eigenvalues of A_ℓ, we know that $\lambda_{1,\ell},\dots,\lambda_{n,\ell} \in [\underset{\sim}{\lambda},\widetilde{\lambda}]$. Then, we are in the position of exploiting Theorem 1.6 (with $A := A_\ell$) and obtain that there exists $M_\ell : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfying (1.14), (1.15) and (1.20) (for any $\varrho \in (0,+\infty]$).

In particular, recalling (1.30) (used here with $H := 0$), we have that

$$M_\ell(x,y) = \frac{N_{A_\ell(x)} + N_{A_\ell(x+y)}}{2}. \tag{4.26}$$

We now recall the notation in (1.16) and denote by $u_{s,\ell} \in H_0^s(\Omega) \cap L^\infty(\Omega)$ the weak solution of

$$\begin{cases} \mathcal{L}_{M_\ell,\varrho,s} u + a h(u) = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

whose existence and uniqueness is guaranteed by Theorem 1.1 (for any $\varrho \in (0, +\infty]$). Bearing in mind (4.26) and the smoothness of A_ℓ, we see that the assumption in (1.18) is satisfied, allowing us to use Theorem 1.4 and conclude that

$$\lim_{s \nearrow 1} u_{s,\ell}(x) = u_{1,\ell}(x),$$

where $u_{1,\ell} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ solves

$$\begin{cases} -\operatorname{div}(A_\ell \nabla u) + a h(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, by (1.23),

$$\begin{aligned} &\|u_{1,\ell}\|_{L^\infty(\Omega)} \leqslant h^{-1}(Q) \\ \text{and} \quad &\|\nabla u_{1,\ell}\|^2_{L^2(\Omega)} \leqslant C\, h^{-1}(Q) \left(\|f\|_{L^1(\Omega)} + Q \|a\|_{L^1(\Omega)} \right). \end{aligned} \tag{4.27}$$

We can therefore assume, up to a subsequence, that $u_{1,\ell}$ converges weakly in $H_0^1(\Omega)$ to some function u_1. Passing to the limit the weak equation satisfied by $u_{1,\ell}$ and the uniform estimate in (4.27), the desired result follows. □

Acknowledgments

The authors would like to thank the referee for their valuable comments and suggestions, which helped us to improve our manuscript.

DA is supported by Ministerio of Ciencia e Innovación (Spain) and European Regional Development Fund (ERDF) PID2021-122122NB-I00 and Junta de Andalucía FQM-116.

SD, CS and EV are members of the Australian Mathematical Society (AustMS). CS and EPL are members of the INdAM–GNAMPA.

CS acknowledges the support of the Juan de la Cierva Fellowship (grant number JDC2023-050365-I).

CS acknowledges the support of the INdAM-GNAMPA through the grant "Borse di studio per l'estero 2023-2024", of which she is the recipient.

CS thanks the Departamento de Análisis Matemático of the Universidad de Granada, where part of this work was carried out in January and February 2025, for the warm and generous hospitality.

This work has been supported by the Australian Laureate Fellowship FL190100081 and by the Australian Future Fellowship FT230100333.

Funding for open access charge: Universidad de Granada / CBUA.

Appendix A. Calculating $\mathcal{L}_K$ in the pointwise sense

Here we show that, for $s \in (0, \frac{1}{2})$, the principal value in definition (1.3) is not needed and the operator $\mathcal{L}_K u$ is well-defined for smooth and compactly supported functions.

Lemma A.1. *Let $s \in (0, \frac{1}{2})$ and assume that (1.1) holds.*

Then, for all $u \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the map $\mathbb{R}^n \ni y \longmapsto \big(u(x)-u(x-y)\big)\, K(x, x-y)$ is in $L^1(\mathbb{R}^n)$ and

$$\|\mathcal{L}_K u\|_{L^\infty(\mathbb{R}^n)} \leqslant C\, C_n\, \|u\|_{C^1(\mathbb{R}^n)} \left(\frac{1}{s} + \frac{1}{1-2s}\right)$$

where $C_n > 0$ depends only on n, and C is as in (1.1).

Proof. We have that

$$\int\limits_{\mathbb{R}^n \setminus B_1} |u(x) - u(x-y)|\, K(x, x-y)\, dy \leqslant 2C \|u\|_{L^\infty(\mathbb{R}^n)} \int\limits_{\mathbb{R}^n \setminus B_1} \frac{dy}{|y|^{n+2s}} = \frac{C\,\widetilde{C}}{s} \|u\|_{L^\infty(\mathbb{R}^n)},$$

for some $\widetilde{C} > 0$ depending only on n.

Moreover, since $s \in (0, \frac{1}{2})$, we infer that

$$\begin{aligned} \int\limits_{B_1} |u(x) - u(x-y)|\, K(x, x-y)\, dy &\leqslant C\, \widehat{C}\, \|\nabla u\|_{L^\infty(\mathbb{R}^n)} \int\limits_{B_1} \frac{dy}{|y|^{n+2s-1}} \\ &= \frac{C\,\widehat{C}}{1-2s} \|\nabla u\|_{L^\infty(\mathbb{R}^n)}, \end{aligned}$$

for some $\widehat{C} > 0$ depending only on n and possibly changing during the computation. □

Here we point out that $\mathcal{L}_K$ can be computed pointwise on smooth functions, provided that condition (1.4) is assumed.

Lemma A.2. *If $s \in \left[\frac{1}{2}, 1\right)$ and condition (1.4) holds true, then the expression in (1.3) is well-defined for all $u \in C_c^\infty(\mathbb{R}^n)$ and*

$$\|\mathcal{L}_K u\|_{L^\infty(\mathbb{R}^n)} \leqslant C_n \|u\|_{C^2(\mathbb{R}^n)} \left(\frac{C}{s(1-s)} + C_\sharp \right),$$

where $C_n > 0$ depends only on n, C is as in (1.1) and $C_\sharp$ denotes the left-hand side of (1.4).

Proof. Let $\varepsilon \in (0,1)$ and observe that

$$\begin{aligned}
&2 \int_{B_1 \setminus B_\varepsilon} \big(u(x) - u(x-y)\big) K(x, x-y)\, dy \\
&\quad = \int_{B_1 \setminus B_\varepsilon} \big(u(x) - u(x+y)\big) K(x, x+y)\, dy + \int_{B_1 \setminus B_\varepsilon} \big(u(x) - u(x-y)\big) K(x, x-y)\, dy \\
&\quad = \int_{B_1 \setminus B_\varepsilon} \big(u(x) - u(x+y)\big) \Big(K(x, x+y) - K(x, x-y)\Big)\, dy \\
&\qquad\quad + \int_{B_1 \setminus B_\varepsilon} \big(2u(x) - u(x+y) - u(x-y)\big) K(x, x-y)\, dy.
\end{aligned} \tag{A.1}$$

Moreover, in view of (1.1), for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
\big|2u(x) - u(x+y) - u(x-y)\big| K(x, x-y) &\leqslant \frac{\|D^2 u\|_{L^\infty(\mathbb{R}^n)}}{2} |y|^2 K(x, x-y) \\
&\leqslant \frac{C \|D^2 u\|_{L^\infty(\mathbb{R}^n)}}{2 |y|^{n+2s-2}}.
\end{aligned}$$

The latter expression belongs to $L^1(B_1)$ and thus

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \int_{B_1 \setminus B_\varepsilon} \big(2u(x) - u(x+y) - u(x-y)\big) K(x, x-y)\, dy \\
&\quad = \int_{B_1} \big(2u(x) - u(x+y) - u(x-y)\big) K(x, x-y)\, dy \\
&\quad \in \left[-\frac{C_n\, C \|D^2 u\|_{L^\infty(\mathbb{R}^n)}}{1-s}, \frac{C_n\, C \|D^2 u\|_{L^\infty(\mathbb{R}^n)}}{1-s} \right],
\end{aligned} \tag{A.2}$$

for some $C_n > 0$.

Furthermore,

$$\begin{aligned}
&\big(u(x) - u(x+y)\big) \Big(K(x, x+y) - K(x, x-y)\Big) \\
&\quad \leqslant \|\nabla u\|_{L^\infty(\mathbb{R}^n)} |y| \Big(K(x, x+y) - K(x, x-y)\Big).
\end{aligned}$$

In view of (1.4), this expression is integrable over $y \in B_1$, with

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \int_{B_1 \setminus B_\varepsilon} \big(u(x) - u(x+y)\big) \Big(K(x, x+y) - K(x, x-y)\Big)\, dy \\
&\quad = \int_{B_1} \big(u(x) - u(x+y)\big) \Big(K(x, x+y) - K(x, x-y)\Big)\, dy \\
&\quad \in \left[-C_\sharp \|\nabla u\|_{L^\infty(\mathbb{R}^n)},\, C_\sharp \|\nabla u\|_{L^\infty(\mathbb{R}^n)}\right].
\end{aligned} \tag{A.3}$$

Also,

$$\big|u(x) - u(x-y)\big|\, K(x, x-y) \leqslant \frac{2C\, \|u\|_{L^\infty(\mathbb{R}^n)}}{|y|^{n+2s}},$$

which is integrable over $y \in \mathbb{R}^n \setminus B_1$, with

$$\left| \int_{\mathbb{R}^n \setminus B_1} \big(u(x) - u(x-y)\big)\, K(x, x-y)\, dy \right| \leqslant \frac{C_n\, C\, \|u\|_{L^\infty(\mathbb{R}^n)}}{s}, \tag{A.4}$$

up to renaming C_n.

The desired result now follows from (A.1), (A.2), (A.3) and (A.4). □

We point out that the inequality in Lemma A.2 can be also achieved so that the L^∞-norm of $\mathcal{L}_K u$ is controlled by a constant independent of s, multiplied by $\|u\|_{C^2(\mathbb{R}^n)}$. To this end, one may use the upper bound for the constant $c_{n,s}$, as introduced for instance in [18, Lemma 2.5].

Appendix B. Bounds on nondegenerate matrices

We now discuss some bounds involving nondegenerate matrices.

Lemma B.1. *Let $\alpha \geqslant \beta > 0$. Let $\mathcal{M}_{\alpha,\beta}$ be the collection of matrices $L \in \mathbb{R}^{n\times n}$ such that, for all $\xi \in \mathcal{S}^{n-1}$,*

$$L\xi \cdot \xi \in [\beta, \alpha].$$

Then,

$$\inf_{\substack{L\in\mathcal{M}_{\alpha,\beta} \\ \xi\in\mathbb{R}^n\setminus\{0\}}} \frac{|L\xi|}{|\xi|} = \min_{\substack{L\in\mathcal{M}_{\alpha,\beta} \\ \xi\in\mathbb{R}^n\setminus\{0\}}} \frac{|L\xi|}{|\xi|} > 0 \tag{B.1}$$

and

$$\sup_{\substack{L\in\mathcal{M}_{\alpha,\beta}\\ \xi\in\mathbb{R}^n\setminus\{0\}}} \frac{|L\xi|}{|\xi|} = \max_{\substack{L\in\mathcal{M}_{\alpha,\beta}\\ \xi\in\mathbb{R}^n\setminus\{0\}}} \frac{|L\xi|}{|\xi|} < +\infty. \tag{B.2}$$

Proof. We focus on the proof of (B.1), since the proof of (B.2) is alike.

We remark that $\mathcal{M}_{\alpha,\beta}\times\mathcal{S}^{n-1}$ is a closed and bounded subset of the Euclidean space $\mathbb{R}^{n\times n+n}$ and therefore the following infimum is attained

$$\inf_{\substack{L\in\mathcal{M}_{\alpha,\beta}\\ \xi\in\mathcal{S}^{n-1}}} |L\xi|.$$

Hence, we can find $L_\star\in\mathbb{R}^{n\times n}$ and $\xi_\star\in\mathcal{S}^{n-1}$ such that

$$|L_\star\xi_\star| = \inf_{\substack{L\in\mathcal{M}_{\alpha,\beta}\\ \xi\in\mathcal{S}^{n-1}}} |L\xi| = \inf_{\substack{L\in\mathcal{M}_{\alpha,\beta}\\ \xi\in\mathbb{R}^n\setminus\{0\}}} \frac{|L\xi|}{|\xi|}. \tag{B.3}$$

We also notice that

$$L_\star\xi_\star \neq 0, \tag{B.4}$$

otherwise $0 = L_\star\xi_\star\cdot\xi_\star \geqslant \beta$, which is a contradiction.

The desired result in (B.1) now follows from (B.3) and (B.4). □

Lemma B.2. *Let $\varrho\in(0,+\infty]$ and $M:\mathbb{R}^n\times B_\varrho\to\mathbb{R}^{n\times n}$ and assume that there exist $\alpha\geqslant\beta>0$ such that, for any $x\in\mathbb{R}^n$, $y\in B_\varrho$, $\xi\in\mathbb{R}^N$,*

$$\beta|\xi|^2 \leqslant M(x-y,y)\xi\cdot\xi \leqslant \alpha|\xi|^2. \tag{B.5}$$

Then, there exist $0<c_{\alpha,\beta,n}\leqslant C_{\alpha,\beta,n}<+\infty$, depending only on α, β and n, such that, for all $x\in\mathbb{R}^n$, $y\in B_\varrho$ and $\xi\in\mathbb{R}^n$,

$$c_{\alpha,\beta,n}|\xi| \leqslant |M(x-y,y)\xi| \leqslant C_{\alpha,\beta,n}|\xi|.$$

Proof. Given x, $y\in\mathbb{R}^n$, we know that $M(x-y,y)$ satisfies (B.5), and therefore $M(x-y,y)$ belongs to $\mathcal{M}_{\alpha,\beta}$, defined in Lemma B.1.

Denoting by $c_{\alpha,\beta,n}$ and $C_{\alpha,\beta,n}$ the quantities in (B.1) and (B.2) respectively, we know by Lemma B.1 that $0<c_{\alpha,\beta,n}\leqslant C_{\alpha,\beta,n}<+\infty$. Thus, the desired inequalities in Lemma B.2 follows from Lemma B.1. □

The following is a useful perturbative estimate regarding the interaction kernel induced by a matrix.

Lemma B.3. *There exists $C_\star>0$, depending only on n, α and β, such that, for all L, $N\in\mathbb{R}^{n\times n}$ for which $|L\xi|$, $|N\xi|\in[\beta,\alpha]$ for all $\xi\in\mathcal{S}^{n-1}$, we have that, for all $y\in\mathbb{R}^n\setminus\{0\}$,*

$$\left|\frac{1}{|Ly|^{n+2s}}-\frac{1}{|Ny|^{n+2s}}\right|\leqslant\frac{C_\star\,\|L-N\|}{|y|^{n+2s}}.$$

Proof. We claim that there exists $C_\star>0$, depending only on n, α and β, such that, for all L, $N\in\mathbb{R}^{n\times n}$ for which $|L\xi|$, $|N\xi|\in[\beta,\alpha]$ for all $\xi\in\mathcal{S}^{n-1}$, we have that

$$\sup_{y\in\mathbb{R}^n\setminus\{0\}}\left|1-\frac{|Ly|^{n+2s}}{|Ny|^{n+2s}}\right|\leqslant C_\star\,\|L-N\|. \tag{B.6}$$

To check this, for every $y\in\mathbb{R}^n\setminus\{0\}$ and N as above, we let

$$\mathbb{R}^{n\times n}\ni L\longmapsto\Phi(L):=1-\frac{|Ly|^{n+2s}}{|Ny|^{n+2s}}$$

and we observe that

$$\begin{aligned}\frac{\partial\Phi}{\partial L_{ij}}&=-(n+2s)\frac{|Ly|^{n+2s-1}}{|Ny|^{n+2s}}\,\frac{\partial}{\partial L_{ij}}|Ly|\\&=-\frac{(n+2s)}{2}\frac{|Ly|^{n+2s-2}}{|Ny|^{n+2s}}\,\frac{\partial}{\partial L_{ij}}|Ly|^2\\&=-\frac{(n+2s)}{2}\frac{|Ly|^{n+2s-2}}{|Ny|^{n+2s}}\,\frac{\partial}{\partial L_{ij}}\sum_{k=1}^n\left(\sum_{m=1}^n L_{km}y_m\right)^2\\&=-(n+2s)\frac{|Ly|^{n+2s-2}}{|Ny|^{n+2s}}\sum_{k,m=1}^n L_{km}y_my_j.\end{aligned}$$

Thus, possibly changing $C_\star$ from line to line, we conclude that

$$\left|\frac{\partial\Phi}{\partial L_{ij}}\right|\leqslant C_\star\,\frac{|Ly|^{n+2s-1}\,|y|}{|Ny|^{n+2s}}\leqslant C_\star\,\frac{|y|^{n+2s-1}\,|y|}{|y|^{n+2s}}=C_\star.$$

Hence,

$$\left|1-\frac{|Ly|^{n+2s}}{|Ny|^{n+2s}}\right|=|\Phi(L)-\Phi(N)|\leqslant C_\star\|L-N\|,$$

which establishes (B.6).

The desired result now follows as a consequence of (B.6), up to renaming constants. □

One of the byproducts of Lemma B.3 is that the kernels induced by sufficiently regular nondegenerate matrices fulfill condition (1.4).

Corollary B.4. *Let $\varrho\in(0,+\infty]$ and $M:\mathbb{R}^n\times B_\varrho\to\mathbb{R}^{n\times n}$ satisfy (1.14) and let K be as in (1.13). Assume that M is Lipschitz continuous.*

Then, condition (1.4) is satisfied, and, in this situation,

$$\int_{B_1} |y| \left|K(x,x+y) - K(x,x-y)\right| dy \leqslant C_\star\, C_M\, s,$$

where $C_\star > 0$ depends only on n, α and β, and C_M denotes the Lipschitz constant of M.

Proof. Given $x \in \mathbb{R}^n$, by virtue of Lemma B.3 we have that

$$\begin{aligned}
&\int_{B_1} |y| \left|K(x,x+y) - K(x,x-y)\right| dy \\
&\quad = c_{n,s} \int_{B_1} |y| \left| \frac{1}{|M(x+y,-y)y|^{n+2s}} - \frac{1}{|M(x-y,y)y|^{n+2s}} \right| dy \\
&\quad \leqslant C_\star\, c_{n,s} \int_{B_1} |y| \frac{\|M(x+y,-y) - M(x-y,y)\|}{|y|^{n+2s}}\, dy \\
&\quad \leqslant C_\star\, C_M\, c_{n,s} \int_{B_1} \frac{dy}{|y|^{n+2s-2}} \\
&\quad \leqslant C_\star\, C_M\, s,
\end{aligned}$$

up to renaming $C_\star$ from line to line. □

Appendix C. Continuity of eigenvalues and eigenprojections

To prove our limit results, we leverage some perturbation results for finite-dimensional linear operators.

For a matrix $A \in \mathbb{R}^{n\times n}$, we denote its eigenvalues by $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$. We also use the standard matrix norm

$$\|A\| := \sup_{\xi \in \mathcal{S}^{n-1}} |A\xi|.$$

We recall that eigenvalues of symmetric matrices satisfy a Lipschitz continuity, according to the following observation:

Lemma C.1. *Given two symmetric matrices A, $B \in \mathbb{R}^{n\times n}$, we have that, for all $i \in \{1,\dots,n\}$,*

$$|\lambda_i(A) - \lambda_i(B)| \leqslant \|A - B\|.$$

Proof. Let $C := B - A$ and note that $C \in \mathbb{R}^{n\times n}$ is symmetric. Hence, we can use formula (4.3.16) in Corollary 4.3.15 of [24] and find that

$$\lambda_i(A) + \lambda_1(C) \leqslant \lambda_i(A+C) \leqslant \lambda_i(A) + \lambda_n(C). \tag{C.1}$$

Also, for all $j \in \{1, \dots, n\}$ we can find $v_j \in \mathcal{S}^{n-1}$ such that $Cv_j = \lambda_j(C)v_j$ and therefore

$$|\lambda_j(C)| = |\lambda_j(C)v_j| = |Cv_j| \leqslant \sup_{\xi \in \mathcal{S}^{n-1}} |C\xi| = \|C\| = \|A - B\|.$$

This and (C.1) entail that

$$\lambda_i(A) - \|A - B\| \leqslant \lambda_i(B) \leqslant \lambda_i(A) + \|A - B\|,$$

which yields the desired result. □

The following is a restatement of a classical result about the Lipschitz continuity of eigenprojections (see Lemma 3 in [14]; see also Lemma 4.3 in [30] and the references therein):

Lemma C.2. *There exist c, $C \in (0, +\infty)$ such that the following statement holds true.*

Let A, $B \in \mathbb{R}^{n\times n}$ be symmetric matrices, with $\|A - B\| \leqslant c$. Let $O \in \mathbb{R}^{n\times n}$ an orthogonal matrix such as $O^T AO$ is diagonal with increasing diagonal entries.

Then, there exists an orthogonal matrix $P \in \mathbb{R}^{n\times n}$ such as $P^T BP$ is diagonal with increasing diagonal entries and

$$\|O - P\| \leqslant C\|A - B\|.$$

Appendix D. Norm equivalency and proof of Lemma 2.1

Here we give the proof of Lemma 2.1.

Proof of Lemma 2.1. We let

$$\Omega_1 := \bigcup_{x\in\Omega} B_1(x).$$

We decompose $\mathbb{R}^n$ into a family of nonoverlapping cubes $\{Q_j\}_{j\in\mathbb{N}}$ with faces normal to the coordinate axes and sides of length $\ell := \frac{\varrho}{8\sqrt{n}}$. We label these cubes in such a way that

$$\Omega_1 \subseteq \bigcup_{j=0}^{N} Q_j, \tag{D.1}$$

with $N \in \mathbb{N}$ depending only on n, ϱ and Ω.

We also denote by $Q_j^\star$ the union of Q_j and its adjacent cubes. In this way, $Q_j^\star$ is a cube with the same center as Q_j and sides of length 3ℓ.

In this framework, we see that

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(z)|^2K(x,z)\,dx\,dz \geqslant \sum_{j=0}^{N} \iint_{Q_j\times\mathbb{R}^n} |u(x)-u(z)|^2K(x,z)\,dx\,dz$$

$$= \sum_{\substack{0\leqslant j\leqslant N\\ i\in\mathbb{N}}} \iint_{Q_j\times Q_i} |u(x)-u(z)|^2K(x,z)\,dx\,dz \geqslant \sum_{j=0}^{N} \iint_{Q_j\times Q_j^\star} |u(x)-u(z)|^2K(x,z)\,dx\,dz. \tag{D.2}$$

Now, if $x\in Q_j$ and $z\in Q_j^\star$, we have that $|x-z|\leqslant 3\sqrt{n}\ell<\varrho$ and accordingly, by (1.1), we have that

$$K(x,z)\geqslant \frac{c}{|x-z|^{n+2s}}.$$

This and (D.2) give that

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(z)|^2K(x,z)\,dx\,dz \geqslant c\sum_{j=0}^{N} \iint_{Q_j\times Q_j^\star} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz. \tag{D.3}$$

Moreover, by virtue of (D.1),

$$\begin{aligned}
\iint_{\Omega_1\times\Omega_1} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz &\leqslant \sum_{j=0}^{N} \iint_{Q_j\times\Omega_1} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz\\
&\leqslant \sum_{j=0}^{N} \iint_{\substack{Q_j\times\Omega_1\\ \{|x-z|<\ell\}}} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz + \sum_{j=0}^{N} \iint_{\substack{Q_j\times\Omega_1\\ \{|x-z|\geqslant\ell\}}} \frac{\big(|u(x)|+|u(z)|\big)^2}{\ell^{n+2s}}\,dx\,dz\\
&\leqslant \sum_{j=0}^{N} \iint_{Q_j\times Q_j^\star} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz + \widetilde{C}\,\|u\|^2_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{D.4}$$

where we denote by $\widetilde{C}>0$ an opportune constant, depending only on n, s, ρ and Ω, and possibly varying from line to line.

Moreover,

$$\begin{aligned}\iint\limits_{\Omega\times(\mathbb{R}^n\setminus\Omega_1)} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz &\leqslant \iint\limits_{\substack{\mathbb{R}^n\times\mathbb{R}^n\\ \{|x-z|\geqslant 1\}}} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz\\ &\leqslant \iint\limits_{\substack{\mathbb{R}^n\times\mathbb{R}^n\\ \{|x-z|\geqslant 1\}}} \frac{\big(|u(x)|+|u(z)|\big)^2}{|x-z|^{n+2s}}\,dx\,dz \leqslant \widetilde{C}\|u\|^2_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{D.5}$$

Now we claim that

$$\|u\|^2_{L^2(\mathbb{R}^n)} \leqslant \widetilde{C} \iint\limits_{\substack{\mathbb{R}^n\times\mathbb{R}^n\\ \{|x-y|\leqslant \varrho\}}} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz. \tag{D.6}$$

To check this, we argue by contradiction and suppose that there exists a sequence of functions u_m such that $u_m = 0$ in $\mathbb{R}^n \setminus \Omega$ and

$$1 = \|u_m\|^2_{L^2(\mathbb{R}^n)} \geqslant m \iint\limits_{\substack{\mathbb{R}^n\times\mathbb{R}^n\\ \{|x-y|\leqslant \varrho\}}} \frac{|u_m(x)-u_m(z)|^2}{|x-z|^{n+2s}}\,dx\,dz.$$

In particular, for each $j \in \{0,\dots,N\}$,

$$\frac{1}{m} \geqslant \iint\limits_{\substack{Q_j\times\mathbb{R}^n\\ \{|x-y|\leqslant \varrho\}}} \frac{|u_m(x)-u_m(z)|^2}{|x-z|^{n+2s}}\,dx\,dz \geqslant \iint\limits_{Q_j\times Q_{j,\varrho}} \frac{|u_m(x)-u_m(z)|^2}{|x-z|^{n+2s}}\,dx\,dz, \tag{D.7}$$

where

$$Q_{j,\varrho} := \bigcup_{x\in Q_j} B_\varrho(x).$$

This gives that

$$\iint\limits_{Q_j\times Q_j} \frac{|u_m(x)-u_m(z)|^2}{|x-z|^{n+2s}}\,dx\,dz + \|u_m\|_{L^2(Q_j)} \leqslant \frac{1}{m} + 1 \leqslant 2$$

and therefore, by compactness (see e.g. [16, Theorem 7.1]), up to a relabeled subsequence, we can suppose that u_m converges to some u_∞ in $L^2(Q_j)$ and a.e. in Q_j, for all $j \in \{0,\dots,N\}$.

As a result,

$$1 = \lim_{m\to+\infty} \|u_m\|^2_{L^2(\mathbb{R}^n)} = \lim_{m\to+\infty} \sum_{j=0}^{N} \|u_m\|^2_{L^2(Q_j)} = \sum_{j=0}^{N} \|u_\infty\|^2_{L^2(Q_j)} = \|u_\infty\|^2_{L^2(\Omega)}. \tag{D.8}$$

However, it follows from (D.7) and Fatou's Lemma that, for all $j \in \{0,\dots,N\}$,

$$\iint_{Q_j\times Q_{j,\varrho}} \frac{|u_\infty(x)-u_\infty(z)|^2}{|x-z|^{n+2s}}\,dx\,dz = 0.$$

This entails that u_∞ is constant in each Q_j and actually in $Q_{j,\varrho}$. As a result, u_∞ is constant, and therefore constantly equal to zero (since $u_\infty = 0$ outside Ω). But this is in contradiction with (D.8) and the proof of (D.6) is complete.

Gathering (D.3), (D.4), (D.5) and (D.6), we conclude that

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz\\
&= \iint_{\Omega_1\times\Omega_1} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz + 2\iint_{\Omega_1\times(\mathbb{R}^n\setminus\Omega_1)} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz\\
&= \iint_{\Omega_1\times\Omega_1} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz + 2\iint_{\Omega\times(\mathbb{R}^n\setminus\Omega_1)} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz\\
&\qquad +2\iint_{(\Omega_1\setminus\Omega)\times(\mathbb{R}^n\setminus\Omega_1)} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz\\
&\leqslant \iint_{\Omega_1\times\Omega_1} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz + \widetilde{C}\|u\|^2_{L^2(\mathbb{R}^n)} + 0\\
&\leqslant \sum_{j=0}^{N} \iint_{Q_j\times Q_j^\star} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz + \widetilde{C}\|u\|^2_{L^2(\mathbb{R}^n)}\\
&\leqslant \frac{1}{c}\iint_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(z)|^2 K(x,z)\,dx\,dz + \widetilde{C}\|u\|^2_{L^2(\mathbb{R}^n)}\\
&\leqslant \frac{1}{c}\iint_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(z)|^2 K(x,z)\,dx\,dz + \widetilde{C}\iint_{\substack{\mathbb{R}^n\times\mathbb{R}^n\\ \{|x-y|\leqslant\varrho\}}} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,dx\,dz\\
&\leqslant \frac{\widetilde{C}}{c}\iint_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(z)|^2 K(x,z)\,dx\,dz,
\end{aligned}$$

as desired. □

Appendix E. Asymptotics as $s \searrow 0$ and $s \nearrow 1$

In this appendix, we provide the details of the proofs of the asymptotic statements in Proposition 1.3.

In the following computations the asymptotic behavior of the constant $c_{n,s}$ will play an important role. For this reason, we recall that

$$\lim_{s\searrow 0} \frac{c_{n,s}}{s} = \frac{2}{\omega_{n-1}} \quad \text{and} \quad \lim_{s\nearrow 1} \frac{c_{n,s}}{1-s} = \frac{4n}{\omega_{n-1}}, \tag{E.1}$$

see e.g. formula (1.50) in [1].

Proof of Proposition 1.3. Without loss of generality, we can suppose that $\varrho = +\infty$, since, for all $\varrho_0 \in (0, +\infty)$, we have that

$$\begin{aligned}
&\lim_{s\nearrow 1} \frac{c_{n,s}}{2} \iint_{\mathbb{R}^n\times(\mathbb{R}^n\setminus B_{\varrho_0})} \frac{|u(x)-u(x-y)||\varphi(x)-\varphi(x-y)|}{|M(x-y,y)y|^{n+2s}}\,dx\,dy \\
&\quad\leqslant \lim_{s\nearrow 1} C(1-s) \iint_{\mathbb{R}^n\times(\mathbb{R}^n\setminus B_{\varrho_0})} \frac{|u(x)-u(x-y)||\varphi(x)-\varphi(x-y)|}{|y|^{n+2s}}\,dx\,dy \\
&\quad\leqslant \lim_{s\nearrow 1} C(1-s) \iint_{\mathbb{R}^n\times(\mathbb{R}^n\setminus B_{\varrho_0})} \frac{(u(x)-u(x-y))^2+(\varphi(x)-\varphi(x-y))^2}{|y|^{n+2s}}\,dx\,dy \\
&\quad\leqslant \lim_{s\nearrow 1} C(1-s) \int_{\mathbb{R}^n\setminus B_{\varrho_0}} \frac{\|u\|^2_{L^2(\mathbb{R}^n)}+\|\varphi\|^2_{L^2(\mathbb{R}^n)}}{|y|^{n+2s}}\,dy \\
&\quad\leqslant \lim_{s\nearrow 1} C(1-s)\Big(\|u\|^2_{L^2(\mathbb{R}^n)}+\|\varphi\|^2_{L^2(\mathbb{R}^n)}\Big) \\
&\quad= 0.
\end{aligned}$$

Now, we remark that

it suffices to prove (1.19) for all $u \in H^1(\mathbb{R}^n)$ which are compactly supported. (E.2)

Indeed, given $u \in H^1(\mathbb{R}^n)$ and $R > 0$, we pick $\tau_R \in C_c^\infty(B_{R+1})$ with $\tau_R = 1$ in B_R. We set $u_R := \tau_R u$ and $v_R := u - u_R = (1-\tau_R)u$. Hence, if (1.19) holds true for all $u \in H^1(\mathbb{R}^n)$ which are compactly supported, we have that, given $\varphi \in C_c^\infty(B_{R_0})$,

$$\begin{aligned}
&\lim_{s\nearrow 1} \frac{c_{n,s}}{2} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_R(x)-u_R(x-y))(\varphi(x)-\varphi(x-y))}{|M(x-y,y)y|^{n+2s}}\,dx\,dy \\
&\quad= \sum_{i,j=1}^n \int_{\mathbb{R}^n} A_{ij}(x)\frac{\partial u_R}{\partial x_i}(x)\frac{\partial \varphi}{\partial x_j}(x)\,dx
\end{aligned}$$

and consequently, keeping in mind (1.14) and assuming $R > R_0$,

$$
\begin{aligned}
&\left| \lim_{s\nearrow 1} \frac{c_{n,s}}{2} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(x-y))(\varphi(x)-\varphi(x-y))}{|M(x-y,y)y|^{n+2s}}\,dx\,dy \right. \\
&\qquad \left. - \sum_{i,j=1}^{n} \int\limits_{\mathbb{R}^n} A_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial \varphi}{\partial x_j}(x)\,dx \right| \\
&= \left| \lim_{s\nearrow 1} \frac{c_{n,s}}{2} \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(v_R(x)-v_R(x-y))(\varphi(x)-\varphi(x-y))}{|M(x-y,y)y|^{n+2s}}\,dx\,dy \right. \\
&\qquad \left. - \sum_{i,j=1}^{n} \int\limits_{\mathbb{R}^n} A_{ij}(x)\frac{\partial v_R}{\partial x_i}(x)\frac{\partial \varphi}{\partial x_j}(x)\,dx \right| \\
&\leqslant \lim_{s\nearrow 1} C(1-s) \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v_R(x)-v_R(x-y)|\,|\varphi(x)-\varphi(x-y)|}{|y|^{n+2s}}\,dx\,dy \\
&\qquad + C \int\limits_{B_{R_0}} |\nabla v_R(x)|\,dx \\
&= \lim_{s\nearrow 1} C(1-s) \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v_R(x)-v_R(z)|\,|\varphi(x)-\varphi(z)|}{|x-z|^{n+2s}}\,dx\,dz + 0 \\
&\leqslant \lim_{s\nearrow 1} C(1-s) \iint\limits_{\mathbb{R}^n\times B_{R_0}} \frac{|v_R(x)-v_R(z)|\,|\varphi(x)-\varphi(z)|}{|x-z|^{n+2s}}\,dx\,dz \\
&= \lim_{s\nearrow 1} C(1-s) \iint\limits_{\mathbb{R}^n\times B_{R_0}} \frac{|v_R(x)|\,|\varphi(x)-\varphi(z)|}{|x-z|^{n+2s}}\,dx\,dz \\
&= \lim_{s\nearrow 1} C(1-s) \iint\limits_{(\mathbb{R}^n\setminus B_R)\times B_{R_0}} \frac{|v_R(x)|\,|\varphi(x)-\varphi(z)|}{|x-z|^{n+2s}}\,dx\,dz \\
&= \lim_{s\nearrow 1} C(1-s) \iint\limits_{(\mathbb{R}^n\setminus B_R)\times B_{R_0}} \frac{(1-\tau_R(x))|u(x)|\,|\varphi(z)|}{|x-z|^{n+2s}}\,dx\,dz \\
&\leqslant \lim_{s\nearrow 1} C(1-s) \int\limits_{\mathbb{R}^n\setminus B_R} \frac{|u(x)|}{|x|^{n+2s}}\,dx \\
&\leqslant \lim_{s\nearrow 1} C(1-s) \sqrt{\int\limits_{\mathbb{R}^n\setminus B_R} |u(x)|^2\,dx \int\limits_{\mathbb{R}^n\setminus B_R} \frac{dx}{|x|^{2n+4s}}} \\
&= 0
\end{aligned}
$$

for some $C > 0$, independent of s and R and possibly varying from line to line. The proof of (E.2) is thereby complete.

Now we point out that

$$\text{it suffices to prove (1.19) for all } u \in C_c^\infty(\mathbb{R}^n). \tag{E.3}$$

To check this, in light of (E.2), we can focus on the case in which $u \in H^1(\mathbb{R}^n)$ with support in B_{R_0} and $\varphi \in C_c^\infty(B_{R_0})$, for some $R_0 > 0$. Thus, we take $u_\varepsilon \in C_c^\infty(B_{R_0})$ with $[u-u_\varepsilon]_1 \leqslant \varepsilon$ and define $v_\varepsilon := u - u_\varepsilon$. In this way, if (1.19) hods true for u_ε it follows that

$$\begin{aligned}
&\lim_{s\nearrow 1} \frac{c_{n,s}}{2} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u_\varepsilon(x)-u_\varepsilon(x-y))(\varphi(x)-\varphi(x-y))}{|M(x-y,y)y|^{n+2s}}\,dx\,dy \\
&\qquad = \sum_{i,j=1}^n \int_{\mathbb{R}^n} A_{ij}(x)\frac{\partial u_\varepsilon}{\partial x_i}(x)\frac{\partial \varphi}{\partial x_j}(x)\,dx
\end{aligned}$$

and therefore, employing [17, Lemma 2.1],

$$\begin{aligned}
&\left|\lim_{s\nearrow 1} \frac{c_{n,s}}{2} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(u(x)-u(x-y))(\varphi(x)-\varphi(x-y))}{|M(x-y,y)y|^{n+2s}}\,dx\,dy \right.\\
&\qquad \left. - \sum_{i,j=1}^n \int_{\mathbb{R}^n} A_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial \varphi}{\partial x_j}(x)\,dx\right| \\
&= \left|\lim_{s\nearrow 1} \frac{c_{n,s}}{2} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(v_\varepsilon(x)-v_\varepsilon(x-y))(\varphi(x)-\varphi(x-y))}{|M(x-y,y)y|^{n+2s}}\,dx\,dy \right.\\
&\qquad \left. - \sum_{i,j=1}^n \int_{\mathbb{R}^n} A_{ij}(x)\frac{\partial v_\varepsilon}{\partial x_i}(x)\frac{\partial \varphi}{\partial x_j}(x)\,dx\right| \\
&\leqslant \lim_{s\nearrow 1} C(1-s) \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v_\varepsilon(x)-v_\varepsilon(x-y)|\,|\varphi(x)-\varphi(x-y)|}{|y|^{n+2s}}\,dx\,dy + C\int_{B_{R_0}} |\nabla v_\varepsilon(x)|\,dx \\
&\leqslant \lim_{s\nearrow 1} C[v_\varepsilon]_s\,[\varphi]_s + C[v_\varepsilon]_1 \\
&\leqslant \lim_{s\nearrow 1} C[v_\varepsilon]_1\,[\varphi]_1 + C[v_\varepsilon]_1 \\
&\leqslant C\varepsilon,
\end{aligned}$$

for some $C > 0$, independent of s and ε and possibly varying from line to line. Accordingly, the claim in (E.3) follows by sending $\varepsilon \searrow 0$.

Now we focus on the proof of (1.19) and, thanks to (E.3), we will assume that u, $\varphi \in C_c^\infty(B_{R_0})$, for some $R_0 > 0$.

We consider a mollification M_ε of M and we remark that, by denoting a standard even mollifier by $\eta_\varepsilon \in C_c^\infty(B_\varepsilon)$, for all $\xi \in \mathcal{S}^{n-1}$,

$$M_\varepsilon(x-y,y)\xi\cdot\xi = \iint_{\mathbb{R}^n\times\mathbb{R}^n} M(x-y-w,y-z)\xi\cdot\xi\,\eta_\varepsilon(w)\,\eta_\varepsilon(z)\,dw\,dz \in [\beta,\alpha], \tag{E.4}$$

due to (1.14).

We now let r be as in (1.18) and (thanks to [31, Theorem 9.8]) conclude that, for every $R>0$, as $\varepsilon \searrow 0$,

$$M_\varepsilon \text{ converges to } M \text{ uniformly in } B_R \times B_{r/2}. \tag{E.5}$$

Furthermore, by [2, equation (3.7) and Appendix W],

$$\begin{aligned}\lim_{s\nearrow 1}\frac{c_{n,s}}{2}&\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u(x)-u(x-y))(\varphi(x)-\varphi(x-y))}{|M_\varepsilon(x-y,y)y|^{n+2s}}\,dx\,dy\\ &=\sum_{i,j=1}^n\int_{\mathbb{R}^n}A_{ij,\varepsilon}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial\varphi}{\partial x_j}(x)\,dx,\end{aligned} \tag{E.6}$$

where

$$A_{ij,\varepsilon}(x) := \frac{n}{\omega_{n-1}}\int_{\mathcal{S}^{n-1}}\frac{\psi_i\psi_j}{|M_\varepsilon(x,0)\,\psi|^{n+2}}\,d\mathcal{H}^{n-1}_\psi. \tag{E.7}$$

This and (1.14) yield that

$$\begin{aligned}\lim_{s\nearrow 1}\frac{c_{n,s}}{2}&\left|\iint_{\mathbb{R}^n\times(\mathbb{R}^n\setminus B_{r/2})}\frac{(u(x)-u(x-y))(\varphi(x)-\varphi(x-y))}{|M_\varepsilon(x-y,y)y|^{n+2s}}\,dx\,dy\right|\\ &\leqslant \lim_{s\nearrow 1}C(1-s)\iint_{\mathbb{R}^n\times(\mathbb{R}^n\setminus B_{r/2})}\frac{|u(x)-u(x-y)|\,|\varphi(x)-\varphi(x-y)|}{|y|^{n+2s}}\,dx\,dy\\ &\leqslant \lim_{s\nearrow 1}C(1-s)\left(\iint_{\mathbb{R}^n\times(\mathbb{R}^n\setminus B_{r/2})}\frac{|u(x)-u(x-y)|^2}{|y|^{n+2s}}\,dx\,dy\right)^{\frac12}\times\\ &\qquad\times\left(\iint_{\mathbb{R}^n\times(\mathbb{R}^n\setminus B_{r/2})}\frac{|\varphi(x)-\varphi(x-y)|^2}{|y|^{n+2s}}\,dx\,dy\right)^{\frac12}\\ &\leqslant \lim_{s\nearrow 1}C(1-s)\int_{\mathbb{R}^n\setminus B_{r/2}}\frac{dy}{|y|^{n+2s}}\end{aligned}$$

$$\leqslant \lim_{s\nearrow 1} \frac{C(1-s)}{s\, r^{2s}}$$
$$= 0,$$

for some $C > 0$ independent of r, s and ε.

As a consequence, we rewrite (E.6) as

$$\lim_{s\nearrow 1} \frac{c_{n,s}}{2} \iint_{\mathbb{R}^n \times B_{r/2}} \frac{(u(x)-u(x-y))(\varphi(x)-\varphi(x-y))}{|M_\varepsilon(x-y,y)y|^{n+2s}}\, dx\, dy$$
$$= \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} A_{ij,\varepsilon}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial \varphi}{\partial x_j}(x)\, dx$$

and therefore, in light of (E.5) and (E.7),

$$\lim_{\varepsilon\searrow 0} \lim_{s\nearrow 1} \frac{c_{n,s}}{2} \iint_{\mathbb{R}^n \times B_{r/2}} \frac{(u(x)-u(x-y))(\varphi(x)-\varphi(x-y))}{|M_\varepsilon(x-y,y)y|^{n+2s}}\, dx\, dy$$
$$= \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} A_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial \varphi}{\partial x_j}(x)\, dx. \tag{E.8}$$

Using (E.4) and Lemma B.3, we obtain that

$$\left| \frac{1}{|M(x-y,y)y|^{n+2s}} - \frac{1}{|M_\varepsilon(x-y,y)y|^{n+2s}} \right|$$
$$\leqslant \frac{C_\star \left\| M(x-y,y) - M_\varepsilon(x-y,y) \right\|}{|y|^{n+2s}}.$$

Accordingly, up to renaming constants we have that

$$\frac{c_{n,s}}{2} \iint_{\mathbb{R}^n \times B_{r/2}} |u(x)-u(x-y)|\, |\varphi(x)-\varphi(x-y)|$$
$$\times \left| \frac{1}{|M(x-y,y)y|^{n+2s}} - \frac{1}{|M_\varepsilon(x-y,y)y|^{n+2s}} \right| dx\, dy$$
$$\leqslant C_\star\, (1-s)$$
$$\times \iint_{\mathbb{R}^n \times B_{r/2}} \frac{|u(x)-u(x-y)|\, |\varphi(x)-\varphi(x-y)|\, \left\| M(x-y,y) - M_\varepsilon(x-y,y) \right\|}{|y|^{n+2s}}\, dx\, dy$$
$$= C_\star\, (1-s)$$
$$\times \iint_{B_{R_0+r} \times B_{r/2}} \frac{|u(x)-u(x-y)|\, |\varphi(x)-\varphi(x-y)|\, \left\| M(x-y,y) - M_\varepsilon(x-y,y) \right\|}{|y|^{n+2s}}\, dx\, dy$$

$$\leqslant C_\star\,(1-s)\,\|M-M_\varepsilon\|_{L^\infty(B_{R_0+2r}\times B_{r/2})}\iint\limits_{B_{R_0+r}\times B_{r/2}}|y|^{2-n-2s}\,dx\,dy$$

$$\leqslant C_\star\,\|M-M_\varepsilon\|_{L^\infty(B_{R_0+2r}\times B_{r/2})},$$

and this is infinitesimal as $\varepsilon\searrow 0$, owing to (E.5).

From this and (E.8) we obtain (1.19), as desired. □

For completeness, we also provide here the asymptotic behavior of the operator $\mathcal{L}_{M,\varrho,s}$ as $s\searrow 0$.

Proposition E.1. *Let $\overline{s}\in(0,1)$ and $u\in H^{\overline{s}}(\mathbb{R}^n)$. Let $\varrho\in(0,+\infty]$ and $M:\mathbb{R}^n\times B_\varrho\to\mathbb{R}^{n\times n}$ be a matrix-valued function with measurable entries that satisfy (1.14) and (1.15).*

If $\varrho=+\infty$, assume also that

$$\begin{aligned}&\text{for any } \psi\in\mathcal{S}^{n-1},\ \text{for any } x\in\mathbb{R}^n \text{ and for any } i,j\in\{1,\dots,n\} \text{ there exists}\\ &M^\infty_{ij}(x,\psi):=\lim_{t\to+\infty}M_{ij}(x-t\psi,t\psi)<+\infty.\end{aligned}\tag{E.9}$$

Then, for any $\varphi\in C^\infty_c(\mathbb{R}^n)$,

$$\begin{aligned}&\lim_{s\searrow0}\frac{c_{n,s}}{2}\iint\limits_{\mathbb{R}^n\times B_\varrho}\frac{(u(x)-u(x-y))(\varphi(x)-\varphi(x-y))}{|M(x-y,y)y|^{n+2s}}\,dx\,dy\\ &\quad=\begin{cases}\dfrac{1}{\omega_{n-1}}\displaystyle\int_{\mathbb{R}^n}u(x)\varphi(x)\int_{\mathcal{S}^{n-1}}\frac{d\mathcal{H}^{n-1}_\psi}{|M^\infty(x,\psi)\,\psi|^n}\,dx & \text{if } \varrho=+\infty,\\ 0 & \text{if } \varrho\in(0,+\infty).\end{cases}\end{aligned}\tag{E.10}$$

To establish Proposition E.1, we will make use of the following observations:

Lemma E.2. *Let $M:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times n}$ be a measurable matrix-valued function satisfying (E.9) for any $x\in\mathbb{R}^n$. Let $R>0$.*

Then,

$$\lim_{s\searrow0}c_{n,s}\int_{\mathbb{R}^n\setminus B_R}\frac{dy}{|M(x-y,y)\,y|^{n+2s}}=\frac{1}{\omega_{n-1}}\int_{\mathcal{S}^{n-1}}\frac{d\mathcal{H}^{n-1}_\psi}{|M^\infty(x,\psi)\,\psi|^n}.$$

Proof. We observe that

$$\int_{\mathbb{R}^n\setminus B_R}\frac{dy}{|M(x-y,y)\,y|^{n+2s}}=\int_R^{+\infty}\int_{\mathcal{S}^{n-1}}\frac{d\rho\,d\mathcal{H}^{n-1}_\psi}{\rho^{1+2s}|M(x-\rho\psi,\rho\psi)\,\psi|^{n+2s}}.$$

Now, by using the change of variables $r:=(\rho/R)^{2s}$, we see that

$$\int\limits_{\mathbb{R}^n\setminus B_R} \frac{dy}{|M(x-y,y)\,y|^{n+2s}} = \frac{1}{2sR^{2s}} \int\limits_1^{+\infty} \int\limits_{\mathcal{S}^{n-1}} \frac{dr\, d\mathcal{H}^{n-1}_\psi}{r^2 |M(x - r^{\frac{1}{2s}}\psi, r^{\frac{1}{2s}}\psi)\,\psi|^{n+2s}}.$$

Thus, by (1.14), we get

$$\frac{1}{r^2 |M(x - r^{\frac{1}{2s}}\psi, r^{\frac{1}{2s}}\psi)\,\psi|^{n+2s}} \leqslant \frac{1}{r^2 c^{n+2s}_{\alpha,\beta,n}} \in L^1\Big((1,+\infty)\times \mathcal{S}^{n-1}; dr\, d\mathcal{H}^{n-1}_\psi\Big)$$

Accordingly, by the Dominated Convergence Theorem, (E.1) and (E.9), we deduce that

$$\begin{aligned}
\lim_{s\searrow 0} c_{n,s} \int\limits_{\mathbb{R}^n\setminus B_R} \frac{dy}{|M(x-y,y)\,y|^{n+2s}} &= \lim_{s\searrow 0} \frac{c_{n,s}}{2sR^{2s}} \int\limits_1^{+\infty} \int\limits_{\mathcal{S}^{n-1}} \frac{dr\, d\mathcal{H}^{n-1}_\psi}{r^2 |M(x - r^{\frac{1}{2s}}\psi, r^{\frac{1}{2s}}\psi)\,\psi|^{n+2s}} \\
&= \frac{1}{\omega_{n-1}} \int\limits_1^{+\infty} \frac{dr}{r^2} \int\limits_{\mathcal{S}^{n-1}} \frac{d\mathcal{H}^{n-1}_\psi}{|M^\infty(x,\psi)\,\psi|^n} \\
&= \frac{1}{\omega_{n-1}} \int\limits_{\mathcal{S}^{n-1}} \frac{d\mathcal{H}^{n-1}_\psi}{|M^\infty(x,\psi)\,\psi|^n},
\end{aligned}$$

as desired. □

Lemma E.3. *Let $\overline{s} \in (0,1)$ and $u, \varphi \in H^{\overline{s}}(\mathbb{R}^n)$. Let $\varrho \in (0,+\infty]$ and $M : \mathbb{R}^n \times B_\varrho \to \mathbb{R}^{n\times n}$ be a measurable matrix-valued function. Let $R \in (0,\varrho)$.*

Then,

$$\lim_{s\searrow 0} c_{n,s} \iint\limits_{\mathbb{R}^n\times B_R} \frac{(u(x)-u(x-y))(\varphi(x)-\varphi(x-y))}{|M(x-y,y)y|^{n+2s}}\, dx\, dy = 0.$$

Proof. Up to scaling, we can suppose that $R = 1$. By (1.14), we infer that

$$\begin{aligned}
&\left| c_{n,s} \iint\limits_{\mathbb{R}^n\times B_1} \frac{(u(x)-u(x-y))(\varphi(x)-\varphi(x-y))}{|M(x-y,y)y|^{n+2s}}\, dx\, dy \right| \\
&\qquad \leqslant \frac{c_{n,s}}{\beta^{n+2s}} \int\limits_{\mathbb{R}^n} \left(\int\limits_{B_1} \frac{|u(x)-u(x-y)||\varphi(x)-\varphi(x-y)|}{|y|^{n+2s}}\, dy \right) dx.
\end{aligned} \tag{E.11}$$

We set

$$g(s) := \int\limits_{\mathbb{R}^n} \left(\int\limits_{B_1} \frac{|u(x)-u(x-y)||\varphi(x)-\varphi(x-y)|}{|y|^{n+2s}}\, dy \right) dx.$$

We notice that $g(s)$ is nondecreasing and that $g(\overline{s}) < +\infty$. From this, we infer that

$$\lim_{s \searrow 0} g(s) \leqslant g(\overline{s}).$$

Using this information, (E.1) and (E.11), we obtain the desired result. □

Proof of Proposition E.1. If $\varrho \in (0, +\infty)$, the desired limit follows from Lemma E.3. Hence, from now on, we suppose that $\varrho = +\infty$.

We observe that

$$\begin{aligned}
&\frac{c_{n,s}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(x-y))(\varphi(x) - \varphi(x-y))}{|M(x-y,y)y|^{n+2s}}\, dx\, dy \\
&\quad = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \left(\int_{B_1} \frac{(u(x) - u(x-y))(\varphi(x) - \varphi(x-y))}{|M(x-y,y)y|^{n+2s}}\, dy \right) dx \\
&\qquad + \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n \setminus B_1} \frac{(u(x) - u(x-y))(\varphi(x) - \varphi(x-y))}{|M(x-y,y)y|^{n+2s}}\, dy \right) dx \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{E.12}$$

In light of Lemma E.3, we have that

$$\lim_{s \searrow 0} |I_1| = 0. \tag{E.13}$$

We now focus on I_2 and we claim that

$$\lim_{s \searrow 0} \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n \setminus B_1} \frac{u(x-y)\varphi(x)}{|M(x-y,y)y|^{n+2s}}\, dy \right) dx = 0. \tag{E.14}$$

Indeed, by (1.14) and the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n \setminus B_1} \frac{|u(x-y)||\varphi(x)|}{|M(x-y,y)y|^{n+2s}}\, dy \right) dx \leqslant \frac{1}{\beta^{n+2s}} \int_{\mathbb{R}^n} |\varphi(x)| \left(\int_{\mathbb{R}^n \setminus B_1} \frac{|u(x-y)|}{|y|^{n+2s}}\, dy \right) dx \\
&\quad \leqslant \frac{1}{\beta^{n+2s}} \int_{\mathbb{R}^n} |\varphi(x)| \left(\int_{\mathbb{R}^n \setminus B_1(x)} |u(z)|^2 dz \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n \setminus B_1} \frac{dy}{|y|^{2n+4s}} \right)^{\frac{1}{2}} dx \\
&\quad \leqslant \frac{1}{\beta^{n+2s}} \left(\frac{\omega_{n-1}}{n+4s} \right)^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)} \|\varphi\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

From this and (E.1), we deduce that

$$\lim_{s\searrow 0}\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n\setminus B_1}\frac{u(x-y)\varphi(x)}{|M(x-y,y)y|^{n+2s}}\,dy\right)dx$$
$$\leqslant \lim_{s\searrow 0}\frac{c_{n,s}}{2\beta^{n+2s}}\left(\frac{\omega_{n-1}}{n+4s}\right)^{\frac{1}{2}}\|u\|_{L^2(\mathbb{R}^n)}\|\varphi\|_{L^1(\mathbb{R}^n)}=0.$$

In the same way, by using the change of variables $z=x-y$, one can prove that

$$\lim_{s\searrow 0}\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n\setminus B_1}\frac{u(x)\varphi(x-y)}{|M(x-y,y)y|^{n+2s}}\,dy\right)dx=0. \tag{E.15}$$

We now claim that

$$\begin{aligned}&\lim_{s\searrow 0}\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n\setminus B_1}\frac{u(x)\varphi(x)}{|M(x-y,y)y|^{n+2s}}\,dy\right)dx\\&=\lim_{s\searrow 0}\frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n\setminus B_1}\frac{u(x-y)\varphi(x-y)}{|M(x-y,y)y|^{n+2s}}\,dy\right)dx\\&=\frac{1}{2\omega_{n-1}}\int_{\mathbb{R}^n}u(x)\varphi(x)\int_{\mathcal{S}^{n-1}}\frac{d\mathcal{H}^{n-1}_{\psi}}{|M^{\infty}(x,\psi)\,\psi|^n}\,dx.\end{aligned} \tag{E.16}$$

Indeed, by using the change of variables $r:=\rho^{2s}$, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n\setminus B_1}\frac{u(x)\varphi(x)}{|M(x-y,y)y|^{n+2s}}\,dy\right)dx=\int_{\mathbb{R}^n}u(x)\varphi(x)\left(\int_{\mathbb{R}^n\setminus B_1}\frac{dy}{|M(x-y,y)y|^{n+2s}}\right)dx\\&=\int_{\mathbb{R}^n}u(x)\varphi(x)\left(\int_1^{+\infty}\int_{\mathcal{S}^{n-1}}\frac{\rho^{-1-2s}\,d\rho}{|M(x-\rho\psi,\rho\psi)\,\psi|^{n+2s}}\,d\mathcal{H}^{n-1}_{\psi}\right)dx\\&=\frac{1}{2s}\int_{\mathbb{R}^n}u(x)\varphi(x)\left(\int_1^{+\infty}\int_{\mathcal{S}^{n-1}}\frac{dr\,d\mathcal{H}^{n-1}_{\psi}}{r^2|M(x-r^{\frac{1}{2s}}\psi,r^{\frac{1}{2s}}\psi)\,\psi|^{n+2s}}\right)dx.\end{aligned}$$

We notice that, by (1.14) and recalling that $u\in H^{\overline{s}}(\mathbb{R}^n)$, we have that

$$\frac{u(x)\varphi(x)}{r^2|M(x-r^{\frac{1}{2s}}\psi,r^{\frac{1}{2s}}\psi)\,\psi|^{n+2s}}\leqslant\frac{|u(x)\varphi(x)|}{r^2\beta^{n+2s}}\in L^1\Big(\mathbb{R}^n\times(1,+\infty)\times\mathcal{S}^{n-1};dx\,dr\,d\mathcal{H}^{n-1}_{\psi}\Big).$$

Hence, in light of Lemma E.2 (here with $R = 1$), we have

$$\begin{aligned}\lim_{s\searrow 0} \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} u(x)\varphi(x) \left(\int_{\mathbb{R}^n\setminus B_1} \frac{dy}{|M(x-y,y)\, y|^{n+2s}} \right) dx \\ = \frac{1}{2\omega_{n-1}} \int_{\mathbb{R}^n} u(x)\varphi(x) \int_{\mathcal{S}^{n-1}} \frac{d\mathcal{H}^{n-1}_{\psi}}{|M^{\infty}(x,\psi)\, \psi|^n}\, dx.\end{aligned} \tag{E.17}$$

Moreover, the change of variables $z := x - y$ leads to

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n\setminus B_1} \frac{u(x-y)\varphi(x-y)}{|M(x-y,y)y|^{n+2s}}\, dx\, dy = \int_{\mathbb{R}^n} u(z)\varphi(z) \left(\int_{\mathbb{R}^n\setminus B_1} \frac{dy}{|M(z,y)y|^{n+2s}} \right) dz.$$

From this and assumption (1.15), setting $t := -y$ we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n\setminus B_1} \frac{u(x-y)\varphi(x-y)}{|M(x-y,y)y|^{n+2s}}\, dx\, dy = \int_{\mathbb{R}^n} u(z)\varphi(z) \left(\int_{\mathbb{R}^n\setminus B_1} \frac{dy}{|M(z+y,-y)y|^{n+2s}} \right) dz \\ &\quad = \int_{\mathbb{R}^n} u(z)\varphi(z) \left(\int_{\mathbb{R}^n\setminus B_1} \frac{dy}{|M(z+y,-y)(-y)|^{n+2s}} \right) dz \\ &\quad = \int_{\mathbb{R}^n} u(z)\varphi(z) \left(\int_{\mathbb{R}^n\setminus B_1} \frac{dt}{|M(z-t,t)t|^{n+2s}} \right) dz.\end{aligned}$$

Bearing in mind (E.17), we thus obtain the claim in (E.16), as desired.

Accordingly, combining (E.12), (E.13), (E.14), (E.15) and (E.16), we obtain the desired result in (E.10). □

Data availability

No data was used for the research described in the article.

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